§ calculus on manifolds

exm $\mathbf{R}^d, \mathbf{S}^d, \mathbf{T}^d$

def d dimensional manifold - a topological space locally homeomorphic to $\mathbf{R}^d$

def chart - a pair (U, ϕ) with U open and ϕ a homeomorphism from U to [an open set in] $\mathbf{R}^d$

purely topological aspects of manifolds are a worthy pursuit, yet our immediate focus of interest is differentiating and integrating functions on manifolds

f-spaces

exm $(\mathbf{R}^v, \mathcal{C}^\infty)$, $(\mathbf{C}^v, \texttt{Hol})$

def an f-space (T, τ, F) consists of a topological space (T, τ) with a map F assigning to each open set $U \in \tau$ a sub-algebra $F(U) \leq \mathcal{C}_\mathbf{C}(U)$ so that $f \in F\left(\bigcup_i V_i\right) \iff f|_{V_i} \in F(V_i)\ \forall i$ whenever $V_i \in \tau$ are open and $f : \bigcup_i V_i \to \mathbf{C}$

def d dimensional smooth manifold - an f-space locally isomorphic to $(\mathbf{R}^d, \mathcal{C}^\infty)$

exr complete the definition of the category of f-spaces / smooth manifolds - what constitutes an arrow?

def a pair of functions, each defined in a neighborhood of a point are *locally equivalent* at that point if there is a neighborhood of said point in which they agree. denote by f_p the equivalence class of a function f about the point p. Furthermore, consider

$$\texttt{Germ}_p = \{f_p\} = \texttt{local functions/local equivalence}$$

obs $\texttt{Germ}_p$ is an algebra

def $\texttt{Germ} = \bigsqcup_p \texttt{Germ}_p$ is endowed with a topology whose open sets are

$$\{f_q \mid q \in V \subseteq U \text{ open }, f \in F(V)\}_{U \in \tau}$$

derivations

exm $\sum g_i \frac{\partial}{\partial x_i}$

def derivation - a linear map D satisfying $D(ab) = aD(b) + D(a)b$ [a.k.a Leibniz's law]

def tangent space to smooth manifold M at p, T_pM - all derivations $D : \texttt{Germ}_p \to \mathbf{C}$

thm [sanity check] dim $T_pM =$ dim M

prf Taylor's theorem

new manifolds from old

so far we've defined smooth manifolds and maps, as well as the tangent space. but all this is cannot be applied before we know how to make, say, $\mathbf{S}^d$ into a smooth manifold!

obs an open sub-space of a manifold is naturally a manifold

to the reader

- you have to think deeply about each idea presented. Furthermore, red underlining means the precise details are left for you to figure out

chapter dependence

analytic number theory	basic number theory, complex analysis
calculus	linear algebra
general topology	basic analysis
differential geometry	calculus
calculus on manifolds	calculus, general topology
algebraic topology	general topology, linear algebra
Lie theory	calculus on manifolds
homological algebra	algebraic topology