phi knight - analysis

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lecture [**] - Arzela Ascoli

(Arzela-Ascoli) given a compact metric space X, a subset $\mathcal{F} \subseteq \mathcal{C}(X)$ is compact iff it is closed, pointwise bounded, and *equicontinuous* - $\forall \varepsilon \exists \delta$ s.t. $d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon \;\; \forall x, y \in X,\; f \in \mathcal{F}$ ⟐

exr. $\implies$ ⌖

exr. compact metric spaces are seperable. ⌖

prf. [given f_n in $\mathcal{F}$, we must show some sub-sequence is (uniformly) convergent.] let x_n denote a dense sequence in X. let $f_{0,n}$ be a sub-sequence of f_n s.t. $f_{0,n}(x_0)$ converges, and successively $f_{k,n}$ denote a sub-sequence of $f_{k-1,n}$ such that $f_{k,n}(x_k)$ converges. the so called diagonal subsequence $g_n = f_{n,n}$ of f_n has $g_n(x_j)$ converging for each j. we'll show g_n is (uniformly) Cauchy. given ε, let δ be as guaranteed by equicontinuity. ergo

$$d(x,y) < \delta \implies |g_n(x) - g_n(y)| < \varepsilon$$

now, let X_δ be a finite sub-set of $\{x_n\}$ that is δ-dense in X. since g_n converges at each of the finitely many points of X_δ,

$$\exists N \text{ s.t.} \forall s \in X_\delta, n, m \geq N \implies |g_n(s) - g_m(s)| < \varepsilon$$

thus

$$n, m \geq N \implies |g_n - g_m| < 3\varepsilon$$

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