Radar, Sonar and Navigation Series 17

# Strapdown Inertial Navigation Technology

2nd Edition

David Titterton
and John Weston

# Strapdown Inertial Navigation Technology

## 2nd Edition

David Titterton
and John Weston

The Institution of Engineering and Technology

Copublished by The Institution of Engineering and Technology, London, United Kingdom and The American Institute of Aeronautics, Reston, Virginia, USA



First published 1996 (0 86341 260 2)
Second edition 2004 (0 86341 358 7)



The Institution of Engineering and Technology
Michael Faraday House
Six Hills Way, Stevenage
Herts, SG1 2AY, United Kingdom

www.theiet.org



**British Library Cataloguing in Publication Data**
Titterton, D.H. (David H.)
Strapdown inertial navigation technology. – 2nd ed. – (Radar, sonar, navigations & avionics)
1. Inertial navigation (Aeronautics) 2. Inertial navigation systems
I. Title II. Weston, J.L. (John L.) III. Institution of Electrical Engineers
629.1’351

**ISBN (10 digit) 0 86341 358 7**
**ISBN (13 digit) 978-0-86341-358-2**

Typeset in India by Newgen Imaging Systems (P) Ltd, Chennai
First printed in the UK by MPG Books Ltd, Bodmin, Cornwall

# Contents

Mr W.D. Coskren of The Charles Stark Draper Laboratories Inc.
Mr G. Vigurs of SULA Systems, UK
Dr P. Rodney, Mr D.P. McRobbie of Sperry-Sun, Halliburton Energy Services Group
Dr Milan Horemuz of the Royal Institute of Technology, Sweden
Dr R. Ebert and Dr P. Lützmann of FGAN-FOM, Germany
Prof. G.E. Stedman of the University of Canterbury, New Zealand
Prof. E.U. Schreiber of Fundamentalstation Wettzell in Bavaria, Germany, Technical University of Munich
Mr B. Waggoner of NSWC, Crane, Indiana, USA
Mr D. Goetze, SEG, Germany

some applications. Moreover, a number of other non-inertial navigational aides are reviewed.

The aim of this book is to provide a clear and concise description of the physical principles of inertial navigation; there is also a more detailed treatise covering recent developments in inertial sensor technology and the techniques for implementing such systems. This includes a discussion of the state-of-the-art of so-called MEMS devices and other novel approaches to sense angular and linear motion.

It is intended that the book should provide an up to date guide to the techniques of inertial navigation, which will be of interest to both the practising engineer and the post-graduate student. The text describes a range of technologies and evaluation techniques to enable informed judgements to be made about the suitability of competing technologies and sensors. Data are provided to give an indication of the range of performance that can be achieved from both component devices and systems.

There is a detailed description of the techniques that may be used to evaluate different technologies, covering a review of testing, characterisation and calibration methods used to ensure optimum performance is achieved from the sensors and the system. Illustrated examples are given to highlight the interaction between competing effects and their impact on performance.

These methods and techniques are drawn together in a detailed design example, which illustrates approaches for defining and analysing the problem, deriving an appropriate specification and designing a solution. The design example also considers the computational requirements, as well as interfaces and evaluation techniques. This study should be of particular interest to technologists tasked with making a system perform to a specification as it illustrates the potential interactions and the compromises that have to be made in the inevitable trade-off between parameters and performance during the design process.

This edition has a chapter describing modern and unusual application of inertial sensors and techniques. The aim is to inspire engineers and technologists to greater innovation with the vast array of technology, methods and techniques that are available to them. This chapter also includes a brief design example to illustrate the issues concerning systems for operation in a hostile environment.

This technology uses many specialised terms and expressions as well as jargon. A comprehensive glossary of terms with an explanation or definition is included to aid the understanding of the subject. Additionally, appendices are included to give further development of techniques and concepts related to strapdown inertial navigation.

It would not be possible to complete a project with this scope without the help of many people, whom we thank. We are particularly grateful to the following for their help and encouragement:

Mr S.K. Davison, Dr P. Groves, Dr K. Brunsen of QinetiQ
Mr E. Whitley, Dr R. Fountain and Mr P.G. Moore of BAE Systems
Mr J.I.R. Owen and Mr P. Marland of DSTL
Mr G. Baker of Systron-Donner Inertial Division of BEI Technologies, Inc.
Mr C.R. Milne, formerly of RAE Farnborough
Mr A.D. King formerly of Ferranti and latterly of GEC Avionics

*Chapter 1*

# Introduction

## 1.1 Navigation

Navigation is a very ancient skill or art which has become a complex science. It is essentially about travel and finding the way from one place to another and there are a variety of means by which this may be achieved [1].

Perhaps one of the simplest forms of navigation is the following of directions, or instructions. For example, a person wishing to travel to a given destination from where they are at the moment may be instructed: turn right at the next junction, turn left at the 'Rose and Crown', keep to the right of a given landmark, . . ., it will be in front of you! Clearly, this method of navigation relies on the observation and recognition of known features or fixed objects in our surroundings and moving between them. In technical narratives, the locations of these features are often referred to as 'way-points'.

An extension of this process is navigation by following a map. In this case, the navigator will determine his or her position by observation of geographical features such as roads, rivers, hills and valleys which are shown on the map. These features may be defined on the map with respect to a grid system or 'reference frame'. For example, positions of terrain features are often defined with respect to the Earth's Equator and the Greenwich meridian by their latitude and longitude. Hence the navigator is able to determine his or her position in that reference frame. As will become clear later in the text, the use of reference frames is fundamental to the process of navigation.

As an alternative method, the navigator may choose to observe other objects or naturally occurring phenomena to determine his or her position. An ancient and well-established technique is to take sightings of certain of the fixed stars to which the navigator can relate his or her position. The fixed stars effectively define a reference frame which is fixed in space. Such a reference is commonly referred to as an 'inertial' reference frame and star sightings enable an observer to determine his or her position with respect to that frame. Given knowledge of the motion of the Earth and the time of the observation, the navigator is able to use the celestial measurements to define his or her position on the surface of the Earth. Navigation systems of this type, which rely upon observation of the outside world, are known as 'position fixing' systems.

One of the principal impediments to accurate navigation at sea was the lack of an accurate time reference to determine longitude. Latitude could be determined from the celestial bodies, provided they could be observed; however, the other reference was dependent on the knowledge of the time of the observation. During the eighteenth century, the astronomer royal in the United Kingdom commissioned a number of studies to solve the 'longitude problem'. The longitude problem was eventually solved by John Harrison, a working class joiner from Lincolnshire with little formal education, who invented an accurate chronometer capable of measuring time on board ship to extraordinary accuracy [2].

An alternative approach is to use the principle of 'dead reckoning' by which present position may be calculated from knowledge of initial position and measurements of speed and direction. The process of dead reckoning is performed by taking the last known position and the time at which it was obtained, and noting the average speed and heading since that time, and the current time. The speed must be resolved though the heading angle to give velocity components north and east. Each is then multiplied by the time which has elapsed since the last position was obtained to give the change in position. Finally, the position changes are summed with initial position to obtain present position.

An equivalent process may be conducted using inertial sensors – gyroscopes and accelerometers – to sense rotational and translational motion with respect to an inertial reference frame. This is known as inertial navigation.

## 1.2 Inertial navigation

The operation of inertial navigation systems depends upon the laws of classical mechanics as formulated by Sir Isaac Newton. Newton's laws tell us that the motion of a body will continue uniformly in a straight line unless disturbed by an external force acting on the body. The laws also tell us that this force will produce a proportional acceleration of the body. Given the ability to measure that acceleration, it would be possible to calculate the change in velocity and position by performing successive mathematical integrations of the acceleration with respect to time. Acceleration can be determined using a device known as an accelerometer. An inertial navigation system usually contains three such devices, each of which is capable of detecting acceleration in a single direction. The accelerometers are commonly mounted with their sensitive axes perpendicular to one another, i.e. mutually perpendicular.

In order to navigate with respect to our inertial reference frame, it is necessary to keep track of the direction in which the accelerometers are pointing. Rotational motion of the body with respect to the inertial reference frame may be sensed using gyroscopic sensors and used to determine the orientation of the accelerometers at all times. Given this information, it is possible to resolve the accelerations into the reference frame before the integration process takes place.

Hence, inertial navigation is the process whereby the measurements provided by gyroscopes and accelerometers are used to determine the position of the vehicle in which they are installed. By combining the two sets of measurements, it is possible to define the translational motion of the vehicle within the inertial reference frame and so to calculate its position within it.

Unlike many other types of navigation system, inertial systems are entirely self-contained within the vehicle, in the sense that they are not dependent on the transmission of signals from the vehicle or reception from an external source. However, inertial navigation systems do rely upon the availability of accurate knowledge of vehicle position at the start of navigation. The inertial measurements are then used to obtain estimates of changes in position which take place thereafter.

## 1.3 Strapdown technology

Whilst the underlying principles of operation are common to all types of inertial navigation system, their implementation may take a variety of different forms. The original applications of inertial navigation technology used stable platform techniques.[1] In such systems, the inertial sensors are mounted on a stable platform and are mechanically isolated from the rotational motion of the vehicle. Platform systems are still in common use, particularly for those applications requiring very accurate estimates of navigation data, such as ships and submarines.

Modern systems have removed most of the mechanical complexity of platform systems by having the sensors attached rigidly, or 'strapped down', to the body of the host vehicle. The potential benefits of this approach are lower cost, reduced size and greater reliability compared with equivalent platform systems. As a result, small, light weight and accurate inertial navigation systems may now be fitted to small guided missiles, for instance. The major penalties incurred are a substantial increase in computing complexity and the need to use sensors capable of measuring much higher rates of turn. However, recent advances in computer technology combined with the development of suitable sensors have allowed such designs to become a reality.

Inertial navigation systems of this type, usually referred to as 'strapdown' inertial navigation systems, are the subject of this book. Whilst there are many books which described the older and well-established platform technology, no similar book exists which deals explicitly with strapdown systems. It was this fact which provided the primary motivation for this publication.

This text describes the basic concepts of inertial navigation and the technological developments which have led to modern strapdown systems. It is intended to provide an introduction to the subject of strapdown inertial navigation which may be read at various levels by both suppliers of inertial sensors and systems and customers for such products and so encourage a more effective two-way dialogue.

By selective reading, the engineer new to the subject may obtain a background understanding of the subject. For those needing to become more closely involved in the various aspects of strapdown system technology, the text provides a more

[1] A major advance occurred in 1953 with the demonstration of the feasibility of all-inertial navigation in flight trials with a system called SPIRE (Space Inertial Reference Equipment) which was 5 ft in diameter and weighed 2700 lb.

extensive description of system configurations, an appreciation of strapdown inertial sensors and computational requirements and an awareness of techniques which may be used to analyse and assess the performance of such systems. References are provided for those seeking more detailed information on different aspects of the subject.

Strapdown inertial navigation systems rely on complex technology and many technology specific terms and jargon are in common usage. Such terminology is defined in the glossary of terms.

Where appropriate, mathematical descriptions of the physical principles and processes involved are presented. The reader new to the subject, who perhaps wishes to gain an appreciation of physical principles without dwelling on the mathematical details of the processes involved, may merely wish to take note of the results of the more mathematical sections, or possibly to skip over these aspects altogether.

## 1.4 Layout of the book

Chapter 2 introduces the underlying concepts of strapdown inertial navigation systems with the aid of simplified examples, and culminates in the definition of the basic functions which must be implemented within such a system. It is shown how the measurements of rotational and translational motion are fundamental to the operation of an inertial navigation system. There follows a brief review of the historical developments which have led to the current state of development of strapdown inertial navigation systems. This is accompanied by an outline discussion of system applications.

The way in which the measurements of rotational and translational motion are combined to form an inertial navigation system are addressed more fully in Chapter 3. This chapter deals at some length with attitude computation and the concept of the navigation equation, both of which are fundamental to the operation of strapdown inertial navigation systems. In addition, a number of possible system configurations are described.

Gyroscope and accelerometer technologies are discussed in some detail in Chapters 4–7. This part of the text provides descriptions of the various instrument types currently in use and some which are likely to become available within the foreseeable future. These include conventional angular momentum gyroscopes, optical rate sensors such as the ring laser gyroscope and the fibre optic gyroscope, pendulous force-feedback accelerometers, solid-state devices and cold-atom sensors, as well as multisensors. The text covers mechanical and electronic aspects of the instruments, measurement accuracy, mathematical descriptions and applications. Chapter 7 is devoted to the description and performance of micro-machined electromechanical systems (MEMS) sensor technology which has, in recent years, found broad application in modern navigation and stabilisation systems.

The testing, calibration and compensation of inertial sensors and systems is addressed in Chapter 8. Predictable errors can be corrected or compensated from observation of performance by a process which involves the implementation of algorithms which are as close as possible to the inverse of the classical sensor error models.

Chapter 9 describes the basic building blocks which combine to form a strapdown inertial system, drawing attention to alternative mechanisations.

A vital factor in the achievement of accurate navigation is the initialisation of the inertial navigation system before the commencement of navigation, prior to take-off in the case of an aircraft navigation system, for instance. This process involves the accurate determination of the position, velocity and attitude of the vehicle navigation system with respect to the chosen reference frame, and is usually referred to as inertial navigation system alignment. Such alignment may have to be undertaken in a vehicle which is moving, as in the case of the in-flight alignment of an airborne inertial navigation system. The difficulties of achieving an accurate alignment in various vehicle applications are highlighted, and techniques for alleviating such problems are described in Chapter 10.

The computer processing of the gyroscope and accelerometer measurements which must be carried out in order to complete the task of navigation is examined in Chapter 11. Computational algorithms are discussed in some detail during the course of this chapter.

Techniques for the analysis of inertial navigation system performance are presented in Chapter 12 to enable the designer to assess system performance. Attention is drawn to a number of errors which are of particular concern in strapdown systems and the use of simulation methods to assess system performance is highlighted.

It is common practice for many applications to combine the outputs of an inertial navigation system with some external measurement data to achieve an overall improvement in navigation accuracy. For example, independent position fixes may be used to aid an inertial navigation system, and so enhance navigation performance beyond that which may be obtained using either the position fixing system or the inertial navigator in isolation. Possible navigation aids are discussed and techniques are presented for mixing inertial and external measurement data to form a so-called integrated navigation system in Chapter 13.

Chapter 14 draws together much of the preceding text through the discussion of a design example. Because the background of the authors is predominantly in the field of guided missile systems, this part of the book is directed at such an application. The design example will be of particular interest to the engineer wishing to specify a system to meet a given requirement and to assess its potential performance.

Finally, Chapter 15 describes a number of applications of inertial navigation systems for both military and civil roles covering systems that operate at sea, in the air, as well as on and below the ground. This chapter aims to describe the particular problems encountered in attempting to design navigation and stabilisation systems fit for the broad range of roles in which this technology has been applied.

The appendices provide descriptions of Kalman filtering techniques, inertial navigation error budgets, inertial system configurations and a comparison of GPS and GLONASS satellite navigation systems. A glossary of principal terms used is given at the end of the book.

## References

1 ANDERSON, E.W.: ‘The principles of navigation’ (Hollis and Carter, 1966)
2 SOBEL, D.: ‘Longitude: the true story of a lone genius who solved the greatest scientific problem of his time’ (Penguin Books, 1996)

*Chapter 2*

# Fundamental principles and historical developments of inertial navigation

## 2.1 Basic concepts

The basic concepts of inertial navigation are outlined here with the aid of some simple examples.

A simple one-dimensional example of navigation involves the determination of the position of a train which is moving along a track between two locations on a perfectly flat plane. It is possible to determine the instantaneous speed of the train and the distance it has travelled from a known starting point by using measurements of its acceleration along the track. Sensors called accelerometers provide such information about their own movement. If an accelerometer is fixed in the train, it will provide information about the acceleration of the train. The time integral of the acceleration measurement provides a continuous estimate of the instantaneous speed of the train, provided its initial speed was known. A second integration yields the distance travelled with respect to a known starting point. The accelerometer together with a computer, or other suitable device capable of integration, therefore constitutes a simple one-dimensional navigation system.

In general, a navigation system is required to provide an indication of the position of a vehicle with respect to a known grid system or reference frame. For instance, it may be required to determine the location of a vehicle in terms of $x$ and $y$ coordinates in a Cartesian reference frame. Considering again the example of a train moving along a track, as depicted in Figure 2.1, it is now necessary to determine the position of the train with respect to the coordinate reference frame shown in the figure.

Given the knowledge of the train's acceleration along the track, and the angle which the track makes with the reference frame, the $x$ and $y$ coordinate positions may be determined. This may be accomplished by resolving the measured acceleration in the reference frame to give $x$ and $y$ components, and by suitable integration of the resolved signals to yield the velocity and position of the train in reference axes.

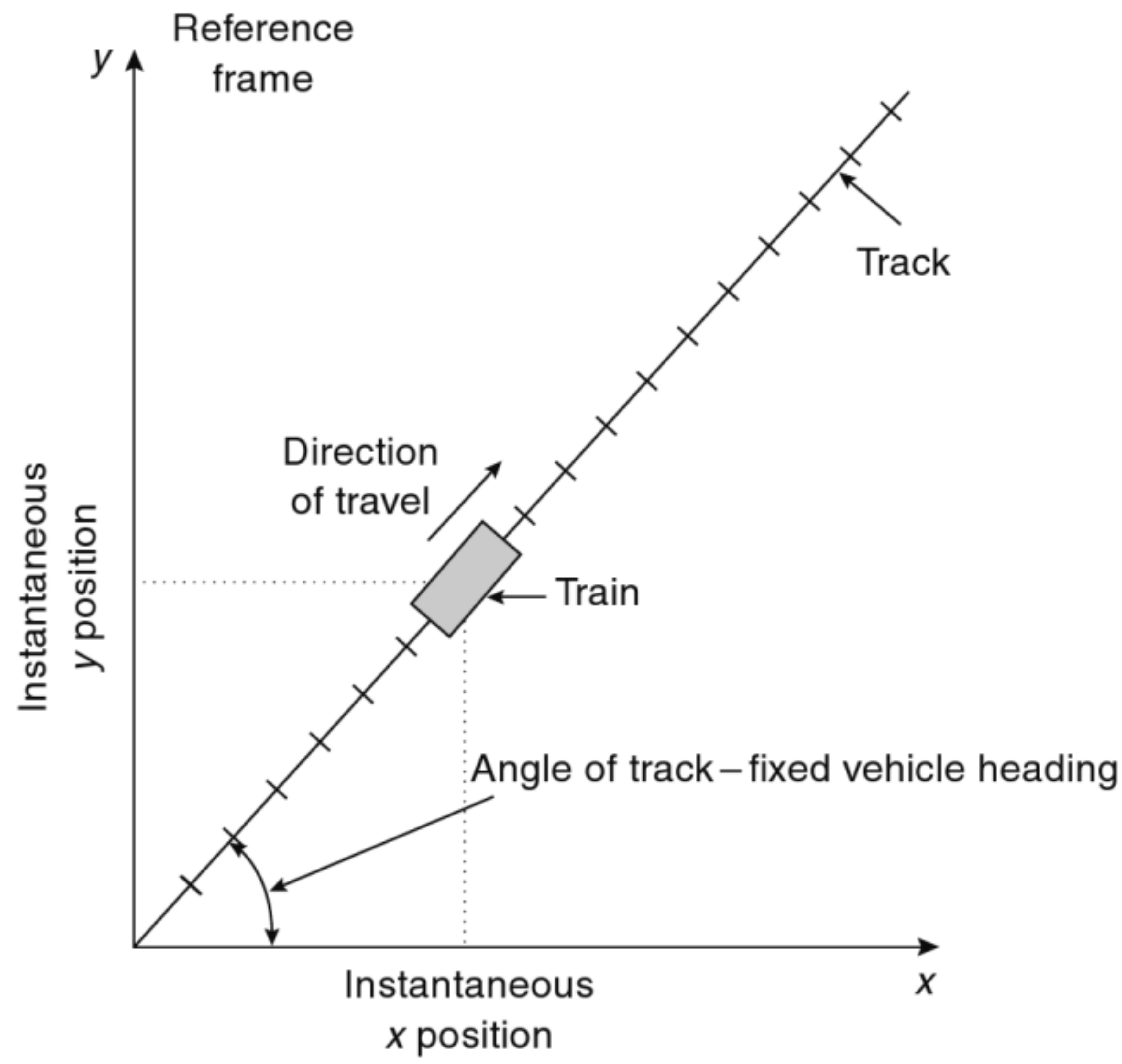


*Figure 2.1 One-dimensional navigation*

In this simple case, the angle of the track defines the heading of the train with respect to the reference frame.

With the more general situation illustrated in Figure 2.2, where the track curves, it is necessary to detect continuously the translational motion of the train in two directions and changes in its direction of travel, that is, to detect the rotations of the train about the perpendicular to the plane of motion as the train moves along the track.

Two accelerometers are now required to detect the translational motion in perpendicular directions along and perpendicular to the track. One sensor suitable for the measurement of the rotational motion is a gyroscope. Depending on the form of construction of this sensor, it may be used to provide either a direct measure of the train's heading with respect to the reference frame, or a measurement of the turn rate of the train. In the latter case, the angular orientation of the train may be calculated by the integration of this measurement, provided the angle is known at the start of navigation. Given such information, it is possible to relate the measurements of acceleration, which are obtained in an axis set which is fixed in the train, to the reference frame. The instantaneous measurements of acceleration may therefore be resolved in the reference frame and integrated with respect to time to determine the instantaneous velocity and position of the vehicle with respect to that frame.

Clearly then, it is possible to construct a simple, two-dimensional, navigation system using a gyroscope, two accelerometers and a computer. In practice, the inertial sensors may be mounted on a platform which is stabilised in space, and hence isolated from the rotation of the vehicle, or mounted directly on to the vehicle to form a strapdown system. The measurements are processed in the computer to provide continuous estimates of the position, speed and the direction of travel or heading of

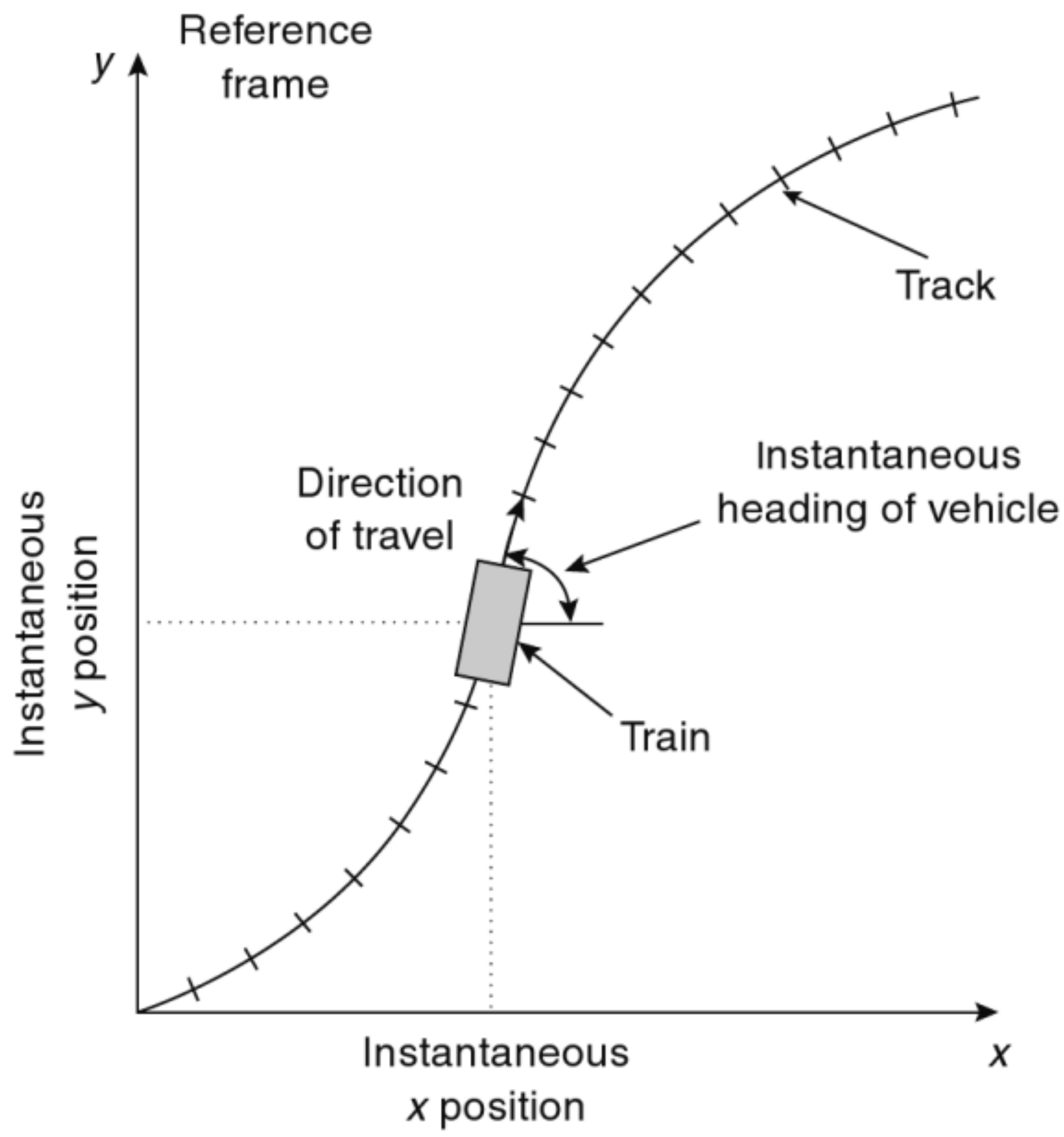


*Figure 2.2* *Two-dimensional navigation*

the train. It must be stressed that inertial navigation is fundamentally dependent on an accurate knowledge of position, speed and heading being available prior to the start of navigation. This is because it uses dead reckoning which relies for its operation on the updating of the system's previous estimates of these navigational quantities, commencing with the initial values input to the system at the start of navigation.

It will be apparent from the preceding discussion that successful navigation of a vehicle can be achieved by using the properties of suitable sensors mounted in the vehicle. In general, it is required to determine a vehicle's position with respect to a three-dimensional reference frame. Consequently, if single-axis sensors are used, three gyroscopes will be required to provide measurements of vehicle turn rates about three separate axes, whilst three accelerometers provide the components of acceleration which the vehicle experiences along these axes. For convenience and accuracy, the three axes are usually chosen to be mutually perpendicular.

In most applications, the axis set defined by the sensitive axes of the inertial sensors is made coincident with the axes of the vehicle, or body, in which the sensors are mounted, usually referred to as the body axis set. The measurements provided by the gyroscopes are used to determine the attitude and heading of the body with respect to the reference frame in which it is required to navigate. The attitude and heading information is then used to resolve the accelerometer measurements into the reference frame. The resolved accelerations may then be integrated twice to obtain vehicle velocity and position in the reference frame.

Gyroscopes provide measurements of changes in vehicle attitude or its turn rate with respect to inertial space. Accelerometers, however, are unable to separate the total acceleration of the vehicle, the acceleration with respect to inertial space, from

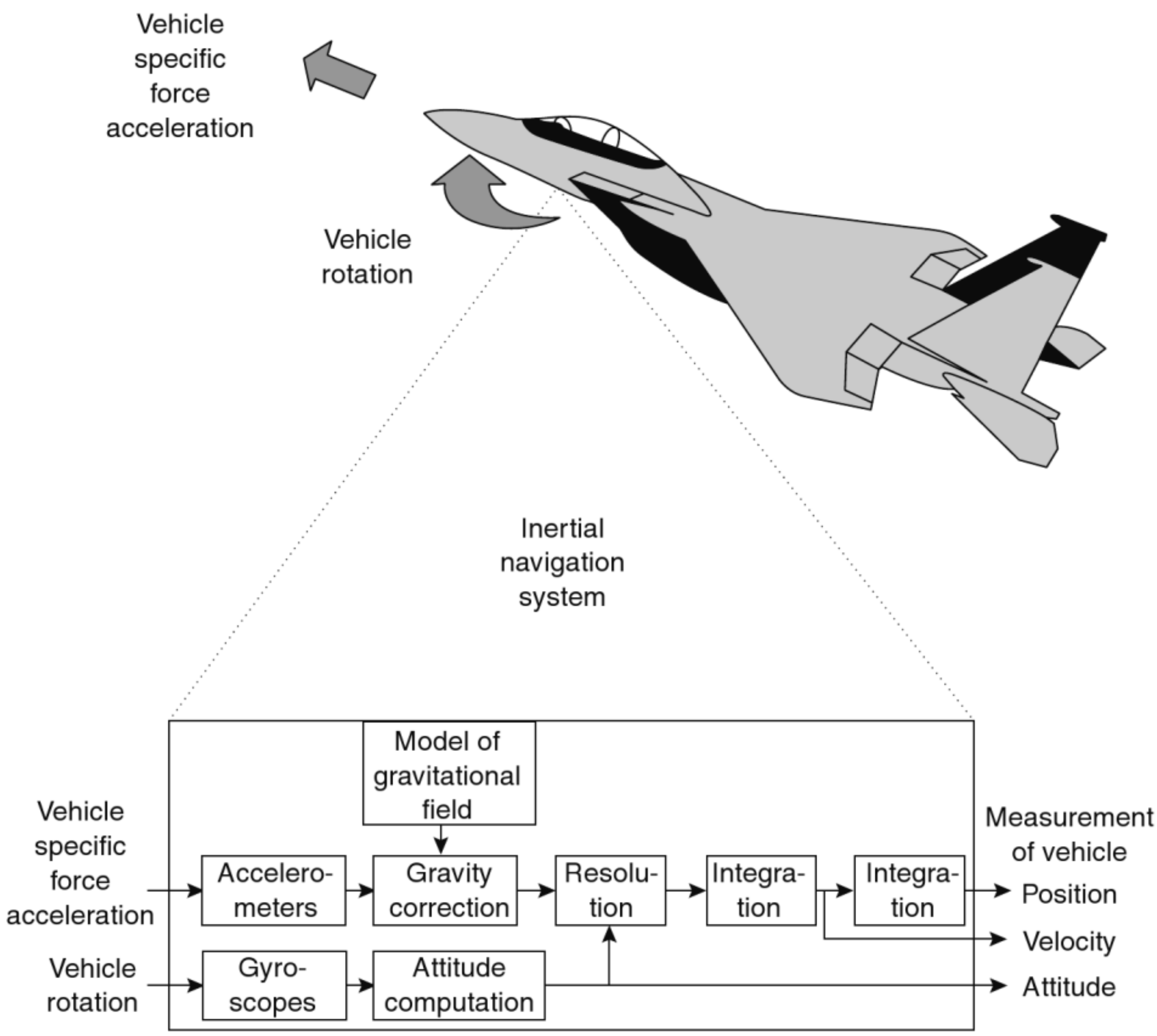


*Figure 2.3 Functional components of an inertial navigation system*

that caused by the presence of a gravitational field. These sensors do in fact provide measurements of the difference between the true acceleration in space and the acceleration due to gravity.[1] This quantity is the non-gravitational force per unit mass exerted on the instrument, referred to in this text for brevity as 'specific force' [1].

Hence, the measurements provided by the accelerometers, especially when close to a large body such as the Earth, must be combined with knowledge of the gravitational field of that body in order to determine the acceleration of the vehicle with respect to inertial space. Using this information, vehicle acceleration relative to the body may be derived.

The navigational function is therefore fulfilled by combining the measurements of vehicle rotation and specific force with knowledge of the gravitational field to compute estimates of attitude, velocity and position with respect to a pre-defined reference frame. A schematic representation of such an inertial navigation system is shown in Figure 2.3.

[1] Algebraically, the sum of the acceleration with respect to inertial space and the acceleration due to gravitational attraction.

of marine or space applications, such systems may be required to provide navigation data to similar accuracy over periods of weeks, months or even longer in the case of interplanetary exploration. One extreme example is the Voyager spacecraft which has been navigating through the solar system and beyond for more than 25 years.

Although the basic principles of inertial navigation systems do not change from one application to another, it will come as no surprise to find that the accuracy of the inertial sensors and the precision to which the associated computation must be carried out varies dramatically over the broad range of applications indicated earlier. It follows therefore that the instrument technologies and the techniques used for the implementation of the navigation function in such diverse applications also vary greatly. Part of the function of this text is to provide some insight into the methods and technologies appropriate to some of the different types of inertial system application outlined earlier.

## 2.5 Trends in inertial sensor development

A number of categories of inertial sensor have made significant progress over the last decade. The development of micro-machined electromechanical systems (MEMS) devices has been spectacular and the performance that can be achieved is approaching inertial grade, so that is likely to be readily achieved in the near term. The development of micro-optical machined electromechanical systems (MOEMS) is expected to provide very high performance sensors in the medium term.

Progress with refinement of the fibre optical gyroscope is likely to continue so it should start to replace the more expensive ring laser gyroscope in the near term.

New techniques, such as the cold atom interferometers, are being researched for highly specialised investigations and very precise measurements, but these approaches are a long way from maturity.

The big drive from the systems applications viewpoint will be further reduction in cost and complexity, with a corresponding leap in reliability.

Inertial sensor performance is discussed in Chapters 4–7.

## References


1 BRITTING, K.: ‘Inertial navigation system analysis’ (Wiley Interscience, New York, 1971)

2 The Bible, Amos 7:7–9

3 SORG, H.W.: ‘From Serson to Draper – two centuries of gyroscopic development’, *Journal of Institute of Navigation*, 1976–77, **23** (4)

4 SCHULER, M.: ‘Die Storung von Pendul-und Kreiselapparaten durch die Beschleunigung der Fahrzeuges’, *Physikalische Zeitschrift B*, 1923, **24**

5 WRIGLEY, W., HOLLISTER, W.M., and DENHARD, W.G.: ‘Gyroscopic theory, design and instrumentation’ (MIT Press, 1969)

6 DRAPER, C., WRIGLEY, W., and HOVORKA, J.: ‘Inertial guidance’ (Pergamon Press, 1960)

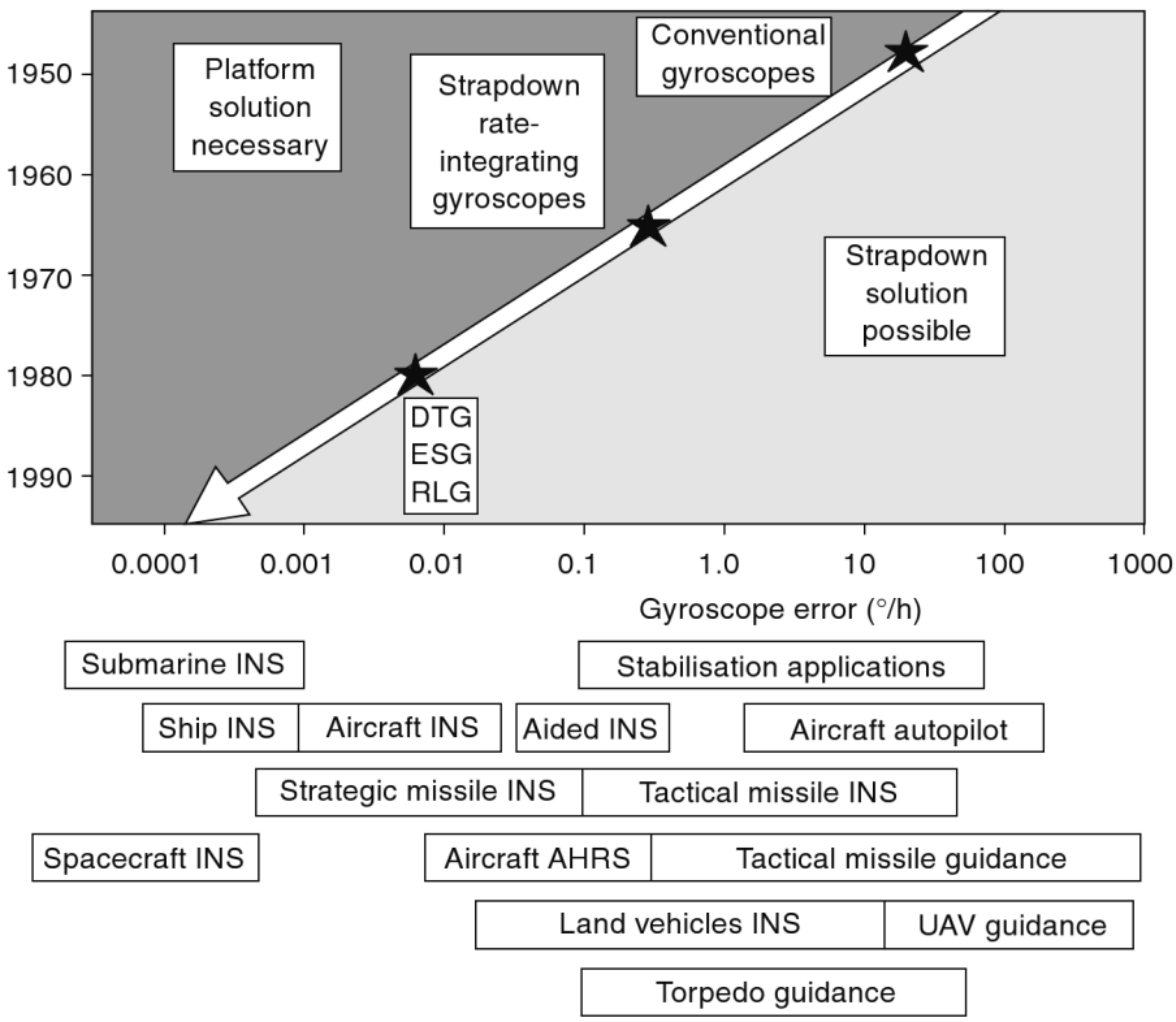


*Figure 2.4 Strapdown sensor development and some applications*

applications as indicated in Figure 2.4. The diagram shows other applications for strapdown technology, the accuracy required from the gyroscopes being related to the position and size of the box in which it is mentioned.

## 2.4 The modern-day inertial navigation system

From the preceding section, it is clear that the range of applications in which inertial navigation systems can and are being used is very extensive, covering navigation of ships, aircraft, tactical and strategic missiles and spacecraft. In addition, there are some more novel applications in the field of robotics, active suspension in racing or high performance motor cars and for surveying underground wells and pipelines.

Such diverse applications call for navigation systems having a very broad range of performance capabilities, as well as large differences in the periods of time over which they will be required to provide navigation data. For instance, tactical missile applications may require inertial navigation and guidance to an accuracy of a few hundred metres for periods of minutes or even a few seconds, whilst other airborne systems are required to operate for several hours whilst maintaining knowledge of aircraft position to an accuracy of one or two nautical miles or better. In the cases

*Chapter 3*

# Basic principles of strapdown inertial navigation systems

## 3.1 Introduction

The previous chapter has provided some insight into the basic measurements that are necessary for inertial navigation. For the purposes of the ensuing discussion, it is assumed that measurements of specific force and angular rate are available along and about axes which are mutually perpendicular. Attention is focused on how these measurements are combined and processed to enable navigation to take place.

## 3.2 A simple two-dimensional strapdown navigation system

We begin this chapter by describing a simplified two-dimensional strapdown navigation system. Although functionally identical to the full three-dimensional system discussed later, the computational processes which must be implemented to perform the navigation task in two dimensions are much simplified compared with a full strapdown system. Therefore, through this introductory discussion, it is hoped to provide the reader with an appreciation of the basic processing tasks which must be implemented in a strapdown system without becoming too deeply involved in the intricacies and complexities of the full system computational tasks.

For the purposes of this discussion, it is assumed that a system is required to navigate a vehicle which is constrained to move in a single plane. A two-dimensional strapdown system capable of fulfilling this particular navigation task was introduced very briefly in Chapter 2 and is shown diagrammatically in Figure 3.1.

The system contains two accelerometers and a single axis rate gyroscope, all of which are attached rigidly to the body of the vehicle. The vehicle body is represented, in the figure, by the block on which the instruments shown are mounted. The sensitive axes of the accelerometers, indicated by the directions of the arrows in the diagram, are at right angles to one another and aligned with the body axes of the

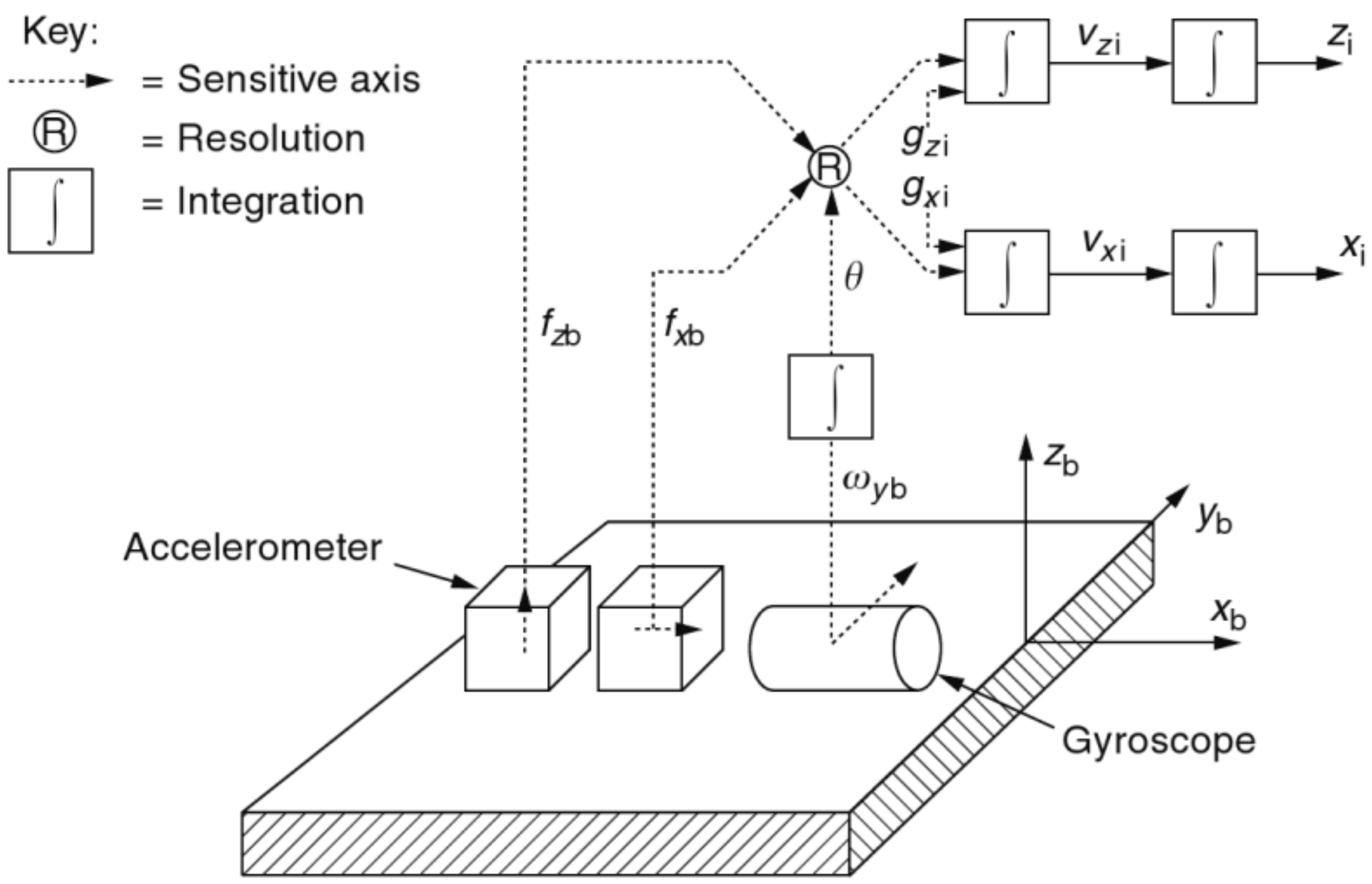


*Figure 3.1 Two-dimensional strapdown inertial navigation system*

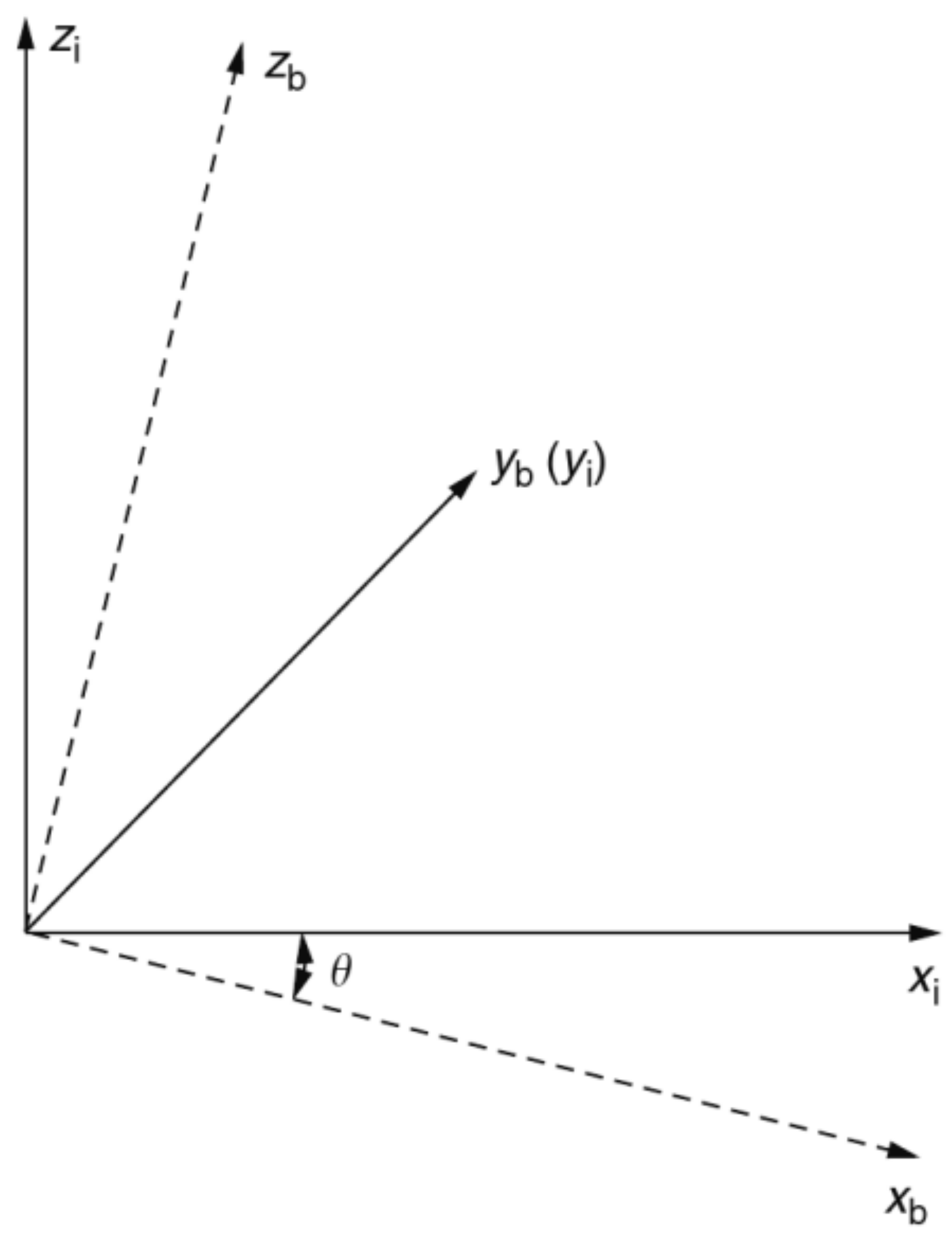


*Figure 3.2 Reference frames for two-dimensional navigation*

vehicle in the plane of motion; they are denoted as the $x_b$ and $z_b$ axes. The gyroscope is mounted with its sensitive axis orthogonal to both accelerometer axes allowing it to detect rotations about an axis perpendicular to the plane of motion; the $y_b$ axis. It is assumed that navigation is required to take place with respect to a space-fixed reference frame denoted by the axes $x_i$ and $z_i$. The reference and body axis sets are shown in Figure 3.2, where $\theta$ represents the angular displacement between the body and reference frames.

## 2.2 Summary

It follows from the introductory discussion that the essential functions which an inertial navigation system must perform may be defined as follows:

- determination of the angular motion of a vehicle using gyroscopic sensors, from which its attitude relative to a reference frame may be derived;
- measure specific force using accelerometers;
- resolve the specific force measurements into the reference frame using the knowledge of attitude derived from the information provided by the gyroscopes;
- evaluate the force resulting from the gravitational field – the gravitational attraction of the Earth in the case of systems operating in the vicinity of the Earth;
- integrate the resolved specific force measurements to obtain estimates of the velocity and position of the vehicle.

The later chapters describe the principles of inertial navigation in some depth and provide detailed information on system mechanisations, inertial sensor technology, computational aspects (including algorithms), design analysis and applications of such systems. However, prior to this, it is instructive to have a brief review of the historical developments which have led to the current state of development of present day inertial navigation systems and their technology.

## 2.3 Historical developments

From the earliest times, people have moved from one place to another by finding or 'knowing' their way; this skill has required some form of navigation. There is an oblique reference to inertial navigation in the Bible [2]. Generally, as in the case of the biblical reference, the earliest applications were on land. Then as the desire developed to explore farther afield, instruments were developed for marine applications. More recently, there have been significant developments in inertial sensors, and systems for inertial navigation on land, in the air, on or under the oceans as well as in space to the planets and beyond.

Our earliest ancestors travelled in search of food, usually on land. As they developed, they crossed rivers generally using landmarks, that is, navigation by observation. Further development of position fixing techniques saw the Polynesians cross the Pacific Ocean about two millennia ago using their understanding of celestial bodies and landmarks. These techniques can only be used in clear weather conditions. During the thirteenth century, the Chinese discovered the properties of lodestone and applied the principles of magnetism to fabricate a compass. They used this instrument to navigate successfully across the south China Sea. This device could be used irrespective of visibility but was difficult to use in rough weather. The other significant development to help the long distance traveller was the sextant, which enabled position fixes to be made accurately on land.

In the seventeenth century, Sir Isaac Newton defined the laws of mechanics and gravitation, which are the fundamental principles on which inertial navigation is based. Despite this, it was to be about another two centuries before the inertial sensors were developed that would enable the demonstration of inertial navigation techniques. However, in the early eighteenth century, there were several significant developments; Serson demonstrating a stabilised sextant [3] and Harrison devising an accurate chronometer, the former development enabling sightings to be taken of celestial objects without reference to the horizon and the latter enabling an accurate determination of longitude. These instruments, when used with charts and reference tables of location of celestial bodies, enabled accurate navigation to be achieved, provided the objects were visible.

Foucault is generally credited with the discovery of the gyroscopic effect in 1852. He was certainly the first to use the word. There were others, such as Bohneberger, Johnson and Lemarle, developing similar instruments. All of these people were investigating the rotational motion of the Earth and the demonstration of rotational dynamics. They were using the ability of the spin axis of a rotating disc to remain fixed in space. Later in the nineteenth century, many fine gyroscopic instruments were made. In addition, there were various ingenious applications of the gyroscopic principle in heavy equipment such as the grinding mill.

A significant discovery was made in 1890 by Professor G.H. Bryan concerning the ringing of hollow cylinders, a phenomenon later applied to solid-state gyroscopes.

The early years of the twentieth century saw the development of the gyrocompass for the provision of a directional reference. The basic principle of this instrument is the indication of true north by establishing the equilibrium between the effect of its pendulosity and the angular momentum of the rotating base carrying the compass. Initially, this instrument was sensitive to acceleration. Professor Max Schuler produced an instrument with a vertical erection system enabling an accurate vertical reference to be defined [4]. This instrument was tuned to the undamped natural period defined by $2\pi\sqrt{R/g}$, approximately equal to 84 minutes, where $R$ is the radius of the Earth and $g$ is the acceleration caused by the Earth's gravitational field. Later, this technique became known as 'Schuler tuning' [5], a phrase originated by Dr Walter Wrigley of MIT. This ingenious method produced a directional instrument insensitive to acceleration for use at sea. Elmer and Lawrence Sperry improved the design of the gyrocompass with further refinements by Brown and Perry. These instruments provided the first steps towards all-weather, autonomous navigation. The Sperry brothers were also at the forefront of the application of the gyroscopic effect to control and guidance in the early twentieth century. They produced navigation and autopilot equipment for use in aircraft and gyroscopes for use in torpedoes.

Rate of turn indicators, artificial horizons and directional gyroscopes for aircraft were being produced in the 1920s. At a similar time, side-slip sensors were being developed, early open-loop accelerometers, and Schuler was demonstrating a north-seeking device for land use giving an accuracy of 22 seconds of arc! There was significant progress during the early part of the twentieth century with the development of stable platforms for fire control systems for guns on ships and the identification of the concept for an inertial navigation system. Boykow identified the use of accelerometers and gyroscopes to produce a full inertial navigation system.

However, at this stage, the quality of the inertial sensors was not suitable for the production and demonstration of such a system.

World War II saw the demonstration of the principles of inertial guidance in the V1 and V2 rockets by German scientists, a prime step forward being the use of a system with feedback leading to accurate guidance. At this time there was much activity in various parts of the world devising new types of inertial sensors, improving their accuracy and, in 1949, the first publication suggesting the concept of the strapdown technique for navigation.

The pace of development and innovation quickened in the 1950s with many significant developments for seaborne and airborne applications. More accurate sensors were produced, with the accuracy of the gyroscope being increased substantially. The error in such sensors was reduced from about 15°/hour to about 0.01°/hour, Professor Charles Stark Draper and his coworkers at MIT being largely responsible for many technical advances with the demonstration of the floated rate-integrating gyroscope [6]. It was also during the 1950s that the principle of force-feedback was applied to the proof mass in an accelerometer to produce an accurate acceleration sensing instrument.

The early part of the 1950s saw the fabrication of a stabilised platform inertial navigation system followed by the first crossing of the United States of America by an aircraft using full inertial navigation. Inertial navigation systems became standard equipment in military aircraft, ships and submarines during the 1960s, all of these applications using the so-called stable platform technology. This era saw further significant developments with increases in the accuracy of sensors, the miniaturisation of these devices and the start of ring laser gyroscope development. Major projects of this period in which inertial system technology was applied were the ballistic missile programmes and the exploration of space.

Similar progress has taken place in the last two decades; one major advance being the application of the micro-computer and development of gyroscopes with large dynamic ranges enabling the strapdown principle to be realised. This has enabled the size and complexity of the inertial navigation system to be reduced significantly for very many applications. The use of novel methods has enabled small, reliable, rugged and accurate inertial sensors to be produced that are relatively inexpensive, thus enabling a very wide range of diverse applications as discussed below. This period has also seen significant advances in the development of solid-state sensors such as optical fibre gyroscopes and silicon accelerometers.

The development of inertial navigation systems in recent years has been characterised by the gradual move from stable platform to strapdown technology as indicated in Figure 2.4. The figure gives an indication of the increasing application of strapdown systems which has resulted from advances in gyroscope technology. Milestones in this continuing development have occurred as a result of the development of the miniature rate-integrating gyroscope, the dynamically tuned gyroscope and more recently, ring laser and fibre optic rate sensors and vibratory gyroscopes, all of which are described in Chapters 4 and 5. MEMS sensors have provided an exciting development that should expand the range of applications of inertial navigation.

Strapdown systems are becoming widely used for aircraft and guided missile applications. More recently this technology has been applied to ship and submarine

$$\dot{\theta} = \omega_{yb}$$
$$f_{xi} = f_{xb}\cos\theta + f_{zb}\sin\theta$$
$$f_{zi} = -f_{xb}\sin\theta + f_{zb}\cos\theta$$
$$\dot{v}_{xi} = f_{xi} + g_{xi}$$
$$\dot{v}_{zi} = f_{zi} + g_{zi}$$
$$\dot{x}_i = v_{xi}$$
$$\dot{z}_i = v_{zi}$$

*Figure 3.3 Two-dimensional strapdown navigation system equations*

Referring now to Figure 3.1, body attitude, $\theta$, is computed by integrating the measured angular rate, $\omega_{yb}$, with respect to time. This information is then used to resolve the measurements of specific force, $f_{xb}$ and $f_{zb}$, into the reference frame. A gravity model, stored in the computer, is assumed to provide estimates of the gravity components in the reference frame, $g_{xi}$ and $g_{zi}$. These quantities are combined with the resolved measurements of specific force, $f_{xi}$ and $f_{zi}$, to determine true accelerations, denoted by $\dot{v}_{xi}$ and $\dot{v}_{zi}$. These derivatives are subsequently integrated twice to obtain estimates of vehicle velocity and position. The full set of equations which must be solved are given in Figure 3.3.

Having defined the basic functions which must be implemented in a strapdown inertial navigation system, consideration is now given to the application of the two-dimensional system, described above, for navigation in a rotating reference frame. For instance, consider the situation where it is required to navigate a vehicle moving in a meridian plane around the Earth, as depicted in Figure 3.4. Hence, we are concerned here with a system which is operating in the vertical plane alone. Such a system would be required to provide estimates of velocity with respect to the Earth, position along the meridian and height above the Earth.

Whilst the system mechanisation as described could be used to determine such information, this would entail a further transformation of the velocity and position, derived in space fixed coordinates, to a geographic frame. An alternative and often used approach is to navigate directly in a local geographic reference frame, defined in this simplified case by the direction of the local vertical at the current location of the vehicle. In order to provide the required navigation information, it now becomes necessary to keep track of vehicle attitude with respect to the local geographic frame denoted by the axes $x$ and $z$. This information can be extracted by differencing the successive gyroscopic measurements of body turn rate with respect to inertial space, and the current estimate of the turn rate of the reference frame with respect to inertial space. For a vehicle moving at a velocity, $v_x$, in a single plane around a perfectly spherical Earth of radius $R_0$, this rate is given by $v_x/(R_0 + z)$ where $z$ is the height of the vehicle above the surface of the Earth. This is often referred to as the transport rate.

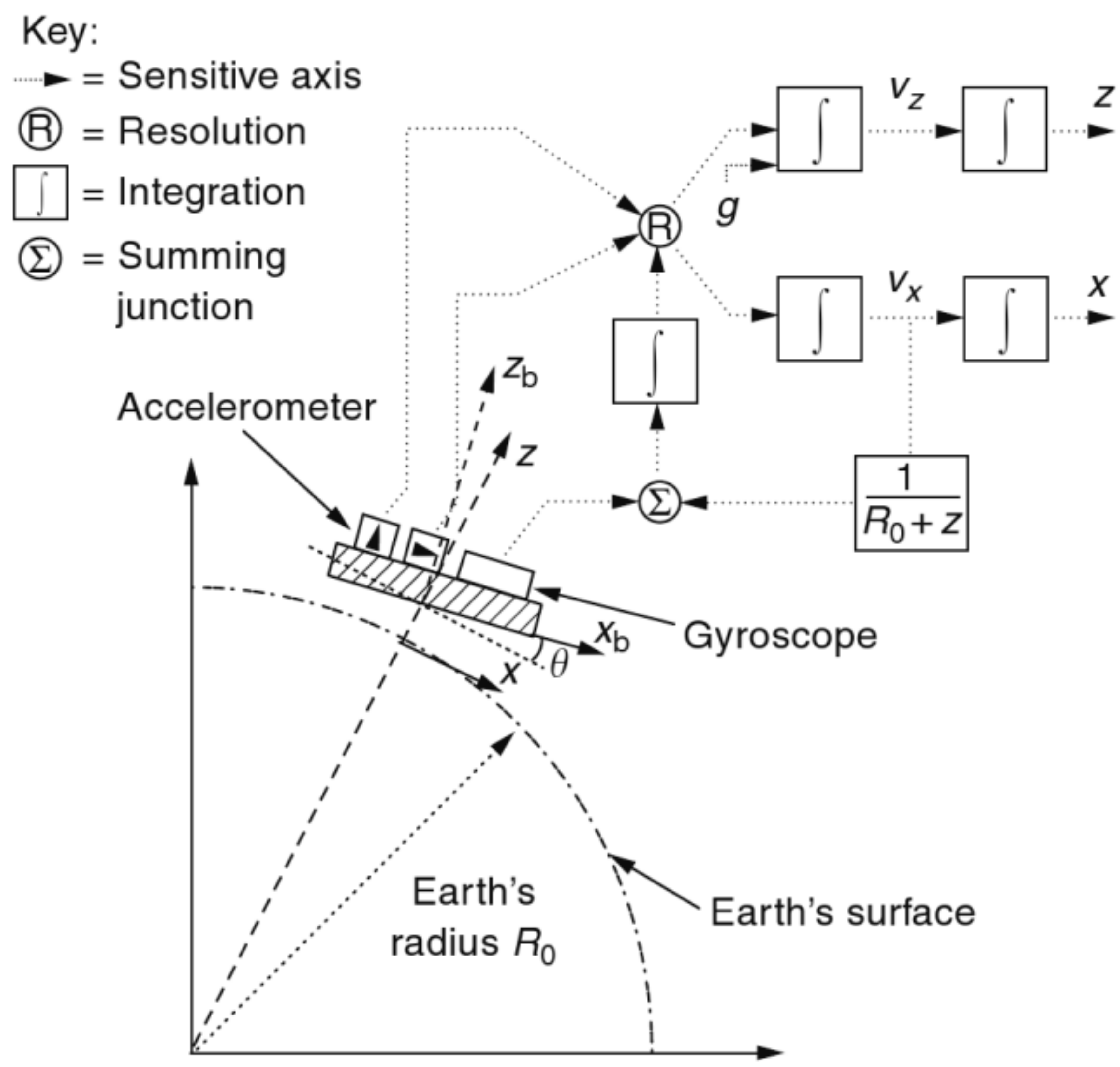


*Figure 3.4 Two-dimensional strapdown inertial system for navigation in a rotating reference frame*

$$\dot{\theta} = \omega_{yb} - v_x / (R_0 + z)$$
$$f_x = f_{xb} \cos\theta + f_{zb} \sin\theta$$
$$f_z = -f_{xb} \sin\theta + f_{zb} \cos\theta$$
$$\dot{v}_x = f_x + v_x v_z / (R_0 + z)$$
$$\dot{v}_z = f_z + g - v_x^2 / (R_0 + z)$$
$$\dot{x} = v_x$$
$$\dot{z} = v_z$$

*Figure 3.5 Simplified two-dimensional strapdown system equations for navigation in a rotating reference frame*

Figure 3.4 shows a modified two-dimensional strapdown system for navigation in the moving reference frame. As shown in the figure, an estimate of the turn rate of the reference frame is derived using the estimated component of horizontal velocity.

The equations which must be solved in this system are given in Figure 3.5.

Comparison with the equations given in Figure 3.3, relating to navigation with respect to a space-fixed axis set, reveals the following differences. The attitude computation is modified to take account of the turn rate of the local vertical reference frame as described above. Consequently, the equation in $\theta$ is modified by the subtraction of the term $v_x/(R_0 + z)$ in Figure 3.4. The terms $v_x v_z/(R_0 + z)$ and $v_x^2/(R_0 + z)$ which appear in the velocity equations are included to take account of the additional forces acting as the system moves around the Earth (Coriolis forces, see Section 3.4). The gravity term ($g$) appears only in the $v_z$ equation as it is assumed that the Earth's gravitational acceleration acts precisely in the direction of the local vertical.

This section has outlined the basic form of the computing tasks to be implemented in a strapdown navigation system using a much simplified two-dimensional representation. In the remainder of this chapter the extension of this simple strapdown system to three dimensions is described in some detail. It will be appreciated that this entails a substantial increase in the complexity of the computing tasks involved. In particular, attitude information in three dimensions can no longer be obtained by a simple integration of the measured turn rates.

## 3.3 Reference frames

Fundamental to the process of inertial navigation is the precise definition of a number of Cartesian co-ordinate reference frames. Each frame is an orthogonal, right-handed, co-ordinate frame or axis set.

For navigation over the Earth, it is necessary to define axis sets which allow the inertial measurements to be related to the cardinal directions of the Earth, that is, frames which have a physical significance when attempting to navigate in the vicinity of the Earth. Therefore, it is customary to consider an inertial reference frame which is stationary with respect to the fixed stars, the origin of which is located at the centre of the Earth. Such a reference frame is shown in Figure 3.6, together with an Earth-fixed reference frame and a local geographic navigation frame defined for the purposes of terrestrial inertial navigation.

The following co-ordinate frames are used in the text:

*The inertial frame* (i-frame) has its origin at the centre of the Earth and axes which are non-rotating with respect to the fixed stars, defined by the axes $\mathrm{O}x_\mathrm{i}$, $\mathrm{O}y_\mathrm{i}$, $\mathrm{O}z_\mathrm{i}$, with $\mathrm{O}z_\mathrm{i}$ coincident with the Earth's polar axis (which is assumed to be invariant in direction).

*The Earth frame* (e-frame) has its origin at the centre of the Earth and axes which are fixed with respect to the Earth, defined by the axes $\mathrm{O}x_\mathrm{e}$, $\mathrm{O}y_\mathrm{e}$, $\mathrm{O}z_\mathrm{e}$ with $\mathrm{O}z_\mathrm{e}$ along the Earth's polar axis. The axis $\mathrm{O}x_\mathrm{e}$ lies along the intersection of the plane of the Greenwich meridian with the Earth's equatorial plane. The Earth frame rotates, with respect to the inertial frame, at a rate $\Omega$ about the axis $\mathrm{O}z_\mathrm{i}$.

*The navigation frame* (n-frame) is a local geographic frame which has its origin at the location of the navigation system, point P, and axes aligned with the directions of north, east and the local vertical (down). The turn rate of the navigation frame, with respect to the Earth-fixed frame, $\omega_\mathrm{en}$, is governed by the motion of the point P with respect to the Earth. This is often referred to as the transport rate.

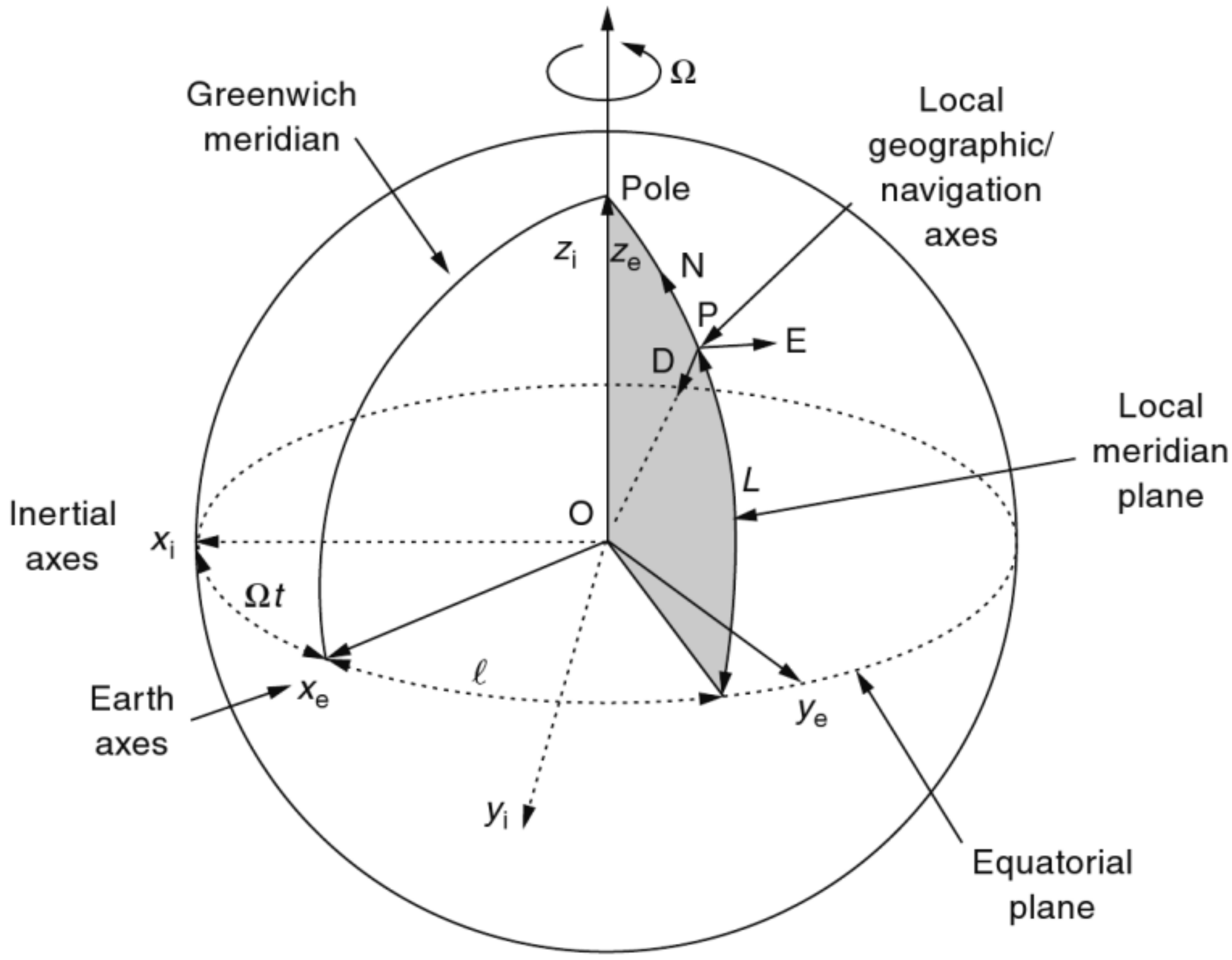


*Figure 3.6 Frames of reference*

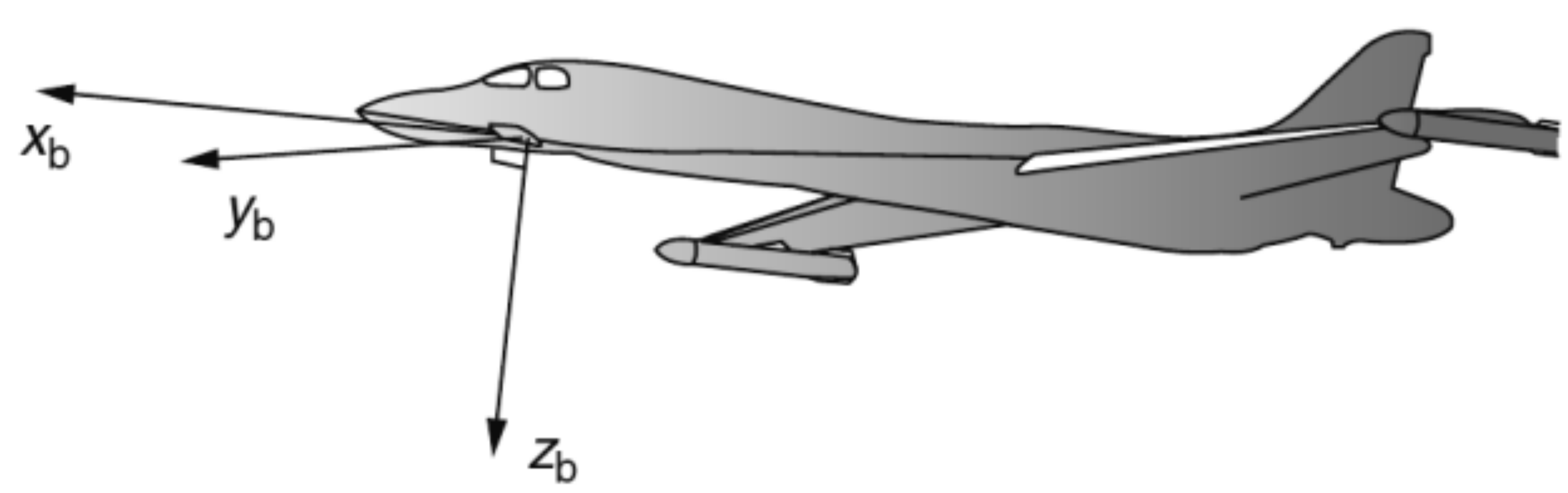


*Figure 3.7 Illustration of a body reference frame*

*The wander azimuth frame* (w-frame) may be used to avoid the singularities in the computation which occur at the poles of the navigation frame. Like the navigation frame, it is locally level but is rotated through the wander angle about the local vertical. Its use is described in Section 3.5.

*The body frame* (b-frame), depicted in Figure 3.7, is an orthogonal axis set which is aligned with the roll, pitch and yaw axes of the vehicle in which the navigation system is installed.

## 3.4 Three-dimensional strapdown navigation system – general analysis

### *3.4.1 Navigation with respect to a fixed frame*

Consider the situation where it is required to navigate with respect to a fixed, or non-accelerating, and non-rotating set of axes. The measured components of specific

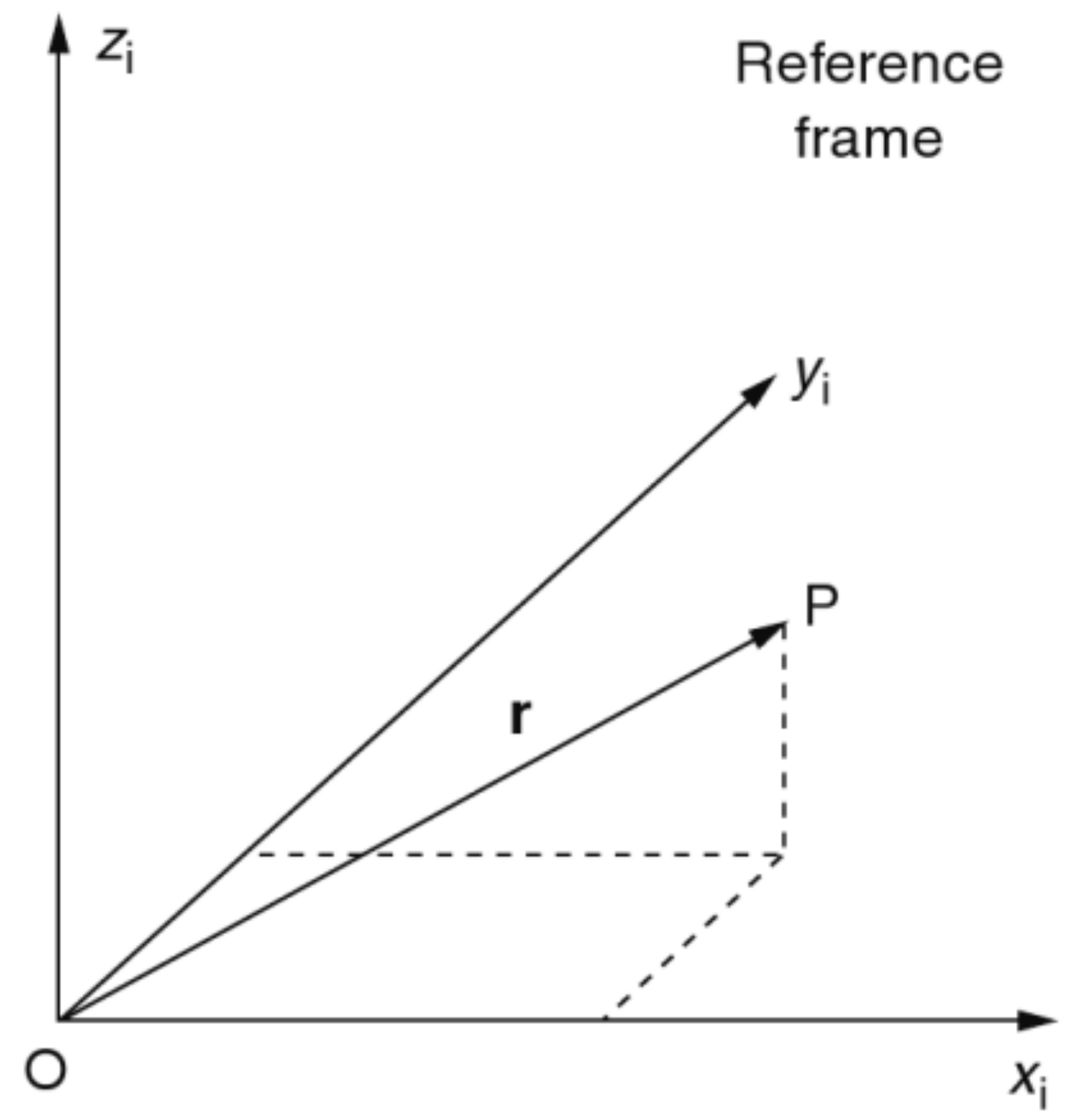


*Figure 3.8 Position vector with respect to reference frame*

force and estimates of the gravitational field are summed to determine components of acceleration with respect to a space-fixed reference frame. These quantities can then be integrated twice, giving estimates of velocity and position in that frame.

This process may be expressed mathematically in the following manner.[1] Let **r** represent the position vector of the point P with respect to O, the origin of the reference frame shown in Figure 3.8.

The acceleration of P with respect to a space-fixed axis set, termed the i-frame and denoted by the subscript i, is defined by:

$$\mathbf{a}_\mathrm{i} = \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} \tag{3.1}$$

A triad of perfect accelerometers will provide a measure of the specific force (**f**) acting at point P where

$$\mathbf{f} = \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} - \mathbf{g} \tag{3.2}$$

in which **g** is the mass attraction gravitation vector.

Rearranging eqn. (3.2) yields the following equation:

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \mathbf{f} + \mathbf{g} \tag{3.3}$$

This is called the navigation equation since, with suitable integration, it yields the navigational quantities of velocity and position. The first integral gives the velocity

[1] Vector and matrix notation is widely used throughout the text for the mathematical representation of strapdown inertial system processes. This notation is adopted both in the interests of brevity and to be consistent with other texts on the subject. Vector and matrix quantities are written in boldface type.

of point P with respect to the i-frame, viz.

$$\mathbf{v}_i = \left.\frac{d\mathbf{r}}{dt}\right|_i \tag{3.4}$$

whilst a second integration gives its position in that frame.

### *3.4.2 Navigation with respect to a rotating frame*

In practice, one often needs to derive estimates of a vehicle's velocity and position with respect to a rotating reference frame, as when navigating in the vicinity of the Earth. In this situation, additional apparent forces will be acting which are functions of reference frame motion. This results in a revised form of the navigation equation which may be integrated to determine the ground speed of the vehicle, $\mathbf{v}_e$, directly. Alternatively, $\mathbf{v}_e$ may be computed from the inertial velocity, $\mathbf{v}_i$, using the theorem of Coriolis, as follows,

$$\mathbf{v}_e = \left.\frac{d\mathbf{r}}{dt}\right|_e = \mathbf{v}_i - \boldsymbol{\omega}_{ie} \times \mathbf{r} \tag{3.5}$$

where $\boldsymbol{\omega}_{ie} = [0 \quad 0 \quad \Omega]^T$ is the turn rate of the Earth frame with respect to the i-frame and $\times$ denotes a vector cross product.

Revised forms of the navigation equation suitable for navigation with respect to the Earth are the subject of Section 3.5.

### *3.4.3 The choice of reference frame*

The navigation equation, eqn. (3.3), may be solved in any one of a number of reference frames. If the Earth frame is chosen, for example, then the solution of the navigation equation will provide estimates of velocity with respect to either the inertial frame or the Earth frame, expressed in Earth coordinates, denoted $\mathbf{v}_i^e$ and $\mathbf{v}_e^e$, respectively.[2]

In Section 3.5, a number of different strapdown system mechanisations for navigating with respect to the Earth are described. In each case, it will be shown that the navigation equation is expressed in a different manner depending on the choice of reference frame.

### *3.4.4 Resolution of accelerometer measurements*

The accelerometers usually provide a measurement of specific force in a body fixed axis set, denoted $\mathbf{f}^b$. In order to navigate, it is necessary to resolve the components of the specific force in the chosen reference frame. In the event that the inertial frame is selected, this may be achieved by pre-multiplying the vector quantity $\mathbf{f}^b$ by the direction cosine matrix, $\mathbf{C}_b^i$, using,

$$\mathbf{f}^i = \mathbf{C}_b^i \mathbf{f}^b \tag{3.6}$$

[2] Superscripts attached to vector quantities denote the axis set in which the vector quantity coordinates are expressed.

where $\mathbf{C}_{\mathrm{b}}^{\mathrm{i}}$ is a $3 \times 3$ matrix which defines the attitude of the body frame with respect to the i-frame. The direction cosine matrix $\mathbf{C}_{\mathrm{b}}^{\mathrm{i}}$ may be calculated from the angular rate measurements provided by the gyroscopes using the following equation:

$$\dot{\mathbf{C}}_{\mathrm{b}}^{\mathrm{i}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{i}} \boldsymbol{\Omega}_{\mathrm{ib}}^{\mathrm{b}} \tag{3.7}$$

where $\boldsymbol{\Omega}_{\mathrm{ib}}^{\mathrm{b}}$ is the skew symmetric matrix:

$$\boldsymbol{\Omega}_{\mathrm{ib}}^{\mathrm{b}} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{3.8}$$

This matrix is formed from the elements of the vector $\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}} = [p \quad q \quad r]^{\mathrm{T}}$ which represents the turn rate of the body with respect to the i-frame as measured by the gyroscopes. Equation (3.7) is derived in Section 3.6.

The attitude of the body with respect to the chosen reference frame, which is required to resolve the specific force measurements into the reference frame, may be defined in a number of different ways. For the purposes of the discussion of navigation system mechanisations in this and the following section, the direction cosine method will be adopted. Direction cosines and some alternative attitude representations are described in some detail in Section 3.6.

### *3.4.5 System example*

Consider the situation in which it is required to navigate with respect to inertial space and the solution of the navigation takes place in the i-frame. Equation (3.3) may be expressed in i-frame coordinates as follows:

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{i}} = \mathbf{f}^{\mathrm{i}} + \mathbf{g}^{\mathrm{i}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{i}} \mathbf{f}^{\mathrm{b}} + \mathbf{g}^{\mathrm{i}} \tag{3.9}$$

It is clear from the preceding discussion that the integration of the navigation equation involves the use of information from both the gyroscopes and the accelerometers contained within the inertial navigation system. A block diagram representation of the resulting navigation system is given in Figure 3.9.

The diagram displays the main functions to be implemented within a strapdown navigation system; the processing of the rate measurements to generate body attitude, the resolution of the specific force measurements into the inertial reference frame, gravity compensation and the integration of the resulting acceleration estimates to determine velocity and position.

## 3.5 Strapdown system mechanisations

Attention is focused here on inertial systems which may be used to navigate in the vicinity of the Earth. It has been shown in Section 3.4 how estimates of position and velocity are derived by integrating a navigation equation of the form given in

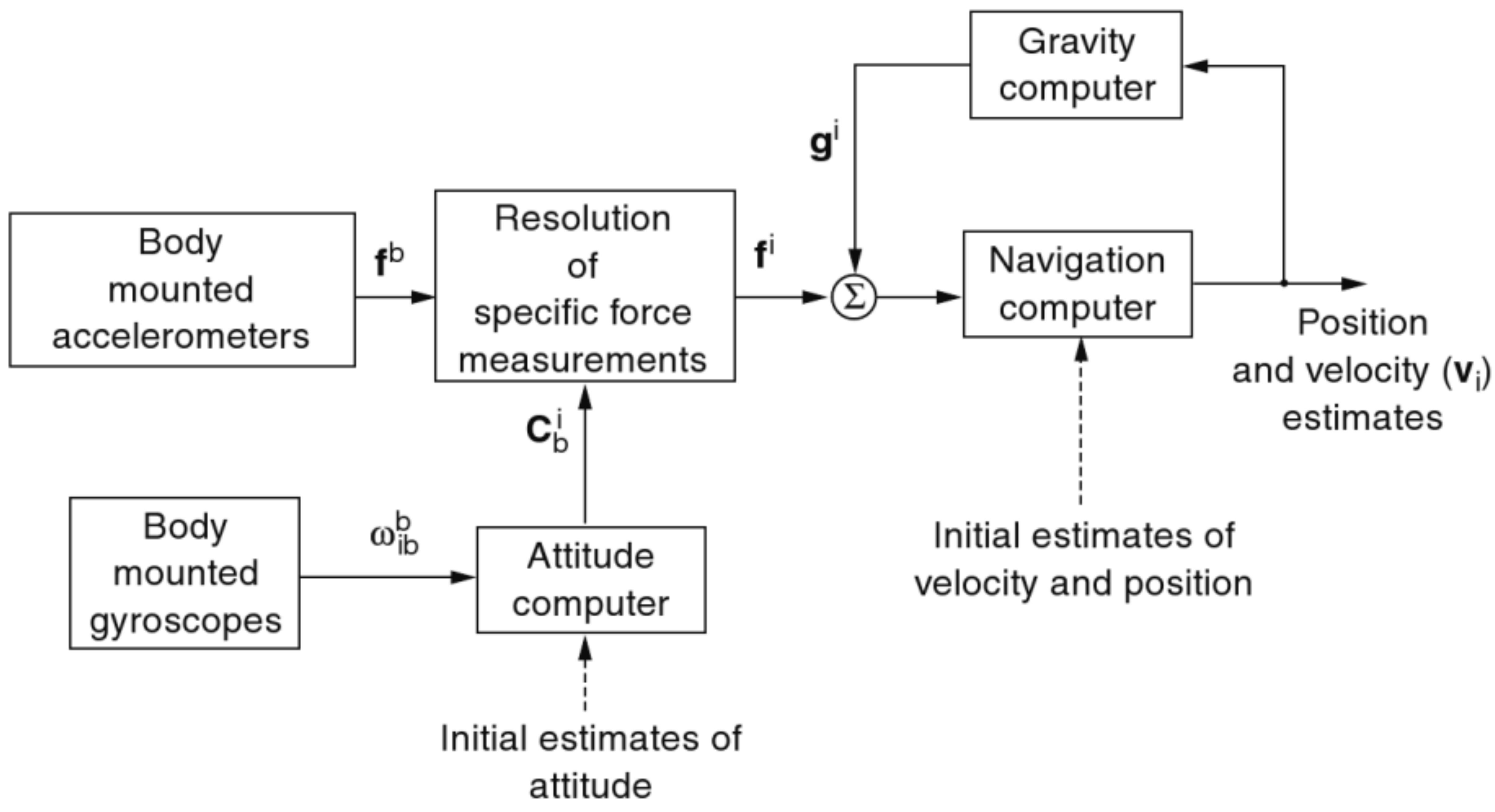


*Figure 3.9 Strapdown inertial navigation system*

eqn. (3.3). In systems of the type described later, in which it is required to derive estimates of vehicle velocity and position with respect to an Earth fixed frame, additional apparent forces will be acting which are functions of the reference frame motion. In this section, further forms of the navigation equation are derived, corresponding to different choices of reference frame [1].

The resulting system mechanisations are described together with their applications. As will become apparent, the variations in the mechanisations described here are in the strapdown computational algorithms and not in the arrangement of the sensors or the mechanical layout of the system.

### 3.5.1 *Inertial frame mechanisation*

In this system, it is required to calculate vehicle speed with respect to the Earth, the ground speed, in inertial axes, denoted by the symbol $\mathbf{v}_{\mathrm{e}}^{\mathrm{i}}$. This may be accomplished by expressing the navigation equation (eqn. (3.3)) in inertial axes and deriving an expression for $\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{e}}$ in terms of ground speed and its time derivatives with respect to the inertial frame.

Inertial velocity may be expressed in terms of ground speed using the Coriolis equation, viz.

$$\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{i}} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{e}} + \boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r} \tag{3.10}$$

Differentiating this expression and writing $\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{e}} = \mathbf{v}_{\mathrm{e}}$, we have,

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{i}} = \left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{e}}}{\mathrm{d}t}\right|_{\mathrm{i}} + \left.\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r}]\right|_{\mathrm{i}} \tag{3.11}$$

Applying the Coriolis equation in the form of eqn. (3.10) to the second term in eqn. (3.11) gives:

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{i}} = \left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{e}}}{\mathrm{d}t}\right|_{\mathrm{i}} + \boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{v}_{\mathrm{e}} + \boldsymbol{\omega}_{\mathrm{ie}} \times [\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r}] \tag{3.12}$$

In generating the above equation, it is assumed that the turn rate of the Earth is constant, hence $\frac{\mathrm{d}\boldsymbol{\omega}_{\mathrm{ie}}}{\mathrm{d}t} = 0$.

Combining eqns. (3.3) and (3.12) and rearranging yields:

$$\left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{e}}}{\mathrm{d}t}\right|_{\mathrm{i}} = \mathbf{f} - \boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{v}_{\mathrm{e}} - \boldsymbol{\omega}_{\mathrm{ie}} \times [\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r}] + \mathbf{g} \tag{3.13}$$

In this equation, **f** represents the specific force acceleration to which the navigation system is subjected, while $\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{v}_{\mathrm{e}}$ is the acceleration caused by its velocity over the surface of a rotating Earth, usually referred to as the Coriolis acceleration. The term $\boldsymbol{\omega}_{\mathrm{ie}} \times [\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r}]$, in eqn. (3.13), defines the centripetal acceleration experienced by the system owing to the rotation of the Earth, and is not separately distinguishable from the gravitational acceleration which arises through mass attraction, **g**. The sum of the accelerations caused by the mass attraction force and the centripetal force constitutes what is known as the local gravity vector, the vector to which a 'plumb bob' would align itself when held above the Earth (Figure 3.10). This is denoted here by the symbol $\mathbf{g}_{\mathrm{l}}$, that is:

$$\mathbf{g}_{\mathrm{l}} = \mathbf{g} - \boldsymbol{\omega}_{\mathrm{ie}} \times [\boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r}] \tag{3.14}$$

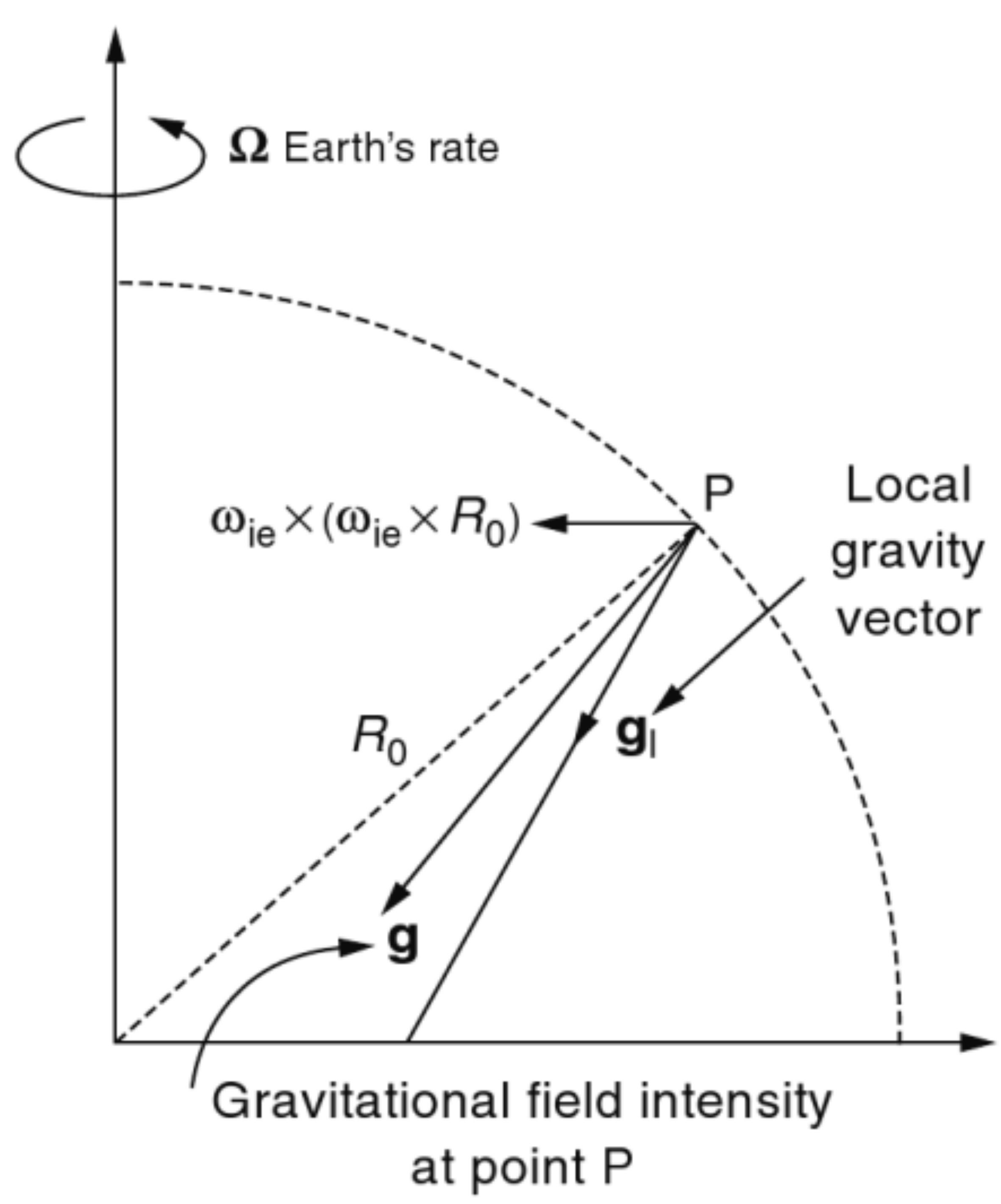


*Figure 3.10 Diagram showing the components of the gravitational field*

Combining eqns. (3.13) and (3.14) gives the following form of the navigation equation:

$$\left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i} = \mathbf{f} - \boldsymbol{\omega}_\mathrm{ie} \times \mathbf{v}_\mathrm{e} + \mathbf{g}_\mathrm{l} \tag{3.15}$$

This equation may be expressed in inertial axes, as follows, using the superscript notation mentioned earlier.

$$\dot{\mathbf{v}}^\mathrm{i}_\mathrm{e} = \mathbf{f}^\mathrm{i} - \boldsymbol{\omega}^\mathrm{i}_\mathrm{ie} \times \mathbf{v}^\mathrm{i}_\mathrm{e} + \mathbf{g}^\mathrm{i}_\mathrm{l} \tag{3.16}$$

The measurements of specific force provided by the accelerometers are in body axes, as denoted by the vector quantity $\mathbf{f}^\mathrm{b}$. In order to set up the navigation eqn. (3.16), the accelerometer outputs must be resolved into inertial axes to give $\mathbf{f}^\mathrm{i}$. This may be achieved by pre-multiplying the measurement vector $\mathbf{f}^\mathrm{b}$ by the direction cosine matrix $\mathbf{C}^\mathrm{i}_\mathrm{b}$ as described in Section 3.4.4 (eqn. (3.6)). Given knowledge of the attitude of the body at the start of navigation, the matrix $\mathbf{C}^\mathrm{i}_\mathrm{b}$ is updated using eqns. (3.7) and (3.8) based on measurements of the body rates with respect to the i-frame which may be expressed as follows:

$$\boldsymbol{\omega}^\mathrm{b}_\mathrm{ib} = [\,p \quad q \quad r\,]^\mathrm{T} \tag{3.17}$$

Substituting for $\mathbf{f}^\mathrm{i}$ from eqn. (3.6) in eqn. (3.16) gives the following form of the navigation equation:

$$\dot{\mathbf{v}}^\mathrm{i}_\mathrm{e} = \mathbf{C}^\mathrm{i}_\mathrm{b}\mathbf{f}^\mathrm{b} - \boldsymbol{\omega}^\mathrm{i}_\mathrm{ie} \times \mathbf{v}^\mathrm{i}_\mathrm{e} + \mathbf{g}^\mathrm{i}_\mathrm{l} \tag{3.18}$$

The final term in this equation represents the local gravity vector expressed in the inertial frame.

A block diagram representation of the resulting inertial frame mechanisation is shown in Figure 3.11.

### *3.5.2 Earth frame mechanisation*

In this system, ground speed is expressed in an Earth-fixed co-ordinate frame to give $\mathbf{v}^\mathrm{e}_\mathrm{e}$. It follows from the Coriolis equation, that the rate of change of $\mathbf{v}_\mathrm{e}$, with respect to Earth axes, may be expressed in terms of its rate of change in inertial axes using:

$$\left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{e} = \left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i} - \boldsymbol{\omega}_\mathrm{ie} \times \mathbf{v}_\mathrm{e} \tag{3.19}$$

Substituting for $\left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i}$ from eqn. (3.15), we have:

$$\left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{e} = \mathbf{f} - 2\boldsymbol{\omega}_\mathrm{ie} \times \mathbf{v}_\mathrm{e} + \mathbf{g}_\mathrm{l} \tag{3.20}$$

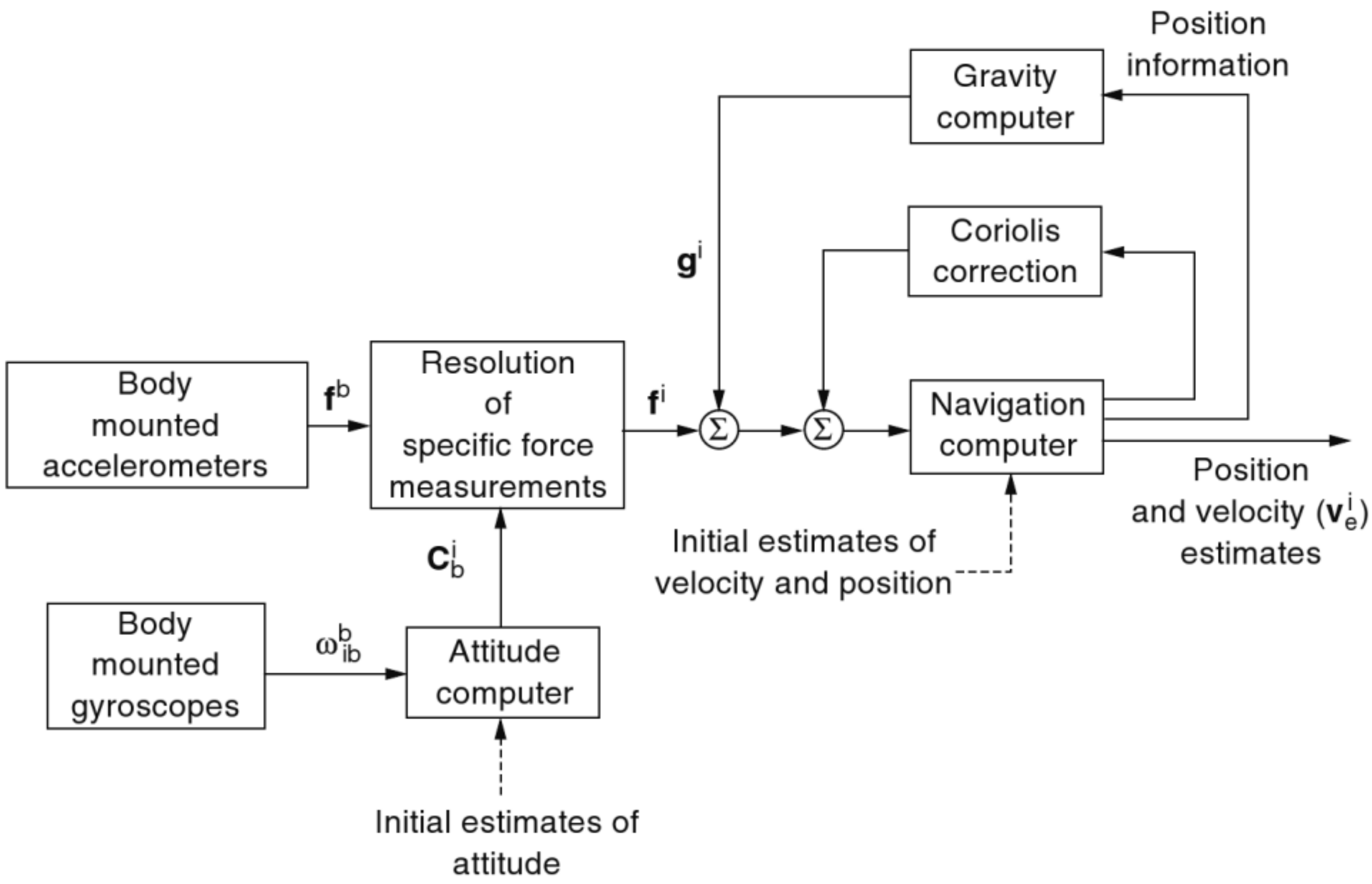


*Figure 3.11 Strapdown inertial navigation system – inertial frame mechanisation*

This may be expressed in Earth axes as follows:

$$\dot{\mathbf{v}}_{e}^{e} = \mathbf{C}_{b}^{e}\mathbf{f}^{b} - 2\boldsymbol{\omega}_{ie}^{e} \times \mathbf{v}_{e}^{e} + \mathbf{g}_{l}^{e} \tag{3.21}$$

where $\mathbf{C}_{b}^{e}$ is the direction cosine matrix used to transform the measured specific force vector into Earth axes. This matrix propagates in accordance with the following equation:

$$\dot{\mathbf{C}}_{b}^{e} = \mathbf{C}_{b}^{e}\boldsymbol{\Omega}_{eb}^{b} \tag{3.22}$$

where $\boldsymbol{\Omega}_{eb}^{b}$ is the skew symmetric form of $\boldsymbol{\omega}_{eb}^{b}$, the body rate with respect to the Earth-fixed frame. This is derived by differencing the measured body rates, $\boldsymbol{\omega}_{ib}^{b}$, and estimates of the components of Earth's rate, $\boldsymbol{\omega}_{ie}$, expressed in body axes as follows:

$$\boldsymbol{\omega}_{eb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{C}_{e}^{b}\boldsymbol{\omega}_{ie}^{e} \tag{3.23}$$

in which $\mathbf{C}_{e}^{b} = \mathbf{C}_{b}^{eT}$, the transpose of the matrix $\mathbf{C}_{b}^{e}$.

A block diagram representation of the Earth frame mechanisation is shown in Figure 3.12.

A variation on this system may be used when it is required to navigate over relatively short distances, with respect to a fixed point on the Earth. A mechanisation of this type may be used for a tactical missile application in which navigation is required with respect to a ground based tracking station. In such a system,

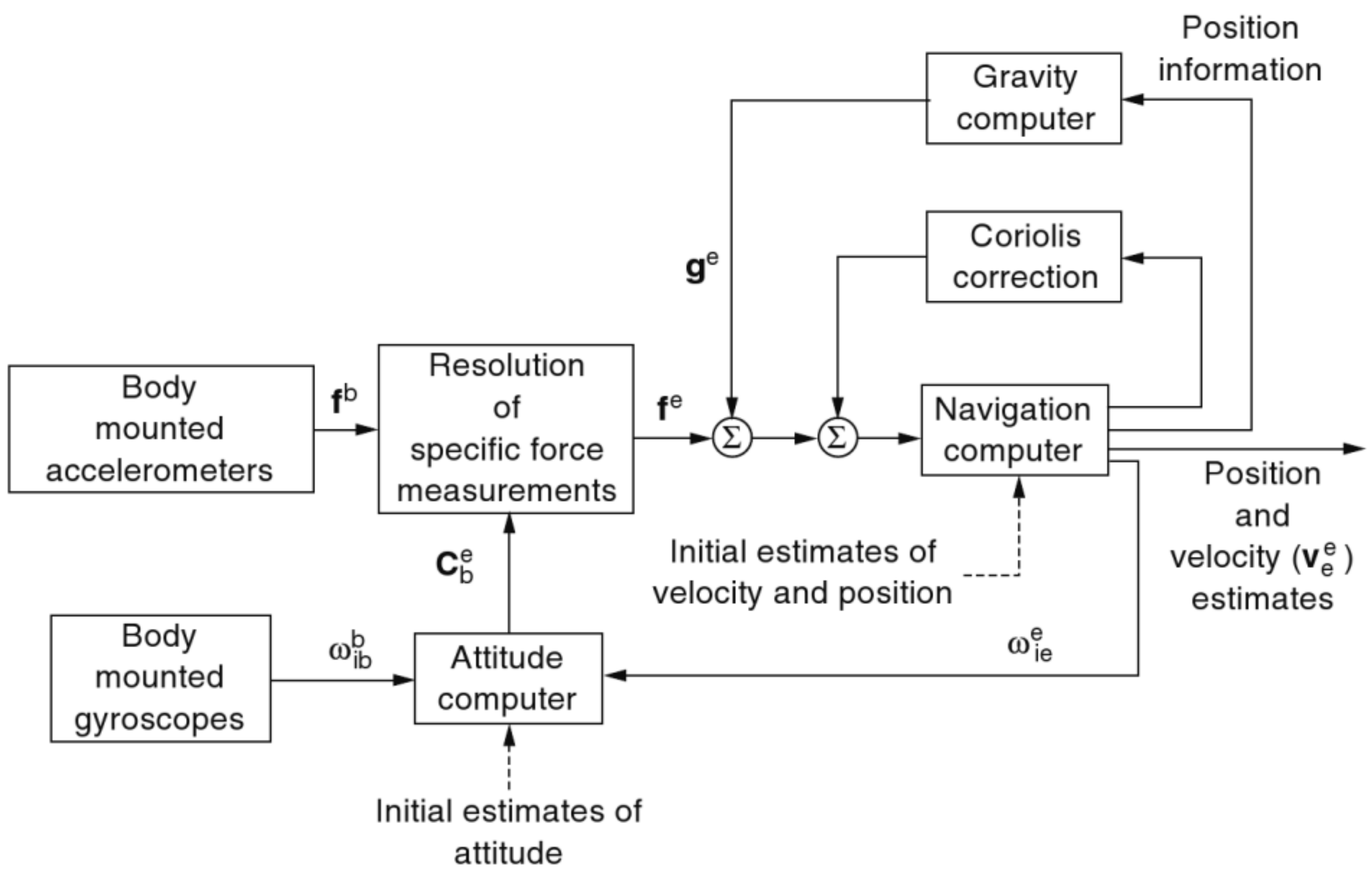


*Figure 3.12 Strapdown inertial navigation system – Earth frame mechanisation*

target tracking information provided by the ground station may need to be combined with the outputs of an on-board inertial navigation system to provide missile mid-course guidance commands. In order that the missile may operate in harmony with the ground systems, all information must be provided in a common frame of reference.

In this situation, an Earth-fixed reference frame may be defined, the origin of which is located at the tracking station, its axes aligned with the local vertical and a plane which is tangential to the Earth's surface, as illustrated in Figure 3.13.

For very short term navigation, as required for some tactical missile applications, further simplifications to this system mechanisation may be permitted. For instance, where the navigation period is short, typically 10 minutes or less, the effects of the rotation of the Earth on the attitude computation process can sometimes be ignored, and Coriolis corrections are no longer essential in the velocity equation to give sufficiently accurate navigation. In this situation, attitude is computed solely as a function of the turn rates measured by the gyroscopes, and eqn. (3.21) reduces to the following:

$$\dot{\mathbf{v}}_e^e = \mathbf{C}_b^e \mathbf{f}^b + \mathbf{g}_l^e \tag{3.24}$$

It is stressed, that such simplifications can only be allowed in cases where the navigation errors, induced by the omission of Earth rate and Coriolis terms, lie within the error bounds in which the navigation system is required to operate. This situation arises when the permitted gyroscopic errors are in excess of the rotation rate of the Earth, and allowable accelerometer biases are in excess of the acceleration errors introduced by ignoring the Coriolis forces.

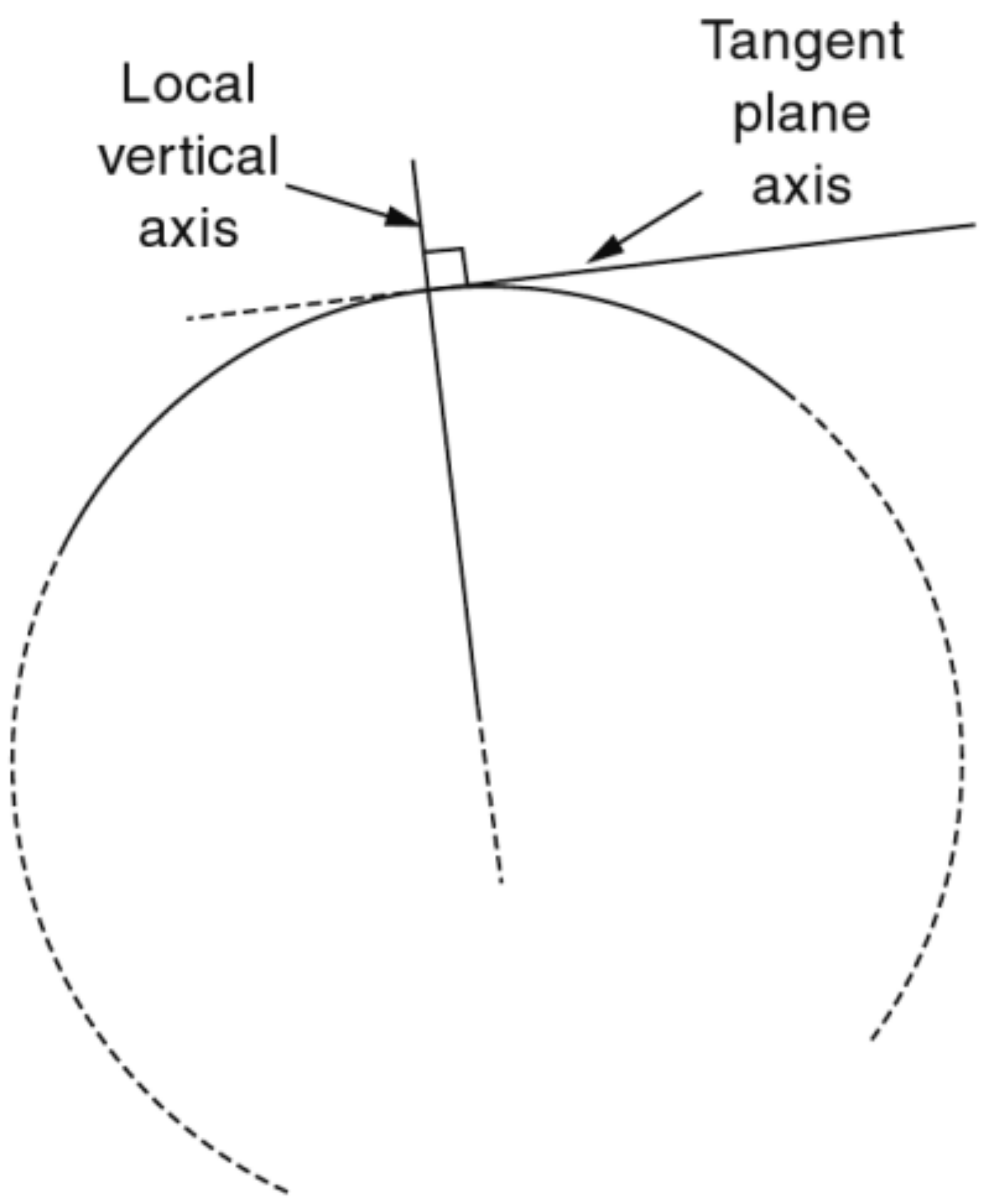


*Figure 3.13* *Tangent plane axis set*

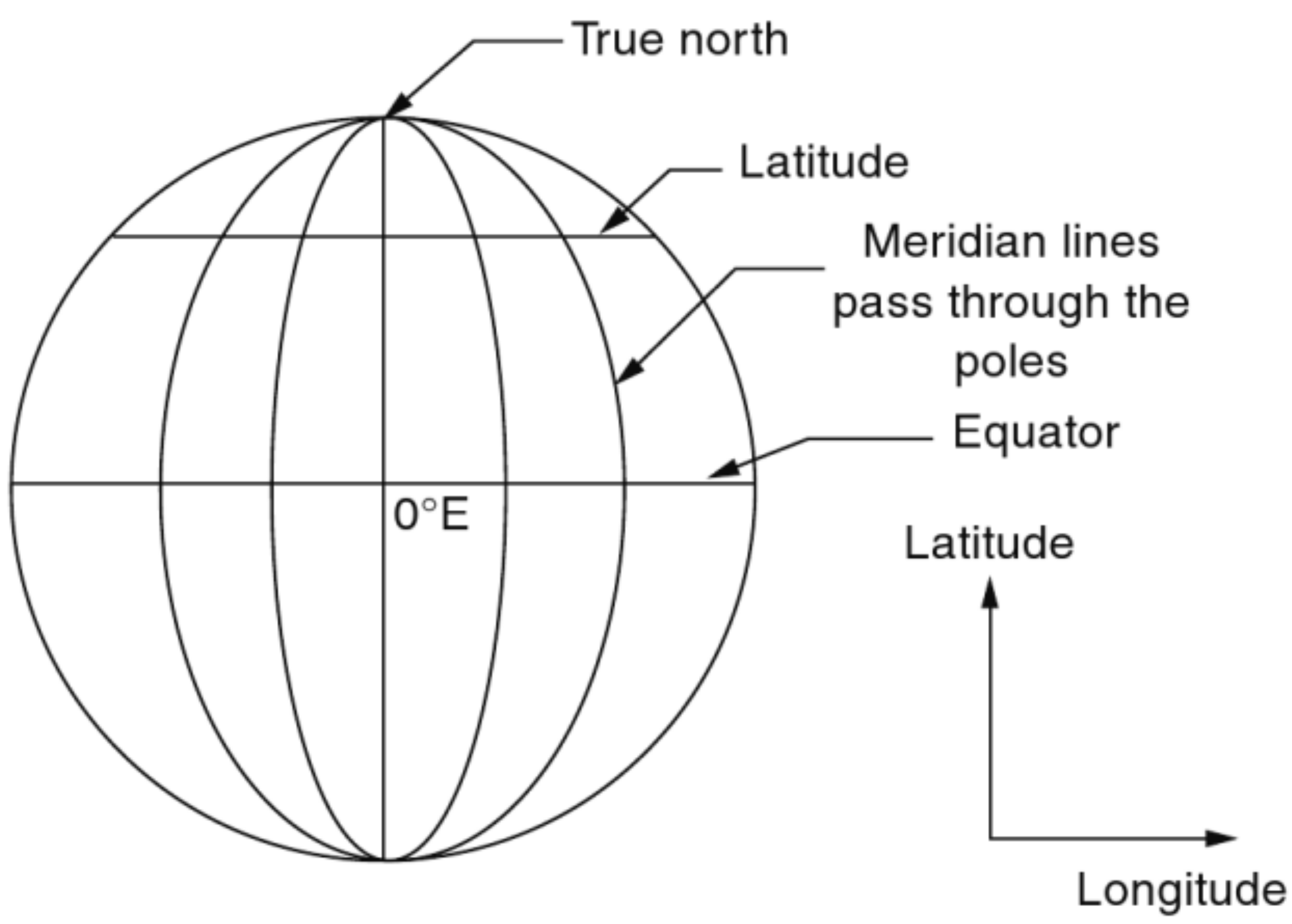


*Figure 3.14* *Geographic co-ordinate system*

### *3.5.3 Local geographic navigation frame mechanisation*

In order to navigate over large distances around the Earth, navigation information is most commonly required in the local geographic or navigation axis set described earlier. Position on the Earth may be specified in terms of latitude (degrees north or south of a datum) and longitude (degrees east or west of a datum). Figure 3.14 shows this geographic co-ordinate system on a globe. Lines of constant latitude and longitude are called parallels and meridians, respectively.

Navigation data are expressed in terms of north and east velocity components, latitude, longitude and height above the Earth. Whilst such information can be computed using the position estimates provided by the inertial or Earth frame mechanisations described before, this involves a further transformation of the vector quantities $\mathbf{v}_e^i$ or $\mathbf{v}_e^e$. Further, difficulties arise in representing the Earth's gravitational field precisely in a computer. For these reasons, the navigation frame mechanisation, described here, is often used when navigating around the Earth.

In this mechanisation, ground speed is expressed in navigation coordinates to give $\mathbf{v}_e^n$. The rate of change of $\mathbf{v}_e^n$ with respect to navigation axes may be expressed in terms of its rate of change in inertial axes as follows:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_n = \left.\frac{d\mathbf{v}_e}{dt}\right|_i - [\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e \tag{3.25}$$

Substituting for $\left.\frac{d\mathbf{v}_e}{dt}\right|_i$, from eqn. (3.15), we have:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_n = \mathbf{f} - [2\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e + \mathbf{g}_l \tag{3.26}$$

This may be expressed in navigation axes as follows:

$$\dot{\mathbf{v}}_e^n = \mathbf{C}_b^n \mathbf{f}^b - [2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n] \times \mathbf{v}_e^n + \mathbf{g}_l^n \tag{3.27}$$

where $\mathbf{C}_b^n$ is a direction cosine matrix used to transform the measured specific force vector into navigation axes. This matrix propagates in accordance with the following equation.

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \boldsymbol{\Omega}_{nb}^b \tag{3.28}$$

where $\boldsymbol{\Omega}_{nb}^b$ is the skew symmetric form of $\boldsymbol{\omega}_{nb}^b$, the body rate with respect to the navigation frame. This is derived by differencing the measured body rates, $\boldsymbol{\omega}_{ib}^b$, and estimates of the components of navigation frame rate, $\boldsymbol{\omega}_{in}$. The latter term is obtained by summing the Earth's rate with respect to the inertial frame and the turn rate of the navigation frame with respect to the Earth, that is, $\boldsymbol{\omega}_{in} = \boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}$. Therefore,

$$\boldsymbol{\omega}_{nb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_n^b[\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n] \tag{3.29}$$

A block diagram representation of the navigation frame mechanisation is shown in Figure 3.15.

It is instructive to consider the physical significance of the various terms in the navigation equation (3.27). From this equation, it can be seen that the rate of change of the velocity, with respect to the surface of the Earth, is made up of the following terms:

1. The specific force acting on the vehicle, as measured by a triad of accelerometers mounted within it.
2. A correction for the acceleration caused by the vehicle's velocity over the surface of a rotating Earth, usually referred to as the Coriolis acceleration. The effect in two dimensions is illustrated in Figure 3.16. As the point P moves away

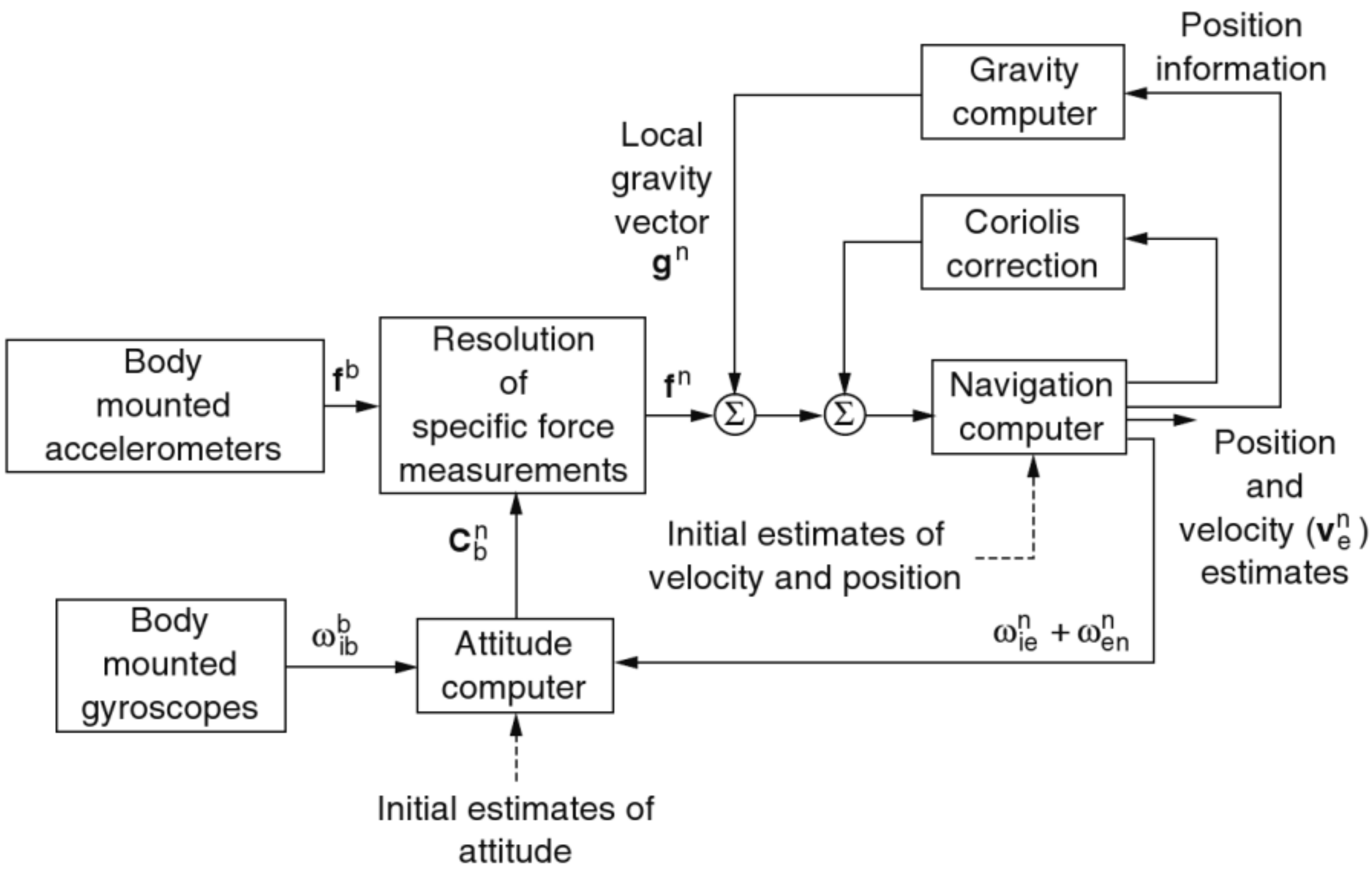


*Figure 3.15 Strapdown inertial navigation system – local geographic navigation frame mechanisation*

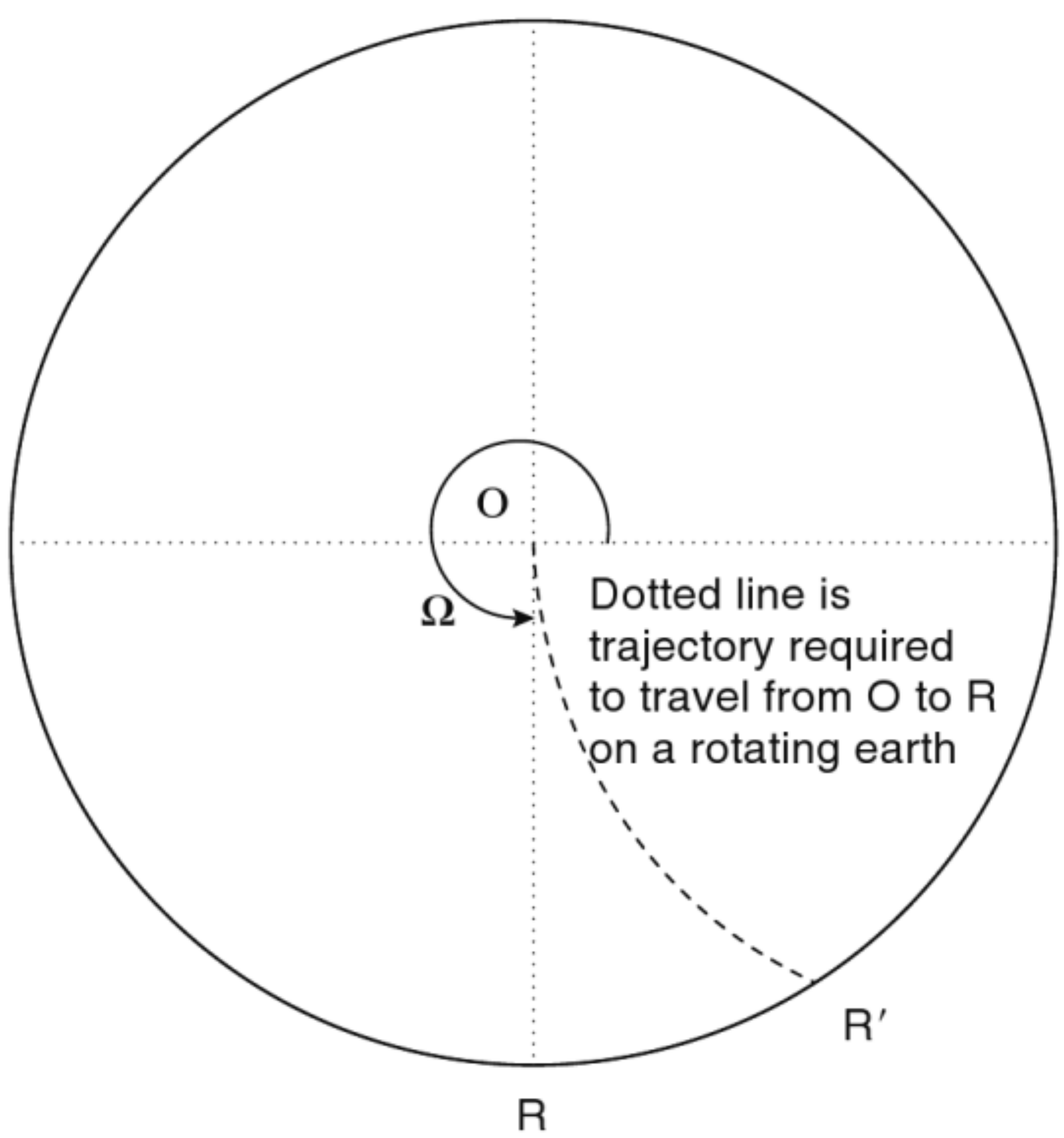


*Figure 3.16 Illustration of the effect of Coriolis acceleration*

from the axis of rotation, it traces out a curve in space as a result of the Earth's rotation.

3. A correction for the centripetal acceleration of the vehicle, resulting from its motion over the Earth's surface. For instance, a vehicle moving due east over

the surface of the Earth will trace out a circular path with respect to inertial axes. To follow this path, the vehicle is subject to a force acting towards the centre of the Earth of magnitude equal to the product of its mass, its linear velocity and its turn rate with respect to the Earth.

4. Compensation for the apparent gravitational force acting on the vehicle. This includes the gravitational force caused by the mass attraction of the Earth, and the centripetal acceleration of the vehicle resulting from the rotation of the Earth. The latter term arises even if the vehicle is stationary with respect to the Earth, since the path which its follows in space is circular.

A simple example serves to illustrate the importance of the Coriolis effect. Consider a vehicle launched from the north pole with the intention of flying to New York city. The vehicle is assumed to travel at an average speed of 3600 miles/h. During the flight, of approximately 1 h, the Earth will have rotated by about 15°, a distance of approximately 900 miles at the latitude of New York. Consequently, if no Coriolis correction was made to the on-board inertial guidance system during the course of the flight, the vehicle would arrive in the Chicago area rather than New York as originally intended.

### *3.5.4 Wander azimuth navigation frame mechanisation*

In the local geographic navigation frame mechanisation described in the previous section, the n-frame is required to rotate continuously as the system moves over the surface of the Earth in order to keep its $x$-axis parallel to true north. In order to achieve this condition worldwide, the n-frame must rotate at much greater rates about its $z$-axis as the navigation system moves over the surface of the Earth in the polar regions, compared to the rates required at lower latitudes. This effect is illustrated in Figure 3.17 which shows a polar view of a near polar crossing. It should be clear from the diagram that the rate at which the local geographic navigation frame must rotate about its $z$-axis in order to maintain the $x$-axis pointing at the pole becomes very large, the heading direction slewing rapidly through 180° when moving past the pole. In the most extreme case, a direct crossing of the pole, the turn rate becomes infinite when passing over the pole.

The effect is illustrated mathematically as follows. The turn rate of the navigation frame, the transport rate, may be expressed in component form as:

$$\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}} = \left[ \frac{v_{\mathrm{E}}}{R_0 + h} \quad \frac{-v_{\mathrm{N}}}{R_0 + h} \quad \frac{-v_{\mathrm{E}} \tan L}{R_0 + h} \right]^{\mathrm{T}} \tag{3.30}$$

where $v_{\mathrm{N}}$ is the north velocity, $v_{\mathrm{E}}$ the east velocity, $R_0$ the radius of the Earth, $L$ the latitude and $h$ the height above ground.

It will be seen that the third component of the transport rate becomes indeterminate at the geographic poles.

One way of avoiding the singularity, and so providing a navigation system with world-wide capability, is to adopt a wander azimuth mechanisation in which the $z$-component of $\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}}$ is set to zero. A wander axis system is a locally level frame

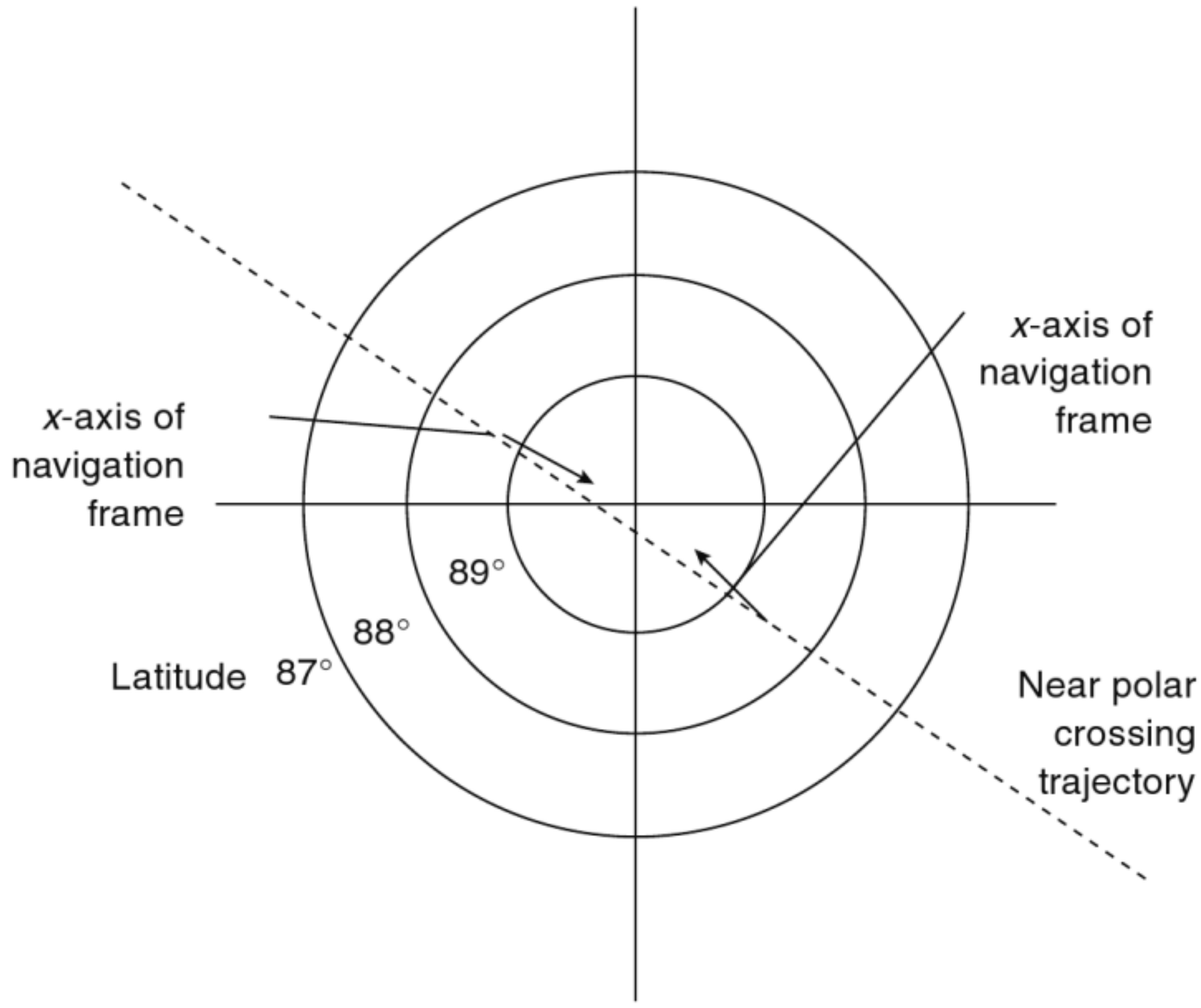


*Figure 3.17 Geographic reference singularity at pole crossings*

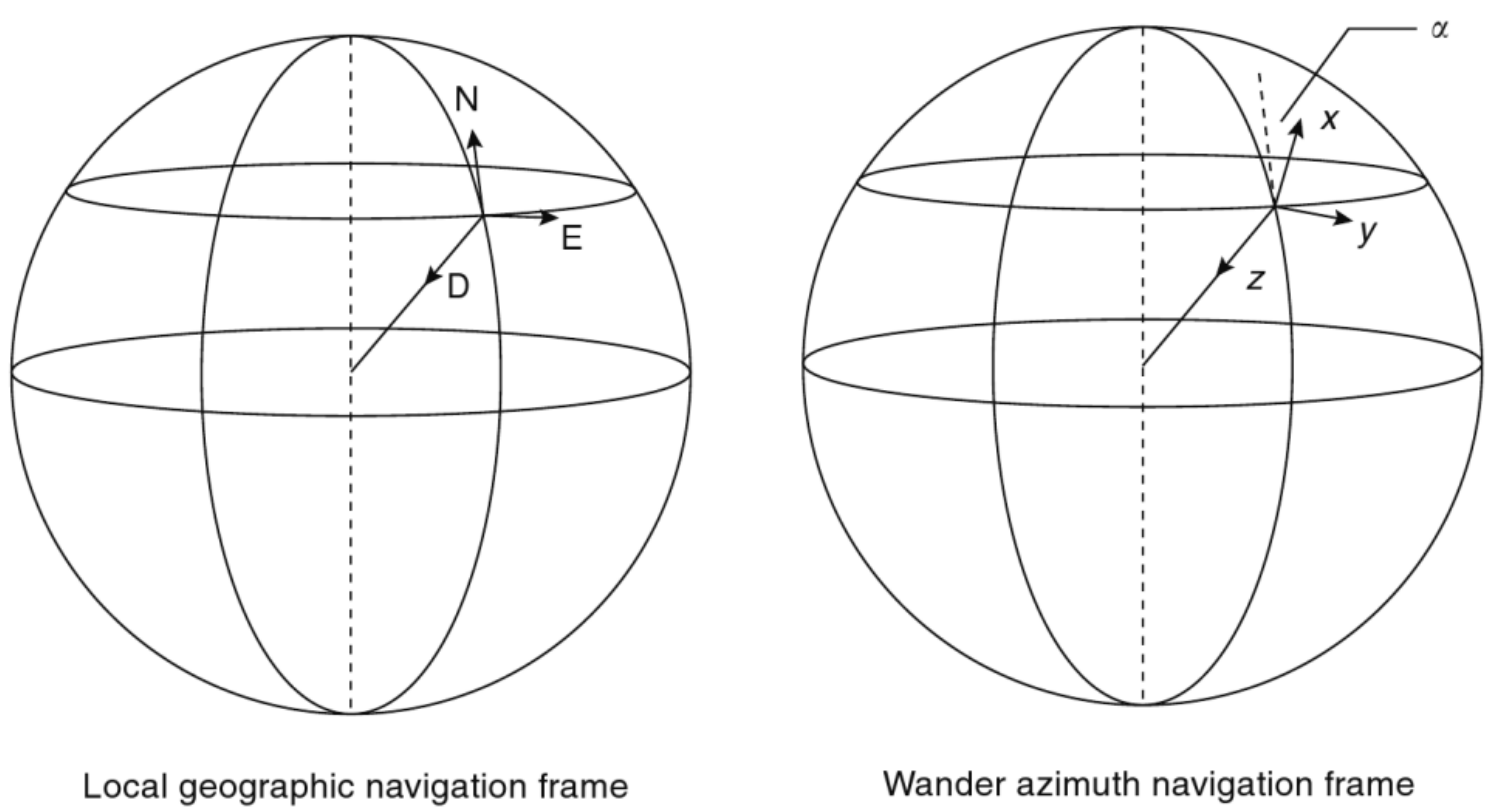


*Figure 3.18 Illustration of wander azimuth frame*

which moves over the Earth's surface with the vehicle, as depicted in Figure 3.18. However, as the name implies, the azimuth angle between true north and the $x$-axis of the wander axis frame varies with vehicle position on the Earth. This variation is chosen in order to avoid discontinuities in the orientation of the wander frame with respect to the Earth as the vehicle passes over either the north or south poles.

A navigation equation for a wander azimuth system, which is similar in form to eqn. (3.27), may be constructed as follows:

$$\dot{\mathbf{v}}_e^w = \mathbf{C}_b^w \mathbf{f}^b - [2\mathbf{C}_e^w \boldsymbol{\omega}_e^e + \boldsymbol{\omega}_{ew}^w] \times \mathbf{v}_e^w + \mathbf{g}_l^w \tag{3.31}$$

This equation is integrated to generate estimates of vehicle ground speed in the wander azimuth frame, $\mathbf{v}_e^w$. This is then used to generate the turn rate of the wander frame with respect to the Earth, $\boldsymbol{\omega}_{ew}^w$. The direction cosine matrix which relates the wander frame to the Earth frame, $\mathbf{C}_e^w$, may be updated using the equation

$$\dot{\mathbf{C}}_e^w = \mathbf{C}_e^w \boldsymbol{\Omega}_{ew}^w \tag{3.32}$$

where $\boldsymbol{\Omega}_{ew}^w$ is a skew symmetric matrix formed from the elements of the angular rate vector $\boldsymbol{\omega}_{ew}^w$. This process is implemented iteratively and enables any singularities to be avoided. Further details concerning wander azimuth systems and the mechanisations described earlier appear in Reference 1.

### *3.5.5 Summary of strapdown system mechanisations*

This section has provided outline descriptions of a number of possible strapdown inertial navigation system mechanisations. Further details are given in Reference 1. The choice of mechanisation is dependent on the application. Whilst any of the schemes described may be used for navigation close to the Earth, the local geographic navigation frame mechanisation is commonly employed for navigation over large distances. The wander azimuth system provides a world-wide navigation capability. These mechanisations provide navigation data in terms of north and east velocity, latitude and longitude and allow a relatively simple gravity model to be used. For navigation over shorter distances, an Earth fixed reference system may be applicable.

## 3.6 Strapdown attitude representations

### *3.6.1 Introductory remarks*

Consider now ways in which a set of strapdown gyroscopic sensors may be used to instrument a reference co-ordinate frame within a vehicle which is free to rotate about any direction. The attitude of the vehicle with respect to the designated reference frame may be stored as a set of numbers in a computer within the vehicle. The stored attitude is updated as the vehicle rotates using the measurements of turn rate provided by the gyroscopes.

The co-ordinate frames referred to during the course of the discussion which follows are orthogonal, right-handed axis sets in which positive rotations about each axis are taken to be in a clockwise direction looking along the axis from the origin, as indicated in the Figure 3.19. A negative rotation acts in an opposite sense, that is, in an anti-clockwise direction. This convention is used throughout this book.

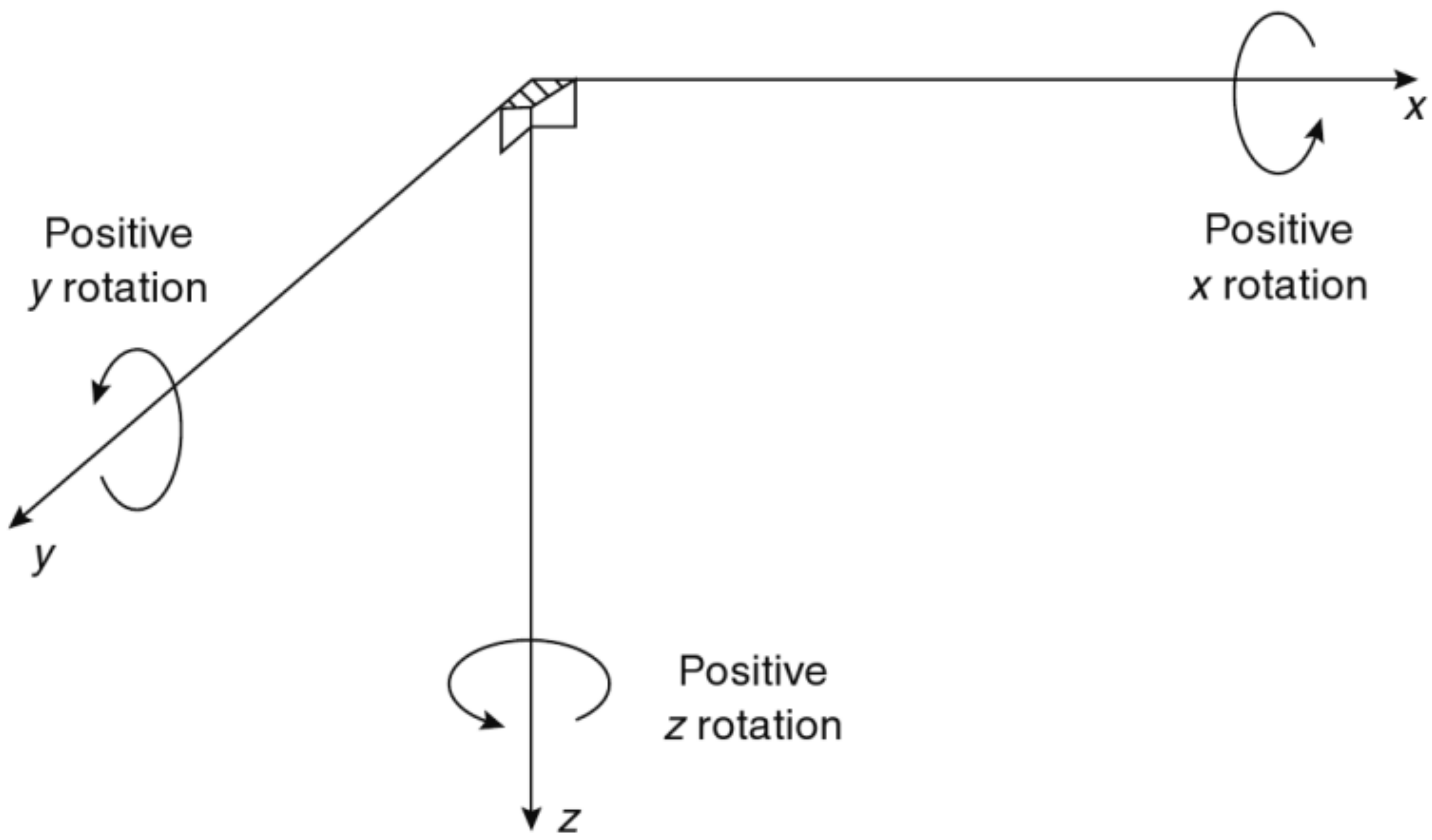


*Figure 3.19 Definition of axis rotations*

It is important to remember that the change in attitude of a body, which is subjected to a series of rotations about different axes, is not only a function of the angles through which it rotates about each of those axes, but the order in which the rotations occur. The illustration given in Figure 3.20, although somewhat extreme, shows quite clearly that the order in which a sequence of rotations occurs is most important.

Rotations are defined here with respect to the orthogonal right-handed axis set, O$xyz$, indicated in the figure. The sequence of rotations shown in the left half of the figure is made up of a 90° pitch, or $y$-axis rotation, followed by a 90° yaw, or $z$-axis rotation, and a further pitch rotation of $-90°$. On completion of this sequence of turns, it can be seen that a net rotation of 90° about the roll ($x$) axis has taken place. In the right hand figure, the order of the rotations has been reversed. Although the body still ends up with its roll axis aligned in the original direction, it is seen that a net roll rotation of $-90°$ has now taken place. Hence, individual axis rotations are said to be non-commutative. It is clear that failure to take account of the order in which rotations arise can lead to a substantial error in the computed attitude.

Various mathematical representations can be used to define the attitude of a body with respect to a co-ordinate reference frame. The parameters associated with each method may be stored within a computer and updated as the vehicle rotates using the measurements of turn rate provided by the strapdown gyroscopes. Three attitude representations are described here, namely:

1. *Direction cosines*. The direction cosine matrix, introduced in Section 3.5, is a $3 \times 3$ matrix, the columns of which represent unit vectors in body axes projected along the reference axes.
2. *Euler angles*. A transformation from one co-ordinate frame to another is defined by three successive rotations about different axes taken in turn. The Euler angle representation is perhaps one of the simplest techniques in terms of physical

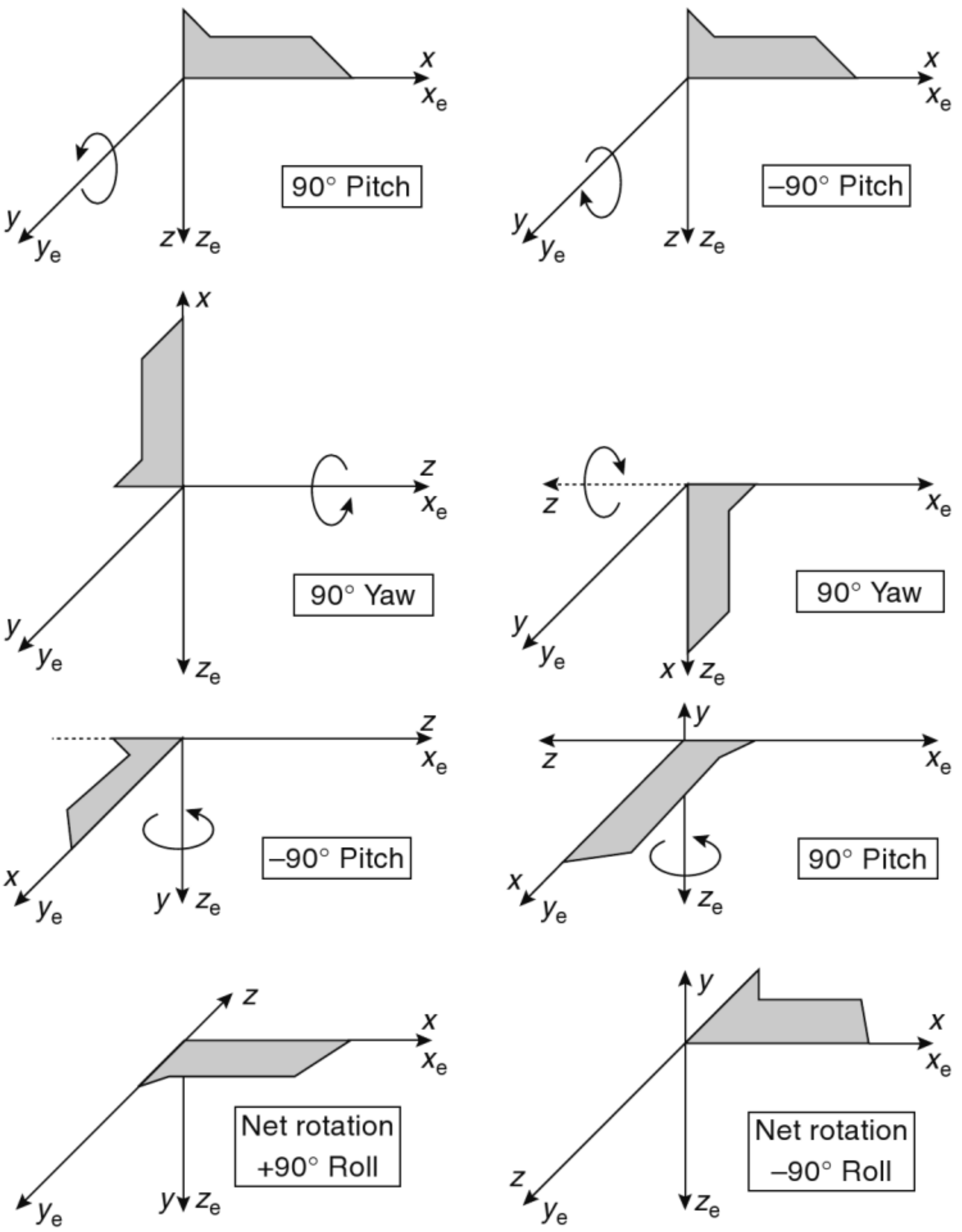


*Figure 3.20 Illustration of effect of order of body rotations*

appreciation. The three angles correspond to the angles which would be measured between a set of mechanical gimbals,[3] which is supporting a stable element, where the axes of the stable element represent the reference frame, and with the body being attached via a bearing to the outer gimbal.

3. *Quaternions*. The quaternion attitude representation allows a transformation from one co-ordinate frame to another to be effected by a single rotation about a vector defined in the reference frame. The quaternion is a four-element vector representation, the elements of which are functions of the orientation of this vector and the magnitude of the rotation.

[3] A gimbal is a rigid mechanical frame which is free to rotate about a single-axis to isolate it from angular motion in that direction. A stable platform can be isolated from body motion if supported by three such frames with their axes of rotation nominally orthogonal to each other.

In the following sections, each of these attitude representations is described in detail.

## *3.6.2 Direction cosine matrix*

### *3.6.2.1 Introduction*

The direction cosine matrix, denoted here by the symbol $\mathbf{C}_b^n$, is a $3 \times 3$ matrix, the columns of which represent unit vectors in body axes projected along the reference axes. $\mathbf{C}_b^n$ is written here in component form as follows:

$$\mathbf{C}_b^n = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{3.33}$$

The element in the $i$th row and the $j$th column represents the cosine of the angle between the $i$-axis of the reference frame and the $j$-axis of the body frame.

### *3.6.2.2 Use of direction cosine matrix for vector transformation*

A vector quantity defined in body axes, $\mathbf{r}^b$, may be expressed in reference axes by pre-multiplying the vector by the direction cosine matrix as follows:

$$\mathbf{r}^n = \mathbf{C}_b^n \mathbf{r}^b \tag{3.34}$$

### *3.6.2.3 Propagation of direction cosine matrix with time*

The rate of change of $\mathbf{C}_b^n$ with time is given by:

$$\dot{\mathbf{C}}_b^n = \lim_{\delta t \to 0} \frac{\delta \mathbf{C}_b^n}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{C}_b^n(t + \delta t) - \mathbf{C}_b^n(t)}{\delta t} \tag{3.35}$$

where $\mathbf{C}_b^n(t)$ and $\mathbf{C}_b^n(t + \delta t)$ represent the direction cosine matrix at times $t$ and $t + \delta t$, respectively. $\mathbf{C}_b^n(t + \delta t)$ can be written as the product of two matrices as follows:

$$\mathbf{C}_b^n(t + \delta t) = \mathbf{C}_b^n(t)\mathbf{A}(t) \tag{3.36}$$

where $\mathbf{A}(t)$ is a direction cosine matrix which relates the b-frame at time $t$ to the b-frame at time $t + \delta t$. For small angle rotations, $\mathbf{A}(t)$ may be written as follows:

$$\mathbf{A}(t) = [\mathbf{I} + \boldsymbol{\delta\Psi}] \tag{3.37}$$

where $\mathbf{I}$ is a $3 \times 3$ identity matrix and

$$\boldsymbol{\delta\Psi} = \begin{bmatrix} 0 & -\delta\psi & \delta\theta \\ \delta\psi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 0 \end{bmatrix} \tag{3.38}$$

in which $\delta\psi$, $\delta\theta$ and $\delta\phi$ are the small rotation angles through which the b-frame has rotated over the time interval $\delta t$ about its yaw, pitch and roll axes, respectively.

In the limit as $\delta t$ approaches zero, small angle approximations are valid and the order of the rotations becomes unimportant.

Substituting for $\mathbf{C}_{\mathrm{b}}^{\mathrm{n}}(t+\delta t)$ in eqn. (3.35) we obtain:

$$\dot{\mathbf{C}}_{\mathrm{b}}^{\mathrm{n}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{n}} \lim_{\delta t \to 0} \frac{\boldsymbol{\delta\Psi}}{\delta t} \tag{3.39}$$

In the limit as $\delta t \to 0$, $\boldsymbol{\delta\Psi}/\delta t$ is the skew symmetric form of the angular rate vector $\boldsymbol{\omega}_{\mathrm{nb}}^{\mathrm{b}} = [\omega_x \quad \omega_y \quad \omega_z]^{\mathrm{T}}$, which represents the turn rate of the b-frame with respect to the n-frame expressed in body axes, that is,

$$\lim_{\delta t \to 0} \frac{\boldsymbol{\delta\Psi}}{\delta t} = \boldsymbol{\Omega}_{\mathrm{nb}}^{\mathrm{b}} \tag{3.40}$$

Substituting in eqn. (3.39) gives:

$$\dot{\mathbf{C}}_{\mathrm{b}}^{\mathrm{n}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{n}} \boldsymbol{\Omega}_{\mathrm{nb}}^{\mathrm{b}} \tag{3.41}$$

where

$$\boldsymbol{\Omega}_{\mathrm{nb}}^{\mathrm{b}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.42}$$

An equation of the form of eqn. (3.41) may be solved within a computer in a strapdown inertial navigation system to keep track of body attitude with respect to the chosen reference frame. It may be expressed in component form as follows:

$$\begin{array}{lll} \dot{c}_{11} = c_{12}\omega_z - c_{13}\omega_y & \dot{c}_{12} = c_{13}\omega_x - c_{11}\omega_z & \dot{c}_{13} = c_{11}\omega_y - c_{12}\omega_x \\ \dot{c}_{21} = c_{22}\omega_z - c_{23}\omega_y & \dot{c}_{22} = c_{23}\omega_x - c_{21}\omega_z & \dot{c}_{23} = c_{21}\omega_y - c_{22}\omega_x \\ \dot{c}_{31} = c_{32}\omega_z - c_{33}\omega_y & \dot{c}_{32} = c_{33}\omega_x - c_{31}\omega_z & \dot{c}_{33} = c_{31}\omega_y - c_{32}\omega_x \end{array} \tag{3.43}$$

### *3.6.3 Euler angles*

#### *3.6.3.1 Introduction*

A transformation from one co-ordinate frame to another can be carried out as three successive rotations about different axes. For instance, a transformation from reference axes to a new co-ordinate frame may be expressed as follows:

rotate through angle $\psi$ about reference $z$-axis
rotate through angle $\theta$ about new $y$-axis
rotate through angle $\phi$ about new $x$-axis

where $\psi$, $\theta$ and $\phi$ are referred to as the Euler rotation angles. This type of representation is popular because of the physical significance of the Euler angles which correspond to the angles which would be measured by angular pick-offs between a set of three gimbals in a stable platform inertial navigation system.

*3.6.3.2 Use of Euler angles for vector transformation*

The three rotations may be expressed mathematically as three separate direction cosine matrices as defined below:

$$\text{rotation } \psi \text{ about } z\text{-axis, } \mathbf{C}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.44}$$

$$\text{rotation } \theta \text{ about } y\text{-axis, } \mathbf{C}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3.45}$$

$$\text{rotation } \phi \text{ about } x\text{-axis, } \mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{3.46}$$

Thus, a transformation from reference to body axes may be expressed as the product of these three separate transformations as follows:

$$\mathbf{C}_{\mathrm{n}}^{\mathrm{b}} = \mathbf{C}_3\mathbf{C}_2\mathbf{C}_1 \tag{3.47}$$

Similarly, the inverse transformation from body to reference axes is given by:

$$\mathbf{C}_{\mathrm{b}}^{\mathrm{n}} = \mathbf{C}_{\mathrm{n}}^{\mathrm{bT}} = \mathbf{C}_1^{\mathrm{T}}\mathbf{C}_2^{\mathrm{T}}\mathbf{C}_3^{\mathrm{T}} \tag{3.48}$$

$$\begin{aligned} \mathbf{C}_{\mathrm{b}}^{\mathrm{n}} &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\psi & \begin{array}{l} -\cos\phi\sin\psi \\ \quad +\sin\phi\sin\theta\cos\psi \end{array} & \begin{array}{l} \sin\phi\sin\psi \\ \quad +\cos\phi\sin\theta\cos\psi \end{array} \\ \cos\theta\sin\psi & \begin{array}{l} \cos\phi\cos\psi \\ \quad +\sin\phi\sin\theta\sin\psi \end{array} & \begin{array}{l} -\sin\phi\cos\psi \\ \quad +\cos\phi\sin\theta\sin\psi \end{array} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \end{aligned} \tag{3.49}$$

This is the direction cosine matrix given by eqn. (3.33) expressed in terms of Euler angles.

For small angle rotations, $\sin\phi \to \phi$, $\sin\theta \to \theta$, $\sin\psi \to \psi$ and the cosines of these angles approach unity. Making these substitutions in eqn. (3.49) and ignoring products of angles which also become small, the direction cosine matrix expressed in terms of the Euler rotations reduces approximately to the skew symmetric form shown below:

$$\mathbf{C}_{\mathrm{b}}^{\mathrm{n}} \approx \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \tag{3.50}$$

This form of matrix is used in Chapter 11 to represent the small change in attitude which occurs between successive updates in the real time computation of body attitude, and in Chapters 10 and 12 to represent the error in the estimated direction cosine matrix.

#### 3.6.3.3 *Propagation of Euler angles with time*

Following the gimbal analogy mentioned earlier, $\phi$, $\theta$ and $\psi$ are the gimbal angles and $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ are the gimbal rates. The gimbal rates are related to the body rates, $\omega_x$, $\omega_y$ and $\omega_z$ as follows:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_3 \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{C}_3\mathbf{C}_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \tag{3.51}$$

This equation can be rearranged and expressed in component form as follows:

$$\begin{aligned} \dot{\phi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\tan\theta + \omega_x \\ \dot{\theta} &= \omega_y \cos\phi - \omega_z \sin\phi \\ \dot{\psi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\sec\theta \end{aligned} \tag{3.52}$$

Equations of this form may be solved in a strapdown system to update the Euler rotations of the body with respect to the chosen reference frame. However, their use is limited since the solution of the $\dot{\phi}$ and $\dot{\psi}$ equations become indeterminate when $\theta = \pm 90°$.

### 3.6.4 *Quaternions*

#### 3.6.4.1 *Introduction*

The quaternion attitude representation is a four-parameter representation based on the idea that a transformation from one co-ordinate frame to another may be effected by a single rotation about a vector $\boldsymbol{\mu}$ defined with respect to the reference frame. The quaternion, denoted here by the symbol $\mathbf{q}$, is a four element vector, the elements of which are functions of this vector and the magnitude of the rotation:

$$\mathbf{q} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \cos(\mu/2) \\ (\mu_x/\mu)\sin(\mu/2) \\ (\mu_y/\mu)\sin(\mu/2) \\ (\mu_z/\mu)\sin(\mu/2) \end{bmatrix} \tag{3.53}$$

where $\mu_x$, $\mu_y$, $\mu_z$ are the components of the angle vector $\boldsymbol{\mu}$ and $\mu$ the magnitude of $\boldsymbol{\mu}$.

The magnitude and direction of $\boldsymbol{\mu}$ are defined in order that the reference frame may be rotated into coincidence with the body frame by rotating about $\boldsymbol{\mu}$ through an angle $\mu$.

A quaternion with components $a$, $b$, $c$ and $d$ may also be expressed as a four-parameter complex number with a real component $a$, and three imaginary components,

$b$, $c$ and $d$, as follows:

$$\mathbf{q} = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \tag{3.54}$$

This is an extension of the more usual two parameter complex number form with one real component and one imaginary component, $x = a + \mathbf{i}b$, with which the reader is more likely to be familiar.

The product of two quaternions, $\mathbf{q} = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ and $\mathbf{p} = e + \mathbf{i}f + \mathbf{j}g + \mathbf{k}h$ may then be derived as shown below applying the usual rules for products of complex numbers, viz:

$$\mathbf{i}\cdot\mathbf{i} = -1 \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\cdot\mathbf{i} = -\mathbf{k} \qquad \ldots \text{ etc.}$$

Hence,

$$\begin{aligned}\mathbf{q}\cdot\mathbf{p} &= (a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d)(e + \mathbf{i}f + \mathbf{j}g + \mathbf{k}h)\\ &= ea - bf - cg - dh + (af + be + ch - dg)\mathbf{i}\\ &\quad + (ag + ce - bh + df)\mathbf{j} + (ah + de + bg - cf)\mathbf{k}\end{aligned} \tag{3.55}$$

Alternatively, the quaternion product may be expressed in matrix form as:

$$\mathbf{q}\cdot\mathbf{p} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \tag{3.56}$$

### *3.6.4.2 Use of quaternion for vector transformation*

A vector quantity defined in body axes, $\mathbf{r}^{\mathrm{b}}$, may be expressed in reference axes as $\mathbf{r}^{\mathrm{n}}$ using the quaternion directly. First define a quaternion, $\mathbf{r}^{\mathrm{b}'}$, in which the complex components are set equal to the components of $\mathbf{r}^{\mathrm{b}}$, and with a zero scalar component, that is, if:

$$\mathbf{r}^{\mathrm{b}} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

$$\mathbf{r}^{\mathrm{b}'} = 0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

This is expressed in reference axes as $\mathbf{r}^{\mathrm{n}'}$ using:

$$\mathbf{r}^{\mathrm{n}'} = \mathbf{q}\mathbf{r}^{\mathrm{b}'}\mathbf{q}^{*} \tag{3.57}$$

where $\mathbf{q}^{*} = (a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d)$, the complex conjugate of $\mathbf{q}$.

Hence,

$$\begin{aligned}\mathbf{r}^{\mathrm{n}'} &= (a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d)(0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z)(a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d)\\ &= 0 + \{(a^2 + b^2 - c^2 - d^2)x + 2(bc - ad)y + 2(bd + ac)z\}\mathbf{i}\\ &\quad + \{2(bc + ad)x + (a^2 - b^2 + c^2 - d^2)y + 2(cd - ab)z\}\mathbf{j}\\ &\quad + \{2(bd - ac)x + 2(cd + ab)y + (a^2 - b^2 - c^2 + d^2)z\}\mathbf{k}\end{aligned} \tag{3.58}$$

Alternatively, $\mathbf{r}^{n'}$ may be expressed in matrix form as follows:

$$\mathbf{r}^{n'} = \mathbf{C}'\mathbf{r}^{b'}$$

where

$$\mathbf{C}' = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{C} \end{bmatrix} \qquad \mathbf{r}^{b'} = \begin{bmatrix} 0 \\ \mathbf{r}^{b} \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \tag{3.59}$$

which is equivalent to writing:

$$\mathbf{r}^{n} = \mathbf{C}\mathbf{r}^{b}$$

Comparison with eqn. (3.34) reveals that $\mathbf{C}$ is equivalent to the direction cosine matrix $\mathbf{C}_b^n$.

### *3.6.4.3 Propagation of quaternion with time*

The quaternion, $\mathbf{q}$, propagates in accordance with the following equation:

$$\dot{\mathbf{q}} = 0.5\mathbf{q} \cdot \mathbf{p}_{nb}^{b} \tag{3.60}$$

This equation may be expressed in matrix form as a function of the components of $\mathbf{q}$ and $\mathbf{p}_{nb}^{b} = [0, \quad \boldsymbol{\omega}_{nb}^{b\,T}]^T$ as follows:

$$\mathbf{q} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = 0.5 \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{3.61}$$

that is,

$$\begin{aligned} \dot{a} &= -0.5(b\omega_x + c\omega_y + d\omega_z) \\ \dot{b} &= 0.5(a\omega_x - d\omega_y + c\omega_z) \\ \dot{c} &= 0.5(d\omega_x + a\omega_y - b\omega_z) \\ \dot{d} &= -0.5(c\omega_x - b\omega_y - a\omega_z) \end{aligned} \tag{3.62}$$

Equations of this form may be solved in a strapdown navigation system to keep track of the quaternion parameters which define body orientation. The quaternion parameters may then be used to compute an equivalent direction cosine matrix, or used directly to transform the measured specific force vector into the chosen reference frame (see eqn. (3.57)).

### 3.6.5 Relationships between direction cosines, Euler angles and quaternions

As shown in the preceding sections, the direction cosines may be expressed in terms of Euler angles or quaternions, viz:

$$\mathbf{C}_\mathrm{b}^\mathrm{n} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \begin{matrix} -\cos\phi\sin\psi \\ +\sin\phi\sin\theta\cos\psi \end{matrix} & \begin{matrix} \sin\phi\sin\psi \\ +\cos\phi\sin\theta\cos\psi \end{matrix} \\ \cos\theta\sin\psi & \begin{matrix} \cos\phi\cos\psi \\ +\sin\phi\sin\theta\sin\psi \end{matrix} & \begin{matrix} -\sin\phi\cos\psi \\ +\cos\phi\sin\theta\sin\psi \end{matrix} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \tag{3.63}$$

By comparing the elements of the above equations, the quaternion elements may be expressed directly in terms of Euler angles or direction cosines. Similarly, the Euler angles may be written in terms of direction cosines or quaternions. Some of these relationships are summarised in the following sections.

#### 3.6.5.1 Quaternions expressed in terms of direction cosines

For small angular displacements, the quaternion parameters may be derived using the following relationships:

$$\begin{aligned} a &= \tfrac{1}{2}(1+c_{11}+c_{22}+c_{33})^{1/2} \\ b &= \frac{1}{4a}(c_{32}-c_{23}) \\ c &= \frac{1}{4a}(c_{13}-c_{31}) \\ d &= \frac{1}{4a}(c_{21}-c_{12}) \end{aligned} \tag{3.64}$$

A more comprehensive algorithm for the extraction of quaternion parameters from the direction cosines, which takes account of the relative magnitudes of the direction cosine elements, is described by Shepperd [2].

*Chapter 4*

# Gyroscope technology 1

## 4.1 Introduction

Gyroscopes are used in various applications to sense either the angle turned through by a vehicle or structure (displacement gyroscopes) or, more commonly, its angular rate of turn about some defined axis (rate gyroscopes). The sensors are used in a variety of roles such as:

- stabilisation,
- autopilot feedback,
- flight path sensor or platform stabilisation,
- navigation.

It is possible with modern gyroscopes for a single sensor to fulfil each of the above tasks, but often two or more separate clusters of sensors are used.

The most basic and the original form of gyroscopes makes use of the inertial properties of a wheel or rotor spinning at high speed. Many people are familiar with the child's toy which has a heavy metal rotor supported by a pair of gimbals [1]. When the rotor is spun at high speed, the rotor axis continues to point in the same direction despite the gimbals being rotated. This is a crude example of a mechanical, or conventional, displacement gyroscope.

Examples of mechanical spinning wheel gyroscopes used in strapdown applications are the single-axis rate-integrating gyroscope and twin axis 'tuned' or flex gyroscopes. An alternative class designation for gyroscopes that cannot be categorised in this way, is not surprisingly called unconventional sensors, some of which are solid-state devices. The very broad and expanding class of unconventional sensors includes devices such as:

- Rate transducers which include mercury sphere and magneto-hydrodynamic sensors;
- Vibratory gyroscopes;
- Nuclear magnetic resonance (NMR) gyroscopes;

# Strapdown Inertial Navigation Technology

## 2nd Edition

Inertial navigation is widely used for the guidance of aircraft, missiles, ships and land vehicles, as well as in a number of novel applications such as surveying underground pipelines in drilling operations. This book sets out to provide a clear and concise description of the physical principles of inertial navigation, the associated growth of errors and their compensation. There is also detailed treatment of recent developments in inertial sensor technology and a description of techniques for implementing and evaluating such systems. This new edition includes a number of refinements covering sensor technology, geodesy and error modelling, plus new chapters on MEMS technology and inertial systems applications.

**David Titterton** is a member of the DSTL college of Fellows (individual merit scientists) and is currently the technical leader of laser systems at this government laboratory. He is also a visiting Professor at the University of Cranfield.

**John Weston** is a Prinipal Scientist at Halliburton Sperry-Sun researching inertial/gyroscopic systems for the surveying of underground pipelines and well bores. He has worked in missile guidance and control, originally with British Aerospace plc.

ISBN 0-86341-358-7

9 780863 413582 >

The Institution of Engineering and Technology
www.theiet.org
0 86341 358 7
978-0-86341-358-2