phi knight - complex analysis

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lecture 1 - derivatives, integrals, and hole detection

def. a function $f : U \to \mathbf{C}$ defined on an open subset U of C is holomorphic if $f'(z)$ exists $\forall z \in U$.

intuitively, $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$ simply says f takes infinitesimal circles around z_0 and maps them to infinitesimal circles around $f(z_0)$, rotated and stretched by $f'(z)$

exr. f holomorphic $\iff$ it is real-differentiable and Df is a scalar times a rotation. ✥

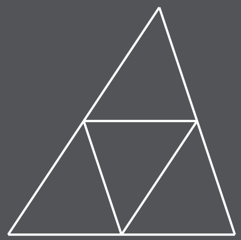
explicitley,

(Cauchy-Riemann) $u_x = v_y$ and $u_y = -v_x$ if $f = u + iv$ is the decomposition of a holomorphic function $f : U \to \mathbf{C}$ into real and imaginary parts. ✥

so, a holomorphic function is one that has a derivative. does it have an anti-derivative? it depends. e^z does. but $1/z$ defined on $\mathbf{C}^\times$ does not. this is very intuitive, as there is no continuous function g with $e^{g(z)} = z$ (i.e. logarithm), because there is no continuous angle function - going in a circle around the origin we get $0 = \tau$. more generally, if f has an anti-derivative, then $\int_{\gamma: a \to b} f(z)\mathrm{d}z = F(b) - F(a)$[1], and in particular $\oint f(z)\mathrm{d}z = 0$. but we have $\oint_{\mathbf{T}} 1/z\mathrm{d}z = \int_0^\tau 1/e^{it}\frac{\mathrm{d}e^{it}}{\mathrm{d}t}\mathrm{d}t = i\tau \neq 0$. the problem, as we'll see, is this "hole" at the origin. without holes in the domain, holomorphic functions always have antiderivaitves.

(Goursat) given $f \in \mathrm{hol}(U)$ and a solid triangle $T \subset U$ we have $\oint_{\partial T} f(z)\mathrm{d}z = 0$ ◈

indeed, we can divide any triangle into four similar parts.



whence $\oint_{\partial T}$ is the sum of four similar integrals for the smaller triangles. ergo, $|\oint_{\partial T} f(z)\mathrm{d}z|$ is at most $4|\oint_{\partial T_1} f(z)\mathrm{d}z|$ for T_1 one of the four triangles of T. continuing, we have $T \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$, each congruent to half the previous, with $|\oint_{\partial T} f(z)\mathrm{d}z| \leq 4^n|\oint_{\partial T_n} f(z)\mathrm{d}z|$. by Cantor's theorem, we have $\{z_0\} = \bigcap T_n$. now, $f(z) = f(z_0) + f'(z_0) + (z - z_0)h(z)$ with h continuous and approaching 0 as $z \to z_0$, and since the linear part has an anti-derivative, we have $\oint f(z)\mathrm{d}z = \oint h(z)(z - z_0)\mathrm{d}z$. therefore $|\oint_{\partial T} f(z)\mathrm{d}z| \leq 4^n\text{perimeter}(T_n)\max_{T_n}|h(z)(z - z_0)| \leq 4^n\text{perimeter}(T_n)\text{diameter}(T_n)\max_{T_n} h$. as T_n is 2^n times smaller than T, and $\max_{T_n} h \to 0$ taking $n \to \infty$ finishes the proof. ◆

from which we deduce

$f \in \mathrm{hol}(U)$ has an anti-derivative if U is convex. in particular, $\int_\gamma f = 0$ for closed rectifiable curves $\gamma \subset U$. ✥

let's demonstrate how to use this to do some non-trivial integrals. we'll show that $e^{-\pi x^2}$ is its own

[1] interpreted as the Riemann integral of a piecewise continuously differentiable, or even rectifiable curve, say. see exercises

Fourier transform.

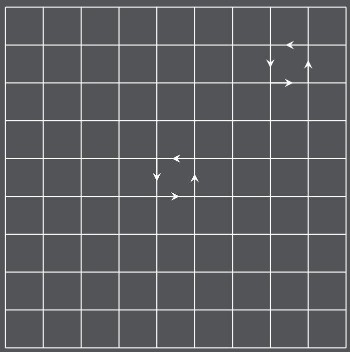
exm. for $f(z) = e^{-\pi z^2}$ and γ the rectangle $-R \to R \to R + i\xi \to -R + i\xi \to -R$ we have the total integral zero. but the part on the real line goes to 1 as $R \to \infty$. on the next segment we have $f(R + iy) = e^{-\pi(R^2+2iRy-y^2}$. since the interval is bounded and so is y, this integral is at most $O(e^{-\pi R^2})$. the same holds for the parallel edge. in total we find $0 = 1 - \lim \int_{-R}^{R} e^{-\pi(x+i\xi)^2 \mathrm{d}x}$ i.e. $\int_{\mathbf{R}} e^{-\pi x^2} e^{-\tau i x\xi} = e^{-\pi\xi^2}$.

it's useful to generalize

thm. if γ_0, γ_1 are homotopic rectifiable curves in U then $\int_{\gamma_0} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z$ for all $f \in \mathrm{hol}(U)$. in particular, if U is simply connected, each $f \in \mathrm{hol}(U)$ has an anti-derivative. ◈

there is a simple proof if the homotopy is rectifiable. it goes morally as follows. assume wlog the curves are sufficiently close, in the sense that there is a sequence of balls B_k, $k = 0, \ldots, n$ contained in U and covering the curves. now pick points z_k, w_k, $k = 1, \ldots, n$ on the curves between D_k and D_{k-1}. then on each segment the integral is just given by the local anti-derivative, and when passing from one ball to the next, the two local anti-derivatives on the intersection differ by a constant. when the homotopy is general, we need a bit more analysis.

wlog $\gamma_t(s) : [0,1] \times [0,1] \to U$ is a homotopy. by compactness, $\exists r > 0$ s.t. $B_r(\gamma_t(s)) \subset U$ $\forall t, s$. since $\gamma_t(s)$ is uniformly continuous, $\exists N$ s.t. $|t - t'| \le 1/N$ and $|s - s'| \le 1/N$ implies $|\gamma_t(s) - \gamma_{t'}(s')| < r/2$. so let us divide the square into $(N + 1)^2$ squares. for each such square $A \to B \to C \to D \to A$ consider the quadrilateral $R = \gamma(A) \to \gamma(B) \to \gamma(C) \to \gamma(D) \to \gamma(A)$. since R is contained in $B_r(\gamma(A))$, we have $\int_R f(z)\mathrm{d}z = 0$ by the above fact. summing over all squares, cancellations, and the fact that $\gamma_t(0)$ and $\gamma_t(1)$ are constant gives us that the integral of f over the straight line segments $\gamma_j(k/N)$ to $\gamma_j((k+1)/N)$ agrees for $j = 0, 1$. but again, by the above fact, and the fact that $\gamma_j(s)$ is inside a ball in U on these intervals, the integral over γ is the integral over these linear segments.



lecture 2 - Cauchy's integral formula and its many corollaries

(Cauchy)
$$f(z)=\frac{1}{\tau i}\oint_{\partial D}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta$$
for each point z in the interior of the closed disk D and f holomorphic in a neighborhood of D. ◈

indeed, $\oint_{\partial D}\frac{f(\zeta)-f(z)}{\zeta-z}\mathrm{d}\zeta = \oint_{z+\varepsilon\mathbf{T}}\frac{f(\zeta)-f(z)}{\zeta-z}\mathrm{d}\zeta$ by homotopy equivalence. but the integrand is bounded around z as it approaches a limit. hence letting $\varepsilon \to 0$ we get that this integral equals zero. we are done by the fact that $\oint_{\mathbf{T}} 1/z = \tau i$ ◆

in particular, f is determined by its values on ∂D. note that we have f in the center of D as a sort of average of f on ∂D. in particular,

cor. given $f \in$ hol(U), $|f|$ cannot have a local maxima at any point $p \in U$ unless it is constant around p. similarly, only if $f(p)=0$ or f constant around p can $|f|$ have a local minima at p. ◈

since integrating / averaging improves the smoothness of a function, it is now not surprising that holomorphic functions are smooth. better yet, holomorphic functions are analytic

thm. let f be holomorphic in $B_r(z_0)$. then a power series about z_0 with radius of convergence at least r equals f on $B_r(z_0)$. ◈

indeed, wlog let $z_0 = 0$, $r = 1$, f holomorphic in a neighborhood of $|z| \le 1$ and $a_n = \frac{1}{i\tau}\oint_{\mathbf{T}}\frac{f(z)}{z^{n+1}}$. if M is a bound for $|f|$ on the disk, then $|a_n| \le M/r^n$ whence $\sum a_n z^n$ converges at least in the open disk. now, for fixed w in that open disk we have
$$i\tau f(w)=\oint_{\mathbf{T}}\frac{f(z)}{z-w}\mathrm{d}z=\oint_{\mathbf{T}}f(z)\sum\frac{w^n}{z^{n+1}}\mathrm{d}z$$
which is interchangable via the Weirestrass M-test.[2] and we are done. ◆

cor. holomorphic functions are infinitely differentiable - $f \in \mathrm{hol}(U) \implies f' \in \mathrm{hol}(U)$[3]. ◈

(Cauchy) $f \in \mathrm{hol}(\overline{B_r(z_0)}) \implies |f^{(n)}(z_0)| \le \frac{n!\max_{\partial B_r(z_0)}|f|}{r^n}$ ◈

(Liouville) $f \in \mathrm{hol}(\mathbf{C})$ non-constant $\implies f$ unbounded ◈

(Gauss) $\mathbf{C}$ algebraically closed ◈

(Riemann) the following are equivalent for $f \in \mathrm{hol}(U - \{p\})$, $p \in U$.

1. f holomorphically extendable to p
2. f continuously extendable to p
3. f bounded around p
4. $(z-p)f(z) \to 0$ as $z \to p$ ◈

indeed, $1 \to 2 \to 3 \to 4$ being trivial, if 4 holds then $h(z) = (z-p)^2 f(z)$ for $z \ne p$ and $h(p) = 0$ is holomorphic in U, since we have $h'(p) = \lim(z-p)f(z) = 0$. so around p we have h as a convergent power series without the two first terms, making $h(z)/(z-p)^2 = f(z)$ is holomorphically extendable to p. ◆

cor. the real and imaginary parts of $u + iv = f \in \mathrm{hol}(U)$ are harmonic, $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$ ◈

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[2] dominated convergence could overkill
[3] up till now, we did not even know that f' is continuous!

cor. let $f_n \in \mathrm{hol}(U)$ converge to $f : U \to \mathbf{C}$ uniformly on compact sets. then $f \in \mathrm{hol}(U)$. ◈

indeed, wlog $U = B_1(0)$. since $0 = \oint_{\partial T} f_n \to \oint_{\partial T} f$, for any triangle T in U, the function $F(z) = \int_{0\to z} f$ is a holomorphic, infinitely differentiable antiderivative for f ◆

cor. if $f, g \in \mathrm{hol}(D)$ and $\{f = g\}$ has an accumulation point in D then $f = g$. ◈

demonstrate that $f \neq g$ is possible if the accumulation point is outside D.

cor. for a non-constant $f \in \mathrm{hol}(D)$, $|f|$ has no local maxima, and its local minima are its roots $\{f = 0\}$. ◈