tomachello

§ rationale

this is a book about mathematics, a topic centered around solving problems. thus, problem solving is our leading focus. to clarify, i consider "abstract theories" as tools to be developed only when neccesary to solve a problem. namely i find one has to gain an intuition before earning the most powerful weapons known. this book was designed so as to be read by my younger self, who so desparately needed just this book. trying to improve on what had been my actual mathematical education, other major differences between the standard teachings and that of this book should be pointed out. rigour does not come first [sometimes not at all], intuition does. each topic tries to be as self contained as possible, not relying on previous topics when unnecessary [in spirit opposite to Bourbaki].

§ subjects

basic nt, analytic nt, algebraic nt, complex analysis, Fourier analysis, diff equations, mechanics, basic diff geometry, advanced calculus / diff topology, Lie theory, algebraic topology, set topology, group theory, representation theory, logic, set theory, game theory, algorithms and data structures, computation, homological algebra, linear algebra, functional analysis, measure theory, advanced probability, basic probability, Galois theory, algebraic geometry, graph theory, matroids, coxeter groups

§ topics to include

covering spaces, fundamental group, singular homology, martingales, Brownian motion, tensor and exterior algebra, origins of algebraic geometry, $G*H$, analytic proof of Weierstrass approximation theorem, symmetric group representation theory, Harr measure, measures are dual to continuous functions, Jacobi's equal areas theorem, many proofs of quadratic reciprocity, "combinatorial topology" (as starting point of alg top)

§ analysis - metric spaces

def a **metric** on a set X is a function $\rho : X^2 \to [0, \infty)$ satisfying

- $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$
- $\rho(x,y) = \rho(y,x)$
- $\rho(x,y) > 0$ iff $x \neq y$

the pair (X, ρ) is then a **metric space**

def a sub-set A of a metric space (X, ρ) is **open** if $\forall x \in A \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq A$

def a sub-set C of a metric space (X, ρ) is **closed** if $c_n \to c$ with $c_n \in C$ implies $c \in C$

thm open and closed sets are complementary notions

§ fundamental topology - topological groups

exm

- $\mathbb{R}$
- $\mathbb{R}_{>0}$, $\mathbb{C}^*$, $\mathbb{H}^*$ and $\mathbb{S}(\mathbb{H}) \equiv \mathbb{S}^3$
- $\mathbb{S}^1$
- $\mathtt{GL}_n(\mathbb{R})$, $\mathtt{SL}_n(\mathbb{R})$, $\mathtt{O}_n$ and complex analogs.

def a **topological group** is a group G with a compatible structure of a topology - group operation and inverse are continuous

exr write a careful rigorous definition, and demonstrate that continuous group operation does not imply the inverse is continuous. however, one can verify the continuity of both operations by the continuity of $(a,b) \mapsto ab^{-1}$

obs translating, a topological group is **homogeneous** - it looks the same at every point

exr prove from axioms that indeed translation is a homeomorphism

exr the quotient map $G \to G/H$ [H any sub-group] is an open map

warning $\dfrac{G}{\mathtt{ker} f}$ might not be homeomorphic to $\mathtt{img} f$

exm $\mathbb{R} \to \mathbb{T}^2$ with $x \mapsto (e^{ix}, e^{i\lambda x})$ for $\lambda \in \mathbb{R} - \mathbb{Q}$

exr if f is an open map or a closed map, we have $\dfrac{G}{\mathtt{ker} f} \equiv \mathtt{img} f$

exr [sanity check]

- $H \lhd G$ implies $\dfrac{G}{H}$ topological group
- $\mathbb{R}/\mathbb{Z} \equiv \mathbb{S}^1$ and $\mathbb{C}^* \equiv \mathbb{R}_{>0} \times \mathbb{S}^1$
- product is topological group
- $H \leq L \leq G$ implies L/H has the same topology as a sub-space of G/H and as a quotient of L
- $\dfrac{\frac{G}{H}}{\frac{L}{H}} \equiv \dfrac{G}{L}$

some more abstract non-sense will be needed

exr tfae for a top group G

- G Hausdorff
- the unit is closed [equiv every point in G is closed]

- $\forall f : H \to G$ the kernel is closed in H
- the intersection of all neighbourhoods of the unit is the unit

§ algebraic topology - homological algebra

exm

- **Set**
- **Top**
- $\mathbf{Vec}_k$
- **Grp**

def a **category** $\mathcal{C}$ consists of

- a collection of *objects* $\mathbf{obj}(\mathcal{C})$
- a collection of *arrows* $\mathbf{arr}(\mathcal{C})$, with each arrow having one object as its *source*, and one object as its *target*. [we write $s \xrightarrow{f} t$ to mean f is an arrow whose source is s and whose target is t. we write $\mathbf{arr}(s,t)$ for the collection of all such arrows.]
- a partial *composition* rule for arrows - given two arrows $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ we have a composition $a \xrightarrow{f \star g} c$.

subject to the following axioms

- associativity - $(f \star g) \star h = f \star (g \star h)$ for any $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$
- identity arrow - $\forall x \in \mathbf{obj}(\mathcal{C}) \exists x \xrightarrow{1_x} x$ so that $f \star 1_x = f$ and $1_x \star g = g$ for any $\xrightarrow{f} x$ and $x \xrightarrow{g}$

generalizing the fundamental group, homology and co-homology, we're interested in ways of transforming a commutative diagram in one category to the same commutative diagram in a different category. that's what's known as a (covariant) functor. now, some transformations reverse the arrows. those are the contravariant functors.

exm fundamental group

def given categories $\mathcal{C}, \mathcal{D}$, a covariant **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

- a mapping $F : \mathbf{obj}(\mathcal{C}) \to \mathbf{obj}(\mathcal{D})$
- as well as a mapping $F : \mathbf{arr}(\mathcal{C}) \to \mathbf{arr}(\mathcal{D})$

subject to

- for arrow $a \xrightarrow{f} b$ in $\mathcal{C}$, $F(a) \xrightarrow{F(f)} F(b)$ in $\mathcal{D}$
- $F(f \star g) = F(f) \star F(g)$
- $F(1_x) = 1_{F(x)}$

exr define contravariant functors

exm + exr

- $\mathfrak{P} : \mathbf{Set} \to \mathbf{Set}$ taking a set to its power set (and a function to?) is a contravariant functor
- ${}^* : \mathbf{Vec}_k \to \mathbf{Vec}_k$ taking a linear space to its dual (and a linear map to?) is a contravariant functor

- Free : **Set** $\to$ **Vec**$_k$ taking a set S to the free linear space generated by the set (and a function to?) is a covariant functor

def a **pre-additive category** is a category $\mathcal{C}$ with

- for each two objects $a, b \in \mathbf{obj}(\mathcal{C})$ the structure of an abelian group on $\mathbf{arr}_{\mathcal{C}}(a, b)$ [written additively]

subject to distributivity - given a diagram

$$\bullet \xrightarrow{f} \bullet \underset{g_1}{\overset{g_2}{\rightrightarrows}} \bullet \xrightarrow{h} \bullet$$

we have $f \star (g_1 + g_2) \star h = f \star g_1 \star h + f \star g_2 \star h$

obs the zero arrow (identity in the abelian group), when composed with any arrow gives the zero arrow.

my goal in this abstract chapter is to convince you this axiomatic approach really can and does encapsulate all the fundamental properties we use. our first example is the following theorem

thm in a pre-additive category, [finite] products and sums are "the same"

but wait. what does product or sum *mean*?

def product, co-product [sum], initial and terminal objects, "universal property"

exm + exr Set, Top, Vec, Grp

exr make sense of and prove the above theorem shorthands

$\Delta^n = \{x \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum x_i = 1\}$

$\Delta^n_i = \Delta^n \cap \{x_i = 0\}$

$\delta^n_i : \Delta^{n-1} \to \Delta^n$

we care about maps $\sigma : \Delta^n \to X$