Positive Geometry and their relation to scattering amplitudes

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Contents

Chapter 1

Introduction

Scattering amplitudes are one of the most fundamental observables in Quantum Field Theory, describing the probability of particle interactions in high-energy physics. Historically they have been computed using Feynman diagrams, which involves summing over vast numbers of diagrams that often involve nonphysical intermediate states that must cancel in the final answer. Results such as the Parke-Taylor formula, that have such simple final results, hinted at the idea that there must be alternative approach.

The on-shell approach, which focuses on physical observables like scattering amplitudes rather than off-shell quantities, bypasses the complexities associated with Feynman diagrams by leveraging symmetries, recursion relations, and other properties inherent to our physical systems. Through the introduction of twistors and Grassmannian, the connection between scattering amplitudes and geometry has brought us insights, such has hidden dual-conformal symmetries which are completely masked behind the complexities of Feynman diagrams. The development of the Amplituhedron[5] gives further credence to the idea that QFTs may have a more fundamental geometric interpretation.

In this paper I will first give a review on the spinor helicity formalism[8], before giving an introduction to superamplitudes, focusing primarily on $\mathcal{N} = 4$ super Yang-Mills, and some the techniques surrounding them including the super-BCFW shift and twistor variables which provide use with a new space to act as the playground of our theory.

In chapter 3 we will then introduce the notion of positive geometries[10] and canonical forms, and build up an intuition for these spaces using ϕ^3 theory and the associahedron[2] to see how we can indeed extract scattering amplitudes from these principles.

Finally we will discuss the Amplituhedron construction and see how through the principles of positive geometries we can extract amplitudes of all loop-levels in super Yang-Mills.

Chapter 2

Mathematical Background

2.0.1 Notation

Before we start I'd like to make a quick comment on the notation I'm following. This paper follows the work of Henriette Elvang and Yu-tin Huang, as such I am following the conventions used by them. This convention is the opposite of what is typically used when talking about spinor-helicity variables, i.e. the roles of λ and $\tilde{\lambda}$ are reversed as well as square brackets will carry an undotted index, while angle brackets will carry a dotted index.

2.1 Spinor-Helicity Formalism Recap

In order to talk about on-shell scattering amplitudes, we must first develop a language to talk about the external data of our particles that encodes the momenta and helicity of our particles. This shall be the language of spinor-helicity variables.

We can construct a (2x2) matrix for momenta using the Pauli matrices:

$$p_a^{\alpha\dot{\alpha}} \equiv p_a^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \begin{pmatrix} p_a^0 + p_a^3 & p_a^1 - ip_a^4 \\ p_a^1 + ip_a^4 & p_a^0 - p_a^3 \end{pmatrix}$$

If a particle is on-shell it satisfies $p^2 = m^2$, hence for a massless particle $\det(p_a^{\alpha\dot{\alpha}}) = 0$, meaning $p_a^{\alpha\dot{\alpha}}$ has rank 1 and can be represented by:

$$p_a^{\alpha\dot{\alpha}} \equiv \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$$

We refer to $\lambda_a^\alpha, \tilde{\lambda}_a^{\dot{\alpha}}$ as the Spinor-Helicity variables. For real momenta $\tilde{\lambda} = \pm\lambda^*$, for complex momenta $\lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$ are independent. A rescaling $\lambda_a^\alpha \to t^{-1}\lambda_a^\alpha$ $\tilde{\lambda}_a^{\dot{\alpha}} \to t\tilde{\lambda}_a^{\dot{\alpha}}$ leaves p_a invariant and represents an action of the little group, i.e., the biggest subgroup of the Lorentz group that leaves 4-momentum invariant. If we denote the wave function for a particle a with helicity $= \pm\sigma_a$, by $|a\rangle^{h_a}$, then under the action of the little group this transforms by:

$$|a\rangle^{h_a} \to t_a^{-2h_a} |a\rangle^{h_a}$$

From this we can construct the Lorentz invariants:

$$[ab] = \lambda_a^\alpha \lambda_{b\alpha} \qquad \langle ab\rangle = \tilde{\lambda}_{a\dot{\alpha}}\tilde{\lambda}_b^{\dot{\alpha}}$$

Using this we can construct:

$$(p_a + p_b)^\mu (p_a + p_b)^\nu \eta_{\mu\nu} \equiv (p_a + p_b)^2 = \langle ba\rangle[ab]$$

note that we can raise and lower indices with $[p|^a = \epsilon^{ab}|p]_b$ and $|p\rangle^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}$, where ϵ is the anti-symmetric Levita-Civita defined as:

$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} = -\epsilon_{\dot{a}\dot{b}}$$

Resulting in our brackets being anti-symmetric with $\langle ij\rangle = -\langle ji\rangle$ and $[ij] = -[ji]$, with all brackets of the form $\langle i|j]$ vanishing. In amplitudes we typically use the short hand $|i\rangle$ to mean $|p_i\rangle$

If we want to construct the Lorentz-invariant phase space of an on-shell particle with complex momenta, we have 4 degrees of freedom modulo the action of the little group, in reality leaving just 3 degrees of freedom. This gives us the differential form:

$$d^3LIPS_a \equiv \frac{d^2\lambda_a d^2\tilde{\lambda}_a}{Vol(GL(1))}$$

The purpose of introducing this form, is that with its addition we can view on-shell functions as forms on the phase space of the external kinematic data, though for now we won't worry to much about this.

Using this formalism we can view on-shell functions and scattering amplitudes as rational function of our external data which is encapsulated by our $\lambda, \tilde{\lambda}, h_a$. We will see that its possible to build larger amplitudes from smaller ones, and in fact we can completely determine the scattering amplitude for three massless particles from just momentum conservation and little group scaling. Firstly we see that for three massless particle, momentum conservation tells us:

$$p_1^\mu + p_2^\mu + p_3^\mu = \tilde{\lambda}_1\lambda_1 + \tilde{\lambda}_2\lambda_2 + \tilde{\lambda}_2\lambda_2 = 0$$

Contracting on the left and right by various spinors gives us a set of simultaneous equations which can be solved to show $\langle 12\rangle[12] = \langle 31\rangle[31] = \langle 23\rangle[23] = 0$, assuming we allow the momenta to be complex, meaning we let $\lambda_i, \tilde{\lambda}_i$ to be independent of each other. From here the only non-zero solutions come from either all the λs being proportional or all the $\tilde{\lambda}$s. In effect meaning either all the angle brackets being zero or all the square brackets being zero. So as A_3 must be a function of only 1 kind of bracket, a consequence is there is no little group invariant of only one kind of bracket and so the scaling property of of amplitudes dictates that either:

$$A_3[h_1, h_2, h_3] \propto [12]^{h_3-h_1-h_2}[23]^{h_1-h_2-h_3}[31]^{h_2-h_3-h_1}$$

$$A_3[h_1, h_2, h_3] \propto \langle 12\rangle^{h_1+h_2-h_3}\langle 23\rangle^{h_2+h_3-h_1}\langle 31\rangle^{h_3+h_1-h_2}$$

In reality there is also a coupling constant, but we shall be ignoring these throughout this paper. A final constraint coming from continuity conditions, requires for the top 3 point amplitude that $(h_1+h_2+h_3) \leq 0$ and for the bottom one $(h_1+h_2+h_3) \geq 0$. So for a 3-point amplitude with 1 negative helicity gluon and 2 positive helicity gluons this becomes:

$$A_3[-++] = \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

2.2 BCFW Shift

Now we'll give a quick overview of one of the methods used to build up higher point amplitudes, before returning to this idea later on with super-BCFW shifts. The general idea is we deform some of the momenta in such a way that a given subset of the n momenta goes on-shell and generates an on-shell internal propagator and as the on-shell momenta are null, locality allows us to split the amplitude up into the product of two smaller amplitudes.

For a general shift we equip each propagator with a vector r_i^μ such that $\hat{p}_i^\mu = p_i^\mu + z r_i^\mu$. Demanding that these vectors satisfy the following properties:

$$\sum_{i=1}^{n} r_i^\mu = 0$$

$$r_i^\mu \cdot r_j^\mu = 0, \qquad \forall i,j \in \{1,...,n\}$$
$$p_i^\mu \cdot r_i^\mu = 0 \qquad \forall i \in \{1,...,n\}$$

ensures that total momentum conservation is preserved by the deformed propagators and the shifted momenta are still all on-shell. Note that while we have defined an r_i for each p_i, they can just be zero, such as in the BCFW shift where in general the majority of them will be.

For a BCFW shift we chose two particles i, j and apply the following deformation known as a $[i, j\rangle$-shift:

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle$$

All other propagators we leave untouched. We can check momentum is still conserved:

$$\sum_{a=1}^{n} |a]\langle a| = |\hat{i}]\langle\hat{i}| + |\hat{j}]\langle\hat{j}| + \sum_{a=1}^{n/\{i,j\}} |a]\langle a|$$

$$= (|i] + z|j])\langle i| + |j](\langle j| - z\langle i|) + \sum_{a=1}^{n/\{i,j\}} |a]\langle a| = |i]\langle i| + |j]\langle j| + \sum_{a=1}^{n/\{i,j\}} |a]\langle a| = 0$$

Given that $|\hat{i}\rangle \to |i\rangle$, and $\hat{p}_i^2$ contains a term like $\langle\hat{i}\hat{i}\rangle$ it is trivially zero, similarly for $|\hat{j}] \to |j]$, so the deformed momenta are still on-shell, so we can now treat the amplitude as a function of the deformed momenta. For a tree-level amplitude, the pole structure is limited to only having simple poles, and these simple poles describe its complete analytic structure.

Now if we take a set of subset of the momenta $\{p_i\}_{i\in I}$ with $2 \le |I| \le (n-2)$, then defining $P_I^\mu = \sum_{i\in I} p_i^\mu$, we find that:

$$\hat{P}_I^2 = \left(\sum_{i\in I} \hat{p}_i^\mu\right)^2 = P_I^2 + 2zP_I \cdot R_I$$

where R_I is the sum of the r_i. By then defining $z_I = -\frac{P_I^2}{2P_I \cdot R_I}$, we can write this as:

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I)$$

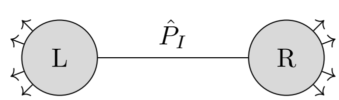
We will start now to think of the shifted amplitude $\hat{A}_n$ as a function of z. We can always return to our unshifted amplitude by just setting $z = 0$. Now as we move around the complex plane we will hit certain z where $\hat{P}_I$ goes on shell, specifically at $z = z_I$. Note that for a tree level diagram we can only ever have 1 power of the $\frac{1}{\hat{P}_I^2}$ propagator. So at the points $z = z_I$ we expect to get an internal $\frac{1}{\hat{P}_I^2}$ propagator, giving us a simple pole at that point. Now if we consider the function $\frac{\hat{A}_n(z)}{z}$, the pole at $z = 0$ will have the reside $\hat{A}_n(0) = A_n$. So by drawing a contour around the pole at the origin and the poles at z_I, we find by Cauchy's theorem that:

$$A_n = -\sum_{z_I} \mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n$$

B_n is the residue at the pole as $z \to \infty$, for a shift to be considered a valid shift we require that $B_n = 0$. We can think of this residue at infinity as being the same condition that $\oint_C dz \frac{\hat{A}_n(z)}{z} = 0$. So finally as this internal propagator is on-shell and light-like, we can exploit locality to factorise the shift amplitude into two parts:

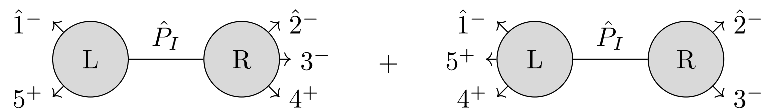
$$\mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = -\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I)$$

By construction $\hat{A}_L(z_I), \hat{A}_R(z_I)$ must be amplitudes of a smaller number of particles. Pictorially this looks like:



We've included a sum over z_I which for any given I, seems like should just be a single term. The sum should really be thought of as a sum over all possible factorisation channels I. For example if we did a [1,2⟩ shift on the amplitude

$A_5[1^-, 2^-, 3^-, 4^+, 5^+]$, we would get two possible factorisation channels:



Note that we should always have the shifted momenta on opposite side of the factorisation channel, as otherwise there would have a been an existing internal propagator, without shifting the momenta. The helicity of the internal propagator is fixed in both diagrams In the first it must be negative on the left and positive on the right as you can't have a 3 negative 4 particle amplitude and in the second diagram 3 negative helicity 3 particle amplitudes also disappear. So the internal particle must have the same orientation as the first.

The BCFW shift is one example of a recursion relation on amplitudes, we can repeat this process of dividing the amplitudes into smaller and smaller pieces until you reach amplitudes you can determine from other principles. One example of this is an inductive proof that can be done for the Parke-Taylor formula, to yield the famous result:

$$A_n[1^+, ..., i^-, ..., j^-, ..., n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle ... \langle n1 \rangle}$$

2.3 Supermultiplets

In theories with supersymmetry, we can combine particles of different helicities into groups of supermultiplets. These particles are then related to one another by the supersymmetry generators Q_I^α and $\tilde{Q}_I^{\dot{\alpha}}$, with SU($\mathcal{N}$) indices I= 1,...,$\mathcal{N}$, defined so that:

$$\{Q_I^\alpha, \tilde{Q}_J^{\dot{\alpha}}\} = 2\delta_{IJ} P^{\alpha\dot{\alpha}}$$

Where $P^{\alpha\dot{\alpha}}$ is our usual momentum operator.

By then acting on external states with out generators it allows us to move through the spectrum of our supermultiplet in the following way:

$$Q_I^\alpha |a\rangle^{h_a} = \tilde{\lambda}_a^\alpha |a\rangle_I^{h_a - 1/2} \text{ and } \tilde{Q}_I^{\dot{\alpha}} |a\rangle^{h_a} = \lambda_a^{\dot{\alpha}} |a\rangle_I^{h_a + 1/2}$$

By Introducing a conjugate spinor $\tilde{\lambda}_{\overline{a}}$ for $\tilde{\lambda}_a$ such that $[\overline{a}a] = 1$, we can simplify our notation by defining $Q_I \equiv \varepsilon_{\dot{\alpha}\dot{\beta}} Q_I^{\dot{\alpha}} \tilde{\lambda}_{\overline{a}}^{\dot{\beta}}$. Now note that we can only act with at most 4 Q_I to a given state. We can also define the highest-helicity state as the state where $\tilde{Q}_I^\alpha |a\rangle^{+\sigma_a} = 0$. Then we can generate the entire spectrum of a given supermultiplet as:

$$|a\rangle^{+\sigma_a} \equiv |a\rangle^{\sigma_a}$$

$$Q_I |a\rangle^{+\sigma_a} \equiv |a\rangle_I^{\sigma_a - 1/2}$$

$$Q_I Q_J |a\rangle^{+\sigma_a} \equiv |a\rangle_{IJ}^{\sigma_a - 1}$$

$$\vdots$$

$$Q_I ... Q_k |a\rangle^{+\sigma_a} \equiv |a\rangle^{\sigma_a - \mathcal{N}/2}_{I...K}$$

Typically we then act on the lowest helicity state with $\tilde{Q}$ operator in a similar manner, to generate a CPT-conjugate supermultiplet, which is necessary for our theory to respect the CPT symmetries. However in $\mathcal{N} = 4$ Super Yang-Mills, the supermultiplet generated by acting as above, is CPT self-conjugate meaning it respects the CPT symmetries by itself.

In particular in $\mathcal{N} = 4$ super Yang-Mills, we have $\sigma_a = 1$, and we can check that continuously applying Q_I to a spin 1 state, will produce 1 positive helicity gluon, 4 gluinos with (h=1/2), 6 scalars, 4 anti-gluinos and the gluon with h=-1. This shall be the supersymmetric theory we focus on from here.

We can equally define our theory with the use of annihilation and creation operators, satisfying the following commutation relations:

$$\begin{aligned}
[\tilde{Q}_A, a(i)] &= 0, & [Q^A, a(i)] &= [i|\ a^A(i), \\
[\tilde{Q}_A, a^B(i)] &= |i\rangle\ \delta^B_A\ a(i), & [Q^A, a^B(i)] &= [i|\ a^{AB}(i), \\
[\tilde{Q}_A, a^{BC}(i)] &= |i\rangle\ 2!\ \delta^{[B}_A a^{C]}(i), & [Q^A, a^{BC}(i)] &= [i|\ a^{ABC}(i), \\
[\tilde{Q}_A, a^{BCD}(i)] &= |i\rangle\ 3!\ \delta^{[B}_A a^{CD]}(i), & [Q^A, a^{BCD}(i)] &= [i|\ a^{ABCD}(i), \\
[\tilde{Q}_A, a^{BCDE}(i)] &= |i\rangle\ 4!\ \delta^{[B}_A a^{CDE]}(i), & [Q^A, a^{1234}(i)] &= 0.
\end{aligned}$$

We can deduce the supersymmetric ward identities from the following:

$$0 = \langle [\tilde{Q}_a, a_1^{\cdots}, ..., a_n^{\cdots}] \rangle$$

$$0 = \langle [Q^a, a_1^{\cdots}, ..., a_n^{\cdots}] \rangle$$

The above commutation relations together with the fact that that supercharges annihilate the vacuum, give us relations between different amplitudes.

One particular example we can show is

$$\langle 0 | [\tilde{Q}_A, a_1^B a_2 ... a_n | 0\rangle = 0 \implies \delta^B_A |1\rangle A[g^+, ..., g^+] = 0$$

And so we can see that the see how all positive gluon amplitudes vanish. Similarly we can show that:

$$A_n[g^+ ... g_i^- ... g_j^- ... g^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle^4} A_n[g^- g^- g^+ ... g^+]$$

Reproducing the Parke-Taylor formula.

We now introduce the on-shell superspace, with the addition of the Grassmann variable η_A labelled by our SU(4) index A=1,...,4, which allows us to package all 16 states into an on-shell superfield:

$$\Omega = |a\rangle^{+1} + \eta_I |a\rangle^{+1/2A} + \frac{1}{2!}\eta_I\eta_J |a\rangle^{0IJ} + \frac{1}{3!}\eta_I\eta_J\eta_K |a\rangle^{-1/2IJK} + \frac{1}{4!}\eta_I\eta_J\eta_K\eta_L |a\rangle^{-1IJKL}$$

We can extract the relevant states using Grassmann differential operators. i.e if we want to extract a certain scalar defined by $\eta_1\eta_2|a_i\rangle^{12}$ we can use the operator $\frac{\partial^2}{\partial\eta_{i1}\partial\eta_{i2}}$

In the on-shell superspace the supercharges are defined as follows:

$$q^{A\alpha} \equiv [p|^{\alpha}\frac{\partial}{\partial\eta_A} \qquad\qquad q_A^{\dagger\dot{\alpha}} \equiv |p\rangle^{\dot{a}}\eta_A$$

where $|p], |p\rangle$ are the spinors associated with the null momentum p of a particle a.

2.4 Superamplitudes

We can think of our previously defined Ω_i as wave functions for the i'th external particle of a superamplitude $A_n(\Omega_1...\Omega_n)$. Each particle will come with a momentum p_i and information about the kind of particle it is encode by a set of $\tilde{\eta}_i$. And in fact when we expand A_n we see that due to the global SU(4) symmetry the amplitude will vanish unless we can gather our external states into a SU(4) singlet. This requires our superamplitude to be polynomials of degree 4(k+2) in the η_As

We can then extract any amplitude from superamplitude by applying the appropriate Grassmann differential operators to the superamplitude. And so we can think of these superamplitudes containing all n-particle amplitudes for the particles in our theory, we just have to select the term we want that will give us the specific amplitude we want. In fact the super amplitude can be written in terms of all N^kMHV amplitudes as follows:

$$A_n = A_n^{MHV} + A_n^{NMHV} + ... + A_n^{\text{anti-}MHV}$$

Where our A_n^{MHV} are η_A of degree 8, the NMHV amplitudes are degree 12 and so on.

As an example say we'd like the amplitude of 2 negative helicity gluons and (n-2) positive helicity gluons, then we extract this by:

$$A_n(1^+, ..., i^-, ..., j^-, ..., n^+) = \prod_{A=1}^{4}\frac{\partial}{\partial\eta_{iA}}\prod_{B=1}^{4}\frac{\partial}{\partial\eta_{jB}}A_n(\Omega_1, ..., \Omega_n)$$

Here we've chosen to extract this using partial derivatives, but the same result could also be achieved with Grassmann integrals. To give a more concrete example, the MHV superamplitude can be written as follows:

$$A_n[g^-, g^-, g^+, ..., g^+](\eta_1)^4(\eta_2)^4 + A_n[g^-, \lambda^{123}, \lambda^4, ...](\eta_1)^4(\eta_{21}\eta_{22}\eta_{23})\eta_{34} + ...$$

There is only one term that contains any given combination of 8 specific $\tilde{\eta}$s so when acting with 8 specific Grassmann derivatives, all other terms drop out. In the above example if we act with $\frac{\partial}{\partial\eta_{34}}$ this will kill the first term. Note there is a slight abuse of notation here, $(\eta_i)^4 = (\eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4})$

So,

$$A_n[g^-, \lambda^{123}, \lambda^4, ...] = \prod_{A=1}^{4} \frac{\partial}{\partial \eta_{1A}} \prod_{B=1}^{3} \frac{\partial}{\partial \eta_{2B}} \frac{\partial}{\partial \eta_{34}} A_n^{MHV}$$

When computing a ward identity we commute our supersymmetry generators through the n-point function adding a new term anytime we have a non zero commutator. In the language of the on-shell superspace the action of these generators can be written more succinctly as the sum of the action of the super charges on the on-shell superspace:

$$Q^A = \sum_{i=1}^{n} [i|^{\alpha} \frac{\partial}{\partial \eta_{iA}} \qquad\qquad \tilde{Q}_A = \sum_{i=1}^{n} |i\rangle^{\dot{\alpha}} \eta_A$$

From which we can simplify how we express our ward identifies to the following:

$$Q^A A_n = 0 \qquad\qquad \tilde{Q}_A A_n = 0$$

If we return to our definition of the anti-commutator and look at its action on the super amplitude $\{Q^A, \tilde{Q}_B\} A_n = Q^A \tilde{Q}_B A_n + \tilde{Q}_B Q^A A_n = 0 = 2\delta^A_B P A_n$. And so we can see that the statement that the supersymmetry generators annihilate the amplitude is the same as the state that total momentum is conserved. The supersymmetric ward identities can be satisfied if we construct all superamplitudes with the following Grassmann δ-function:

$$\delta^{(8)}(\tilde{Q}) = \frac{1}{2^4} \prod_{A=1}^{4} \tilde{Q}_{A\dot{\alpha}} \tilde{Q}^{\dot{\alpha}}_{A} = \frac{1}{2^4} \prod_{A=1}^{4} \sum_{i,j}^{n} \langle ij \rangle \eta_{iA} \eta_{jA}$$

This is thought as enforcing supermomentum conservation. Note that this is a polynomial of degree 8 in the ηs with the property $\tilde{Q}\delta^{(8)}(\tilde{Q}) = 0$. We can also use momentum conservation to show $Q\delta^{(8)}(\tilde{Q}) = 0$ in the following way:

$$Q^B \delta^{(8)}(\tilde{Q}) = \sum_{k=1}^{n} [k| \frac{\partial}{\partial \eta_{kB}} \prod_{A=1}^{4} \sum_{i,j}^{n} \langle ij \rangle \eta_{iA} \eta_{jA}$$

If we consider what happens when k equals either i or j, then we will always be able to pull a factor of $\sum_{i=1}^{n} |i\rangle [j|$ out, which is just the total momentum of our system, so must be zero. For $k \neq i, j$ the the Grassmann derivative will simply remove all such terms, so in the end we see that the action of Q^B will indeed annihilate the delta function.

If we combine this δ-function, with the construction:

$$A_n^{N^K MHV} = \delta^{(8)}(\tilde{Q}) P_{4k}$$

Where P_{4k} is a polynomial of degree $4k$ in η, and has the property that $Q^A P_{4k} = 0$, then all the constraints placed by our ward identities are satisfied.
For the $k = 0$ MHV case, P_{4k} must be solely a rational function of the particle's

momenta as the ηs are completely fixed by supermomentum conservation, and in fact all MHV amplitudes can be written as:

$$A_n^{MHV}[12...n] = \frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle ... \langle n1\rangle}$$

Specific amplitudes can then be extracted as we said earlier from applying appropriate derivatives to this formula.

2.5 Super BCFW

Now while we can determine relations between different amplitudes using ward identities, and fully fix MHV amplitudes, in order to determine N^kMHV amplitudes for $k \geq 1$ we need new machinery. One such method is know as a super-BCFW shift.

While a normal BCFW shift preserves total momentum conservation and keeps the external momenta on-shell, it doesn't preserve supermomentum conservation, so in addition to the normal shifts we must also include a shift in one of the Grassmann variables. For a $[i, j\rangle$ super-shift we then have the following transformations:

$$|\hat{i}] = |i] + z|j] \qquad |\hat{j}\rangle = |j\rangle - z|j\rangle \qquad \hat{\eta}_{iA} = \eta_{iA} + z\eta_{jA}$$

After this, the process for splitting the amplitude up is very similar, the only extra consideration is that for, regular BCFW shifts we had to sum over the possible factorisation channels for the internal propagator, while for super-BCFW shifts we must sum over all 16 possible states that can be exchanged over the internal line. The possible particle exchange is dependent on the external particles in question as each sub-amplitude must still have 4(2+k) powers of η. The simplest kind of exchange is a gluon exchange, but we must still sum over the two possible options for the helicity of the particle. For gluino exchange we 8 possible configurations of how to split up the ηs of the internal propagator and 12 for a scalar particle exchange. We can write all these possible sums concisely as:

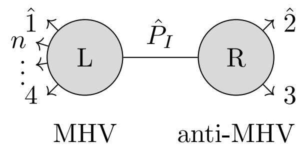
$$\left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{\hat{P}A}}\right) \left[\hat{A}_L \frac{1}{P^2} \hat{A}_R\right]\Bigg|_{\eta_{\hat{P}A}=0}$$

As both $\hat{A}_L, \hat{A}_R$ are dependent on the $\eta_{\hat{P}A}$s, by the product we will get the summation of every possible combination of how we could apply the derivatives to the sub-amplitudes. This is equivalent to taking integrals of the same quantity and integrating by parts over each $\eta_{\hat{P}A}$,

$$\int d^4\eta_{\hat{P}} \hat{A}_L \frac{1}{P^2} \hat{A}_R$$

When we're trying to determine what diagrams to consider, we really just have to keep track of how many ηs are in each diagram and the total number of

ηs in the whole amplitude. For example when we look at an MHV n particle amplitude, we know the final amplitude can only have 8 ηs and the factorisation channel remove 4 powers of η. So combined, the two sub-amplitudes must have 12 ηs. Given that every MHV amplitude has at least 8, this limits us to only one possible diagram including the anti-MHV amplitude which just has just 4 powers of η. If we specify we're using an $[1, 2\rangle$ super-shift, then the only diagram we have to consider is:



Corresponding to the factorisation:

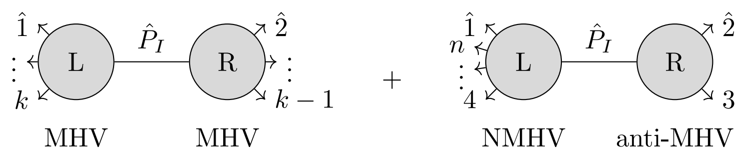
$$\int d^4\eta_{\hat{P}} \frac{\delta^{(8)}(\sum_{i\in L} |\hat{i}\rangle\hat{\eta}_i)}{\langle 1\hat{P}\rangle\langle\hat{P}4\rangle...\langle n1\rangle} \frac{1}{P^2} \frac{\delta^{(4)}([\hat{P}2]\eta_3 + [23]\eta_{\hat{P}} + [3\hat{P}]\eta_2)}{[23][3\hat{P}][\hat{P}2]}$$

From here we can use the δ-function on the RHS to fix $\eta_{\hat{P}} = \frac{-([\hat{P}2]\eta_3 + [3\hat{P}]\eta_2)}{[23]}$, and by letting $P = P_{23} = (p_2 + p_3)$ we'll find that the δ-function on the LHS in the end gives us back $\delta^{(8)}(\tilde{Q})$ as we'll remove any $\eta_{\hat{P}}$ dependence from it, yielding the result:

$$\frac{\delta^{(8)}(\tilde{Q})}{\langle 1\hat{P}\rangle\langle\hat{P}4\rangle...\langle n1\rangle} \frac{1}{P^2} \frac{[23]^4}{[3\hat{P}][\hat{P}2][23]}$$

Expressing P^2 as $\langle 23\rangle[23]$ and using a couple identities $\langle\hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] = -\langle 12\rangle[23]$ and $\langle\hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] = -\langle 34\rangle[23]$ will from here return our original formula for the MHV amplitude.

For The NMHV case, we can take the same approach of counting the ηs to decide what diagrams to consider. Given we need 12 of them by the end we find the following 2 diagrams:



Note that we can only have NMHV amplitudes for at least 5 external particles, so we can only have the the diagram on the RHS for $n \geq 6$ as the left amplitude needs at least 5 external legs. To get the amplitude we then have to sum the LHS diagram from $k = 5$ to n in order to consider all factorisations into two MHV amplitudes. I'm simply going to state the final result here for evaluating the LHS diagram, but for a full calculation you can see [8].

$$A_n^{MHV} \times \frac{\langle 23\rangle\langle k-1, k\rangle\ \delta^{(4)}\left(\langle 1|y_{1k}\cdot y_{k3}|\theta_{31}\rangle + \langle 1|y_{13}\cdot y_{3k}|\theta_{k1}\rangle\right)}{y_{3k}^2\langle 1|y_{13}\cdot y_{3k}|k\rangle\langle 1|y_{13}\cdot y_{3k}|k-1\rangle\langle 1|y_{1k}\cdot y_{k3}|3\rangle\langle 1|y_{1k}\cdot y_{k3}|2\rangle}$$

Where $y_{ij} \equiv p_i + p_{i+1} + ... + p_{j-1}$ and $|\theta_{ij,A}\rangle \equiv \sum_{r=i}^{j-1} |r\rangle \eta_{rA}$. These kind of results appear over and over leading us to define what we call R-invariant, defined as:

$$R_{1jk} = \frac{\langle j-1, j\rangle\langle k-1, k\rangle \; \delta^{(4)}\left(\Xi_{1jk}\right)}{y_{jk}^2 \langle 1|y_{1j} \cdot y_{jk}|k\rangle\langle 1|y_{1j} \cdot y_{jk}|k-1\rangle\langle 1|y_{1k} \cdot y_{kj}|j\rangle\langle 1|y_{1k} \cdot y_{kj}|j-1\rangle}$$

with Ξ defined as follows:

$$\Xi_{1jk,A} = \langle 1|y_{1k} \cdot y_{kj}|\theta_{j1,A}\rangle + \langle 1|y_{1j} \cdot y_{jk}|\theta_{k1,A}\rangle$$

Then the total contribution of the first diagram to the NMHV amplitude is then the sum over all k that allow us to decompose our amplitude in this way:

$$A_n^{MHV} \times \sum_{k=5}^{n} R_{13k}$$

For $n \geq 6$ the second diagram can also be written in terms of our R-invariants:

$$A_n^{MHV} \times \sum_{j=4}^{n-2} \sum_{k=j+2}^{n} R_{1jk}$$

So finally the final full NMHV super-amplitude is the sum of both diagrams giving:

$$A_n^{MHV} \times \sum_{j=3}^{n-2} \sum_{k=j+2}^{n} R_{1jk}$$

The idea of R-invariants extends to $N^k MHV$ amplitudes with $k \geq 2$. Using what are called generalised R-invariants you can construct all tree-level amplitudes for $\mathcal{N} = 4$ super Yang-Mills. This was accomplished in 2009 by J. M. Drummond and J. M. Henn[7].

2.6 Projective Spaces

We will see later on the need for projective spaces both in how we wish to describe our on-shell data in twistor space, but also in the construction of positive geometries in chapter 3. As such I want to give a quick summary of what exactly projective spaces are.

A projective space $\mathbb{P}^n$ is associated with the space $\mathbb{R}^n$ in the following way: Given any line passing through the origin in $\mathbb{R}^n$ we can write this line as a scalar multiple of some point along this line. We associate this line with a single point in our projective space. As we can choose any point from this line to represent it in projective space, points in projective space do not have a unique set of co-ordinates defining them. Instead they are defined up to a rescaling by a constant. Suppose we have a point $(p_1, p_2, p_3) \in \mathbb{P}^3$ this rescaling property to write this instead as the point $(1, p_4, p_5)$, the only exception to this would be points with first co-ordinate equal to 0.

A consequence of this is when we define a line, we typically define it $ax + by + c = 0$, but in projective space we can define it simply as:

$$W^I \cdot Z_I = 0, \qquad W^I = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

Where the Z_I are points in our projective space. Any point satisfying this equation we can rescale such that $Z_I = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$, giving us back our original equation. This holds for inequalities involving lines as well will will prove to be useful when defining boundaries of spaces later on.

2.7 Twistor-Space

One such projective space which will be important to us later is Twistor space. First introduced by Penrose in 1967, it has a close analogue called momentum-twistor space which will become the manner which we describe our external data later on.

To form twistor-space, we first consider the linear equation:

$$\mu_{\dot{\alpha}} = x_{\dot{\alpha}\alpha}\lambda^{\alpha}, \qquad \alpha, \dot{\alpha} = 1, 2$$

where $x_{\dot{\alpha}\alpha}$ is a space time co-ordinate, and $\mu_{\dot{\alpha}}$, λ^{α} are given just given data. This equation defines a null-ray in our spacetime. To see this consider two points satisfying this equation, and see they satisfy:

$$(x^{(1)} - x^{(2)})_{\dot{\alpha}\alpha}\lambda^{\alpha} = 0$$

so we see the matrix $(x^{(1)} - x^{(2)})_{\dot{\alpha}\alpha}$ has a zero eigenvector telling us the determinant of $(x^{(1)} - x^{(2)})_{\dot{\alpha}\alpha}$ is 0, and therefore $(x^{(1)} - x^{(2)})^2 = 0$ implying this is indeed a null ray.

Now we can define the 4-vector, $z = (\mu_{\dot{\alpha}}\ , \lambda^{\alpha})$, associating this point to the null ray defined by those coefficients. Notice that an arbitrary re-scaling of z corresponds to the same null ray in our spacetime. Hence this space of z is a projective $\mathbb{P}^3$ space, which we call twistor-space.

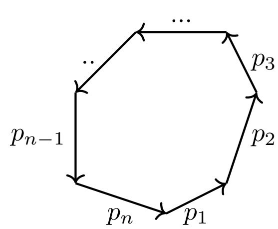
On the other hand suppose we have two points in twistor space that are associated with null rays in space time and we want to know if these null rays intersect. This is the same as asking if there exists an $x_{\dot{\alpha}\alpha}$, such that it satisfies both $\mu^{(1)}_{\dot{\alpha}} = x_{\dot{\alpha}\alpha}\lambda^{(1)\alpha}$ and $\mu^{(2)}_{\dot{\alpha}} = x_{\dot{\alpha}\alpha}\lambda^{(2)\alpha}$ simultaneously. What we find is that $x_{\dot{\alpha}\alpha}$ is given by:

$$x_{\dot{\alpha}\alpha} = \frac{\mu^1_{\dot{\alpha}}\lambda^2_{\alpha} - \mu^2_{\dot{\alpha}}\lambda^1_{\alpha}}{\langle\lambda_1\lambda_2\rangle}$$

Note that in our equations involving $\mu^{(1)}, \mu^{(2)}, \lambda^{(1)}, \lambda^{(2)}$, we have 4 unknowns, and 4 linear equations, so we can always find such a solution.
As we said before though, up to rescaling the λ, μ define the same null ray. So we can take any linear combination of the equations for $\mu^{(1)}, \mu^{(2)}, \lambda^{(1)}, \lambda^{(2)}$ and it will give us the same point $x_{\dot{\alpha}\alpha}$. Geometrically all linear combinations of $z^{(1)}$ and $z^{(2)}$ correspond to a plane in the 4 dimensional space of zs that passes through the origin, however when we look at this projectively this plane becomes a line in twistor space. And so we see that the line $(z_1 z_2)$ has the relation to dual-momentum space:

$$(z_1 z_2) \qquad \longleftrightarrow \qquad x_{\dot{\alpha}\alpha} = \frac{\mu^1_{\dot{\alpha}}\lambda^2_{\alpha} - \mu^2_{\dot{\alpha}}\lambda^1_{\alpha}}{\langle \lambda_1 \lambda_2 \rangle}$$

To motivate why we might want to use twistors to describe our scattering data, consider that for any scattering process we always have that momentum is conserved, i.e. that $\sum_{a=1}^{n} p_a^{\mu} = 0$. So lets draw these momenta out as vectors, top to tail so that they form a closed polygon.



For massless particles these momenta are all null, but the points we use to define our momentum twistor space are actually the points y_a where $p_a^{\dot{\alpha}\alpha} = y_a^{\dot{\alpha}\alpha} - y_{a+1}^{\dot{\alpha}\alpha}$, which in a sense are space-time co-ordinates, but on momentum space. These y form what is known as dual-momentum space. The fact though that the lines connecting our various ys are all null means can represent them in momentum twistor space.

Using the equation $p_a^{\dot{\alpha}\alpha} = y_a^{\dot{\alpha}\alpha} - y_{a+1}^{\dot{\alpha}\alpha}$, and contracting on both sides by $\langle i|_{\dot{\alpha}}$, we can see that we get the relation $\langle i|_{\dot{\alpha}} y_i^{\dot{\alpha}\alpha} = \langle i|_{\dot{\alpha}} y_{i+1}^{\dot{\alpha}\alpha}$. This will form the basis of what are called our incidence relations for our dual-momentum co-ordinates. Explicitly our incidence relations are:

$$[\mu_i|^{\alpha} = \langle i|_{\dot{\alpha}} y_i^{\dot{\alpha}\alpha} = \langle i|_{\dot{\alpha}} y_{i+1}^{\dot{\alpha}\alpha}$$

which are the momentum-twistor equivalent of the relation for defining space-time twistors I described earlier. And so we define our momentum twistors $Z_i^I = (|i\rangle, [\mu_i)$ where I is an SU(2,2) index. Using the same analysis as for momentum-space twistors we can derive the relation for $y^{\dot{\alpha}\alpha}$ in terms of the twistor variables, which gives us:

$$y_i^{\dot{\alpha}\alpha} = \frac{|i\rangle^{\dot{\alpha}}[\mu_{i-1}|^{\alpha} - |i-1\rangle^{\dot{\alpha}}[\mu_i|^{\alpha}}{\langle i, i-1\rangle}$$

Given the relation between p_i to $|i\rangle$ and y_i, it stands to reason we can find a connection between $[i|$ and our momentum twistor co-ordinates, and indeed using the Schouten identity we can find the relation:

$$[i|^\alpha = \frac{\langle i-1,i\rangle[\mu_{i+1}|^\alpha + \langle i,i+1\rangle[\mu_{i-1}|^\alpha + \langle i+1,i-1\rangle[\mu_i|^\alpha}{\langle i-1,i\rangle\langle i,i+1\rangle}$$

Another observation we can make is that two points, lets say $y_1^{\dot\alpha\alpha}, y_2^{\dot\alpha\alpha}$, if we consider how inversions act on $(y_1^{\dot\alpha\alpha} - y_2^{\dot\alpha\alpha})^2$ we see this goes to $\frac{(y_1^{\dot\alpha\alpha}-y_2^{\dot\alpha\alpha})^2}{y_1^2 y_2^2}$. Meaning if y_1 and y_2 are null separated, under inversions they are also null separated. So this leads us to examine how conformal transformations act on our twistor space.

Under a translation $y_i^{\dot\alpha\alpha} \to y_i^{\dot\alpha\alpha} + \Delta^{\dot\alpha\alpha}$, our incidence relation transforms to $([\mu_i|^\alpha - \langle i|_{\dot\alpha}\Delta^{\dot\alpha\alpha}) = \langle i|_{\dot\alpha} y_i^{\dot\alpha\alpha}$, so we see under a translation our momentum twistors transform like $[\mu_i|^\alpha \to [\mu_i|^\alpha - \langle i|_{\dot\alpha}\Delta^{\dot\alpha\alpha}$. The SU(2) x SU(2) Lorentz symmetry carries on to our momentum twistors and finally under the transformation $y^{\dot\alpha\alpha} \to \frac{y^{\dot\alpha\alpha}}{y^2}$ our incidence relation becomes $[\mu_i|^\alpha y_{i\dot\alpha\alpha} = \langle i|_{\dot\alpha}$ and so we see an inversion acts on our momentum twistors by $|i\rangle \longleftrightarrow [\mu_i|$. The key point being that these are all linear transformations on our momentum twistor co-ordinates, and so we can combine them into the symmetry group SL(4), and we uncover that our dual-momentum coordinates have a conformal symmetry that in momentum-twistor space has a simple representation.

Now given we're interested in super-amplitudes, the final extension we can make to our momentum-twistors is to equip them with a Grassmann parameter and raise them to momentum super-twistors defined as:

$$\mathcal{Z}_i^{\mathbf{A}} = (|i\rangle^{\dot\alpha}, [\mu_i^\alpha|, \chi_{iA}), \qquad \mathbf{A} = (\dot\alpha, \alpha, A)$$

Now that we have seen the correspondence between lines and points in twistor space, to null rays and points in momentum space, we can now return to our picture of a closed polygon in momentum space and imagine how this polygon will appear in twistor space. We already have that the momenta p_a themselves will all be associated with specific points in twistor space, but on top of that, the momenta p_a is associated with two ys one at either end. So in twistor space it is the intersection of two lines each of which will have another point associated with momenta p_{a-1}, p_{a+1}. Continuing around in this way, we see that our polygon in momentum space becomes another closed polygon in twistor space with vertices at the Z_a associated with our different momenta.

The advantage of expressing this now in twistor space is twistor space is projective. As points can be rescaled arbitrarily, there is no notion of distance and therefore no metric. We only have to do geometry.

2.8 Super-Amplitudes in Twistor Space

Now that we have a new way to express our external data $(|i\rangle, |i])$ we can examine how our superamplitudes now look in this new representation.

We can use an ϵ to contract the indices on a set of Z_i in order to construct something that is Lorentz invariant. We use the following 4-bracket to denote this invariant:

$$\langle ijkl\rangle = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$$

Two particular 4 brackets we can construct in this way are $\langle k, j-1, j, r\rangle$ and $\langle j-1, j, k-1, k\rangle$

Expanding these two 4-brackets we find the following two relations:

$$\langle k, j-1, j, r\rangle = \langle j-1, j\rangle\langle k|y_{kj}y_{jr}|r\rangle, \qquad \langle j-1, j, k-1, k\rangle = \langle j-1, j\rangle\langle k-1, k\rangle y_{jk}^2$$

Comparing these two expressions with our R-invariants from earlier, we can see that these are exactly the kind of expressions that appeared from our BCFW-super shift. And in fact we can replace the majority of the elements making up our R-invariants:

$$R_{njk} = \frac{\langle j-1, j\rangle^4 \langle k-1, k\rangle^4 \delta^{(4)}\left(\Xi_{njk}\right)}{\langle n, j-1, j, k-1\rangle\langle j-1, j, k-1, k\rangle ... \langle k, n, j-1, j\rangle}$$

To get the numerator in terms of our 4-brackets we choose to define the Grassmann parameter of our momentum super-twistors as $\chi_i^A = \langle i\theta_{iA}\rangle$ where θ_{iA} is the same that we defined earlier during our original discussion on R-invariants. This yields the identity:

$$\langle k|y_{kr}y-rj|\theta_{jA}\rangle = -\frac{\langle k, r-1, r, j-1\rangle\chi_{jA} - \langle k, r-1, r, j\rangle\chi_{j-1,A}}{\langle r-1, r\rangle\langle j-1, j\rangle}$$

Which given how we defined Ξ originally gives us exactly the factors we need to cancel the two angled brackets in the numerator of our expression for R_{njk}:

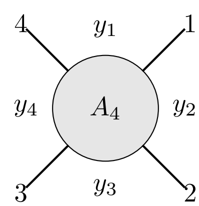
$$R_{njk} = \frac{\delta^{(4)}\left(\langle j-1, j, k-1, k\rangle\chi_{nA} + ... + \langle n, j-1, j, k-1\rangle\chi_{kA}\right)}{\langle n, j-1, j, k-1\rangle\langle j-1, j, k-1, k\rangle ... \langle k, n, j-1, j\rangle}$$

Due to the cyclic nature of both the numerator and the denominator, it is convenient to define the whole R-invariant as a single 5 bracket, such as: $R_{njk} = [n, j-1, j, k-1, k]$

We will see later how these R-invariants can be interpreted more geometrically, but for now we'll just make a comment on how a BCFW-shift looks in twistor space. Returning to the picture of the momenta making up a closed polygon in twistor space. If we shift a subset of the momenta such that we now have an internal propagator go on-shell, then a certain set of sequential momenta will also be on-shell, meaning there is now a null ray connecting the start and end of that string of momenta in our dual momentum space. The same will be true in our corresponding momentum-twistor space, splitting our larger polygon into two smaller closed polygons. We'll see later the relevance of such geometries.

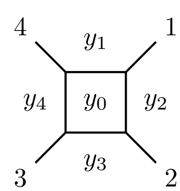
2.9 Loops in Twistor Space

Finally we want to consider how loops look in our twistor space formalism. General loop integrands such as the kind you might obtain from looking at Feynman diagrams in general do not have a unique form, they depend how the loop momentum is parameterised. This means in general they can have different pole structures. So our first objective will be to find a more canonical way of writing down these loop integrands. Our dual momentum coordinates provide an intuitive approach to this. An alternative name for our these coordinates is zone variables, as the external momenta can be thought of as splitting up the plane in which they're defined and the dual variables can then be imagined as labelling the zones they're split up into. Below I've included a diagrammatic representation:



where A_4 is a tree level amplitude.

To extend this concept to include loop level we will label the regions not labelled in this way by new loop variables. We will show an example for a single loop variable we'll call y_0. Note that this construction is only valid with the following two conditions, firstly that we have an ordering to our external particles and secondly we're are only considering planar diagrams. An 4 particle example for such a loop diagram could be:



Defining y_{0i} as $(y_0 - y_i)^2$, the box integrand becomes $\frac{1}{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2}$ with an integration measure of d^4y_0 Now recall that points in our dual momentum space are in fact lines in momentum twistor space. So we can think of the integral $\int d^4y_0$ as an integral over all distinct lines (AB). Then using an identity we stated earlier on, but re-expressing it in a more appropriate form:

$$y_{0i}^2 = \frac{\langle A, B, i-1, i\rangle}{\langle AB\rangle\langle i-1, i\rangle}$$

we can express this as $\frac{\langle AB\rangle^4\langle 41\rangle...\langle 34\rangle}{\langle AB41\rangle...\langle AB45\rangle}$. Now typically the integrand of a loop calculation will come with other factors, such as in this particular case for a 4 particle amplitude we will get factors of $s = y_{13}^2$ and $u = -y_{24}^2$ in the numerator as well, which will end up canceling the majority of our numerator, leaving just $\frac{\langle AB\rangle^4 \langle 1234\rangle^2}{\langle AB41\rangle ... \langle AB45\rangle}$. The $\langle AB\rangle^4$ will later be absorbed into our integration measure, leaving behind an object that is dual conformally invariant. We can associate the line (AB) with two twistor coordinates $\mathcal{Z}_A, \mathcal{Z}_B$, so the integral over lines (AB) can be thought of as an integrals over the these twistor variables instead. The requirement that they be distinct lines requires us to mod out a GL(2) invariance. giving the integration measure $\int \frac{d^4 Z_A d^4 Z_b}{\mathrm{Vol}[GL(2)]}$. The final addition to the measure we need to make is to insure its projectively invariant. To achieve this we need as many powers of A and B on the bottom as on the top. Leaving the final measure:

$$\int \frac{d^4 Z_A d^4 Z_b}{\mathrm{Vol}[GL(2)] \langle AB\rangle^4}$$

This is as far as we'll take the topic of super-amplitudes, my main aim for this section has been to reach a point where we can recognise the structure of a loop integral expressed in this formalism as we shall later see them reappear, but from an entirely different angle.

Chapter 3

Positive Geometry and the Grassmannian

3.1 Positive Geometry

We will leave behind the heavy machinery we've just been looking at for a while to discuss the notion of positive geometries and canonical forms. First we'll look at how to define the simplest objects in these geometries known as simplices, how we can extract their canonical forms, before looking at how these simplices can combines to form the basis of more complicated geometries.

A positive geometry $\mathcal{A}$, can be roughly described as a real, orientated, closed geometry which has boundaries of all co-dimensions, meaning the boundary of any positive geometry is also a positive geometry. Crucially positive geometries have an associated unique differential form $\Omega(\mathcal{A})$ known as the canonical form, satisfying the following properties:

1. $\Omega(\mathcal{A})$ is meromorphic and has logarithmic poles on the boundary of the space and nowhere else.

2. For any Hyper-surface H containing a boundary $\mathcal{B}$ of $\mathcal{A}$, taking the residue of the $\Omega(\mathcal{A})$ along H gives:

$$Res_H \Omega(\mathcal{A}) = \Omega(\mathcal{B})$$

3. If $\mathcal{A}$ is a point, then $\Omega(\mathcal{A}) = \pm 1$ where the sign depends on the orientation of the space.

In order for the canonical form to be unique for each geometry, we must assume the ambient space doesn't admit non-zero holomorphic top forms. Because of this assumption we normally treat positive geometries as being embedded in

$\mathbb{P}^n(\mathbb{R})$, but as $\mathbb{R}^n$ can be easily embedded into $\mathbb{P}^n(\mathbb{R})$, it is often easier to visualise as lines in Euclidean space.

The simplest examples of positive geometries are projective simplices, essentially the generalisation of a triangle to different dimensions. An N-simplex Δ, in $\mathbb{P}^N$ can be completely defined by its vertices. If we let Z_i denote the N+1 vertices of Δ, with $Z_i \in \mathbb{R}^{N+1}/\{0\}$ and $i \in \{1, 2, ..., N+1\}$
Then the N-simplex $\mathcal{N}$ is given by:

$$\mathcal{N} = \left\{ \sum_{i=1}^{i=N+1} c_i Z_i \in \mathbb{P}^N | c_i \geq 0, i = 1, 2, ..., N+1 \right\}$$

Explicitly, N=1 simplices are given as line segments, N=2 simplices as triangles and N=3 simplices as tetrahedra.

We can extend the idea of projective simplices to convex polytopes by allowing more than N+1 vertices to define our Geometry. i.e.

$$\mathcal{P} = \left\{ \sum_{i=1}^{i=M} c_i Z_i \in \mathbb{P}^N | c_i \geq 0, i = 1, 2, ..., M \right\}, M > N+1$$

This definition is known as the convex Hull of the set of vertices points.
A convex polytope is one where the line segment connecting any two points in the space is completely contained within the polytope itself. This is an important condition given how we've defined our polytope in projective space, as we'd like all positive linear combinations of our vertices to define its interior. With simplices this is a given, but for polytopes if we want to keep the same notion of interior we require the Z_as to form a convex polytope. However this convex Hull definition is not enough to specify the facet structure of the vertices and therefore additional conditions are required to completely describe the space. One way to achieve this is to group the M Z_is into an M x (N+1) matrix and fixing the signs of all the ordered minors, in the following way:

$$\langle i_1, i_2, ..., i_{N+1} \rangle \qquad for \quad 1 \leq i_1 < i_2 < ..., i_{N+1} \leq M$$

Here $\langle i_1, i_2, ..., i_{N+1} \rangle$ is the determinant of the minor created from columns associated with the vertices $\{Z_{i_1}, Z_{i_2}, ..., Z_{i_{N+1}}\}$

Essentially we have to order the points in such a way that their ordered minors are always positive.

An alternate definition is given by a system of linear inequalities that define the space instead. If a projective N-simplex is defined by a set of N+1 linear inequalities and Y is a point in $\mathbb{P}^n$, then any linear inequality can be written as $Y \cdot W \geq 0$ in terms of a vector $W \in \mathbb{R}^n$, then our positive space is defined as:

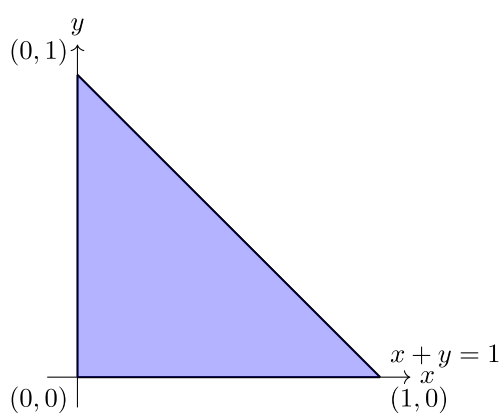
$$\mathcal{A} = \{Y \in \mathbb{P}^n | Y \cdot W_i \geq 0, \text{ for } i = 1, ..., N+1\}$$

These W_i we can interpret as being the facets of our simplex, for triangles they are the edges of the triangle, for tetrahedra they are the faces. Using these examples we see that this satisfies the condition that every boundary, defined here by $Y \cdot W_i = 0$, is a projective simplex in one dimension lower. A tetrahedron is bounded by faces which are triangle, triangles are bounded by line segments, which are in turn bounded by points.

Lets us see exactly how we can determine these canonical forms for some simple examples.
First lets look at the a line segment $[a, b] \subset \mathbb{R}$. Our differential form must contain simple poles only at a and b. Just from this we can deduce that the correct canonical form for this line segment must be :

$$\Omega([a,b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \text{dlog}(x-a) - \text{dlog}(x-b) = \text{dlog}\frac{(x-a)}{(x-b)}$$

Somethings to note here is the choice of a minus sign in front of one term and not the other. Demanding that our form be projectively invariant requires us to have opposite signs for the $\frac{dx}{x-a}$ and $\frac{dx}{x-b}$ terms so that the final dlog form was only given as a ratio of two points. The canonical form must be unique only up to an overall sign, so the choice of which one we made negative was fairly arbitrary and affects only the orientation of the geometry. Note we used a change of variables $\frac{dx}{x-a} = \text{dlog}(x-a)$ to make the logarithmic nature of the poles apparent. We can also check the the residues at the boundaries a and b. In fact we see $\text{Res}_{x=a}(\Omega([a,b]) = 1$ and $\text{Res}_{x=b}(\Omega([a,b]) = -1$, which is what we expected as the boundaries are just points, where the difference in sign relates to the orientation of the line segments,
Next lets consider the example of a a triangle. In particular consider the triangle bounded by the lines, $x = 0$, y=0 and $x + y = 1$,



then the choice for our form would be:

$$\Omega(\mathcal{A}) = \frac{dx \wedge dy}{xy(1-x-y)} = \text{dlog}\frac{x}{1-x-y} \wedge \text{dlog}\frac{y}{1-x-y}$$

Again a change of variables is used to make the logarithmic singularities along the boundaries more apparent. To show the equality of the two forms its easier to expand the RHS as follows:

$$= (\text{dlog}x - \text{dlog}(1-x-y)) \wedge (\text{dlog}y - \text{dlog}(1-x-y))$$

$$= \text{dlog}x \wedge \text{dlog}y - \ \text{dlog}(1-x-y) \wedge \text{dlog}y - \text{dlog}x \wedge \text{dlog}(1-x-y)$$

Due to the negative sign in $(1-x-y)$, $\text{dlog}(1-x-y) = \frac{-dx}{1-x-y}$ similarly for going to dy. Also note that for any vector v, $v \wedge v = 0$, resulting in:

$$= \frac{dx}{x} \wedge \frac{dy}{y} + \frac{dx}{1-x-y} \wedge \frac{dy}{y} + \frac{dx}{x} \wedge \frac{dy}{1-x-y}$$

$$= \frac{(1-x-y)+x+y}{xy(1-x-y)} dx \wedge dy \quad = \quad \frac{dx \wedge dy}{xy(1-x-y)}$$

Again we can also check that this satisfies the condition that the residue along the boundaries of our space should yield another positive geometry. Indeed we see that $Res_{y=0}(\Omega(\mathcal{A})) = \frac{dx}{x(1-x)}$, which letting b=1 and a=0 in the line segment example above we'll see this is the same form.

We can generalize these results, to D-dimensional simplex. The poles are defined by D+1 linear equations, however we can treat this projectively in the following way. A simplex Δ in $\mathbb{P}^N$ defined by its vertices $Z_i \in \mathbb{R}^{N+1}$ has the canonical form:

$$\Omega(\Delta) = \frac{\langle Z_1 Z_2 ... Z_{N+1} \rangle^N \langle Y d^N Y \rangle}{N! \langle Y Z_1 Z_2 ... Z_N \rangle \langle Y Z_2 Z_3 ... Z_{N+1} \rangle ... \langle Y Z_{N+1} Z_1 ... Z_{N-1} \rangle}$$

Here we see that Y $\to Z_i$ is a pole of this differential form, as any angle bracket including Z_i will go to zero. The factor of $\langle Z_1 Z_2 ... Z_{N+1} \rangle^N$ is there to balance out the bottom and keep this expression projectively invariant under the rescaling $Z_i \to tZ_i$ and the term $\frac{\langle Y d^N Y \rangle}{N!}$ is the natural measure on $\mathbb{P}^N$. It should also be mentioned that we are taking the wedge product between the dYs in the measure term here. For example for N=2, $\langle Y d^2 Y \rangle = \epsilon_{abc} Y^a dY^b \wedge dY^c = 2Y^1 dY^2 \wedge dY^3 + 2Y^2 dY^3 \wedge dY^1 + 2Y^3 dY^1 \wedge dY^2$, where in this calculation I've used that the wedge product is anti-symmetric i.e. $(a \wedge b) = -(b \wedge a)$.

While we had a vertex-centric definition of the interior of a positive simplex, we also had a facet-centric definition. So how exactly can we derive a canonical form from this definition? We'll begin by simply claiming for a convex simple polytope $\mathcal{A}$ of dimension m, its canonical form can be expressed in the following way:

$$\Omega(\mathcal{A}) = \sum_{\text{vertex Z}} \text{sign(Z)} \bigwedge_{a=1}^{m} \text{dlog}(Y \cdot W_a)$$

where for each vertex Z, the W_a are the adjacent facets to the vertex and the sign(Z) is there to encode the orientation of the facets.

In order to show this relation we can preform induction on the dimension m. For $m = 0$, $\mathcal{A}$ is a single point and depending on its orientation, its canonical form is ± 1. For $m > 0$ we assume our formula holds for all $a = 1, ..., m-1$.

Now returning to our formula we see that in m dimensions the poles of this differential form are given by the facets and if we take a given facet F, defined by $Y \cdot W_F = 0$, we see

$$\mathrm{Res}_F \Omega(\mathcal{A}) = \sum_{Z'} \mathrm{sign(Z')} \bigwedge_{a=1}^{m-1} \mathrm{dlog}(Y \cdot W_a)$$

where Z' are the vertices adjacent to F. But by induction this is the canonical form on F. Since canonical forms are determined uniquely by their poles and residues, the fact the differential form has the correct poles and residues implies this is a canonical form on $\mathcal{A}$.

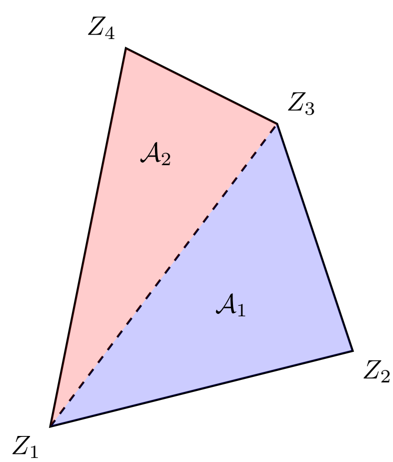
This formula holds not only for simplices but indeed for all simple polytopes, meaning all polytopes $\mathcal{A}$ where the vertices are the intersection of exactly $\dim(\mathcal{A})$ facets.

3.2 Triangles

We now want to know how to build up the geometries of more complicated polytopes from simpler ones. One method of doing this is known as triangulation[3]. Intuitively this can be understood in 2 dimensions as the sub-dividing of polygons into triangles, but can be extended into higher dimensions as the partitioning of polytopes into simplices. The canonical forms of our more complicated geometries are given in the following way, if we have a positive geometry $\mathcal{A}$ which can be partitioned into a set $\{\mathcal{A}_a\}$ of geometries where $\Omega(\mathcal{A}_a)$ are known, then:

$$\Omega(\mathcal{A}) = \sum_a \Omega(\mathcal{A}_a)$$

Its important to note that while there isn't a unique triangulation for any given polytope, the canonical form we can extract from this triangulation is not dependent on the choice of triangulation and will indeed be unique to our geometry.



Consider the above example of a quadrilateral $\mathcal{A}$ in $\mathbb{P}^2$ with vertices Z_1, Z_2, Z_3, Z_4, then one possible way to triangulate $\mathcal{A}$ is into the triangles $\mathcal{A}_1$ with vertices Z_1, Z_2, Z_3 and $\mathcal{A}_2$ with vertices Z_1, Z_3, Z_4 then,

$$\Omega(\mathcal{A}) = \Omega(\mathcal{A}_1) + \Omega(\mathcal{A}_2) = \frac{\langle 123 \rangle^2 \langle Y d^2 Y \rangle}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle} + \frac{\langle 134 \rangle^2 \langle Y d^2 Y \rangle}{\langle Y13 \rangle \langle Y34 \rangle \langle Y41 \rangle}$$

Each of the simplices has a boundary which is not a boundary of $\mathcal{A}$ namely the line (Z_1, Z_3). This is reflected in the individual forms of $\mathcal{A}_1$ and $\mathcal{A}_2$, both having a pole at $\langle Y13 \rangle$. However these spurious boundaries and poles cancel out in the sum of the simplices. By expressing Y as $aZ_1 + bZ_2 + cZ_3 + dZ_4$ and using the Schouten identity in $\mathbb{P}^2$:

$$Z_a \langle b, c, d \rangle + Z_b \langle c, d, a \rangle + Z_c \langle d, a, b \rangle + Z_d \langle a, b, c \rangle = 0$$

We see that the form on $\mathcal{A}$ becomes:

$$\Omega(\mathcal{A}) = \frac{\langle Y d^2 Y \rangle [\langle 123 \rangle \langle 234 \rangle \langle Y41 \rangle + \langle 124 \rangle \langle 134 \rangle \langle Y23 \rangle]}{\langle Y12 \rangle \langle Y23 \rangle \langle Y34 \rangle \langle Y41 \rangle}$$

And so we see the pole $\langle Y31 \rangle$ has disappeared from our expression.

3.3 Grassmannians

The real Grassmannian $Gr_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$ which contain the origin[1].

To specify an element of $Gr_{k,n}$ we can think of needing $k$linearly independent n-vectors, and packaging them into a single k x n matrix. This representation is not unique however and any two k x n matrices that are related through invertible row operations represent the same element of $Gr_{k,n}$. So specifically an elements of $Gr_{k,n}$ is represented by a k x n matrix of rank k, modulo GL(k).

Concretely suppose $A \in Gr_{k,n}$ then A has a representation:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & x_{k3} & \dots & x_{kn} \end{pmatrix}$$

Then through a series of row operations we can transform it to:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x_{1k+1} & \dots & x_{1n} \\ 0 & 1 & \dots & 0 & x_{2k+1} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & x_{kk+1} & \dots & x_{kn} \end{pmatrix}$$

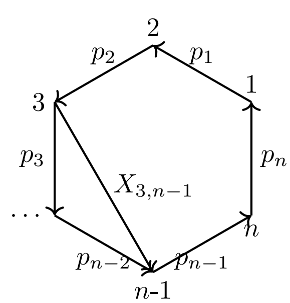
From this we can see the dimensions of an $Gr_{k,n}$ are$k(n-k)$ and the only SL(k) invariants of $Gr_{k,n}$ are its maximal minors. This means the only GL(k) invariants are the ratios of any two maximal minors, or alternatively the maximal minors up to a simultaneous re-scaling of all minors by the same amount. We can use this invariance of the minors up to rescaling to define a set of projective co-ordinates for the Grassmannian known as the Plücker co-ordinates. First let $[n]$ denote the set $1, 2, ..., n$ and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. Then for I $= \{i_1 < i_2 < ... < i_k\} \subset \binom{[n]}{k}$. $p_I(V)$ is the maximal minor of the matrix V with column set I, meaning the minor constructed with the columns of V with columns labeled by I. e.g. Let I = 1,3,4 and V the matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \textit{then,} \qquad p_I(V) = \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

3.4 Associahedron

To finish this section on positive geometries we'll look at an object called the kinematic associahedron. The associahedron itself is a well known mathematical object, but we shall see that within the space of kinematic variables lives a particular associahedron which completely describes tree-level amplitude in ϕ^3 theory. We shall use this as an example to put into practice some of the ideas we have developed in this chapter, as well see the remarkable result, that all n-particle tree-level amplitudes in this theory can be described by the combinatorics of triangulating a given n-gon. ϕ^3 theory acts as a great playground for such ideas due to its simplicity and the only external particle data we have to consider is momentum.

Firstly I'd like to introduce the space $\mathcal{K}_n$, the set of Mandelstam invariants for an n particle scattering, $s_{i_1,i_2,...,i_k} = (p_{i_1} + p_{i_2} + ... + p_{i_k})^2 \in \mathcal{K}_n$. One convenient way to represent this is by introducing planar variables, we draw the n momenta as a closed convex polygon, with the sides being the n momenta and the vertices labeled {1,2,...,n}.



The vertex i, will be the vertex that p_i leaves from. Then finally we describe the momenta by a new set of co-ordinates X_{ij}, associated to the vertices i, j, where $1 \leq i < j \leq n$, by $X_{ij} = (p_i + p_{i+1} + ... + p_{j-1})^2$ or $s_{i,i+1,...,j-1}$, which can be visualised in this picture of a convex polygon, as the diagonal connecting vertex i to vertex j. Within $\mathcal{K}_n$ there is positive subspace where all the non zero X_{ij} are positive, $X_{i,i+1} = 0$ and $X_{1,n} = 0$. The condition that $X_{i,i+1} = 0$ is demanding all the momenta are on-shell and $X_{1,n} = 0$ is demanding total momentum conservation.

In terms of Feynman diagrams in an n particle tree level scattering there are $(n-3)$ internal propagators. So we can can equip each graph Γ_n with a form as follows:

$$\Omega(\Gamma_n) = \text{Sign}(\Gamma_n) \bigwedge_{a=1}^{n-3} \text{dlog} X_{i_a,j_a}$$

$\text{Sign}(\Gamma_n) = \pm 1$ and flips when you exchange two propagators in a Feynman diagram. The total scattering form is then the sum of all n particle Feynman diagrams, and so the planar scattering form for an n-particle scattering form is given by:

$$\Omega_n^{(n-3)} = \sum_{\text{planar } \Gamma_n} \Omega_{\Gamma_n}^{(n-3)}$$

For n=3 we define $\Omega_{n=3}$=± 1. In general for an n particle scattering there are C_{n-2} different Feynman diagrams contributing to the scattering amplitude. Here C_k is the k^{th} Catalan number and is defined as $C_k = \frac{1}{k+1}\binom{2k}{k}$.The term $\text{Sign}(\Gamma_n)$ leads us into a bit of bother as depending on what the sign is for each graph we could potentially get many different forms associated with each diagram, in order to obtain a unique form from the scattering amplitude we need to demand the form be projectively invariant. I.e if under the transformation $\{X_{ij}\} \to \Lambda(X)\{X_{ij}\}$.

We can use this condition that the form be projectively invariant to define the sign flip rule between graphs.

Consider to graphs g,g' related by a mutation, where we swap two of the internal propagators. Then they will still share $(n-4)$ propagators, but will differ by propagators which we'll denote X_{ij} and $X_{i'j'}$. Now consider that the forms on g and g' will contain terms as follows:

$$\Omega(g) = \text{sign}(g)\text{dlog}(\Lambda(X)X_{ij}) \wedge \left(\bigwedge_{b=1}^{n-4} \text{dlog}(\Lambda(X)X_{i_b j_b}) \right)$$

$$\Omega(g') = \text{sign}(g')\text{dlog}(\Lambda(X)X_{i'j'}) \wedge \left(\bigwedge_{b=1}^{n-4} \text{dlog}(\Lambda(X)X_{i_b j_b}) \right)$$

We simply need to split our $\text{dlog}(\Lambda(X)X_{ij})$ into $(\text{dlog}(\Lambda(X)) + \text{dlog}(X_{ij}))$ and our final scattering form will contain terms that look like:

$$... + (\text{sign}(g) + \text{sign}(g'))\text{dlog}\Lambda(X) \wedge \left(\bigwedge_{b=1}^{n-4} \text{dlog} X_{i_b j_b} \right) + ...$$

So in order for the Λ dependence to be removed we require that:

$$\text{sign}(g) = -\text{sign}(g')$$

As we've described it above it looks like there could be Λ dependence on some of the $X_{i_b j_b}$, but upon expanding everything we will always have dlog(Λ) wedged with exactly $(n-4)$ Xs, and so the two graphs that share those $(n-4)$ Xs, must always have opposite signs to cancel out.

Now how can we manifest these results by looking at our space of Mandelstam invariants? Well firstly, one thing to notice is that the dimensions of $\mathcal{K}_n$ is $\frac{n(n-3)}{2}$ whereas the scattering forms arising from looking at the Feynman diagrams are only $(n-3)$ dimensional. However we can carve out a particular subspace of $\mathcal{K}_n$ with the following set of constraints and a relation between the s_{ij} and our planar variables:

$$-s_{ij} = c_{ij} > 0, \text{for i, j} \neq n, \quad \text{i,j are not adjacent}$$

$$s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}$$

This gives us a set of $\frac{(n-2)(n-3)}{2}$ relations, meaning the subspace of $\mathcal{K}_n$ satisfying these constraints has dimension $= \frac{n(n-3)}{2} - \frac{(n-2)(n-3)}{2} = (n-3)$, and we call the subspace carved out by these constraints H_n

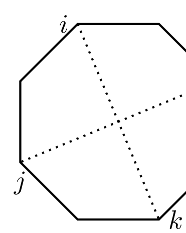
These constraints also give us an interesting geometric result when combined with our positivity conditions on the X_ij. To see this lets us take preform a summation on our relation between the s_{ij} and the planar variables:

$$\sum_{i \leq a < j, k \leq b, l} s_{a,b} = \sum_{i \leq a < j, k \leq b, l} X_{a,b+1} + X_{a+1,b} - X_{a,b} - X_{a+1,b+1}$$

If we were to keep b constant and just preform over a, we see on the RHS as we move through a, the first and the last term will cancel except the first term when $a = i$ and the last term when $a = j-1$, similarly for the two middle terms, then preforming the b summation after gives another similar result, leaving just:

$$-\sum_{i \leq a < j, k \leq b, l} c_{a,b} = X_{i,l} + X_{j,k} - X_{i,k} - X_{j,l}$$

Now while in general a condition for our positive space is just that $X_{i,j} \geq 0$, we see here that both $X_{i,k}$ and $X_{j,l}$ cannot be zero as the LHS is strictly negative and the RHS is non-negative. What this corresponds to geometrically is the picture:



So we see a consequence of our constraints is that crossing diagonals cannot both go to zero simultaneously. Note that how $X_{i,j}$ was defined with $i < j$ this ordering of i, j, k, l is enforced.

Now that we've set up the kinematics of our spaces, I'd like to give the full definition of an associahedron. Given a convex polytope of dimension $(n-3)$ for $n \geq 3$, if it satisfies the following properties then it is an associahedron:

1. For d = 0,1,...,$(n-3)$, there exists a one-to-one correspondence between the codimension d boundaries and the d-diagonal partial triangulations of an n-gon

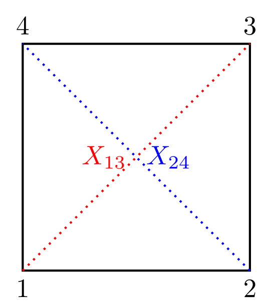
2. A codimension d boundary $F-1$ is adjacent to a codimension $d+k$ boundary F_2, iff the partial triangulation of F_2 can be obtained by adding k diagonals to the partial triangulation of F_1

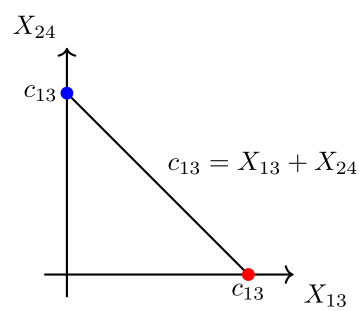
Property 1 has two immediate consequences; the polytopes interior, which is essentially a codimension 0 boundary, corresponds to a triangulation with zero diagonals and vertices, which are the highest codimensional boundaries, correspond to full triangulations.

Now the number of full triangulation of a given n-gon and hence the number of vertices of a (n-3) dimensional associahedron is C_{n-2}, which hints at its connection to ϕ^3 scatterings as we already saw that, that is the same number of planar Feynman diagrams that contribute to a n particle scattering.

Now the claim is the the positive region defined by $X_{ij} \geq 0$ and our H_n subspace of $\mathcal{K}_n$ is an associahedron, but before we look at why, lets first examine this positive region for $n = 4$ and $n = 5$ and see how we can interpret this correspondence between boundaries and triangulations.

First lets start by defining $\mathcal{A}_n := H_n \cap \{X_{ij} \geq 0\}$. Now for $n = 4$, we have 2 non-zero X_{ij} to consider, X_{13} and $X_{2,4}$ as well as the relation $c_{13} = X_{13} + X_{24}$. Now we see the positive region defined in kinematic space is the line segment defined by our relation, from $X_{13} = 0, X_{24} = c_{13}$ to $X_{13} = c_{13}, X_{14} = 0$. Since for $n = 4$ our region $\dim(\mathcal{A}_4) = 1$ we only have co-dimension 1 boundaries and indeed for a convex quadrilateral there are only 2 triangulations with 1 diagonal



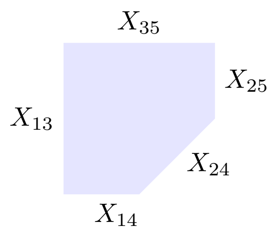


Now there's not a lot we extract from this as the 4-gon has too few triangulations to really appreciate the combinatorics of this space, but we want to start thinking of the addition of these diagonals as the associated momenta going on-shell.

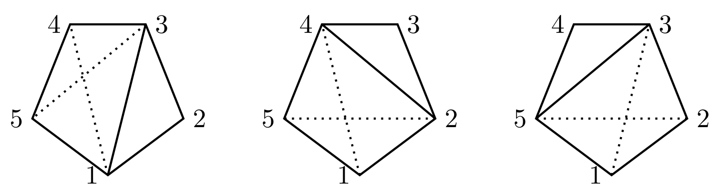
Now for the $n = 5$ case we have 5 non-zero variables and 3 non-zero c_{ij} constraints, so while the kinematic space is 5-dimensional, $\mathcal{A}_5$ is two dimensional so we can use a basis (X_{13}, X_{14}) to describe our other planar variables transforming our constraints to the space:

$$
\begin{array}{rcl}
 & & X_{13} \geq 0 \\
-c_{13} = X_{14} - X_{13} - X_{24} & & X_{24} = X_{14} - X_{13} + c_{13} \geq 0 \\
-c_{14} = X_{24} - X_{14} - X_{25} & \rightarrow & X_{25} = -X_{13} + c_{14} + c_{13} \geq 0 \\
-c_{24} = X_{25} - X_{24} - X_{35} & & X_{35} = -X_{14} + c_{24} + c_{14} \geq 0 \\
 & & X_{14} \geq 0
\end{array}
$$

Defining the region:

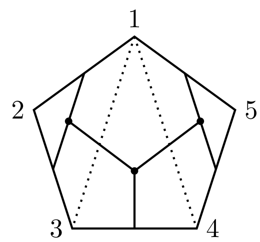


Now lets consider a pentagon with vertices 1,...,5 and draw in the diagonal 1,3. What we see is there are two potential diagonals we could draw in to get a full triangulation. Comparing the potential diagonals we could add, to the vertices along the X_{13} boundary, we see they are exactly the diagonals that correspond to the X_{ij} that join to our X_{13} boundary at those vertices. Doing the same for a couple other edges of $\mathcal{A}_n$, we see this pattern continues.



And so we see the result that the facet structure of our space is entirely described by triangulations of a convex pentagon. And indeed this continues to higher n. In $n = 6$, $\mathcal{A}_6$ is a 3-dimensional polytope in kinematic space, who's faces are the boundaries where the X_{ij} go to zero and who's facet structure, meaning which faces are connected which faces, is completely described by triangulations of a convex hexagon.

We can further connect our positive space to ϕ^3 amplitudes by noticing the connection between triangulations of an n-gon and n-particle tree level Feynman diagrams. As previously stated an n-particle tree level diagram has $(n-3)$ internal propagators, which in a massless theory, by momentum conservation all our internal propagators will be zero and on-shell. Using the example of a convex pentagon we'll see how these internal propagators relate to the diagonals of a triangulation on an n-gon. We can imagine drawing a Feynman diagram such that the external particles enter through the edges of the pentagon, and our vertices are the midpoints of the internal triangles, connected through the diagonals involved in our triangulation.



Now considering how we originally set up our planar variables with the edges of our convex polygon being the momenta of the individual particles, we associate the momentum $X_{i,i+1}$ to the particle that crosses that edge of the pentagon in our picture. What we see by momentum conservation at the vertices is the internal propagators will have momenta corresponding to the diagonal they cross.

We immediately see another consequence of restricting ourselves to the space of H_n. Limiting ourselves to X_{ij} that don't allow simultaneously crossing diagonals, means we consider solely planar graphs in ϕ^3 theory.

As we continue we see that all tree level Feynman diagrams can be constructed in this way, a result that may not be surprising as given the number of independent n-particle Feynman diagrams and full triangulations of a n-gon are the same. It should be stated though that this construction enforces a cyclic ordering on the particles in our scattering.

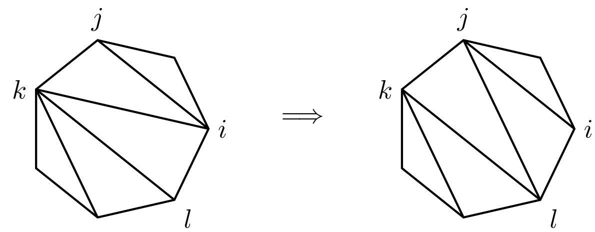
We now want to determine the canonical form on $\mathcal{A}_n$. Note that the vertices of $\mathcal{A}_n$ are the intersection of $(n-3)$ of the non-zero X_{ij} which define the facets of our space, and the dimension of $\mathcal{A}_n$ is also $(n-3)$, so our polytope is in fact a simple polytope. From this we can use a formula we proved earlier, to show the canonical form on $\mathcal{A}_n$ is given as follows:

$$\Omega(\mathcal{A}_n) = \sum_Z \text{sign(Z)} \bigwedge_{a=1}^{n-3} \text{dlog} X_{i_a j_a}$$

Where Z denotes the set of all vertices of $\mathcal{A}_n$ and $X_{i_a j_a}$ are the facets that meet at a given vertex. Sign(z) $= \pm 1$, is given relative to the orientation of a chosen vertex.

Now given that both the vertices of our polytope and the set of a n-particle planar Feynman graphs are in one to one correspondence with triangulations of an n-gon, thinking of these correspondences as bijective mappings between sets implies there is also a one to one correspondence between the vertices of $\mathcal{A}_n$ and the n-particle Feynman graphs.

One thing we ought to be sure of though, is that the signs flip in the same way for the Feynman graphs as for the vertices. For Feynman graphs we defined the sign flip to be when you exchange one internal propagator for another you must introduce a relative minus sign, which we did to ensure GL(1) invariance of the amplitude. The equivalent procedure for vertices is a mutation in the associated triangulations, where one diagonal is swapped for another. In practice for any $n \geq 4$ we can always identify a quadrilateral with the n-gon and exchanging its diagonal, such as:



Here we are assuming that $1 \leq i < j < k < l \leq n$, and since the diagonals we are always exchanging are always crossing they satisfy: $-\sum_{i \leq a < j, k \leq b, l} c_{a,b} = X_{i,l} + X_{j,k} - X_{i,k} - X_{j,l}$. If we take the exterior derivative of this equation we get:

$$dX_{jk} + dX_{il} = dX_{ik} + dX_{jl} \implies dX_{jk} + dX_{il} - dX_{ik} = dX_{jl}$$

In both triangulations the propagators X_{jk} and X_{il} appear, so using the fact that $\text{dlog}(X_{ij}) = dX_{ij}/X_{ij}$, when we substitute the expression above into the wedge product of the propagators we see that:

$$\bigwedge_{a=1}^{n-3} dX_{i_a j_a} = -\bigwedge_{a=1}^{n-3} dX_{i'_a j'_a}$$

as on the RHS we will have a term involving $dX_{jk} \wedge dX_{il} \wedge (dX_{jk} + dX_{il} - dX_{ik})$, but on expanding the bracket the extra terms involving $dX_{jk} \wedge dX_{jk}$ and similar term for X_{il} will disappear as in general $dx \wedge dx = 0$. So we can conclude that under a mutation $Z \to Z'$, we can conclude:

$$\text{sign}(Z) = -\text{sign}(Z')$$

With all that we can finally say the canonical form on $\mathcal{A}_n$ is indeed the scattering form we get from looking at the planar Feynman diagrams.

Though we can take this one step further by defining the following quantity:

$$d^{n-3}X := \text{sign}(g) \bigwedge_{a=1}^{n-3} dX_{i_a j_a}$$

This quantity arises from the fact that our space is (n-3) dimensional and so we can choose a basis of (n-3) propagators to describe the rest with. And so by choosing a basis, the relative sign of any other graph comes from how the variables expressing the form are orientated relative to this basis. By re-expressing the form in terms of the basis we lose the negative sign. So continuing to use our relation between $\text{dlog}(X_{ij})$ and dX_{ij} we see that:

$$\Omega(\mathcal{A}_n) = \sum_Z \text{sign}(Z) \frac{1}{\prod_{a=1}^{n-3} X_{i_a j_a}} \bigwedge_{a=1}^{(n-3)} dX_{i_a j_a}$$

$$= \sum_g \frac{1}{\prod_{a=1}^{n-3} X_{i_a j_a}} d^{n-3}X = m_n d^{n-3}X$$

And so we arrive at the result we expected, the canonical form on $\mathcal{A}_n$ gives us exactly the amplitude m_n

For $n = 4$ we have previously stated the non-zero X_{ij} and constraints, but taking the exterior derivative of our constraint leads us to the relation, $dX_{13} = -dX_{24}$. Then plugging what we know into of equation for the form we get:

$$\Omega(\mathcal{A}_4) = \text{dlog}X_{13} - \text{dlog}X_{24} = \frac{dX_{13}}{X_{13}} + \frac{dX_{13}}{X_{24}}$$

We see how despite the fact we had a negative sign coming from the mutation between vertices, upon re-expressing dX_{24} in terms of dX_{13} we lost our negative sign. Now for 4 particle interactions X_{13} and X_{24} have an interpretation in terms of the usual s,t and u variables. So in the end what we are left with is:

$$\Omega(\mathcal{A}_4) = \left(\frac{1}{s} + \frac{1}{t}\right) ds$$

Which is exactly the form we wanted.

For n=5 from our associahedron point of view we'll get the form:

$$\Omega(\mathcal{A}_5) = \text{dlog}X_{13} \wedge \text{dlog}X_{14} - \text{dlog}X_{13} \wedge \text{dlog}X_{35} +$$

$$\text{dlog}X_{25} \wedge X_{35} - \text{dlog}X_{25} \wedge \text{dlog}X_{24} + \text{dlog}X_{14} \wedge \text{dlog}X_{2}4$$

Which rewriting in a more digestible way yields the result:

$$\Omega(\mathcal{A}_5) = \left(\frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{14}X_{24}} + \frac{1}{X_{24}X_{25}} + \frac{1}{X_{24}X_{35}}\right) d^2X$$

I leave it as an exercise to the reader to calculate the 5-particle amplitude via Feynman diagrams to confirm that it is indeed the result we should obtain.

Chapter 4

The amplituhedron

4.1 Introducing the Amplituhedron

The second example of positive geometry we're going to look at is the amplituhedron. We shall take the approach in this chapter of first constructing it mathematically, before then interpreting it in the context of $\mathcal{N}=4$ super Yang-Mills and seeing how the result compares to object we saw arise in our discussion on superamplitudes earlier on.

To begin we're going to return to our definition of a convex polytope and in particular focus on polygons in $\mathbb{P}^2$; given a set of n vertices $Z_1, Z_2, ..., Z_n$ then any point in the interior can be written in the form:

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + ... + c_n Z_n^I$$

This is a linear combination of constants and vectors and therefore can be written more concisely as the product:

$$Y^I = C^a Z_a^I$$

where due to the the C n-tuples being GL(1) invariant we have:

$$(c_1, ..., c_n) \subset Gr_+(1,n) \text{ and } (Z_1, ..., Z_n) \subset M_+(3,n)$$

Where $M_+(k,n)$ is the space of k x n matrices with postive ordered minors or alternatively, we can say that its Plücker co-ordinates of all ordered index sets are positive. This is similar space to $Gr_+(k,n)$ the only difference is it doesn't have the GL(k) invariance to it.

Hence ranging over all possible C^a is enough to define the whole interior of the polygon. The product of the two will give us a (1x3) matrix which although we can't say anything about its ordered minors, it will retain the GL(1) invariance of the Cs and therefore lives in $Gr(1,3)$

We can generalise this construction firstly by considering the dimension of our vertices. So far we've been thinking about points in $\mathbb{P}^2$, which is convenient as it has a simple visual representation, but in the end we want to describe the real world, which is not 2 dimensional. So we can take our Z_a to be in $\mathbb{P}^m$, then our collection $(Z_1, ..., Z_n) \subset M_+(1+m, n)$.

A second generalisation we could make, is letting our Z_as be $(k+m)$ vectors and looking at the space of k-planes. Now rather than Y being a subspace of $Gr(1, 1+m)$ it'll be a subspace of $Gr_{k,k+m}$. We can see this by again considering Y as the positive linear combinations of our vertices, however now our C is are subsets of $Gr_+(k, n)$. So our Y is now the space $\mathrm{Gr}(k, k+m)$. This space is then denoted $\mathcal{A}_{n,k,m}(Z)$, which we call the tree amplituhedron. The m=2 and k=1 is the simplest case and is nicely represented as 2d polygons, while the m=4 is the applicable case for physics and it takes momentum 4 twistors as its inputs.

4.2 Boundaries of the Amplituhedron

Lets now look at the boundaries of the m=2 and k=1 Amplituhedron and see how demanding positivity for the Cs leads us to a natural way of talking its boundaries.
As we've already seen we can describe the internal points of a convex polygon by Y $= c_1Z_1 + ... + c_nZ_n$, now let us try and compute $\langle YZ_iZ_j\rangle$ for some $i < j$.

$$\langle YZ_iZ_j\rangle = \langle(c_1Z_1 + ... + c_nZ_n)Z_iZ_j\rangle = c_1\langle Z_1Z_iZ_j\rangle + ... + c_n\langle Z_nZ_iZ_j\rangle$$

Now considering that Z $\subset M_+(3, n)$ we have that its ordered minors are all positive. Additionally in any matrix swapping 2 columns flips the sign of the determinant. So any $\langle Z_aZ_iZ_j\rangle$ where $i < a < j$ will be strictly negative as its one column exchange away from an ordered minor, while clearly if $a < i < j$ or $i < j < a$ then $\langle Z_aZ_iZ_j\rangle$ will be strictly positive, from the definition of $M_+(3, n)$ and the cyclic invariance of 3x3 determinants. Now if as we sweep through all the c_a, $\langle YZ_iZ_j\rangle$ changes from being positive to negative, then at some point $\langle YZ_iZ_j\rangle \to 0$ telling us Y is on the line (Z_iZ_j). Now given that Y defines the interior of our shape, (Z_iZ_j) should not be a boundary of our shape. Alternatively if we never have the same sign everywhere for $\langle YZ_iZ_j\rangle$, then we have that (Z_iZ_j) is indeed a boundary for our shape.

Now to see precisely what this implies for the boundaries of our shape, lets look again at the sum for $\langle YZ_iZ_j\rangle$:

$$\langle YZ_iZ_j\rangle = \sum_a c_a\langle Z_aZ_iZ_j\rangle$$

Now if i, j are not consecutive, this sum will include terms with $i < a < j$ whose angled bracket will be negative and as we consider all c_a, we can scale up this negative angle bracket to make it the dominant term or scale it down completely, making it negligible in comparison to the positive terms. So overall we will have a sign flip for some Cs in this sum.
On the other hand for i, j being consecutive, all the non zero brackets have either $a < i < j$ or $i < j < a$, so regardless of our choice of the c_a we will get a

positive sum:

$$\langle YZ_iZ_{i+1}\rangle = \sum_a c_a\langle Z_aZ_iZ_{i+1}\rangle > 0$$

Therefore we get that the boundaries of our shape are the lines (Z_iZ_{i+1}), which in the case of a polygon we could have probably guessed.

However if we take the same approach with the k =1 and m=4 case we'll find a similar result. Lets us try and compute the determinant $\langle YZ_iZ_jZ_kZ_l\rangle$ for $i < j < k < l$. By decomposing Y into the sum over all the vertices we will arrive at:

$$\langle YZ_iZ_jZ_kZ_l\rangle = \sum_a c_a\langle Z_aZ_iZ_jZ_kZ_l\rangle$$

Now if $a < i$, $j < a < k$ or $l < a$ then $\langle Z_aZ_iZ_jZ_kZ_l\rangle$ is either an ordered minor of an element of $M_+(5, n)$ or an even number of column transposition away from an element of $M_+(5, n)$. Either way this would result in something strictly positive. It is again only when $i < a < j$ or $k < a < l$ that this sum can be negative. So with this in mind and using the idea that we want $\langle YZ_iZ_jZ_kZ_l\rangle$ to be strictly greater than 0 if the plane $(Z_iZ_jZ_kZ_l)$ forms a boundary of $\mathcal{A}_{n,k,m}$, we can conclude that our boundaries are planes of the form $(Z_iZ_{i+1}Z_jZ_{j+1})$ as we have:

$$\langle YZ_iZ_{i+1}Z_jZ_{j+1}\rangle = \sum_a c_a\langle Z_aZ_iZ_{i+1}Z_jZ_{j+1}\rangle > 0$$

Indeed for any even m the boundaries are the planes $(Z_iZ_{i+1}...Z_{i_{m/2}-1}Z_{i_{m/2}})$

If we want to consider $k > 1$, we have to be careful to respect the positivity of C by keeping its minors ordered, but apart from that we follow the same approach just with the k-plane $(Y_1...Y_k)$ in place of the single Y. Now we have the result:

$$\langle Y_1...Y_kZ_iZ_{i+1}Z_jZ_{j+1}\rangle = \sum_{a_1<...<a_k}\langle c_{a_1}...c_{a_k}\rangle\langle Z_{a_1}...Z_{a_k}Z_iZ_{i+1}Z_jZ_{j+1}\rangle > 0$$

4.3 Triangulating the Amplituhedron

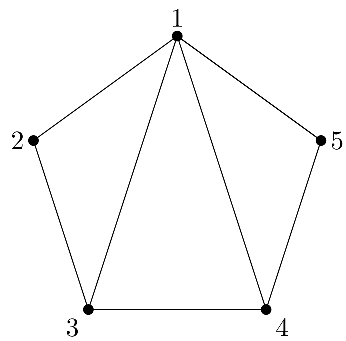
The amplituhedron can be viewed as a map from $Gr_{\geq}(k, n) \to Gr(k, k + m)$. Now given that dim $Gr_{\geq}(k, n) = k(n-k)$ and dim $Gr(k, k+m) = mk$, we normally only consider $n \geq m+k$ meaning this map is highly redundant. Therefore if our aim is to map into a mk dimensional space, then we want to find spaces of equal dimension to map into it. In our $m = 2$, k=1 example, such a space could correspond to letting choosing (Z_i, Z_j, Z_k) and letting the associated (c_i, c_j, c_k) be non-zero, while the others vanish, and in the end describes the triangle constructed from the vertices (Z_i, Z_j, Z_k)

Now our aim should be to triangulate our entire interior with a collection of such "cells", the only real conditions we should place on this collection, is that they cover the whole interior of our Amplituhedron and they should only overlap on their boundaries. As with other positive geometries, in general we don't have a unique triangulation.

For our polygon case we can construct the triangulation, where the individual cells are constructed with the vertices $(1,i,i+1)$ and the entire space is simply:

$$\sum_i (1,i,i+1)$$

Pictorially the n=5 case with this triangulation is as follows:



Now while harder to visualise, for k=1 and m=4, we can see the relevant cells, must be constructed from 5 Cs. We can then show the following is a triangulation for this case:

$$\sum_{i+1<j}(1,i,i+1,j,j+1)$$

Why have we chosen this particular triangulation to represent our space? We can interpret the labels of the vertices as R-invariants, then this expression becomes the factor between the $\mathcal{A}_n^{\mathrm{MHV}}$ and $\mathcal{A}_n^{\mathrm{NMHV}}$

Now as our cells have dimension mk we can associate a collection of positive co-ordinates $C^\Gamma(\alpha_1^\Gamma ... \alpha_{mk}^\Gamma)$. We now let our C depend on these α co-ordinates. Then define the cell Γ to be the space defined by $Y = C(\alpha)Z$ and letting alpha range over all positive values.

4.4 Canonical Form

Like other positive geometries we want to associate a canonical form to our space. Using our m=2 example lets consider the triangle defined by $Y = Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3$. This is how we describe a single cell of our triangulation of the $\mathcal{A}_{n,1,2}$. To this cell, we associate the form:

$$\Omega_{123} = \frac{d\alpha_2}{\alpha_2} \wedge \frac{d\alpha_3}{\alpha_3}$$

While the logarithmic singularities as we approach the lines $(Z_1 Z_2)$ and $(Z_1 Z_3)$ can be seen as $\alpha_3 \to 0$, $\alpha_2 \to 0$ respectively, the boundary as $\alpha_2, \alpha_3 \to \infty$ due to the projective nature of the space, can be understood as the coefficient of Z_1

going to 0 instead and hence is the boundary $(Z_2 Z_3)$. This is apparent if we rewrite this form in the the following way:

$$\Omega_{123} = \frac{\langle Y dY dY \rangle \langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle}$$

Now we see that if for example, Y becomes purely a linear combination of Z_1 and Z_2, then the $\langle Y12 \rangle$ bracket will go to 0. This is how the logarithmic singularities are captured by the form.

Now once we have a canonical form for a single cell, extending this form to the whole amplituhedron is simple. We already have a decomposition of the whole amplituhedron into simpler spaces, which we have defined forms for, so the form for the entire space becomes the sum of the forms of our smaller spaces. For example if we let Ω_{ijk} be the form over the triangle with vertices Z_i, Z_j, Z_k, then using the triangulation we described earlier, the form over the whole of $\mathcal{A}_{n,1,2}$ is:

$$\Omega = \sum_i \Omega_{1ii+1}$$

As with previous examples, this form will be independent of our choice of triangulation and also includes singularities along internal lines of our space, which should be present in the final form. However these spurious singularities will cancel in the sum and therefore not be a problem in the end.

Now lets extend this idea to the m=4 tree amplituhedron. Once again we want to associate a form with logarithmic singularities on all boundaries. Again we can do this by first finding a triangulation of the whole $\mathcal{A}_{n,k,4}$, whose individual cells are $4k$ dimensional, and taking their sum to find the final form on the entire space. For any k, a cell Γ would have the following form associated to it:

$$\Omega^{\Gamma}_{n,k} = \prod_{i=1}^{4k} \frac{d\alpha^{\Gamma}_i}{\alpha^{\Gamma}_i}$$

For k=1 one way of defining a cell in $Gr_+(1,n)$ is defining the space Y = $Z_1 + \alpha_{a_2} Z_2 + ... + \alpha_{a_5} Z_5$, with the associated form:

$$\frac{d\alpha_{a_2}}{\alpha_{a_2}} \wedge ... \wedge \frac{d\alpha_{a_5}}{\alpha_{a_4}} = \frac{\langle Y d^4 Y \rangle \langle Z_{a_1} Z_{a_2} Z_{a_3} Z_{a_4} Z_{a_5} \rangle^4}{\langle Y Z_{a_1} Z_{a_2} Z_{a_3} Z_{a_4} \rangle ... \langle Y Z_{a_5} Z_{a_1} Z_{a_2} Z_{a_3} \rangle}$$

Specifically lets use the triangulation we found earlier, $\sum_{i+1<j} (1, i, i+1, j, j+1)$.

The individual cells Γ will have the form:

$$\Omega_{1,i,j} = \frac{\langle Y d^4 Y \rangle \langle Z_1 Z_i Z_{i+1} Z_j Z_{j+1} \rangle^4}{\langle Y Z_1 Z_i Z_{i+1} Z_j \rangle ... \langle Y Z_{j+1} Z_1 Z_i Z_{i+1} \rangle}$$

And therefore the form for the whole amplituhedron will be:

$$\Omega = \sum_{1<i<j-1} \Omega_{1,i,j}$$

4.5 Extracting the Superamplitude

Having defined the amplituhedron, we now want to know how we can extract the superamplitude from it. Firstly we must decide what external data to associate with our vertices. Note that we will be focusing exclusively on the m=4 amplituhedron from this point on as it is the most relevant to physics. We will associate the top 4 components of our Zs with the usual bosonic momentum-twistor variables. For the remaining k entries we specifically choose the following anti-commuting Grassmann parameters, $\phi_1, ...\phi_k$ and χ_a, and let our Zs be defined as follows:

$$Z_a = \begin{pmatrix} z_a \\ \phi_1^A \cdot \chi_{a_1 A} \\ \vdots \\ \phi_k^A \cdot \chi_{a_k A} \end{pmatrix}$$

Where z_a is a 4 vector and A = 1,...,$\mathcal{N}$

Now we can use a GL(4+k) transformation to send Y $\rightarrow Y_0$, where:

$$Y_0 = \begin{pmatrix} 0_{4\times k} \\ 1_{k\times k} \end{pmatrix}$$

Now the tree super-amplitude can be interpreted straight from the canonical form on the Amplituhedron, by:

$$\mathcal{M}_{n,k}(z_a, \chi_a) = \int d^{\mathcal{N}}\phi_1 ... d^{\mathcal{N}}\phi_k \int \Omega_{n,k}(Y; Z)\delta^{4k}(Y; Y_0)$$

Where the delta function is known as a projective δ-function and is defined as follows:

$$\delta^{4k}(Y; Y_0) = \int d^{k\times k}\rho_\alpha^\beta \det(\rho)^4 \delta^{k\times(k+4)}(Y_\alpha^I - \rho_\alpha^\beta Y_{0\beta}^I)$$

localising Y to Y_0 by integrating over GL(k) transformations.

For general k we know that any element of Gr$(k, k+m)$ can be gauge fixed such that it has the form $(1_{k\times k}|\alpha_{k\times m})$, so for each row $Y_1, ..., Y_k$ we have m free variables. So we should get a 4-form coming from each row of Y. Hence any form on Gr$(k, k+4)$ should be of the form:

$$\Omega = \langle Y_1...Y_k d^4 Y_1\rangle ... \langle Y_1...Y_k d^4 Y_k\rangle \times \omega_{n,k}(Y; Z)$$

Which gives us the final expression for $\mathcal{M}$:

$$\mathcal{M}_{n,k}(z_a, \chi_a) = \int d^{\mathcal{N}}\phi_1 ... d^{\mathcal{N}}\phi_k \omega_{n,k}(Y_0; Z)$$

Where ω will encode the singularities of the form and as we will see, be our super-amplitude.

If we specifically look at a n=5, k=1 case we only have 1 cell to consider, when we then localise the canonical form to Y_0 we're left with:

$$\int d^{\mathcal{N}}\phi_1 \frac{\langle Z_1 Z_2 Z_3 Z_4 Z_5\rangle^4}{\langle Y_0 Z_1 Z_2 Z_3 Z_4\rangle ... \langle Y_0 Z_5 Z_1 Z_2 Z_3\rangle}$$

The bottom part of this fraction becomes just the cyclic product of momentum twistor variables. The top part expands to:

$$(\phi \cdot \chi_5\langle 1234\rangle + ... + \phi \cdot \chi_4\langle 5123\rangle)^4 = ((\phi \cdot (\chi_1\langle 2345\rangle + ... + \chi_5\langle 1234\rangle)^4$$

Where we've suppressed the index for k from ϕ and χ for convenience, and the $\langle ijkl\rangle$ represents $\langle z_i z_j z_k z_l\rangle$, the determinant of the matrix of just the momentum-twistor variables.

Also to keep it neat we'll define $\xi = \chi_1\langle 2345\rangle + ... + \chi_5\langle 1234\rangle$, which will still be a Grassmann variable. So we have:

$$(\phi^A \cdot \xi_A)^4 = (\phi^1\xi_1 + ... + \phi^4\xi_4)^4$$

Due to ϕ and ξ being Grassmann variables when expanding completely any terms involving any squares of ξ or ϕ will be zero so the only term that survives will be $\phi^1\xi_1\phi^2\xi_2\phi^3\xi_3\phi^4\xi_4$.

From here the final ϕ integrals are fairly trivial as when we integrate Grassmann variables we use the identity:

$$\int d\eta f(\eta) = \int d\eta\ (a\eta + b) = a$$

So as $\phi^1...\phi^4$ is a common term for the whole expression, when computing the ϕ integral, we simply remove all the ϕs from the expression and are left with:

$$\mathcal{M}_{5,1}(z_a, \chi_a) = \frac{\xi_1\xi_2\xi_3\xi_4}{\langle 1234\rangle ... \langle 5123\rangle}$$

or:

$$= \frac{(\chi_{1,1}\langle 2345\rangle + ... + \chi_{5,1}\langle 1234\rangle) \times ... \times (\chi_{1,4}\langle 2345\rangle + ... + \chi_{5,4}\langle 1234\rangle)}{\langle 1234\rangle ... \langle 5123\rangle}$$

4.6 Creating loops in the Amplituhedron

So far we have been working with the tree amplituhedron, but we want to try and extend this to loops as well. Loops in the amplituhedron are encoded by the geometry. In order to achieve this we "hide" the virtual particles in the following way. Suppose we have the following positive matrix C:

$$\begin{pmatrix} A_1 & B_1 & 1 & \dots & m & A_2 & B_2 & m+1 & \dots & n \\ 1 & 0 & c_{11} & \dots & c_{1m} & 0 & 0 & c_{1,m+1} & \dots & c_{1n} \\ 0 & 1 & c_{21} & \dots & c_{2m} & 0 & 0 & c_{2,m+1} & \dots & c_{2n} \\ 0 & 0 & c_{31} & \dots & c_{3m} & 1 & 0 & c_{3,m+1} & \dots & c_{3n} \\ 0 & 0 & c_{41} & \dots & c_{4m} & 0 & 1 & c_{4,m+1} & \dots & c_{4n} \\ 0 & 0 & c_{51} & \dots & c_{5m} & 0 & 0 & c_{5,m+1} & \dots & c_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots & 0 & 0 & \vdots & \vdots & \vdots \end{pmatrix}$$

Since this is a representation of a Grassmannian we can always gauge fix our matrix such that the columns A_1, B_1, A_2, B_2 always have this form. To hide these particles we then simply remove the corresponding columns, i.e. the A and B columns.
From there we are left with a matrix which we can split up into the following pieces:

$$\begin{pmatrix} D_{(1)} \\ D_{(2)} \\ C \end{pmatrix}$$

Where $D_{(1)}$ corresponds to the first two rows and $D_{(2)}$ to rows three and four.

The positivity of our original matrix leads to the minors of the the following matrices also being positive:

$$\begin{pmatrix} D_{(1)} \\ D_{(2)} \\ C \end{pmatrix}, \begin{pmatrix} D_{(1)} \\ C \end{pmatrix}, \begin{pmatrix} D_{(2)} \\ C \end{pmatrix}, (C)$$

As if we calculate a maximal minor of our original matrix involving all the AB columns we'd be left with just minors of C and any minors not involving our AB columns correspond to minors of C together with the 2-planes corresponding to the columns we didn't involve. And so the lost columns leave behind a residual positivity in C.

So far we've always been hiding pairs of columns at any given time, if we had hidden individual columns, the positivity of the remaining minors wouldn't be guaranteed. Instead they would have depended on the ordering of the removed columns. So in order to avoid having to impose additional structure to our data, we instead choose to remove pairs of columns. In the above example I have used 2 pairs A_1B_1 and A_2B_2, but this can be extended to any number of pairs.

Now given that all the maximal minors of C together with our 2-planes $D_{(i)}$ are all non zero, this implies linear independence between the rows and therefore the vectors that define our 2-planes must live in the $(n-k)$ compliment of C, which defines a k-plane in n dimensional space.

So we can now define the space $\mathrm{Gr}(k, n; L)$, the space of our usual C, together with a collection of L 2-planes $D_{(i)}$. We denote the collection (C, $D_{(i)}$) as $\mathcal{C}$, and demand positivity not only of the ordered minors of C, but also of all the ordered $(k+2l) \times (k+2l)$ minors of C together with any collection of the $D_{(i)}$, with $0 \leq l \leq L$

We now need to decide how to incorporate these 2-planes into our definition of the Amplituhedron. In the same way we defined the space Y, as $Y = C \cdot Z$ or the positive linear combination of our external data. We can use our collection of 2-planes to create a new collection of 2-planes in the compliment space of Y, defined as follows:

$$\mathcal{L}^I_{\gamma(i)} = D_{\gamma a(i)} Z^I_a$$

Here γ = 1,2, and denotes the two rows of the the 2-planes, i = 1,...,L and tells us which plane we're talking about.

Again these 2 planes live in the compliment space of Y. So the whole loop Amplituhedron, which we call $\mathcal{A}_{n,k,L}(Z)$, live in $\mathrm{Gr}(k, k+4; L)$: the space of k-planes in $(k+4)$ dimensions together with a collection of 2-planes in the 4 dimensional compliment of Y.

This new collection of (Y, $\mathcal{L}_{(i)}$) we denote as $\mathcal{Y}$ and can be concisely defined as:

$$\mathcal{Y} = \mathcal{C} \cdot Z$$

Now we need to extend the idea of cells and canonical forms to the loop Amplituhedron. We already know Y lives in $\mathrm{Gr}(k,\ k+4)$ so has dimension $4k$. The $\mathcal{L}_{(i)}$ on the other hand are 2 planes living in a 4-d space, so can be represented as a 2×4 matrix and by gauge fixing we can choose its representation such that:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_3 & \alpha_4 \end{pmatrix}$$

so is 4 dimensional. From this we can associate a set of positive co-ordinates $(\alpha_1^\Gamma, ..., \alpha_{4(k+L)}^\Gamma)$ to any cell Γ of $\mathcal{A}_{n,k,L}(Z)$, such that $\mathcal{C}(\alpha) = (D_{(i)}(\alpha), C(\alpha))$ is in $Gr_+(k; n; L)$. Demanding that such a cell has logarithmic singularities on its boundary gives us a canonical form:

$$\Omega^\Gamma = \prod_{i=1}^{4(k+L)} \frac{d\alpha_i^\Gamma}{\alpha_i^\Gamma}$$

We then need a collection of non-intersecting cells T that under $\mathcal{Y} = \mathcal{C} \cdot Z$ cover the entire Amplituhedron. Which gives us the final form on $\mathcal{A}_{n,k,L}(Z)$:

$$\Omega_{n,k,L}(\mathcal{Y}; Z) = \sum_{\Gamma \in T} \prod_{i=1}^{4(k+L)} \frac{d\alpha_i^\Gamma}{\alpha_i^\Gamma}$$

We chose Y_0 such that the first 4 rows were all 0s. So given that our 2-planes $\mathcal{L}_{(i)}$ live in the compliment of Y, when we localise Y to Y_0, we have to demand the opposite on the $\mathcal{L}_{(i)}$, so we take them to be non vanishing in the first 4 entries and zero for all entries after that, such that $\mathcal{L}_{(i)}^I = (\mathcal{L}_{(i)2\times 4} | 0_{2\times k})$. The two rows here for each $\mathcal{L}_{(i)}$, define a line $(\mathcal{L}_{\gamma=1}\mathcal{L}_{\gamma=2})$ in projective space, specifically in $\mathbb{P}^3$. Now while Y is fully fixed by delta functions, the only restrictions on $\mathcal{L}_(i)$ are that it lives in the compliment of Y_0, so these free variables we treat as our loop integration variables. Then including our $\mathcal{L}_{(i)}$ into our expression for the amplitude yields:

$$\mathcal{M}_{n,k}(z_a, \eta_a; \mathcal{L}_{\gamma(i)}) = \int d^{\mathcal{N}}\phi_1 ... d^{\mathcal{N}}\phi_k \int \Omega_{n,k,L}(Y, \mathcal{L}_{\gamma(i)}; Z)\delta^{4k}(Y; Y_0)$$

In the same way we looked at the free variables of Y to see how the differential form must look, we'll do the same with our $\mathcal{L}_{(i)}$. We know after we've gauge fixed our 2-planes, they're left with two free variables in each of the rows, so each row will give us an associated 2-form. Now while $\mathcal{L}_{1(i)}$ and $\mathcal{L}_{2(i)}$ are related by the fact that together they specify a particular plane and they're both related to Y as they're defined to live in its compliment, any $\mathcal{L}_i$ is not tied to any other $\mathcal{L}_j$, so any given $\mathcal{L}_{(i)}$ will contribute a differential term like:

$$\langle Y\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{1(i)}\rangle\langle Y\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{2(i)}\rangle$$

meaning any form on $\mathrm{Gr}(k, k+4; L)$ can be written as:

$$\Omega = \langle Y d^4 Y_1\rangle ... \langle Y d^4 Y_k\rangle \prod_{i=1}^{L}\langle Y\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{1(i)}\rangle\langle Y\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{2(i)}\rangle\times\omega_{n,k,L}(Y,\mathcal{L}_{(i)})(Z)$$

with Y $= Y_1...Y_k$
and therefore we get the L-loop integrand amplitude:

$$\mathcal{M}_{n,k}(z_a,\eta_a,\mathcal{L}_{\gamma(i)}) = \int d^4\phi_1...d^4\phi_k\prod_{i=1}^{L}\langle\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{1(i)}\rangle\langle\mathcal{L}_{1(i)}\mathcal{L}_{2(i)}d^2\mathcal{L}_{2(i)}\rangle\omega_{n,k}(Y_0,\mathcal{L}_{\gamma(i)};Z)$$

To see an example of this let us look at the simplest loop-level case, the MHV 1-loop case. Now we have no Y to consider and a single 2-plane. A 4-dimensional cell must have 3 non-zero coordinates in the top row of the matrix D and 3 non-zero entries in the bottom row. Now the following 2 matrices:

$$\begin{pmatrix} 1 & a & b & 0 \\ -1 & 0 & x & y \end{pmatrix}, \qquad \begin{pmatrix} 1 & a & b & 0 & 0 \\ -1 & 0 & 0 & x & y \end{pmatrix}$$

for positive coordinates a, b, x, y both matrices will only have positive non zero minors, and will form the basis for our cell decomposition of the amplituhedron in this case. Let $[a, b, c; x, y, z]$ the cell in $Gr_+(2, n)$ with non zero entries in top row in columns (a, b, c) and non zero entries in the bottom row in columns (x, y, z). Then the cell decomposition[4]:

$$\sum_{i<j}[1, i, i+1; 1, j, j+1]$$

covers the amplituhedron and in the case where $i+1 = j$ we have situation like the matrix on the left described above and for all other cases a situation like the matrix on the right.

These cells map into Gr(2,4) in the following way:

$$A = Z_1 + \alpha_1 Z_i + \alpha_{i+1}Z_{i+1}, B = -Z_1 + \alpha_j Z_j + \alpha_{j+1}Z_{j+1}$$

which gives us the associated form:

$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(1,i,i+1)\cap(1,j,j+1)\rangle^2}{\langle AB1i\rangle\langle AB1,i+1\rangle\langle AB,i,i+1\rangle\langle AB1j\rangle\langle AB1,j+1\rangle\langle ABj,j+1\rangle}$$

We can see here that the bottom of the fraction is just the boundaries of our cell as we expected while the top gives us our integration measure for a scattering process described via twistor coordinates.

4.7 Conclusion

So in conclusion we have seen how to build up amplitudes from ones which are completely determined from first principles and how in the context of supersymmetry, these amplitudes can be expressed by twistor coordinates. Twistor coordinates gives us a hint that these amplitudes might be able to be viewed as entirely geometric objects. Indeed in the planar limit of $\mathcal{N} = 4$ super Yang-Mills and in planar ϕ^3 theory we've been able to achieve this. In the case of the Amplituhedron, using the principles of positive geometry by carefully defining boundaries of our space and decomposing it into manageable pieces, we were able to define a canonical form on our space that can describe loop amplitudes to all orders.

Certain areas to still be explored from this geometric point of view of scattering amplitudes include topics such as the relation between cluster algebras and the Amplituhedron as are explored in[11]. From the associahedron point of view this is also an area that can still be explored as well, as triangulations of a given n-gon can be described by mutations of certain quivers[9]. The Idea of triangulations can be extended to quadrangulations for ϕ^4 theory.[6]

I thank you for reading my project and hope you've enjoyed it.

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