Algorithms and Data Structures
(AED — Algoritmos e Estruturas de Dados)
LEI, MIEC, 2018/2019 (40437)

— TP.01 —

Summary:

- What AED is all about
- Rules of the game
- List of planned lectures
- Recommended bibliography for the entire course
- The C programming language

Teachers:

- **TP1**, **P3**, and **P4**: Tomás Oliveira e Silva, tos@ua.pt, DETI 4.2.37
- **P1** and **P2**: Diogo Gomes, dgomes@ua.pt, IT

When and where:

- **TP1** room 11.1.3 (DMat) Monday, 14h00m-16h00m
- **P1** room 4.2.11 (DETI), Monday, 9h00m-11h00m
- **P2** room 4.2.11 (DETI), Monday, 11h00m-13h00m
- **P3** room 4.2.03 (DETI), Monday, 16h00m-18h00m
- **P4** room 4.2.03 (DETI), Monday, 18h00m-20h00m

Recommended bibliography for this lecture:

- **C in a Nutshell, A Desktop Quick Reference**, Peter Prinz and Tony Crawford, O'Reilly, 2006.
- **C, A Reference Manual**, Samuel P. Harbison III and Guy L. Steele Jr., fifth edition, Prentice Hall, 2002.
- **C Programming (A Comprehensive Look at the C Programming Language and Its Features)**, wiki book.
- **The C Programming Language**, Brian W. Kernighan and Dennis M. Ritchie, second edition, Prentice Hall, 1988.

This document was last updated on October 21, 2018.

How to navigate these lecture notes

Links to other parts of this document and to other documents are displayed in dark orange.

To avoid an excessive use of that color, in the summary of each lecture the links to the various parts of the lecture are located in the filled dark orange circles (●).

At the bottom of each page, on the right hand side, there are links that go

- to the first page of this document (AED link)
- to the first page of the current lecture (TP.N or P.N links)
- to the previous lecture (◀)
- to the next lecture (▶)

The list of planned lectures pages has links to all the other lectures. The first page of this document has a direct link to that page.

What AED is all about

Program = Algorithm + Data Structures

$$\textbf{Good Program} = \begin{cases} \textbf{Algorithm + Good Data Structures} \\ \quad \textbf{or} \\ \textbf{Good Algorithm + Data Structures} \end{cases}$$

Very good Program = Good Algorithm + Good Data Structures

Exceptional Program = State-of-the-art Algorithm + State-of-the-art Data Structures

A good algorithm/data structure has a "small" (i.e., as small as theoretically possible) computational complexity.

The computational complexity measures the time or memory resources (space) needed to run the program.

A state-of-the-art algorithm/data structure does the job better than any other algorithms/data structures available to solve the same problem.

It may be advantageous to trade time for space (or to trade space for time).

A good programmer knows many good algorithms and data structures (and knows where to look for more), and is capable of determining which ones are best for the job at hand.

An exceptional programmer is capable of devising new algorithms or data structures to solve a new problem.

Example: computing Fibonacci numbers

The Fibonacci numbers are defined by the recursive formula

$$F_n = F_{n-1} + F_{n-2}, \qquad n > 1,$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. We present below some possible ways to compute F_n in C (without detection of bad inputs or of arithmetic overflow):

- **Recursive implementation:**

```
int F_v1(int n)
{
  return (n < 2) ? n : F_v1(n - 1) + F_v1(n - 2);
}
```

- **"Memoized" recursive implementation:**

```
int F_v2(int n)
{
  static int Fv[50] = { 0,1 };

  if(n > 1 && Fv[n] == 0)
    Fv[n] = F_v2(n - 1) + F_v2(n - 2);
  return Fv[n];
}
```

- **Non-recursive implementation:**

```
int F_v3(int n)
{
  int a,b,c;

  if(n < 2)
    return n;
  for(a = 0,b = 1;n > 1;n--)
  {
    c = a + b; // c = F(n-2) + F(n-1)
    a = b;     // a = F(n-1)
    b = c;     // b = F(n)
  }
  return b;
}
```

- **"Clever" implementation (Binet's formula):**

```
int F_v4(int n)
{
  const double c1 = 0.44721359549995793928;
  const double c2 = 0.48121182505960344750;
  return (int)round(c1 * exp((double)n * c2));
}
```

Note that $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Rules of the game (part 1)

Grading in AED will abide by the following rules:

- Grading has two components: theoretical part, G_1, and computer lab part, G_2, with $0 \leqslant G_1, G_2 \leqslant 20$.
- A grade below 7.5 in either of the two parts means course failure (RNM).
- Presence in the theoretical lectures is not mandatory (but it is strongly recommended).
- Presence in the computer lab classes is mandatory for ordinary students. Failure to attend more than N computer lab classes means course failure (RPF), without possibility of attending the recourse epoch, N being equal to 3 for ordinary students and to ∞ for working students.
- The tentative final grade G is given by $G = \mathrm{round}\left(\frac{G_1+G_2}{2}\right)$. If $G \leqslant 16$, then G will be the final grade. Otherwise, the final grade will also take in consideration one oral presentation, to be done on the first two weeks of January 2019, of an extra individual work assigned to each eligible student. In that case the final grade will be $\geqslant 16$ and $\leqslant 20$.
- The G_1 grade can be obtained in one of two ways: either during the semester as the weighted average of three tests to be performed in the first hour of some theoretical lectures, or by a final exam. In the first case, $G_1 = 0.40t_1 + 0.35t_2 + 0.25t_3$, where t_1 is the **best** grade and t_3 is the **worst** grade. Each student has to chose by October ?? which of the two choices she/he desires. It is **very highly recommended** that the students take the **first** choice.

  For example, if the grades of the three tests are 14, 17, 12, then

  $$G_1 = 17 \times 0.40 + 14 \times 0.35 + 12 \times 0.25 = 14.7$$

  The final exam will have a duration of at least 3 hours.

Rules of the game (part 2)

- The G_2 grade is the weighted average of three reports of work done in the classroom (or outside classes for working students); $G_2 = 0.40p_1 + 0.35p_2 + 0.25p_3$, where p_1 is the **best** grade and p_3 is the **worst** grade. Each report will be graded based on

  1. clarity of exposition,
  2. quality of the results obtained,
  3. code quality,
  4. originality,
  5. amount of work done in the practical class, and
  6. amount of work done **before** the practical class.

  Plagiarism will be severely punished. Each report can be done by groups of at most 3 students. Grades may be different for the students of each group, according to

  1. how much each contributed to the work (stated in the report), and
  2. the teacher's perception of how much each student appeared to work.

  This example gives an idea of how a report should be structured.
- In the recourse and special epochs the G_2 grade is the grade of a report of the work done in an all day (8 hours) computer lab session.
- Students wishing to raise their grades must do so either in the recourse epoch or in the next school year.

List of planned lectures (part 1)

Class	Summary
17-09-2018 TP.01	What AED is all about. Rules of the game. The C programming language (overview, preprocessor directives, comments, data types, declaration, definition, and scope of variables).
P.01	Study and simple modifications of some `C` programs.
24-09-2018 TP.02	The C programming language (assignments and expressions, statements, functions, and standard library functions). The C++ programming language (classes, templates, and exceptions).
P.02	Study and simple modifications of some `C` and `C++` programs.
01-10-2018 TP.03	Algorithms. Abstract data types. Computational complexity. The RAM model of computation. Formal and empirical algorithm analysis. Worst, best, and average cases. Asymptotic notation ($O()$, $\tilde{O}()$, $\Omega()$, and $\Theta()$). Examples.
P.02a	Conclusion of the study and simple modifications of some `C++` programs.
08-10-2018 TP.04	Classes of problems. Polynomial-time algorithms and non polynomial-time algorithms. Analysis of the time/space needed by an algorithm. Examples (for non-recursive and for recursive algorithms).
P.03	Computational complexity exercises (paper and pencil).
15-10-2018 TP.05	**Test (TP.01 to TP.04).** Elementary data structures, part 1. Arrays. Linked lists (singly- and doubly-linked). Stacks. Queues. Circular buffers.
P.04	Computational complexity examples. First practical assignement (the traveling salesman problem).
22-10-2018 TP.06	Elementary data structures, part 2. Heaps. Priority queues. Binary trees (unordered, ordered, and balanced). Tries. Hash tables.
P.05	Study of C++ code that implements some elementary data structures.
29-10-2018 TP.07	Searching unordered data in i) an array, ii) a linked list, iii) a binary tree. How to improve search times. Searching ordered data in i) an array, ii) a binary tree. Sorting methods (Bubble sort and shaker sort, Insertion and Shell sort, Quicksort, Merge sort, Heap sort). Other sorting methods (count sort, rank sort, selection sort, bitonic sort).
P.06	To be defined.

List of planned lectures (part 2)

Class	Summary
05-11-2018 TP.08	Common algorithmic techniques: divide-and-conquer and dynamic programming. Examples. Specialized algorithmic techniques (meet-in-the-middle, ...)
P.07	To be defined.
12-11-2018 TP.09	**Test (TP.05 to TP.08).** Finding all possibilities (exhaustive search), part 1. Depth-first search. Breadth-first search. Backtracking. Pruning.
P.08	To be defined.
19-11-2018 TP.10	Finding all possibilities (exhaustive search), part 2. Examples.
P.09	To be defined.
26-11-2018 TP.11	Graphs, part 1. Data structures for graphs. Graph traversal (depth-first and breadth-first). Connected components.
P.10	To be defined.
03-12-2018 TP.12	Graphs, part 2. All paths and all cycles. Shortest paths. Minimal spanning tree.
P.11	To be defined.
10-12-2018 TP.13	**Test (TP.09 to TP.12).** Extra stuff: (to be decided.)
P.12	To be defined.
17-12-2018 TP.14	Extra stuff: Topics in computational geometry (point location, quad-trees and oct-trees, convex hull, Voronoi diagrams, Delaunay triangulations, Steiner trees).
P.14	**Final Presentations.**

All programming will be done in either `C` or `C++`.

Recommended bibliography for the entire course

Algorithms, Robert Sedgewick and Kevin Wayne, fourth edition, Addison Wesley, 2011
Análise da Complexidade de Algoritmos, António Adrego da Rocha, FCA.
Analysis of Algorithms, Jeffrey J. McConnell, second edition, Jones and Bertlett Publishers, 2008.
C in a nutshell, a desktop quick reference, Peter Prinz and Tony Crawford, O'Reilly, 2006.
Estruturas de Dados e Algoritmos em C, António Adrego da Rocha, terceira edição, FCA.
Programming Pearls, Jon Bentley, second edition, Addison Wesley, 2000.
Thinking in C++. Volumes One and Two, Bruce Eckel and Chuck Allison, Prentice Hall, 2000 and 2003.

Books that each serious programmer should have (incomplete list)

Algorithm Design, Jon Kleinberg and Éva Tardos, Addison Wesley, 2006.
Algorithms, Robert Sedgewick and Kevin Wayne, fourth edition, Addison Wesley, 2011
Computational Geometry. Algorithms and Applications, M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, second edition, Springer, 2000.
Concrete Mathematics, Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, second edition, Addison Wesley, 1994.
Handbook of Data Structures and Applications, Dinesh P. Mehta and Sartaj Sahni (editors), Chapman and Hall/CRC, 2005.
Introduction to Algorithms, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, third edition, The MIT Press, 2009.
Numerical Recipes. The Art of Scientific Computing, William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, third edition, Cambridge University Press, 2007.
Object-Oriented Software Construction, Bertrand Meyer, second edition, Prentice-Hall, 1997.
The Algorithm Design Manual, Steven S. Skiena, second edition, Springer, 2008.
The Art of Computer Programming, Volume 1 (Fundamental Algorithms), Donald E. Knuth, third edition, Addison Wesley, 1997.
The Art of Computer Programming, Volume 2 (Seminumerical Algorithms), Donald E. Knuth, third edition, Addison Wesley, 1998.
The Art of Computer Programming, Volume 3 (Sorting and Searching), Donald E. Knuth, third edition, Addison Wesley, 1998.
The Art of Computer Programming, Volume 4A (Combinatorial Algorithms, Part 1), Donald E. Knuth, Addison Wesley, 2011.

The C programming language

Summary:

- C language overview
- Preprocessor directives
- Comments
- Data types
- Declaration, definition, and scope of variables
- Assignments and expressions
- Statements
- Functions
- Standard library functions
- Coding style

[Remark: due to space and time limitations, we will omit many details.]

Fact. According to an IEEE spectrum survey, in 2018 the top five most popular programming languages were:

- `Python` (100.0, 100.0 in 2017)
- `C++` (99.7, 97.4 in 2017)
- `Java` (97.5, 99.4 in 2017)
- `C` (96.7, 99.7 in 2017)
- `C#` (89.4, 88.8 in 2017)

An example:

```
1  /*
2  ** Hello world program
3  */
4
5  #include <stdio.h>
6
7  int main(void)
8  {
9    puts("Hello world!");
10   return 0;
11 }
```

(The numbers on the left are not part of the code.) Lines 1 to 3 are a comment. Line 5 instructs the compiler to replace line 5 by the contents of the file named `stdio.h` (one of the files of the C compiler's standard libraries). Line 7 declares and defines a function named `main`, which is the entry point of the program and, in this case, takes no arguments and returns an integer. Lines 8 to 11 constitute the body of the function. Line 9 is a call to a function named `puts` (declared in `stdio.h`), belonging to the C standard library, that outputs its string argument to the terminal. Line 10 forces a return from the main function with a return value of 0 (since `main` is the entry point of the program, this actually specifies the error code of the entire program; 0 means all is well, non-zero means some error occurred).

C language overview

Each C source code file may contain

- preprocessor directives
  Preprocessor directives tell the compiler to manipulate the source code in certain ways. For example, the `#include` directive tells the compiler to replace the entire include directive line by the text of the file whose name appears after the include directive. There are also directives that allow us to define replacement text for a given word and to do conditional compilation of code.
- comments
  A comment is a chunk of text that is ignored by the compiler
- declaration of new data types
  New data types agglomerate one or more existing data type into a new type, and give it a name that we can use from that point on to refer to that new type.
- declaration of variables and of functions
  A declaration of a variable is a description of the type and memory storage attributes of the variable. A declaration of a function is a description of the arguments and return value (if any) and their type of the function. It **does not** reserve space in memory for a variable and **no code** is produced for a function. After a declaration, the variable or function can be used in our code, even if its definition (see next item) is elsewhere in the code (it can even be missing in our code if it resides in a code library).
- definition of variables and of functions
  When the compiler encounters a definition it reserves space for a variable and generates code for a function. It is possible to declare and define a variable (or a function) at the same time. Variable and function names have to be unique.

Preprocessor directives (part 1)

The C preprocessor can be used to modify on the fly the text that is going to be fed to the C compiler. Each preprocessor directive must be placed on a line that begins with the character #. The most important of them are:

- `#include <filename>`
  `#include "filename"`
  This directive instructs the preprocessor to replace the directive by the entire text of the file whose name follows the include directive. The first form looks for files only in compiler directories (standard library header files). The second form looks for user files (in the current directory).
- `#define NAME substitution_text`
  `#define NAME(arg1,arg2,...) substitution_text`
  This directive defines a preprocessor macro named `NAME`. On subsequent lines, each time the text `NAME` appears in the source code it is replaced by the substitution text. In the first form, the macro does not have any arguments. In the second form it can have one or more arguments. If the name of an argument appears in the substitution text it gets replaced by the text that was placed in the argument when the macro was invoked. For example, the code fragment

  ```
  #define C       (int)
  #define X(i)    x[i]
  #define Y(i,j)  i * j
  C X(3) + Y(i,7);
  ```

  gets transformed into the code

  ```
  (int) x[3] + i * 7;
  ```

Preprocessor directives (part 2)

- `#undef NAME`
  It is not possible to redefine a macro (with a different replacement text) without first removing its previous definition. This directive makes sure that a previous definition of the macro (if any) is removed.
- `#if EXPRESSION`
  `#elif EXPRESSION`
  `#else`
  `#endif`
  If the integer expression, which must use only constants known to the preprocessor (macros with replacement text that are integers), is non-zero, then the text in the lines following an `#if` directive and up to an `#elif`, an `#else` or an `#endif` directive gets fed to the compiler; `#elif` and `#else` directives are treated in the logical way. Symbols unknown to the preprocessor are replaced by zeros. For example, in the code fragment

```
#if N == 1
line1
#elif N == 2
line2
#else
line3
#endif
```

  `line1` is passed to the compiler only if the macro `N` is defined and evaluates to 1, `line2` is passed to the compiler only if the macro `N` is defined and evaluates to 2, and `line3` is passed to the compiler if the macro `N` is not defined (and so is replaced by 0) or if it does not evaluate to either 1 or 2.

Comments

Comments are annotations placed in the source code of a program. A good comment explains a non-obvious thing, such as how some piece of code works, or what trade-offs (between, say, execution time and memory usage) were made in that part of the code and why. Comments are also usually used to indicate who wrote a part of a program, and to record significant changes in the source code.

In C there are two kinds of comments: single line comments, which begin with `//` and end at the end of the line, such as in

```
d = (d + 1) | 1; // if d is even increment it by one, otherwise, increment it by 2
```

and comments that can span multiple lines, which begin with `/*` and end with **the first** `*/`, such as in

```
/*
** in the following loop d takes the values 2, 3, 5, 7, 9, 11, 13, 15, 17, ...,
** up to (and including) the square root of n
**
** we would have liked for d to be the prime numbers 2, 3, 5, 7, 11, 13, 17, ...,
** but that is much more difficult to achieve
**
** FIX ME: there is arithmetic overflow if n is a prime number close to the largest representable
** signed integer; for 32-bit integers this can be fixed by exiting the loop as soon as d > 46340
*/
for(d = 2;d * d <= n;d = (d + 1) | 1)
```

Comments are removed by the preprocessor. The preprocessor joins a line terminated by `\` with the next line, so single line comments may actually span more than one line if they are terminated in that way).

A simple and fast way to force the compiler to ignore a large continuous piece of code, even it is has comments, is to put it between `#if 0` and `#endif` lines.

Data types (part 1a, integer data types)

The C language has the following fundamental integer data types (in non-decreasing order of size): `char`, `short`, `int`, `long` and `long long`. Each of these types can be either `signed` (the default with the possible exception of the `char` type) or `unsigned`. In the following code fragment we present an example of the simultaneous declaration, definition, and initialization of variables for each one of these types.

```
             char  c0 = 'A';  // by default signed on most compilers
  signed     char  c1 = 'B';  // make sure the type is signed
unsigned     char  c2 = 'C';

             short s0 = 1763;  // the same as signed short
unsigned     short s1 = 1728;

             int   i0 = -1373762;  // the same as signed int
unsigned     int   i1 = 8382382U;  // the trailing U signals that the integer constant is unsigned

             long  l0  = 82781762873L;  // the same as signed long and signed long int
unsigned     long  l1 = 38273827322UL;  //   the int is optional, so we do usually do not put it

        long long  L0  = 8278176284398439847 3LL;  // the same as signed long long int
unsigned long long  L1 = 3827382733493498332 2ULL;  //   the int is optional
```

Unfortunately, the designers of the C language did not specify the size (number of bytes) of most of these types. So, an `int` may have two bytes if the compiler is producing code for a very old processor, and four bytes if it is targeting a modern processor. The types `int8_t`, `uint8_t`, `int16_t`, `uint16_t`, `int32_t`, `uint32_t`, `int64_t`, and `uint64_t`, defined in the header file `stdint.h`, should be used whenever a specific size is desired.

Data types (part 1b, integer data types)

Let $b_0, b_1, b_2, \ldots, b_{n-2}, b_{n-1}$ be the bits on an n-bit integer B, b_0 being the least significant bit and b_{n-1} being the most significant bit. On virtually all contemporary processors, for an **unsigned integer data type** the value of B is given by

$$B = b_0 \times 2^0 + b_1 \times 2^1 + b_2 \times 2^2 + \cdots + b_{n-2} \times 2^{n-2} + b_{n-1} \times 2^{n-1} = \sum_{i=0}^{n-1} b_i 2^i.$$

Therefore, to increase the number of bits of an unsigned integer the new bits get a value of 0 since doing that does not change the value represented by the bits. To reduce the number of bits, in C the most significant bits are simply discarded (no error is raised if the result does not represent the original unsigned integer; it is assumed that the programmer knows that she/he is doing).

For a **signed integer data type** the value of B is given by (two's complement!)

$$\begin{aligned} B &= b_0 \times 2^0 + b_1 \times 2^1 + b_2 \times 2^2 + \cdots + b_{n-2} \times 2^{n-2} - b_{n-1} \times 2^{n-1} \\ &= -b_{n-1} 2^{n-1} + \sum_{i=0}^{n-2} b_i 2^i = -b_{n-1} 2^n + \sum_{i=0}^{n-1} b_i 2^i. \end{aligned}$$

This last equality, that expresses B with $n+1$ bits in which $b_n = b_{n-1}$, shows that to increase the number of bits of a signed integer the new bits get the value of the most significant bit of the original signed integer, since doing that does not change the value represented by the bits. An for unsigned integers, to reduce the number of bits, in C the most significant bits are simply discarded (again, no error is raised if the result does not represent the original signed integer).

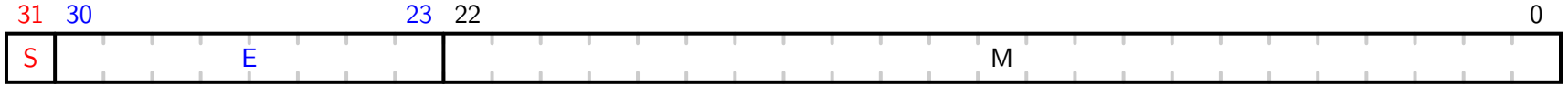
To convert from a n-bit signed to an n-bit unsigned integer, and vise-versa, the C compiler simply does nothing (does not change any bit). Again, no error is raised if the result does not represent the original integer.

Data types (part 2, floating point)

The C language has two fundamental floating point data types: `float` (single precision, 4 bytes, about 7 significant decimal digits) and `double` (double precision, 8 bytes, about 16 significant decimal digits). In the following code fragment we present an example of the simultaneous declaration, definition, and initialization of variables of each one of these types.

```
 float f =  1.23e3f;  // the same as 1230.0f (f denotes a 4-byte floating point constant)
double d = -1.23e6;   // the same as -1230000.0 (no f, so a 8-byte floating point constant)
```

All contemporary processors represent floating point numbers using the IEEE 754 standard. For example, a single precision float in encoded in 32 bits according to the following format:

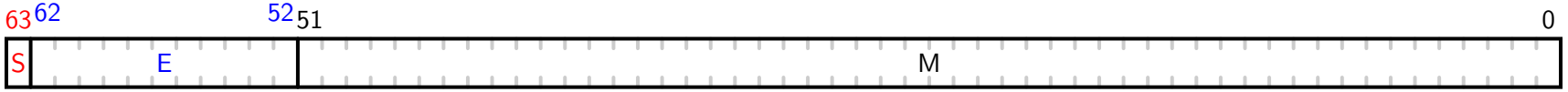


One bit encodes the sign of the number, 8 bits encode an exponent, and 23 bits encode a so-called mantissa. In this format, assuming that both E and M are unsigned integers with the appropriate number of bits, a non-zero real number x is given by

$$x = (-1)^S \times \left(1 + \frac{M}{2^{23}}\right) \times 2^{E-127}.$$

(The real number 0.0 is encoded by all zeros; 0 and 255 are special values of E.)

The double precision floating point format is similar, but the number of bits used to represent the exponent and the mantissa are larger (11 bits for the exponent and 52 bits for the mantissa):



Data types (part 3, pointer data types)

The C language has only one more fundamental data type: a pointer. A pointer is an integer that represents the memory address where a variable of a given type is stored. One can, using the appropriate syntax (see below) read or write the contents of the memory location whose address is stored in the pointer. (Note that, to the processor, the pointer itself is an integer variable.) Given that it is possible to do arbitrary modifications to a pointer, it is possible in C to manipulate the contents of arbitrary memory locations (dangerous, but powerful!). Pointers are declared and used by putting a * before the pointer name. To get the address of a variable put an & before the variable name. For example, in the code fragment

```
float f = 1.0f,*pf = &f;
*pf = 2.0;
```

the single precision floating point variable `f` gets the value $\mathbf{1.0}$ at the end of the first line and the value of $\mathbf{2.0}$ at the end of the second. As long as `pf` is not modified, it is possible to change the contents of `f` through the pointer. Incrementing (decrementing) a pointer makes it point to the next (previous) adjacent variable in memory of the same type. If that memory location does not hold a variable of that type, changing the memory through the pointer leads to all kinds of problems. Assigning `NULL` to a pointer is the standard way in C to say that the pointer points to nothing.

We also have pointers to functions: they store the starting address of a function.

Data types (part 4, literal values)

Literal values are the numeric constants we put in our programs. Integer literals can be encoded in several ways:

- as a character, as in `'3'`, `'\''`, `'"','\n'`, or `'\t'`
- as a decimal integer, as in `123` or `-17`
- as an hexadecimal integer (base 16), as in `0x01F3`
- as an octal integer (base 8), as in `0173`. Constants beginning with with a `0` **are specified in octal**!
- as an binary integer (base 2), as in `0b10101` (`gcc` compiler).

With the exception of character encodings, append `U` at the end to mark an unsigned integer, append `L` to mark a `long` integer, and append `LL` to mark a `long long` integer. All of these can also be in lower case.

Floating points constants can have an optional sign (plus or minus), can have zero or more digits before an optional decimal point and zero or more digits after the decimal point (in all, at least one digit must be present). It can also have an optional exponent part, composed by the letter e (upper or lower case), followed by a decimal signed integer. For example, `-1.2`, `.2e-13`, `+12.`, `1e-8` are all valid floating point literals. By default, floating point literals are in double precision; append `f` to get a single precision literal, as in `1.23e4f`.

String literals, enclosed by double quotation marks, as in `"12\"9348"`, are of type `const char *`, i.e., they cannot be overwritten. Note, however, that in the first line of the following code

```
char str[] = "292348"; // a char array with 7 elements (why?); str[0] = 'T' is ok
char *pstr = "2983";   // a string literal; *pstr = '4' gives a runtime error
```

the string is used to initialize the array, and so it is not a string literal.

Data types (part 5, arrays)

It is possible to extend the fundamental data types in two ways: using arrays and creating new data types.

An array is just a contiguous group of variables of the same type. For example, the following code

```
double d[10],D[10][10];
```

declares an (unidimensional) array `d` of 10 double precision floating point numbers and declares a bi-dimensional array (a matrix) `D` of 10 by 10 (i.e., 100) double precision floating point numbers. Accessing the array elements is done using square brackets. Indices start at 0. Usually, **no run-time tests** are performed to verify if the index being used to access an array element has a valid value. Using an out-of-range value does not result in any compiler error but will usually lead to a hard to discover run-time error.

It is important to realize that C does not allow you to manipulate an entire array as a single entity. This is so because an array name is a pointer to its first element. For example, in the code

```
double d[10],*pd = d;
```

This implies that `i[d]` is the same as `d[i]`, but no one sane writes an array access in that twisted way.

it is possible to perform accesses to the array using either `d[i]`, which is the same as `*(d+i)`, or `pd[i]`, which is the same as `*(pd+i)`. Both are equivalent, as long as `pd` is not modified. On the other hand, in the code

```
double d[10],*pd = &d[9];
```

an access to `pd[i]` is equivalent to an access to `d[i+9]`, i.e., `pd[-9]` is the same as `d[0]` (assuming that `pd` has not been modified).

In C a string is an array of characters, terminated by a 0. For example, the code

```
char str[20] = "AB"; // same as char str[20] = { 'A','B',0 }; or char str[20] = { 65,66,0 };
```

defines a string that can hold up to 19 characters (space **must** be reserved for the 0 terminator), initialized with the two character string `AB`.

Data types (part 6a, strings and character sets)

As mentioned in the previous page, a string is an array of characters terminated by a byte with value 0 (`'\0'`). The actual text of the string depends on the character set used. For example, in the ASCII character set a byte with a (unsigned) value smaller than 32 is a control character, and a byte with a value between 32 and 126 represent letters, numbers, and various symbols; values larger than 127 and undefined (127 is the delete symbol, usually not represented graphically).

The following table presents some control characters (under GNU/Linux, use the command

```
man ascii
```

on a terminal to get the complete list).

value	name	meaning	escape sequence
0	NUL	null character (end of string)	`\0`
7	BEL	terminal bell	`\a`
8	BS	backspace	`\b`
9	HT	horizontal tab	`\t`
10	LF	new line	`\n`
12	FF	form feed (new page)	`\f`
13	CR	carriage return	`\r`
27	ESC	escape	`\e`

Data types (part 6b, strings and character sets)

The following table presents the so-called printable ASCII characters (range 32 to 126, i.e., 0x20 to 0x7E). The encoding (the byte values) are presented in hexadecimal (sum of the value of the first row with the value of the first column), because that makes the way the letters and numbers are organized in the ASCII code cristal clear.

	0x00	0x01	0x02	0x03	0x04	0x05	0x06	0x07	0x08	0x09	0x0A	0x0B	0x0C	0x0D	0x0E	0x0F
0x20		!	"	#	$	%	&	’	(	)	*	+	,	-	.	/
0x30	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
0x40	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0x50	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
0x60	‘	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
0x70	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	

For example, to get the character corresponding to a single decimal digit stored in a valiable named `digit` all we need to do is write

```
char c = (char)(0x30 + digit); // '0' + digit
```

To encode letters with accents it is possible to use the (now deprecated) so-called iso-latin character set, which encodes them using byte values in the range 160 to 255. (Under GNU/Linux, use the command

```
man iso_8859-1
```

on a terminal to get the complete list of encodings.)

Data types (part 6c, strings and character sets)

Nowadays the preferred encoding is the ISO 10646 Universal Character Set (UCS), which can encode any character in any written language. In UCS, each character is encoded in a 31-bit integer (the often mentioned unicode is a 20-bit subset of the the UCS). To avoid wasting a lot of memory space, it is common to store the UCS/unicode code using the so-called `utf-8` encoding (use "`man utf8`" to get more details about this). In the `uft-8` encoding of UCS, each ASCII character is encoded in only one byte, and letters with accents are encoded in two bytes. The following small table gives some examples of the iso-latin and utf-8 encodings.

character	iso-latin	utf-8
á	0xE1	0xC3,0xA1
é	0xE9	0xC3,0xA9
í	0xED	0xC3,0xAD
ó	0xF3	0xC3,0xB3
ú	0xFA	0xC3,0xBA
ã	0xE3	0xC3,0xA3
ç	0xE7	0xC3,0xA7

Irrespective of the encoding used, a string always terminates with a single byte with the value 0, and that value cannot occur anywhere inside the string.

Unicode (spoiler)

I'm excited about the proposal to add a "brontosaurus" emoji codepoint because it has the potential to bring together a half-dozen different groups of pedantic people into a single glorious internet argument.

Data types (part 7, pointer arithmetic)

A pointer is the address of a memory location. In C, adding an integer to a pointer **does not**, in general, change the address by that amount: it changes the address by that amount **times** the size in bytes of the type that the pointer points to. Things are done in this way to make working with pointers easier to the programmer, in particular when dealing with arrays. For example, in the code

```
int a[100];
int *pa = &a[30]; // same as int *pa = a + 30;
int *pA = &a[-2]; // same as int *pA = a - 2;
```

the pointer `pa` points to the element with index 30 of the array `a`. The address of this element is the sum of the address of the beginning of the array (`a`, or, what is the same, `&a[0]`) with the number of bytes required to store 30 integers (120 bytes if each integer occupies 4 bytes). In C we only need to add 30 to the pointer to get the correct address; the multiplication by the size of the type pointed to is done automatically by the compiler.

There are two exceptions to the above rule of pointer arithmetic:

- Adding an integer to a pointer to a function is meaningless, because the size of a function (the number of bytes of its code) is not constant and is not known at compile time. So, if one attempts to do this the compiler generates an error.
- Adding a constant to a pointer to `void` (the `void` type will be formally introduced soon), adds that many bytes to the pointer (so `sizeof(void)` is 1 and not 0, as would be more natural). This is done in order to make life easier to the programmer, since manipulating pointers to `void` is sometimes useful.

[**Homework:** study carefully how pointer arithmetic in done in C.]

Data types (part 8, structures and unions)

A structure is a contiguous group of variables of the same or different types, each with its own name. It is declared using the keyword `struct`. In the following code,

```
struct dot
{
  double x;
  double y;
  int color;
  struct dot *next;
};
```

a new data type, named `struct dot`, is declared. It has 4 fields named `x`, `y`, `color`, and `next`, respectively of types `double`, `double`, `int`, and pointer to a dot structure. (One can also put arrays, and even other structures, inside a structure.) The following code fragment gives an example of how structure fields are accessed:

```
struct dot d;   // a dot structure (RESERVES SPACE FOR THE STRUCTURE)
struct dot *pd; // a pointer to a dot structure (DOES NOT RESERVE SPACE FOR THE STRUCTURE)
pd = &d;        // set the pointer to point to the structure (now it can be safely used)
d.x = 3;        // set the x field to 3
pd->color = 5;  // set the color field to 5
d.next = NULL;  // set the next field to NULL (a special address that points to nothing)
```

Unlike arrays, the name of the structure represents the entire structure (it is **not** a pointer to its position in memory; to get that use an `&`). The abbreviation `struct struct_name;` tells the compiler that a structure named `struct_name` will be fully specified later on.

Unions are like structures, except that all its fields are superimposed in memory. (Believe it or not that is sometimes useful.) Only one field can be in use at any given time.

Data types (part 9, enum and void)

It is possible to create a data type which has a discrete set of constant integer values, each one with its own name. Such a type is an enumerated type. For example, the code

```
enum color { black,red,green,blue = 4 };
```

declares an enumerated type called `enum color`, that can have 4 values: `black`, `red`, `green`, and `blue`, respectively with numerical values 0, 1, 2, and 4. (These numerical values are how the names are internally represented in the program; the program code should use the names.) The following code defines a variable of this type and initializes it with the value `red`:

```
enum color dot_color = red;
```

An enumerated data type is an integer data type. The compiler is free to choose the number of bytes needed to internally represent its values.

There is one final data type that can be used in programs: `void`. This data type cannot have a value. It can be used to tell the compiler that a function does not return anything, or that a function does not have argument, as exemplified in the following code:

```
void f(int x); // returns nothing, has an integer argument
int g(void);   // returns an integer, does not have arguments
void h(void);  // returns nothing, does not have arguments
```

A pointer to `void` is a popular way to declare a pointer to something which has a type that is not explicitly given. Of course it is not possible to dereference (access the memory) a pointer to void; one has to first cast it (see next page) to a pointer to a specific type.

Data types (part 10, casts)

It is possible to tell the compiler to explicitly convert one data type to another compatible type. Such conversions, called casts, are implicitly done by the compiler in arithmetic expressions. For example, if `i` if an `int` and `d` is a `double`, the result of the expression `i + d`, which is the sum of the integer variable `i` with the double precision floating point variable `d` is a double precision floating point number. Before doing the addition, `i` is converted (cast) to the double precision floating point number, and only then is the addition performed. To make the conversion explicit, use `(double)i + d`. [My personal opinion is that mixing different data types in an expression without explicit type conversions is a bad programming practice.]

In C, to explicitly convert `x` to the data type `T` one writes `(T)x`. In C it is only possible to convert from one numeric type to another (that the compiler knows about), because the language does not offer any mechanism to tell the compiler how to perform more complex conversions. For example, it is not possible to convert from an integer to a structure. You will need to write your own function, and call it explicitly, in order to do that.

It is possible to cast a pointer to a data type to a pointer to another data type. That is a very dangerous thing to do (you **must** know what you are doing!) unless one of the two is a pointer to `void`. Indeed, a pointer to `void` is the standard way to go when one desires to perform some action on a memory region without needing to known what it contains (for example, reading or writing it to a file):

```
void write_data(void *ptr,int size);     // function to write size bytes starting at address ptr
double array[100];
write_data((void *)array,sizeof(array)); // write the 100 doubles
```

Since a pointer is represented by an integer it is also possible to convert a pointer to a sufficiently wide integer, and vice versa, but that is not recommended (and not needed in any sane program).

Data types (part 11, typedef)

In C it is also possible to give another name to an existing data type using the `typedef` keyword. For example, the following code fragment declares a data type named u64 that is supposed to be a 64-bit unsigned integer:

```
#ifdef IS_A_32_BIT_CPU
typedef unsigned long long u64; // A 64-bit data type on a 32-bit CPU
#endif
#ifdef IS_A_64_BIT_CPU
typedef unsigned long u64;      // A 64-bit data type on a 64-bit CPU
#endif
```

The rest of our code can now use the type u64. Switching from a 32-bit to a 64-bit CPU requires only **two** very small changes in the code (and a recompilation), namely, undefining the symbol `IS_A_32_BIT_CPU` and defining the symbol `IS_A_64_BIT_CPU`. (This can be done without modifying the code by defining the appropriate symbol on the command line that invokes the compiler.)

A `typedef` can also be used in conjunction with the declaration of a `struct`:

```
typedef struct dot // "struct dot" is now known to exist
{
  double x,y; int color;
  struct dot *next; // the type dot is not yet known, so we have to use struct dot, which is known
}
dot; // the name of the new type is dot; it is the same as struct dot
```

Data types (part 12, sizeof)

The number of bytes used by any variable of type `T` is given by `sizeof(T)`. The parenthesis are optional (to avoid confusion due to the precedence of operators, please always use parenthesis). For example, in the code

```
int i = sizeof(dot); // same as i = sizeof dot;
```

the variable `i` will be initialized with the number of bytes required to store a `dot` in memory.

The number of bytes used by a specific variable named, say, `var`, is given by `sizeof(var)`; again, the parenthesis are optional. The `sizeof` operator does the right thing when the variable is an array (it returns the number of bytes needed to store the entire array), despite the fact that in other places the array name is actually a pointer to its first element.

The argument of the `sizeof` operator may also be an expression. For example, `sizeof(1 + 2)` is the number of bytes needed to store the result of the expression @1 + 2@. Here the parenthesis **must** be used; `sizeof(1 + 2)` is not the same as `sizeof 1 + 2` (why?).

The `sizeof` operator returns an integer with type `size_t`, which is usually a 32-bit data type in 32-bit processors and a 64-bit data type in 64-bit processors (it is usually the number of bytes needed to store a pointer variable). Thus, to avoid an implicit cast, the first example above should have been written as follows:

```
int i = (int)sizeof(dot);
```

[**Homework:** since `a[i]` is converted by the compiler into `*(a + i)` before generating code, what should the type of `i` be on i) 32-bit processors, ii) 64-bit processors?]

Declaration, definition, and scope of variables (part 1)

Variables can be defined either outside or inside a function body. Variables defined outside a function body, called global variables, can be used by more than one function. Variables defined inside a function body, called local variables, can only be used directly by that function; they can, however, be made accessible to other functions **called** by that function via pointers (it is a serious mistake to return the address of a local variable).

In the case of variables declared or defined **outside** a function body there are several cases to consider:

- Declaration of a variable that may, or may not, be defined in the same source code file, as in

  ```
  extern int global_var;
  ```

  The `extern` keyword tells the compiler that the variable may be defined elsewhere.
- Declaration of a variable that is defined in the same source code file, as in

  ```
  int global_var;
  ```

  Without initialization, this is actually a declaration (it will be transformed into a definition if a definition is not found elsewhere in the source code; in that case the memory location where the variable resides is initialized with all zeros).
- Definition of a variable that is defined in the same source code file, as in

  ```
  int global_var = 0;
  ```
- Declaration and definition of a global variable that is visible only to the functions of the same source code file, as in

  ```
  static int global_var;
  ```

  The same line of code in a different source code file gives rise to a **different** variable (it is considered very bad practice to use variables with the same name in this way).

Declarations remains in effect until the end of the source code file.

Declaration, definition, and scope of variables (part 2)

A code block (a compound statement) begins with { and ends with }. A function body has one outermost code block. Inside this outermost code block there may exist more, possibly nested, code blocks.

In the case of variables declared or defined **inside** a code block there are also several cases to consider:

- Declaration of a variable that may, or may not, be defined in the same source code file, as in

  ```
  extern int global_var;
  ```

- Definition of a variable that lives only inside the code block, as in

  ```
  int local_var_1;       // uninitialized
  int local_var_2 = 123; // initialized
  ```

  Without initialization, the variable initially has an unspecified value.
  The variable is created whenever the program enters the code block and is destroyed when it leaves the code block.

- Definition of a persistent variable, visible only on the code block, that retains its value even outside the code block, as in

  ```
  static int local_var_1;       // initialized
                                //   with zeros
  static int local_var_2 = 123; // initialized
  ```

  Each invocation of the function uses the **same** variable.

In all cases, the declaration/definition is forgotten at the end of the code block. It is considered bad practice to give the same name to a local and a global variable.

The C99 language standard allows declaration of variables anywhere inside a code block. In previous versions of the C language standard, variables could only be declared at the beginning of a code block.

Declaration, definition, and scope of variables (part 3)

Each data type may be modified by the use of so-called qualifiers. A qualifier may be used by the programmer in a declaration or definition to tell the compiler what operations or optimizations it is allowed to do when copying that variable to or from memory. Of the three qualifiers that the C language currently knows of, `const`, `volatile`, and `restrict`, only the `const` qualifier will be described here. [**Homework:** find you what the other two are for.] An object qualified as `const` is constant; the program cannot modify it. In the following code

```
const double pi = 3.14159265358979323846;
```

the value of `pi` can be used in an expression as if it were a `double` variable, but it cannot be changed.

For a pointer type, a qualifier to the right of the asterisk qualifies the pointer itself; a qualifier to the left of the asterisk qualifies the type of object it points to. For example, in the following code some things are allowed and some are not (read the comments).

```
int i;                    // an integer
const int c = 3;          // an integer constant
const int *p;             // a (variable) pointer to a constant integer
                          // p = &i; is allowed, but i cannot be changed via the pointer
                          // *p = 1; is not allowed
                          // p = &c; is allowed
int * const q = &i;       // constant pointer (it must always point to i)
                          // *q = 1; is allowed
int * const r = &c;       // not allowed
const int * const z = &c; // allowed
```

The `const` qualifier is particularly useful to qualify function arguments that are pointers (many functions receive a pointer to a memory region that will not be modified by the function, in which case a `const` will help the compiler produce better code).

Assignments and expressions (part 1, assignments)

Assignments take the form `LHS = RHS;`, where `LHS` is the so-called lvalue (left value, or, more accurately, location value) and where `RHS` is an expression. The lvalue represents the place where the value of the `RHS` expression will be stored. The following code presents examples of legal and illegal `LHS` lvalues:

```
int a;            a = 3;     // legal
                  a + 1 = 7; // illegal (what is the location of a+1?)
int *pa;          pa = &a;   // legal (the pointer itself has a location)
                  *pa = 7;   // legal
int A[10];        A[3] = a;  // legal
const int c = 3; c = 4;      // illegal (c has a location, but it is not writable)
```

When a pointer is used on the `LHS` to specify a write address, the `LHS` may also modify the pointer if the `++` or `--` operators are used. For example, the following code

```
*++pa = 3; // same as *(++pa) = 3; increment the pointer, and use its new value as the write address
*pa-- = 3; // same as *(pa--) = 3; use the pointer as the write address, and then decrement the pointer
```

is legal, while the code

```
pa++ = &a; // nonsense (are we really trying to store &a in pa and then attempting to increment pa?)
```

is not.

An assignment is in itself an expression, with a value equal to that of the `RHS` (after conversion to the type of the `LHS`). So, it is possible to put several assignments in a chain, as in the following code:

```
int i,j,k;
i = j = k = 3; // same as i = (j = (k = 3));
```

Assignments and expressions (part 2, expressions)

As expression is a sequence of constants, variables, and function calls intertwined with operators that combine them. An expression may perform a mathematical calculation, in which case either we will be interested in recording its value by saving it in a variable (as assignment), or we may be interested to test its value with the purpose of deciding what the program should do next (a conditional jump). An expression may also be useful because of its side effects, as when a function that returns nothing (`void`) is called to do something. The type of an expression is the type of the value that is the result of the expression; it may be `void` if the expression has no value (as in a call to a function that does not return anything). The following are examples of expressions (`i` is an `int`, `d` is a `double` and `exit` is a function than has one `int` argument and that does not return anything):

```
i                   // int
i + '0'             // int
(i << 3) ^ (i & 7)  // int   (the parentheses force i<<3 and i&7 to be evaluated first)
(double)i * d       // double, same as i * d
exit(1)             // void
i && (d == 3.0)     // int
"abc"               // const char *
"abc"[i]            // char
i = 3               // int
d = i = 5           // double
i = d = 2.5         // int
```

The binary arithmetic operators perform an automatic type conversion whenever their two arguments are not of the same type: the argument having the type with smaller range of values is converted to the other. For example, a `char` is converted to an `int` (`char + int` becomes `int + int`), and an `int` is converted to a `double` (`int * double` becomes `double * double`).

Assignments and expressions (part 3a, operators)

C has the following operators, in decreasing order of priority:

1. **postfix operators, left to right associativity**
   - `[]` array access
   - `()` function call
   - `.` structure field
   - `->` structure field, from a pointer
   - `++` post increment; use value, then increment
   - `--` post decrement; use value, then decrement
2. **unary operators, right to left associativity**
   - `++` pre increment; increment, then use value
   - `--` pre decrement; decrement, then use value
   - `!` logic negation (0 gives 1, non-zero gives 0)
   - `~` bitwise negation
   - `+` does nothing (+1 is just 1)
   - `-` arithmetic negation
   - `*` pointer dereference
   - `&` address of
3. **cast operator, right to left associativity**
   - `(T)` type conversion; T is a data type
4. **multiplicative operators, left to right associativity**
   - `*` multiplication
   - `/` division
   - `%` remainder
5. **additive operators, left to right associativity**
   - `+` addition
   - `-` subtraction
6. **shift operators, left to right associativity**
   - `<<` shift left
   - `>>` shift right

   Note: since these are usually used to perform multiplications and divisions by powers of 2 they should have been given a higher priority than addition and subtraction!
7. **relational operators, left to right associativity**
   - `<` less than
   - `<=` less than or equal to
   - `>` larger than
   - `>=` larger than or equal to

   Note: 1 when true, 0 when false
8. **equality operators, left to right associativity**
   - `==` equal to
   - `!=` different from

   Note: 1 when true, 0 when false
9. **bitwise and, left to right associativity**
   - `&` bitwise and

Assignments and expressions (part 3b, operators)

10. **bitwise exclusive or, left to right associativity**
    - `^` bitwise exclusive or
11. **bitwise or, left to right associativity**
    - `|` bitwise or
12. **logical and, left to right associativity**
    - `&&` logical and

    Note: if the argument on the left is zero, the result is 0 and the argument on the right **is not** evaluated; otherwise the result is 0 if the argument on the right is zero and is 1 if not.
13. **logical or, left to right associativity**
    - `||` logical or

    Note: if the argument on the left is nonzero, the result is 1 and the argument on the right **is not** evaluated; otherwise the result is 1 if the argument on the right is nonzero and is 0 if not.
14. **conditional operator, right to left associativity**
    - `?:` `a ? b : c` evaluates to `b` if `a` is nonzero and evaluates to `c` if `a` is zero
15. **assignment operators, right to left associativity**
    - `=` simple assignment
    - `+=` compound assignment (add)
    - `-=` compound assignment (subtract)
    - `*=` compound assignment (multiply)
    - `/=` compound assignment (divide)
    - `%=` compound assignment (remainder)
    - `&=` compound assignment (bitwise and)
    - `^=` compound assignment (bitwise exclusive or)
    - `|=` compound assignment (bitwise or)
    - `<<=` compound assignment (left shift)
    - `>>=` compound assignment (right shift)

    Note: the compound assignment `a op= b` where `op` is one of the operators above is equivalent to `a = a op (b)`.
16. **comma operator, right to left associativity**
    - `,` discard an expression; start a new one

When in doubt about the priority of an operator, use parentheses! Right to left associativity means that things on the right have precedence over things on the left. For example, `a = b = 3;` means `a = (b = 3);`. Left to right associativity is just the opposite. Expressions with side-effects that affect the same variable, like `j = i++ + ++i;`, are ill-defined because different compilers may choose different orders of evaluation.

Statements (part 1, expression, compound, go to, and return statements)

Statements come in many guises:

- **expression statements**

As expression statement is an expression, possibly empty, followed by a semicolon. The following are valid expression statements:

```
;
i = 2;
i = 3, j = 4;
exit(1);
```

- **compound statements (block statements)**

A compound statement (what we called before a code block) starts with a {, is followed by zero of more declarations or definitions of variables and zero of more declarations of functions, is then followed by zero or more statements, and is terminated by a }. The following code is a valid compound statement:

```
{
  int i = 3 + x;        // x declared elsewhere
  {
    int j = i * i + y; // y declared elsewhere
    k += i + j;         // k declared elsewhere
  }
}
```

- **go to statements**

The go to statement causes an unconditional jump to another statement in the same function. It should be used with **care** and **parsimony**, if at all. The destination of the jump is specified by the name of a label, as in

```
goto x_marks_the_spot;
```

The label itself is a name followed by a colon, as in

```
x_marks_the_spot:
```

The break and continue statements, to be presented below, are disciplined (and disguised) go to statements.

- **return statements**

Return statements are used to end the execution of the current function. It has the form

```
return expression;
```

The expression must be missing if the function does not return anything (declared as returning a `void`). The value of the expression is returned to the caller of the function.

Statements (part 2, if statements)

- **if statements**

An if statement has two possible forms: either

```
if(expression)
  statement_t  // to be executed if the
               //   expression is non-zero
```

or

```
if(expression)
  statement_t  // to be executed if the
               //   expression is non-zero
else
  statement_f  // to be executed if the
               //   expression is zero
```

Care must be taken if several if statements are nested and the else part is present in some of them. For example, in the following code

```
if(i >= 0)
  if(i > 0)
    j = 1;
  else
    j = 0;
else
  j = -1;
```

the first else belongs to the second if and the second else belongs to the first if, as suggested by the indentation of the code. (**Advice:** always indent correctly your code.) This is so because the `statement_t` of the first if is `if(i > 0) j = 1; else j = 0;`. If in doubt use curly braces to transform a statement into a compound statement:

```
if(i >= 0)
{
  if(i > 0)
    j = 1;
  else
    j = 0;
}
else
  j = -1;
```

- **loop statements**

There are three kinds of loop statements: for, while, and do-while statements. There is one way to quickly get out of a loop: break statement. There is one way to quickly jump to the next loop iteration: continue statement.

Statements (part 3, for, while, do-while, break, and continue statements)

- **for statements**

A for statement has the following form:

```
for(expression1;expression2;expression3)
  body_statement
```

It is equivalent to

```
  {
      expression1;
loop: if((expression2) == 0) goto end;
      body_statement
next: expression3;
      goto loop;
end:  ;
  }
```

- **while statements**

A while statement has the following form:

```
while(expression)
  body_statement
```

It is equivalent to

```
  {
loop: if((expression) == 0) goto end;
      body_statement
next: goto loop;
end:  ;
  }
```

- **do-while statements**

A do-while statement has the following form:

```
do
  body_statement
while(expression);
```

It is equivalent to

```
  {
loop: body_statement
next: if((expression) != 0) goto loop;
end:  ;
  }
```

Each loop statement will get its own private label names. A break statement inside the body of a loop statement amounts to a `goto end;`, i.e., it forces an exit of the loop statement. A continue statement inside the body of a loop statement amounts to a `goto next;`, i.e., the remaining statements of the body of the loop are skipped.

Statements (part 4, switch statements)

- **labeled statements**

All statements can be labeled, i.e., they can start with a label. There are three forms of a labeled statement:

```
label_name:             statement
case const_expression: statement
default:                statement
```

The first form is used by go to statements. The other two are used by switch statements.

- **switch statements**

A switch statement has the form:

```
switch(int_expression)
  body_statement
```

It is usual, but not necessary, for `body_statement` to be a compound statement. The switch statement works as follows. First, `int_expression` is evaluated. If its value matches the `const_expression` value of one of the case statements, the program jumps to that statement. If none of the cases match, and if there is a default label, the program jumps to the corresponding default statement. Otherwise, the program skips the entire switch statement. A break statement transfers execution to the end of the switch statement. The following code presents an example of a switch statement:

```
switch(c)
{
  case 't':
    k = 1;
    do_t();
    break;  // terminate the switch
  default:  // the default can be anywhere
    k = 2;
    break;  // terminate the switch
  case 'x':
    do_x(); // no break; do next statements
  case 'X':
    k = 3;
    break;  // terminate the switch
}
```

The following code is valid but a bit weird (the switch can be replaced by an if statement):

```
for(i = j = 0;i < 10;i++)
  switch(i % 3)
    case 1: j += i;
```

Functions (part 1, prototypes)

A function definition is composed of two parts, a function header, which specifies the function name, its return type, and its arguments and their types, and a function body, which must take the form of a compound statement. A function declaration (a so-called function prototype) is just the function header, followed by a semicolon, and possibly preceded by the keyword `extern`.

It is not possible to define a function inside another function. Just like variables, it is possible to declare functions at the beginning of a compound statement.

The modern form of the header of a function (there exists an older form, but nowadays no one uses it) has the following form

```
qualifier type function_name( parameter_declarations )
```

The `qualifier` may be absent. The `parameter_declarations` may either be `void`, if the function does not take any arguments, or be composed by one or more individually declared arguments (type and name), separated by comas. The following are examples of function prototypes (headers terminated by a semicolon):

```
int main(void);
int main(int argc,char **argv); // same as int main(int argc,char *argv[]);
extern double sqrt(double x);
static int F(int n);
inline static double sqrt_2(void);
```

(Actually, it is possible to omit the argument names in function prototypes, but we prefer to include them, as their name may shed some light about what they stand for.)

Functions (part 2, parameters)

The parameters of a function behave (in the function body) as if they were ordinary variables, i.e., as if they had been declared and initialized at the very beginning of the compound statement that constitutes the function body. They are passed to the function **by value**, i.e., a copy of each argument is made when the function is called. Thus, changes to the function arguments inside the function, which is allowed, are made on **the copy**. For example, the call `swap(x,y)` to the function

```
void swap(int x,int y)
{
  int tmp;

  tmp = x;
  x = y;
  y = tmp;
}
```

will exchange `x` with `y` only inside the `swap` function. To actually reflect the exchange outside of this function, in C one has to use pointers and pass to the (modified) function the addresses of what we want to exchange. In this case, the function call becomes `swap(&x,&y)` and the function becomes

```
void swap(int *x,int *y)
{
  int tmp;

  tmp = *x;
  *x = *y;
  *y = tmp;
}
```

Note that since the name of an array is a pointer to its first element, in the case of arrays what is passed to the function is a pointer. So, arrays are automatically passed **by reference**. No copy of the entire array is made. On the other hand, structures and unions are passed by value, so if you put an array inside a structure you can actually pass (in disguise) an array by value to a function.

Functions (part 3, qualifiers)

It is possible to declare that a function is `static`. This means that its name is only known to the current file being compiled. A function with the same name can be defined in another source code file. It is very bad practice to have static functions (or one non-static and the others static) with the same name in different source code files. If the source code of a program is distributed among several source code files, using a static function is a great method to hide it from the rest of the source code (for example, to make sure that it is not used inappropriately).

It is also possible to declare that a function is `inline`. This means that the compiler will try to replace all calls to the function by copies of its code. The program will be somewhat larger, as the function code may appears in several places, but it will also be faster, as there will be no function call overhead. If the function body is too large, the compiler may silently refuse to inline a function.

The function called `main` is the entry point of the program. It can be defined to either have no arguments, as in

```
int main(void);
```

or it can be defined to have two arguments, as in

```
int main(int argc,char **argv); // same as int main(int argc,char *argv[]);
```

In the second case, the first argument is the number of command line arguments with which the program was invoked and the second is an array of pointers to strings with the text of the arguments. For example:

```
#include <stdio.h>
int main(int argc,char **argv)
{                                // this program prints its arguments, one per line
  for(int i = 0;i < argc;i++)    // definition of a variable anywhere (a la C++) is a c99 feature
    printf("%s\n",argv[i]);      // compile this using "cc -Wall -O2 -std=c99 main.c"
  return 0;
}
```

Functions (part 4, variable number of arguments)

It is possible for a function to have a variable (i.e., optional) number of arguments. It is the responsibility of the programmer to determine how many arguments were actually provided in every function call. There exists a standard mechanism (c.f. `stdarg.h`) to fetch the value of the next variable argument given its type. The programmer has to know the type of the argument. For example, this can be provided by a previous argument (as done, for example, in the `printf` function; see next slide).

A function has a variable number of arguments if its last argument (the optional part) is specified as `...`. It must have at least one standard argument. The following code illustrates how a variable number of arguments is specified and used (in this example all extra arguments all integers, with a special value to signal the end):

```
#include <stdio.h>
#include <stdarg.h>

int sum(int terminator, ...)
{
  va_list a;                                // standard way to access the extra arguments
  int sum,n;

  va_start(a,terminator);                   // the extra arguments start after the terminator argument
  for(sum = 0;;sum += n)
    if((n = va_arg(a,int)) == terminator)   // get the next int from the argument list
      break;
  va_end(a);
  return sum;
}

int main(void)
{
  printf("%d\n",sum(-1,3,4,7,3,-1));        // should print 17 (3+4+7+3)
  return 0;
}
```

Standard library functions

The C language comes equipped with a relatively large set of predefined functions, declared in so-called header files, and stored in library archives. Among them are functions to read and write data, declared in `stdio.h`, such as

```
int printf(const char *format, ...);                                    // write formated data
int scanf(const char *format, ...);                                     // read formated data
FILE *fopen(const char *path, const char *mode);                        // open a file
int fclose(FILE *fp);                                                   // close a file
size_t fread(void *ptr, size_t size, size_t nmemb, FILE *stream);       // raw read
size_t fwrite(const void *ptr, size_t size, size_t nmemb, FILE *stream); // raw write
int fprintf(FILE *stream, const char *format, ...);                     // formated write
```

functions to allocate and free memory, and to terminate a program, declared in `stdlib.h`, such as

```
void *malloc(size_t size);                                              // allocate memory
void free(void *ptr);                                                   // free memory
void *calloc(size_t nmemb, size_t size);                                // zero-allocate memory
void *realloc(void *ptr, size_t size);                                  // resize an allocation
void exit(int status);                                                  // terminate
```

and functions to compute transcendental mathematical function, declared in `math.h`, such as

```
double sqrt(double x);
double sin(double x);
double cos(double x);
```

Use the help system of your computer (`man` command on GNU/Linux), to get a full description of what a given function does. This web page, which has links to documents describing in full the GNU implementation of the C standard library functions, is also quite useful.

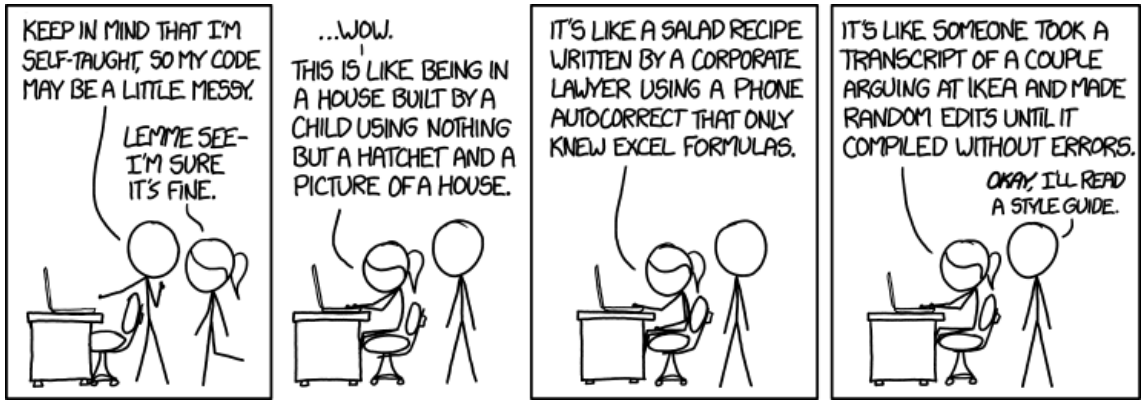
Coding style

Some advice:

- use a consistent coding style; in particular, always indent properly your code,
- do not put too much stuff in a single function,
- use reasonable function and variable names (are you a camelCase fan or a snake_case fan?),
- don't comment obvious code,
- explain, with a comment, each clever trick used in the program (see, for example, the explanation of what the expression `(d+1)|1` does in the `factor.c` program),
- try very hard not to write code what would be admired and envied in the international obfuscated C code contest, and
- try very hard not to be like the guy in the three xkcd cartoons on the next page.

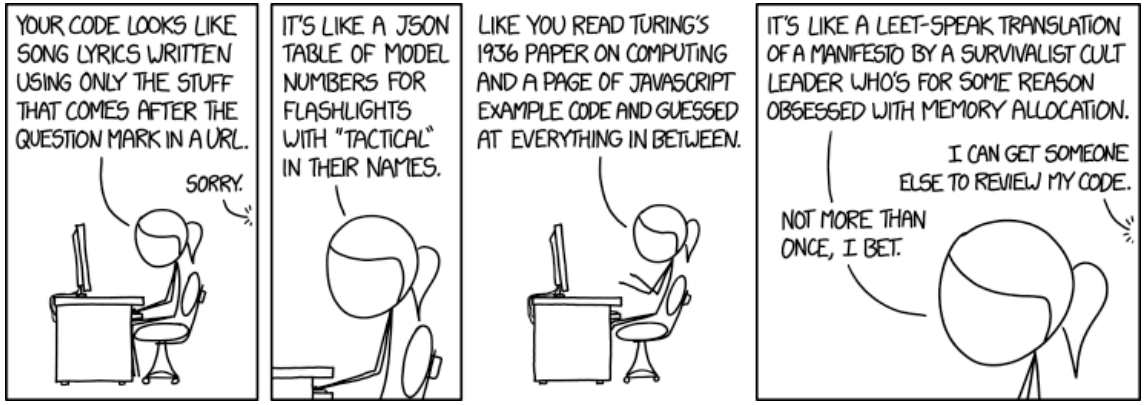
Coding style (xkcd cartoons)

Code Quality (spoiler)



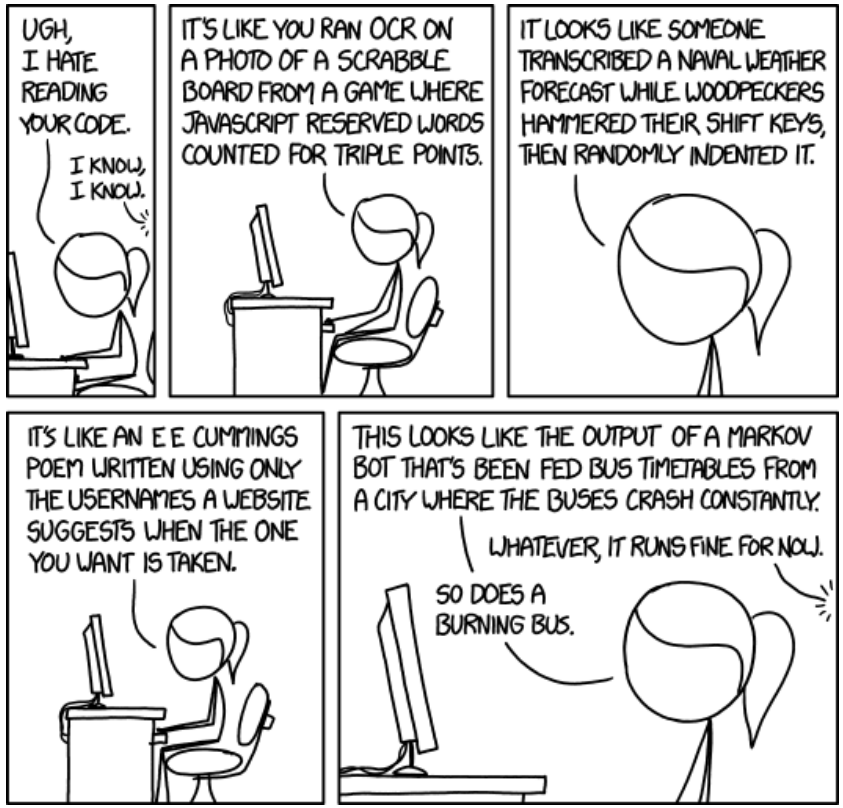
It's like you tried to define a formal grammar based on fragments of a raw database dump from the QuickBooks file of a company that's about to collapse in an accounting scandal.

Code Quality 3 (spoiler)



It's like a half-solved cryptogram where the solution is a piece of FORTH code written by someone who doesn't know FORTH.

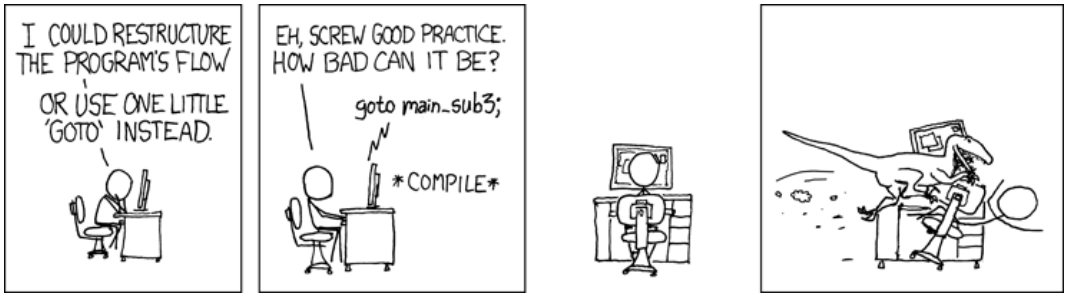
Code Quality 2 (spoiler)



I honestly didn't think you could even USE emoji in variable names. Or that there were so many different crying ones.

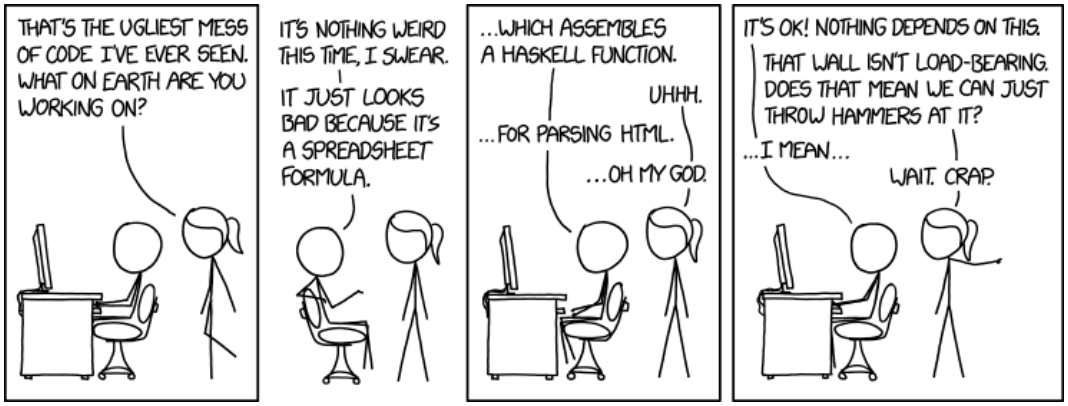
Three more xkcd cartoons

Goto (no spoiler)



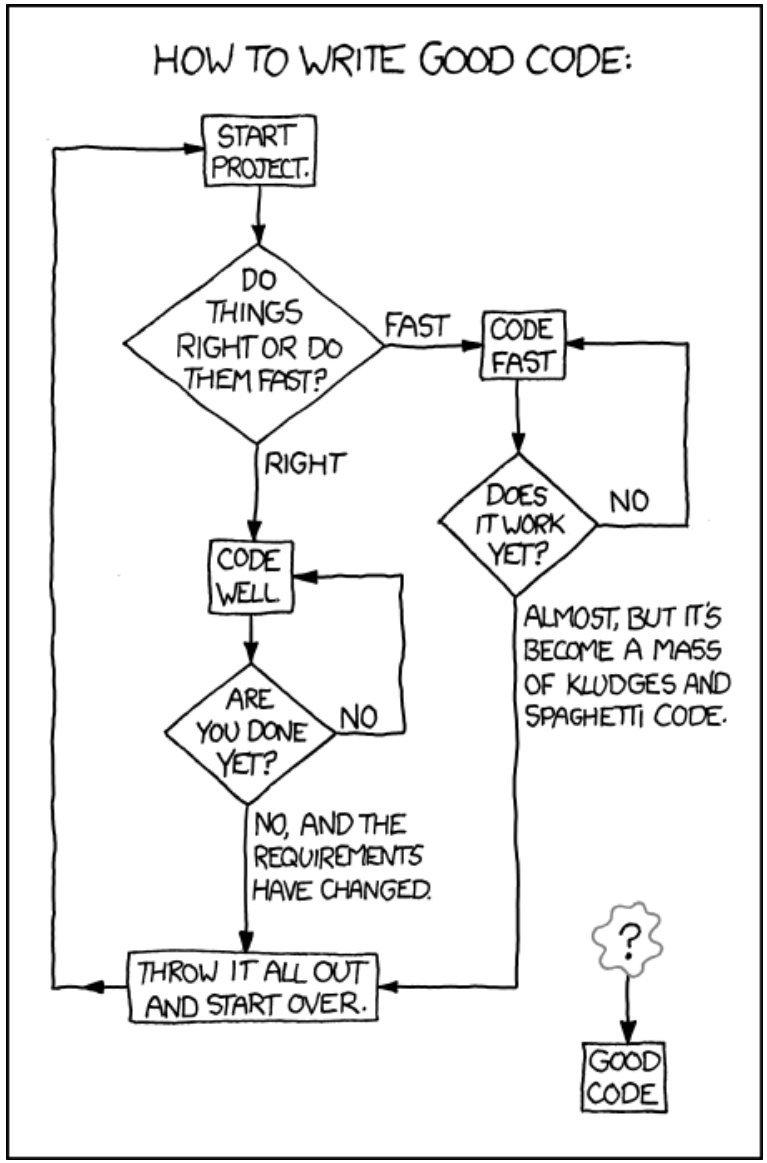
Neal Stephenson thinks it's cute to name his labels 'dengo'.

Bad code (spoiler)



"Oh my God, why did you scotch-tape a bunch of hammers together?"
"It's ok! Nothing depends on this wall being destroyed efficiently."

Good code (spoiler)



You can either hang out in the Android Loop or the HURD loop.

The C programming language (exercises)

— P.01 —

To compile `C` and `C++` programs, Windows 10 users should install the Windows Subsystem for Linux. As an alternative, they can install instead an Integrated Development Environment (IDE), such as Visual Studio or Eclipse. GNU/Linux users need to install the `C++` compiler. (A VirtualBox disk image containg a 18.04 Ubuntu distribution is also available.)

Summary:

- How to compile and run a program (GNU/Linux)
- How to manage archives
- The "Hello World" program
- Program to print some numbers
- Program to print the size in bytes of the fundamental data types
- Computation of Fibonacci numbers
- A more elaborate example (integer factorization)
- Final example (rational approximation)

Study the provided `C` source code carefully. The syntax of the `Java` programming language was inspired by that of the `C` language.

How to compile and run a program (GNU/Linux)

A C program is composed by one or more `.c` source files and by zero or more `.h` files (included by the `.c` source files). To compile the program under GNU/Linux, the following command should be used:

```
cc -Wall -O2 source_files... -o executable_name -lm
```

Replace `source_files...` by the list of the `.c` source files, and replace `executable_name` by the name you desire to give to the executable file. If `-o executable_name` is omitted from the command line, the executable will get by default the name `a.out`. The option `-Wall` instructs the compiler (`cc`) to give you a warning whenever you use dubious C code (such as using an uninitialized variable). The option `-O2` instructs the compiler to optimize the program for speed. The linker option `-lm`, that must be placed at the end, instructs the compiler to link the program with the math library (so that functions like `sin()` and the like are properly taken care of). For example, if your program is composed by the files `source1.c`, `source2.c`, if they include the file `source.h`, and if you desire the executable to be named `prog`, the command should be

```
cc -Wall -O2 source1.c source2.c -o prog -lm
```

and you can run it on a terminal using the command

```
./prog
```

You can automate the process of compiling the program using a `makefile`. Put the text (beware of the tab, denoted below by an arrow)

```
prog:   source1.c source2.c source.h
⟶cc -Wall -O2 source1.c source2.c -o prog -lm
```

in a file named either `Makefile` or `makefile`. Running the command

```
make
```

will recompile your program if at least one of the source files has changed since the last compilation.

How to manage archives

All source code files for this class, together with the `makefile` needed to (re)compile all programs, is distributed in the compressed tar archive `P01.tgz`. On a GNU/Linux system, to extract the files it holds on a terminal, go to the directory (folder) where you want to extract the files (use the `cd` command to do this), and then use the command

```
tar xzvf P01.tgz
```

to extract the files. They should appear in a new directory named `P01`. To create an archive use the command

```
tar czvf name_or_your_archive.tgz list_of_files_and_directories_to_put_in_the_archive
```

The "Hello World" program

The `hello.c` file contains the C code

```c
/*
** Hello world program
*/

#include <stdio.h>

int main(void)
{
  puts("Hello world!");
  return 0;
}
```

Compile it and run it. Modify the code to print "Hello X", where X is your name.

Program to print some numbers

The following program (`table.c`) prints the first ten positive integers, their squares, and their square roots.

```
#include <math.h>
#include <stdio.h>

void do_it(int N)
{
  int i;

  printf(" n n*n      sqrt(n)\n");
  printf("-- --- ----------------\n");
  for(i = 1;i <= N;i++)
    printf("%2d %3d %17.15f\n",i,i * i,sqrt((double)i));
}

int main(void)
{
  do_it(10);
  return 0;
}
```

Compile and run it. Modify the program to print the sine and cosine of the angles $0, 1, 2, \ldots, 90$ (in degrees) and to place the output in a file named `table.txt`.

Hints:

- Use the help system (on GNU/Linux use the command `man`) to study how the formatting string of the `printf` function is specified. Study also the functions `fopen`, `fclose` and `fprintf`.
- The argument of the `sin` and `cos` functions are in radians; the value of π is given by the symbol `M_PI`, which is defined in `math.h`. To convert from degrees to radians multiply the angle by `M_PI/180.0`.

Program to print the size in bytes of the fundamental data types

The file `sizes.h` contains the code

```
#ifndef SIZES_H
#define SIZES_H
void print_sizes(void);
#endif
```

The file `sizes.c` contains the code

```
#include <stdio.h>
#include "sizes.h"
void print_sizes(void)
{
  printf("sizeof(void *) ..... %d\n",(int)sizeof(void *)); // size of any pointer
  printf("sizeof(void) ........ %d\n",(int)sizeof(void));
  printf("sizeof(char) ........ %d\n",(int)sizeof(char));
  printf("sizeof(short) ....... %d\n",(int)sizeof(short));
  printf("sizeof(int) ......... %d\n",(int)sizeof(int));
  printf("sizeof(long) ........ %d\n",(int)sizeof(long));
  printf("sizeof(long long) ... %d\n",(int)sizeof(long long));
  printf("sizeof(float) ....... %d\n",(int)sizeof(float));
  printf("sizeof(double) ...... %d\n",(int)sizeof(double));
}
```

The file `main.c` contains the code

```
#include "sizes.h"
int main(void)
{
  print_sizes();
  return 0;
}
```

Study the way the program is split among the three files. Compile and run the program. On a 64-bit machine, add either -m32 or -m64 to the compilation flags and check if any of the sizes reported by the program changes.

Computation of Fibonacci numbers

The program in the file `fibonacci.c` computes Fibonacci numbers using the four different methods briefly described in this slide. Compare the code of this file with the one presented in the slide.

Compile and run the program. Make graphs of the execution times of each or the four functions reported by the program. Explain why the function `F_v1` is extremelly slow (hint: compare the execution time of that function with the value it returns). Estimate how long your computer will take to compute F_{60} using the `F_v1` function.

The code in the file `fibonacci_with_a_macro.c` is almost identical to the code in the file `fibonacci.c`. In the original code, inside the `for` loop of the `main` function, there are four lines of code, with several statements, that are almost identical. It the modified code they are replaced by a single macro definition and four macro invocations. This guarantees consistency. Use, for example,

```
vim -d fibonacci.c fibonacci_with_a_macro.c
```

or

```
meld fibonacci.c fibonacci_with_a_macro.c
```

to see the differences))between the two files. Note that inside a macro replacement text, a single # stringifies the next token, and ## concatenates the tokens on its left and right hand sides into a single token.

A more elaborate example (integer factorization)

The following program (`factor.c`) computes the factorization of an integer.

```
#include <stdio.h>
#include <stdlib.h>

int factor(int n,int *prime_factors,int *multiplicity)
{
  int d,n_factors;

  n_factors = 0;
  for(d = 2;d * d <= n;d = (d + 1) | 1) // d = 2,3,5,7,9,...
    if(n % d == 0)
    {
      prime_factors[n_factors] = d; // d is a prime factor
      multiplicity[n_factors] = 0;
      do
      {
        n /= d;
        multiplicity[n_factors]++;
      }
      while(n % d == 0);
      n_factors++;
    }
  if(n > 1)
  { // the remaining divisor, if any, must be a prime number
    prime_factors[n_factors] = n;
    multiplicity[n_factors++] = 1;
  }
  return n_factors;
}
```

```
int main(int argc,char **argv)
{
  int i,j,n,nf,f[16],m[16]; // 16 is more than enough...

  for(i = 1;i < argc;i++)
    if((n = atoi(argv[i])) > 1)
    {
      nf = factor(n,f,m);
      printf("%d = ",n);
      for(j = 0;j < nf;j++)
        if(m[j] == 1)
          printf("%s%d",(j == 0) ? "" : "*",f[j]);
        else
          printf("%s%d^%d",(j == 0) ? "" : "*",f[j],m[j]);
      printf("\n");
    }
  return 0;
}
```

Study the program. Compile and run it (for example, `./factor 30`). Why is the program slow when n is a prime number larger than $\mathbf{46339^2}$, such as $\mathbf{2147483647}$? (Hint: arithmetic overflow in the signed integer data type.) Get rid of that programming bug.

Homework challenge: Modify the program so that it outputs all possible divisors of `n`. [Hint: there are `(1+m[0])*(1+m[1])*...*(1+m[nf-1])` divisors.]

Final example (rational approximation)

The program in the file `rational_approximation.c` computes the best rational approximation to a given real number using two different approaches. One is based on a slow but straightforward brute force search for the best solution, and the other (fast, but not the fastest possible!) is based on some interesting mathematical properties of best rational approximations.

Study the program. Pay attention to the data types that are defined and how preprocessor directives can enable or disable (at compile time) parts of the program. Try to understand how the (slow) brute force search works. Attempting to understand the mathematics behind the fast method lies outside the scope of AED (for the curious, it uses the so-called Stern-Brocot tree).

Compile the program and run it. Confirm that the two methods give the same result. Experiment with other real numbers. For example, if the line "`x = M_PI;`" is replaced by the line "`x = exp(1.0);`" or by the line "`x = M_E;`" the program computes best rational approximations to e (base of the natural logarithms).

Modify the program to count and print, if `DEBUG` is negative, the number of tests `e < best_e` that are performed by each of the two functions. Do this for several interesting real numbers, such as π, e, $\sqrt{2}$, and, say, $(1+\sqrt{5})/2$. What can you say about the growth of the number of tests as a function of `max_den` for each of the two functions?

Measure approximately the time it takes your computer to compute

```
best_rational_approximation_slow(M_PI,100000000u)
```

and to compute

```
best_rational_approximation_fast(M_PI,100000000u)
```

Try also other values of the second argument and make graphs of the execution time versus this second argument.

The C++ programming language
— TP.02 —

Summary:

- Overview of the C++ programming language
- Some differences between C and C++
- Classes
- Templates
- Exceptions

[Remark: C++ is a very large and complex programming language (some say* that it is far to much complex). For AED we will only need a relatively small subset of what it has to offer. The rest, although important, will not even be mentioned in these slides.]

* "When its 3 A.M., and you've been debugging for 12 hours, and you encounter a virtual static friend protected volatile templated function pointer, you want to go into hibernation and awake as a werewolf and then find the people who wrote the C++ standard and bring ruin to the things that they love." (Except from The Night Watch, by James Mickens).

Recommended bibliography for this lecture:

- **Thinking in C++. Volume One: Introduction to Standard C++**, Bruce Eckel, second edition, Prentice Hall, 2000.
- **Thinking in C++. Volume Two: Practical Programming**, Bruce Eckel and Chuck Allison, Prentice Hall, 2003.
- **Online reference documentation about C++**
- **C++ Annotations**, Frank B. Brokken, 2015.
- **C++ Primer**, Stanley B. Lippman, Josée Lajoie, and Barbara E. Moo, fifth edition, Addison-Wesley, 2013.

Overview of the C++ programming language

The C++ programming language is a large and complex programming language. It is a superset of the C language (hence the name C++). Most programs written in C are also valid C++ programs. The C++ language has the following features, which are not present in the C language (this list is not exhaustive):

- It is possible to pass values by reference to a function, without making the use of pointers explicit.
- It is possible to define functions with the same name but with different lists of arguments (function overloading).
- The last arguments of a function may have default values (great if one wants to add a new parameter to a function without modifying the code already written that calls the function).
- One can create a data type and the functions that manipulate it (a `class`) in a much more elegant way. It is even possible to make the standard arithmetic operators, such as + and −, work with arguments of the new data type. It is possible to hide the internal details of the data type, so that we can change them without affecting the parts of the program that use the data type.
- One can create so-called name spaces to better control which symbols (such as variables and function names) are visible in each part of our code.
- It is possible to create templates of functions and classes (generic programming).
- It is possible to handle exceptions in a disciplined way (in C one would have to use `goto` statements or, in some cases, the ugly `setjump` and `longjump` functions)

It is customary to use the extension `.cpp` (meaning C-plus-plus) in the names of the source files of a C++ program (`.cc`, `.cxx`, `.c++`, and even `.C`, are also sometimes used).

Some differences between C and C++ (part 1)

C++ allows arguments to be passed by reference. This is accomplished by placing an & before the argument name. In the following code, we show on the left how this has to be done in C and on the right we show how this can be done in C++ (we may also use the left version, but the right one is more elegant, as there are no explicit pointer dereferences):

```
// C; called as follows: swap(&var1,&var2);
void swap(int *x,int *y)
{
  int tmp = *x;
  *x = *y:
  *y = tmp;
}
```

```
// C++; called as follows: swap(var1,var2);
void swap(int &x,int &y)
{
  int tmp = x;
  x = y:
  y = tmp;
}
```

C++ allows functions with the same name but different argument lists to coexist. For example, the following code is valid in C++:

```
int    square(int    x) { return x * x; }
double square(double x) { return x * x; }
```

The code `square(1)` will call the first function, because its argument is an `int`, and the code `square(1.0)` will call the second function, because its argument is a `double`. It is, however, illegal to have two functions with the same name and with the same argument list (same number of arguments and same types), but with different return types. For example, the function

```
double square(int x) { return double(x) * double(x); }  // double(x) does the same as (double)x
```

cannot coexist with the first function defined above.

Some differences between C and C++ (part 2)

C++ allows the specification of default values for the last arguments of a function. This is done by providing initializations (with the default values) in the argument list of the function. It is actually better to put the initializations in the function prototype, as in the following example:

```
int f(int x,int y = 2,int z = 3); // function prototype (usually placed in a header file)

int f(int x,int y,int z) // actual definition of the function
{
  return x + 2 * y + 3 * z;
}
```

In this case we not only can call the `f` function with three arguments as usual, but we can also call it with two (the third, `z`, will get the value $\mathbf{3}$, as specified in the function prototype), and with one (the second and third will get, respectively, the values $\mathbf{2}$ and $\mathbf{3}$). [**Warning:** this feature of the C++ language should be used with extreme care when the function is also overloaded.]

C++ allows the definition of variables in almost any place inside a compound statement. In (old) C, that is only allowed at the beginning of a compound statement. The following code is valid in C++ (it is also valid in modern C dialects):

```
int i;                      // i defined here
for(int j = 0;j < 10;j++)   // j defined here; it will cease to exist at the end of the for cycle
{
  i = 2 * j;
  int k = i + 2 * j;        // k defined here (definition after a statement is allowed)
  cout << k;                // print the value of k
}
```

Some differences between C and C++ (part 3)

In C++ it is possible to control the visibility of symbols by placing them is a name space. In the following code we define two name spaces and put in each of them a global variable and a function (same names but different name spaces; of course this is not recommended but sometimes it cannot be avoided):

```
namespace NEW
{
  static int t_bytes;
  int f(int x) { return 2 * x; }
}
namespace OLD
{
  static int t_bytes;
  int f(int x) { return 3 * x; }
}
```

To get a specific variable or function, place the name of the name space followed by `::` before the variable or function name. For example, `NEW::t_bytes` is the `t_bytes` of the `NEW` name space. It is also possible to say

```
using NEW::t_bytes;
```

and from that point on `t_bytes` will be synonymous with `NEW::t_bytes` (or course, for this to work no symbol with the name `t_bytes` can already exist in the current name space). It is also possible to import all symbols from a name space, thus making all of them available without the `name_space::` prefix. For example,

```
using namespace OLD;
```

will make `t_bytes` a synonym of `OLD::t_bytes` and `f` a synonym of `OLD::f`.

The `std` name space is reserved for standard library functions.

Some differences between C and C++ (part 4)

It is possible to call C functions from a C++ program (the calling conventions are the same, as are the fundamental data types), so that is a question of using the proper function names. This is actually a problem that has to be solved, because the function name that the compiler uses internally is not simply the name of the function: it has also to encode the types of its arguments (this has to be done because a function may be overloaded). The internal names are said to be "mangled." C functions do not have mangled names, so the compiler has to be told to use (or generate) unmangled function names. The following example shows how this is done:

```
extern "C" int f(int x);
extern "C"
{
  int g(int x);
  int h(int x);
}
```

Type casts, in C++, although they can be done just as in C, should be done in the form of a function call, as illustrated in the following code:

```
int i = (int)1.0;  // a C-style cast   (try not to use)
int j = int(1.0);  // a C++-style cast (use)
```

This allows the compiler to better check if the cast makes sense.

In C++, use `nullptr` instead of `NULL`. It serves the same purpose but its use is safer, because in C++ `NULL` is defined to be the constant **0** (and not a pointer to `void` with the value 0 as it is in C).

Some differences between C and C++ (part 5)

Memory can be allocated with the `new` operator and deallocated with the `delete` operator. When used to allocate an instance of a class `new` calls automatically its constructor. Likewise, `delete` calls its destructor. The argument of the `new` operator is a data type. Its return value is a pointer to that type. Note, however, that as in C when one specifies an array what one gets is a pointer to its first element (the constructor of the element type is called for each one of elements of the array). The argument of the `delete` operator should be a pointer received from the `new` operator; if it is not all hell can break loose. `delete` does not have a return value. For array types, the operator `delete[]` calls the destructor for each of the elements of the array (the `delete` operator calls the destructor only for the first element).

The following example shows how `new` and `delete` can be used.

```
int *p_i = new int;              // get memory to an integer
*p_i = 3;                        // give it the value 3
delete p_i;                      // free its memory
p_i = new int(10);               // get memory to another integer at initialize it with the value 10
double *p_d = new double[100];   // get memory for an array of 100 doubles
delete[] p_d;                    // free its memory
class abc;                       // class fully declared elsewhere
abc *pc = new abc;               // pointer to an instance of class abc
```

If there is not enough memory the `new` operator throws a `std::bad_alloc` exception.

Classes (part 1)

Roughly, a C++ class is the combination of a C structure with a set of functions that manipulate the structure. It is a great way to compartmentalize our code, safely hiding the details of how the structure and associated functions are actually implemented. A class is declared just like a structure, but with some extra ingredients:

- while the members of structures have to be data types, members of classes may also be functions.
- when an instance (an object) of a class is created, a constructor member function is called to initialize the data fields of the instance.
- when an instance of the class is destroyed, a destructor member function is called to do any necessary cleanup work.
- some of the members of the class, be they data fields or functions, can be made public, i.e., visible to the entire program, or they can be made private, i.e., visible only by the code that implements the class.
- some data fields may act like global variables, existing only one instance of them irrespective of the number of instances of the class that were created (in C one would have to use a separate global variable to get the same effect; in C++ it is an integral part of the class).

A class with name `CLASS_NAME` is declared as follows:

```
class CLASS_NAME
{
  private:  // private members part
    // put declarations (of functions) and definitions (of functions or data fields) here
  public:   // public members part
    // put declarations (of functions) and definitions (of functions or data fields) here
};
```

We may have several private and public parts. Class member functions may be defined outside the class declaration.

Classes (part 2)

The following example presents the code of a very simple class:

```
class dot
{
  private:
    static int n_dots;  // counts the number of dots created
    double d_x;         // x coordinate
    double d_y;         // y coordinate
  public:
    dot(double x,double y) { n_dots++; d_x = x; d_y = y; } // constructor
    ~dot(void) { n_dots--; }                               // destructor
    double x(void) { return d_x; }                         // definition
    double y(void);                                        // declaration (prototype)
    int number_of_dots(void) { return n_dots; }            // definition
};
double dot::y(void) { return d_y; }                        // definition
int main(void)
{
  dot d(0.0,0.0);    // create a dot; almost the same as "dot d = dot(0.0,0.0);"
  double x = d.x();  // get the x coordinate of d
  // more code
}
```

Note that the name of the constructor function is the name of the class and that the name of the destructor is the name of the class preceded by ~. Note also that inside member functions the names of the data members of the class can be used without reference to the class instance (see discussion of the `this` pointer in the next slide).

Classes (part 3)

When a member function of a class is called it receives an extra hidden argument named `this`. This argument is a pointer to the memory area that holds the data of the class instance that is being used. For example, the member function `dot::y` of the previous slide, shown on the left hand side of the following code, is transformed by the compiler into the code shown on the right hand side:

```
// C++ code
double dot::y(void)
{
  return d_y;
}
```

```
// possible C implementation
double dot_y_implementation(dot * const this)
{
  return this->d_y;
}
```

We can use the `this` pointer in our code (in non static member functions!) without declaring it.

The syntax used to access `struct` data fields is also used to access `class` data members and `class` member functions. For example, if `d_x` had been made `public` in the previous slide, we could have used it as follows:

```
dot d;
d.d_x = 3.0; // set the d_x field of d to 3
```

Since it was made `private` that is not allowed, and we need to provide member functions to set and get its value (if we want to make that data member available to the rest of the program). Calling a member function is done in the same way:

```
double y = d.y();
```

will call the public member function `dot::y` with the `this` pointer set to `&d`.

Templates

Templates are a way to write code in a generic way, without specifying beforehand the data types or other parameters that will be used in a data structure or function. The idea is to write code once, and to use it many times. One writes a template for the code, keeping some data types, and possibly other parameters, unspecified. For a function, this is done as in the following example:

```
template <typename T> T f(T x)
{
  return T(7) * x;  // multiply x by 7; 7 is cast to type T (it must be possible to do that)
}
```

Here, the function template has one generic type named `T` (it can have more), and describes a family of functions, named `f`, whose purpose it to multiply its argument by 7. (Of course this could also be done in a far simpler way, but our purpose here is the describe how a template works.) To use the template to define an actual function, do as follows:

```
int    i = f<int>(3);     // i = 7 * 3
double d = f<double>(5.0) // d = 7.0 * 5.0
```

Class templates are done similarly (here using two generic types):

```
template <typename T1,typename T2> class XYZ
{
  private:
    T1 a_member_variable_of_type_T1;
    T2 a_member_variable_of_type_T2;
    // ...
};
```

and used similarly: `XYZ<int,double> a_variable_of_class_XYZ_int_double;`.

Exceptions

In a program, one possible way to deal with an unexpected case (such as trying to compute the square root of a negative number, when that was thought not to be possible to happen) is just to terminate it. In mission critical applications that is not desirable. What one needs is a way to handle gracefully the unexpected condition (after all, it may have been the result of memory corruption due to a very rare cosmic ray, and not the fault of the program). C++ implements a mechanism (try–catch) that can do that. The idea is to surround the program area we want to protect with a "safety net," that catches these unexpected events. We put our normal code in a `try` block, and we put the recovery code in one or more `catch` blocks. Exceptions (the unexpected events) are signaled by "throwing" an exception, using a `throw` statement. The following example will make things clear:

```
double sqrt(double x)
{
  if(x < 0.0) throw 0;  // throw an integer exception with the value 0
  return sqrt(x);
}
int main(void)
{
  try
  {
    cout << sqrt(-1.0) << endl;
  }
  catch(int i)
  {
    cout << "integer exception number " << i << " caught" << endl;
    exit(1);
  }
}
```

The C++ programming language (exercises)

— P.02 —

Summary:

- How to compile a C++ program (linux)
- The "Hello World" program
- Program to print some numbers
- Program that uses function overloading
- Program that uses a class
- Program that uses a function template
- Program that uses a class template
- Program that uses an exception handler
- Homework

How to compile a C++ program (linux)

A C++ program is composed by one or more `.cpp` source files and by zero or more `.h` files (included by the `.cpp` source files). To compile the program under linux, the following command should be used:

```
c++ -Wall -O2 source_files... -o executable_name -lm
```

Replace `source_files...` by the list of the `.cpp` source files, and replace `executable_name` by the name you desire to give to the executable file. All that was said about compiling C programs also holds for compiling C++ programs (except that now the compiler program is called `c++` and not `cc`).

The "Hello World" program

Extract the file `hello.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Compare it with the `hello.c` given in the P.01 class (you can find it in the `P01.tgz` archive). Modify the program to print the numbers $\mathbf{1, 2, 3, \ldots, 10}$.

Program to print some numbers

Extract the file `table.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Compare it with the C program given in the P.01 practical class. Modify the program to print in another column the cubic roots of the numbers of the second column. (Hint: the function `cbrt` computes a cubic root.)

Program that uses function overloading

Extract the file `overload.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Add two other `show` functions to it, to print i) a char, and ii) an array of **3** integers (fixed array size). For example,

```
show('a');
```

should print

```
char: a
```

and

```
int a[3] = { 2,7,-1 };
show(a);
```

should print

```
array: [2,7,-1]
```

Test your new program.

Program that uses a class

Extract the file `person.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Set `debug` to **1** and recompile and rerun the program. Is the output the same as before? Why? Change the program so that the class `person` also stores the phone number of a person.

Program that uses a function template

Extract the file `f_template.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Add another function to the program than computes the mean of the elements of the array. The return type of that new function should be `double`.

Program that uses a class template

Extract the file `c_template.cpp` from the archive `P02.tgz`. Study, compile, and run the program.

Program that uses an exception handler

Extract the file `exception.cpp` from the archive `P02.tgz`. Study, compile, and run the program. Modify the value of the `special_value` constant so that the exception of type `int` is triggered before the exception of type `double` has a change to be triggered.

Homework

Add exceptions to the `c_template` program. (**Hint:** create an enumerated type with values `stack_full` and `stack_empty` and use them as the exception values.)

Computational complexity
— TP.03 —

Summary:

- Algorithms
- Abstract data types
- Computational complexity
- Algorithm analysis
- Asymptotic notation
- An example
- Useful formulas

Recommended bibliography for this lecture:

- **Analysis of Algorithms**, Jeffrey J. McConnell, second edition, Jones and Bertlett Publishers, 2008.
- **The Algorithm Design Manual**, chapter 2, Steven S. Skiena, second edition, Springer, 2008.
- **Introduction to Algorithms**, chapters 1 and 3, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, third edition, The MIT press, 2009.
- **Análise da Complexidade de Algoritmos**, chapter 1, António Adrego da Rocha, FCA.

Algorithms (part 1, definition)

An algorithm is a detailed description of a method to solve a given problem. To properly specify the problem it is necessary to specify its input, which encodes the information necessary to uniquely identify each instance of the problem, and its output, which encodes the solution to the problem. It is also necessary to specify the properties that the output must have in order for it to be a solution to the problem for the given input (in other words, we must also specify what the problem actually is). An algorithm is a finite sequence of instructions that explain, in painstaking detail, how to produce a valid output (a solution to the problem) given a valid input. These instructions can be given in a natural language, such as English, or in a form closer to how an algorithm may be actually implemented in a given programming language (pseudocode).

An algorithm:

- must be **correct**, i.e., it must produce a valid output for **all** possible valid inputs;
- must be **unambiguous**, i.e., its steps must be precisely defined;
- must be **finite**, i.e., it must produce a valid output in a finite number of steps for **all** possible valid inputs (we are not interested in "algorithms" that may take forever to produce a valid output);
- should be **efficient**, i.e., it should do the job as quickly as possible;
- can be **deterministic**, if the sequence of steps that produces the output depends only on the input, or it
- can be **randomized**, if the sequence of steps that produces the output depends on the input and on random choices.

Algorithms (part 2, example)

The following algorithm is a possible way to sort a sequence of numbers:

Algorithm: Generic exchange sort.

Input: a sequence of n numbers $a_1, a_2, \ldots, a_n$, with $n > 0$.

Output: a permutation (reordering) $b_1, b_2, \ldots, b_n$ of the input sequence such that $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$.

Steps:

1. [Copy input.] For $i = 1, 2, \ldots, n$, set b_i to a_i.
2. [Deal with a special case.] If $n = 1$ terminate the algorithm.
3. [Are we done?] Find a pair of indices (i, j) such that $i < j$ and $b_i > b_j$. Terminate the algorithm if such a pair does not exist.
4. [Exchange.] Exchange b_i with b_j. Return to step 3.

Is the algorithm correct? Yes. For $n > 1$ the algorithm can only terminate when the current b sequence is sorted in non-decreasing order.

Is the algorithm unambiguous? No, because step 3 does not specify how the pair of indices (i, j) is to be found. That is a subproblem that also has to be fully specified. However, no matter how this is done, the algorithm is correct.

Is the algorithm finite? Yes. Since the maximum possible number of exchanges is $n(n-1)/2$ — that is the number of different pairs (i, j) with $i < j$ — sooner or latter the search in step 3 to find an acceptable pair (i, j) will fail.

Is the algorithm efficient? That depends on how the pair of indices is found in step 3. (If that is done from scratch every time, it will be inefficient.)

Abstract data types

Most algorithms need to organize the data they work with in certain ways. So, all modern programming languages allow the programmer to define new data types that are not available natively in the language. More often than not each particular kind of data used by an algorithm can be stored in more than one way but is used in essentially the same way. In those cases the programmer should choose the more efficient storage organization for each kind of data. What constitutes the best storage organization can change during the development of the program. For example, at first it might be thought that the space required to store the data is more important that the time it takes to query or modify it. Later on it may turn out that it is the other way around. So, a good programmer will pay considerable attention to the operations (transformations, queries) that the algorithm needs to perform on its data, and will define data types not by the specific way they store their information but by what operations are allowed to be performed on the information that is supposed to be stored in each one of them. He/she will design abstract data types.

An abstract data type is a data type that exposes to the rest of the program the ways the data it stores can be queried or modified (the interface), without exposing to the rest of the program its internal workings (the implementation), be it how the data is actually stored or how the queries/modifications are actually performed. This will make the program more modular, because as long as the interface of an abstract data type is not changed, changes in its implementation will not affect the rest of the program.

A proper specification of the interface defines not only the names and arguments of the operations that can be performed on instances of the data type (that has to be placed in the source code) but also what their side effects are (that should be placed in the source code in the form of comments). Some programming languages provide facilities (assertions, and pre- and post-conditions) that help ensure that the interface of a data type is used correctly.

Computational complexity (part 1, the RAM model of computation)

To be able to compare the efficiency of different algorithms one needs a model of computation. A model of computation quantifies how much work is needed to perform an elementary task, such as performing an arithmetic operation or a memory access. The RAM model of computation is one of the simplest we can use. It is based on the notion of a Random Access Machine, abbreviated as RAM. Under this model of computation

- an elementary arithmetic operation, such as +, − and the like, a comparison, and an assignment, of numbers with a given fixed number of bits, uses one time step,
- the operation of calling a subroutine (just the call, not the actual work done by the subroutine), of evaluating a numerical transcendental function such as `sin(x)`, and of following a conditional branch, also uses one time step,
- loops, and subroutines, have to be broken down into their individual constituents,
- a memory access, be it a read or a write access, also uses one time step, and
- the memory space used to store a number with a given fixed number of bits uses one unit of space.

This is, of course, a very simplistic model of what happens on a true processor. For example, on a modern processor a division takes much more time than an addition. It is nonetheless a useful model, because in the worst case it deviates (above and below) from what happens on a modern processor by a constant factor.

The computational complexity of an algorithm gives the number of time steps, and the number of units of space, required by the algorithm to solve a given problem. It is a function of the size of the algorithm's input. The size of the input is usually either the number of its data items (say, the number of elements of an array), or one of the inputs itself (say, the exponent in an exponentiation algorithm).

Computational complexity (part 2, a simple example)

Consider the following very simple algorithm:

Algorithm: Mean.
Input: a sequence of n numbers $a_1, a_2, \ldots, a_n$, with $n > 0$.
Output: the arithmetic mean $\mu = \frac{1}{n} \sum_{i=1}^{n} a_i$
Steps:

1. [Initialization.] Set s to a_1.
2. [Sum.] For $i = 2, 3, \ldots, n$, add a_i to s.
3. [Return.] Terminate the algorithm, outputting s/n as the value of μ.

To determine the computational complexity of this algorithm we need to count the number of elementary operations it performs. Step 1 requires one time step to read a_1 (it is read directly into s, so no assignment is needed). For each value of i, step 2 requires five time steps: one to read a_i, another one to add it to s, and another three to increment i, to compare it with n, and to perform a conditional jump according to the result of the comparison. Step 3 requires two time steps, one to perform the division and another to terminate the algorithm (we consider it to be a subroutine return, so we also count it). Since there are $n - 1$ values in step 2, the total number of time steps used by the algorithm is $T(n) = 1 + 5(n - 1) + 2 = 5n - 2$.

Doing such a detailed analysis is usually not necessary. All that one is usually interested in is knowing how fast $T(n)$ grows when n grows. For large n, in this case $T(n)/n$ is almost constant (linear complexity). Knowing that is usually more important than knowing that $T(n) \approx 5n$; knowledge of the growth constant, 5 in this case, although useful, is in most cases an overkill. (After all, the true constant will depend on the processor where the algorithm will be run and on the optimizations of the code made by the compiler.)

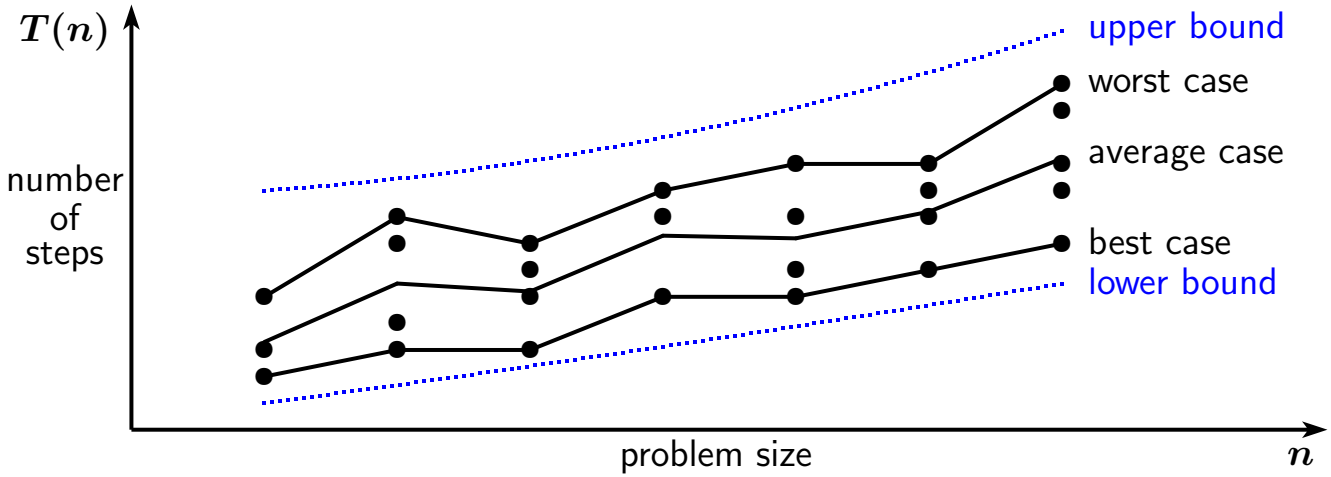
Algorithm analysis (part 1, formal and empirical analysis)

In the last slide we did a formal complexity analysis of a simple algorithm. We have taken the trouble to account for every elementary operation performed by the algorithm. That was easy to do because the algorithm was very easy. For more complex algorithms, specially for those with a flow of execution that depends on the input, that is hard to do, even if one takes a probabilistic approach to the problem. (In a probabilistic approach every conditional branch that depends on the input data has a specific probability of being taken. Assigning these probabilities usually requires a great deal of mathematical sophistication on the part of the person doing the formal complexity analysis.)

When a formal complexity analysis it too difficult to perform, or when we are just too lazy to do it (or when we do not know enough to do it), one can do an empirical complexity analysis. An empirical analysis is based on experimentation. One implements the algorithm using a suitable programing language, and one either adds code to count the number of operations done by the program (this is called instrumenting the program), or one just measures its execution time. In both cases, one has to select a good set of typical inputs of various sizes. The experimental values of $T(n)$ measured in this way can then be plotted as a function of n to see how they grow. With luck, it will be possible to find out a mathematical formula that fits the experimental observations reasonably well.

Algorithm analysis (part 2, worst, best, and average cases)

The worst case time of an algorithm is the function defined by the maximum number of time steps used by the algorithm to deal with any valid input of size n. The best case and average case times are defined in the same way. (Best, worst, and average spaces can also be defined.) As shown in the following figure, these functions may on occasion decrease when the problem size increases.



Best and worst case times are usually easier to determine than the average case time. Furthermore, exact best and worst times are usually more difficult to determine, and to use, than smooth bounds of these functions (in blue in the figure). Obviously, we need a lower bound for the best case and an upper bound for the worst case.

We are usually interested in knowing how fast the lower and upper bounds grow when the size of the problem increases. For example, $T_1(n) = 3n^2$ and $T_2(n) = 10n \log n$ grow at clearly distinct rates, while $T_3(n) = 4n^2$ and $T_4(n) = 3n^2 + 10n$ do not. Mathematicians have a way to express succinctly these differences: asymptotic notation. Using asymptotic notation we talk about the best, average, and worst time **complexity** of an algorithm.

Asymptotic notation (part 1, definitions)

Asymptotic notation allows us to hide irrelevant details about how fast a function grows. For example, when n is a very large number it is usually overkill to know exactly that $T_1(n) = 2n^2 + 3000n + 5$ and that $T_2(n) = 10n^2 + 100n - 23$. For large numbers all that matters is that $T_1(n)$ is approximately $2n^2$ and that $T_2(N)$ is approximately $10n^2$, so that $T_2(n)$ will be about 5 times larger than $T_1(n)$. In asymptotic notation the only detail that is kept about $T_1(n)$ and $T_2(n)$ is that they grow like a square function; even the constant factor that multiplies n^2 is hidden behind the mathematical notation.

Asymptotic notation comes in one of several forms (all functions and constants are assumed to be positive):

- [**Big Oh notation**] The Big Oh notation is useful to deal with **upper bounds**. The notation
  $$f(n) = O(g(n))$$
  means that there exists an n_0 and a constant C such that, for all $n \geqslant n_0$, $f(n) \leqslant Cg(n)$.
- [**Big Omega notation**] The Big Omega notation is useful to deal with **lower bounds**. The notation
  $$f(n) = \Omega(g(n))$$
  means that there exists an n_0 and a constant C such that, for all $n \geqslant n_0$, $f(n) \geqslant Cg(n)$.
- [**Big Theta notation**] The Big Theta notation is useful to deal with **upper and lower bounds** that have the **same form**. The notation
  $$f(n) = \Theta(g(n))$$
  means that there exists an n_0 and two constants C_1 and C_2 such that for all $n \geqslant n_0$ one has $C_1g(n) \leqslant f(n) \leqslant C_2g(n)$.
- [**small oh notation**] The notation
  $$f(n) = o(g(n))$$
  means that $f(n)$ grows slower than $g(n)$, i.e., that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Asymptotic notation (part 2, properties)

The notation introduced in the previous slide can be used in an expression. For example, $f(n) = n^2 + O(n)$ means that $f(n)$ deviates from n^2 by a quantity that is $O(n)$, that is, whose absolute value is bounded by a multiple of n. (The definitions introduced before can be extended to general functions by using absolute values.) We may, for example, say that $O(2n^2 - \log n) = O(3n^2 + 100n - 23)$, because the absolute value of both functions can be bounded by a quadratic, i.e., both are $O(n^2)$.

What can we say about $O\big(f(n) + g(n)\big)$? There are three cases to consider:

- $f(n) = \Theta\big(g(n)\big)$, which implies that $g(n) = \Theta\big(f(n)\big)$. In this case $O\big(f(n) + g(n)\big) = O\big(f(n)\big)$.
- $f(n) = o\big(g(n)\big)$. In this case $O\big(f(n) + g(n)\big) = O\big(g(n)\big)$.
- $g(n) = o\big(f(n)\big)$. In this case $O\big(f(n) + g(n)\big) = O\big(f(n)\big)$.

In all cases the function that grows faster "wins" (remember, upper bounds).

How about $\Omega\big(f(n) + g(n)\big)$? Here the function that grows slower "wins" (remember, lower bounds).

How about $\Theta\big(f(n) + g(n)\big)$? If $f(n) = \Theta\big(g(n)\big)$ then $\Theta\big(f(n) + g(n)\big) = \Theta\big(f(n)\big)$. Otherwise, the lower and upper bounds of $f(n) + g(n)$ do not grow in the same way, and so the Big Theta notation cannot be used.

How about a multiplication by a (positive) constant c? Easy. The constant is discarded, as it is implicit in the notation: $O\big(cf(n)\big) = O\big(f(n)\big)$, $\Omega\big(cf(n)\big) = \Omega\big(f(n)\big)$, and $\Theta\big(cf(n)\big) = \Theta\big(f(n)\big)$.

How about the product of two functions? Easy. Products are retained: $O\big(f(n)g(n)\big) = O\big(f(n)\big)O\big(g(n)\big)$, $\Omega\big(f(n)g(n)\big) = \Omega\big(f(n)\big)\Omega\big(g(n)\big)$, and $\Theta\big(f(n)g(n)\big) = \Theta\big(f(n)\big)\Theta\big(g(n)\big)$.

Asymptotic notation (part 3, useful information)

To determine the truth or falsewood of each of the statements $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$, $f(n) = o(g(n))$, one usually only needs to find out how $\frac{f(n)}{g(n)}$ behaves when n grows to infinity. In pathological cases where $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist, one will need to compute $\liminf_{n\to\infty} \frac{f(n)}{g(n)}$ and $\limsup_{n\to\infty} \frac{f(n)}{g(n)}$. We will not encounter these cases in AED. We have:

$$\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| = \begin{cases} 0, & \Rightarrow\ f(n)=o(g(n)) & f(n)=O(g(n)) \\ >0, & \Rightarrow & f(n)=O(g(n)) \quad f(n)=\Theta(g(n)) \quad f(n)=\Omega(g(n)) \\ \infty, & \Rightarrow & f(n)=\Omega(g(n)) \end{cases}$$

Informally, $f(n) = O(g(n))$ when $f(n)$ does not grow faster than $g(n)$.

While mentally comparing two functions, one can discard all constants and all lower order terms. Note that n^a grows faster than $\log n$ for any $a > 0$, that n^a grows faster that n^b when $a > b \geqslant 0$, that a^n grows faster that b^n when $a > b \geqslant 1$, that a^n grows faster that n^b for any $a > 1$, and that $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + \frac{1}{12n}\right)$ (this is part of Stirling's asymptotic formula for $n!$) grows faster than any n^a or a^n. Note, however, that $n!$ grows slower than n^n. For example, in the comparison of

$$f(n) = 100n^3 + 10^{1000}n^{14/5} \qquad \text{with} \qquad g(n) = 10^{-1000}n! + 2^n$$

it is enough to compare

$$n^3 \qquad \text{with} \qquad n!$$

This is so because n^3 grows faster that $n^{14/5} = n^{2.8}$, and because $n!$ grows faster than any power. In this case $n!$ wins by a huge margin, and so $f(n) = o(g(n))$. Of course, this also means that $f(n) = O(g(n))$. In this particular case, when $n \leqslant 810$, $f(n)$ is actually larger than $g(n)$.

Asymptotic notation (part 4, extra notation)

In the previous slide it is stated that $\log n$ grows slower than any power of n. Mathematically, we say that $\log n = o(n^\epsilon)$ for any $\epsilon > 0$, or that $\log n = n^{o(1)}$. Sometimes, logarithmic factors are a nuisance, as they are minor details in a computational complexity expression. For example, the difference between $O(n^6)$ and $O(n^6 \log n)$ is relatively small. In those cases it is possible to also hide the logarithmic factors behind the asymptotic notation. The $\tilde{O}$ notation was created for that purpose.

- **[Big Soft Oh notation]**

  $$f(n) = \tilde{O}\big(g(n)\big)$$

  means that

  $$f(n) = O\big(g(n) \log^k g(n)\big)$$

  for some $k > 0$.

Asymptotic notation (part 5, examples)

The following examples may help understand better the properties of the Big Oh notation:

- $10n^2 + 30n + 13 = O(n^2)$, because for $n \geqslant 31$ we have $10n^2 + 30n + 13 \leqslant 11n^2$. We have chosen $C = 11$, thus forcing n_0 to be at least 31. Any value of C larger than 10 would also work, but the closer it gets to 10 the larger n_0 has to be.
- $10n^2 + 30n + 13 = O(n^3)$, because for $n \geqslant 13$ we have $10n^2 + 30n + 13 \leqslant n^3$. We have chosen $C = 1$. In this case any positive value of C would also work.
- $10n^2 + 30n + 13 \neq O(n)$, because no matter how C is chosen there exists an n for which $10n^2 + 30n + 13 > Cn$.

We also have [**Homework:** explain why]:

- $10n^2 + 30n + 13 = \Omega(n^2)$.
- $10n^2 + 30n + 13 \neq \Omega(n^3)$.
- $10n^2 + 30n + 13 = \Omega(n)$.

and

- $10n^2 + 30n + 13 = \Theta(n^2)$.
- $10n^2 + 30n + 13 \neq \Theta(n^3)$.
- $10n^2 + 30n + 13 \neq \Theta(n)$.

Asymptotic notation (part 6, dominance relations)

The following functions are ordered according to their growth rate: 1, $\log n$, $\sqrt{n}$, n, $n \log n$, n^2, n^3, 2^n, $n!$.

Consider a processor that can do 10^{10} arithmetic operations per second. (That lies within the capabilities of top end contemporary processors.) The following table presents the time it takes that processor to solve a problem requiring $\log n$, n, $n \log n$, n^2, n^3, 2^n, $n!$, and n^n arithmetic operations (s means seconds, h means hours, d means days, and y means years):

n	$\log n$	n	$n \log n$	n^2	n^3	2^n	$n!$	n^n
10	0.2ns	1.0ns	2.3ns	10ns	100ns	102ns	363μs	1s
20	0.3ns	2.0ns	6.0ns	40ns	800ns	105μs	7.7y	3.3×10^8y
30	0.3ns	3.0ns	10ns	90ns	2.7μs	107ms	8.4×10^{14}y	
40	0.4ns	4.0ns	15ns	160ns	6.4μs	110s		
50	0.4ns	5.0ns	20ns	250ns	12μs	1.3d		
60	0.4ns	6.0ns	25ns	360ns	22μs	3.7y		
10^2	0.5ns	10ns	46ns	1.0μs	100μs	4.0×10^{12}y		
10^3	0.7ns	100ns	691ns	100μs	100ms			
10^4	0.9ns	1.0μs	9.2μs	10ms	100s			
10^5	1.2ns	10μs	115μs	1.0s	1.2d			
10^6	1.4ns	100μs	1.4ms	100s	3.2y			
10^7	1.6ns	1.0ms	16ms	2.8h	3.2×10^3y			
10^8	1.8ns	10ms	184ms	11.6d				
10^9	2.1ns	100ms	2.1s	3.2y				

Asymptotic notation (part 7, wisdom of a master)

> I also must confess a bit of bias against algorithms that are efficient only in an asymptotic sense, algorithms whose superior performance doesn't begin to "kick in" until the size of the problem exceeds the size of the universe. ... I can understand why the contemplation of ultimate limits has intellectual appeal and carries an academic cachet; but in The Art of Computer Programming I tend to give short shrift to any methods that I would never consider using myself in an actual program.
>
> — Donald E. Knuth, The Art of Computer Programming, preface of volume 4A (2011)

An example

We will now determine the computational complexity of the following simple algorithm:

Algorithm: Linear search of unordered data.

Input: a sequence of n distinct numbers $a_1, a_2, \ldots, a_n$, with $n > 0$, and a number b.

Output: the smallest index i such that $a_i = b$, or 0 if no such index can be found.

Steps:

1. [Initialize index.] Set k to 1.
2. [Test.] If $a_k = b$ terminate the algorithm, with k as the output.
3. [Advance.] If $k < n$ then increase k and return to step 2. Otherwise terminate the algorithm with 0 as the output.

Let $f(n)$ denote the number of steps taken by the algorithm, and let $g(n)$ denote the number of average steps. The best case occurs when $b = a_1$. So, $f(n) = \Omega(1)$. The worst case occurs when b is different for all the a_i. In that case steps 2 and 3 will be done n times. Hence, $f(n) = O(n)$.

The average case depends on the probabilities p_i of the events $b = a_i$. Let $p_0 = 1 - \sum_{i=1}^{n} p_i$ be the probability of the remaining event, that b is different for all the a_i. When the algorithm terminates in step 2 for a certain value of i the total number of steps it performs is $1 + 2i + 3(i-1)$. That event has probability p_i. When the algorithm terminates in step 3, with probability p_0, the total number of steps it performs is $1 + 2n + 3(n-1) + 2$. We then have $g(n) = (5n)p_0 + \sum_{i=1}^{n}(5i-2)p_i$. When $p_i = \frac{1}{n}$ we have $p_0 = 0$ and $g(n) = \frac{1}{n}\sum_{i=1}^{n}(5i-2)$. This can be simplified (see next slide) to $g(n) = \frac{5n+1}{2}$, and so in this case $g(n) = O(n)$. In the general case $g(n)$ is as small as possible when $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$.

We could have counted only the number of times the body of the loop (steps 2 and 3) is performed. The result, $np_0 + \sum_{i=1}^{n} ip_i$, is similar to what we get from the more detailed analysis (but without the factor of 5).

Useful formulas

The following formulas are useful to analyze algorithms:

- $\sum_{k=1}^{n} 1 = n$
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\sum_{k=1}^{n} \frac{1}{k} = \log n + \underbrace{\gamma}_{\substack{\text{Euler's constant} \\ \approx 0.577216}} + \frac{1}{2n} + O(n^{-2})$
- $\sum_{k=n}^{m} f(k) = \sum_{k=1}^{m} f(k) - \sum_{k=1}^{n-1} f(k)$

A summation of the form $\sum_{k=a}^{b-1} f(k)$, with a and b integers, can be approximated by an integral of the form $\int_a^b f(x)\,dx$. More precisely, we have (Euler-Maclaurin summation formula):

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m,$$

where B_k are the Bernoulli numbers ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $\ldots$), are where

$$R_m = (-1)^{m+1} \int_a^b \frac{B_m(x - \lfloor x \rfloor)}{m!} f^{(m)}(x)\,dx$$

(here $\lfloor x \rfloor$ is the greatest integer less than or equal to x, so that $x - \lfloor x \rfloor$ is the fractional part of x, and $B_m(x)$ is the m-th order Bernoulli polynomial. [**Exercise**: confirm the first four formulas given above.]

Computational complexity (exercises)

— P.03 —

Summary:

- Paper and pencil exercises (with solutions and computer verification)
- Extra problems (without solutions)
- Empirical study of the computational complexity of three algorithms

Paper and pencil exercises (part 1a, code)

Give a formula for the value returned by each of the following functions, and give their running time in Big Theta notation. Write a program to confirm your results (you can find these functions in the file `functions.c`, stored in the archive `P03.tgz`).

- ```
  int f1(int n)
  {
    int i,r = 0;

    for(i = 1;i <= n;i++)
      r += 1;
    return r;
  }
  ```

- ```
  int f2(int n)
  {
    int i,j,r = 0;

    for(i = 1;i <= n;i++)
      for(j = 1;j <= i;j++)
        r += 1;
    return r;
  }
  ```

- ```
  int f3(int n)
  {
    int i,j,r = 0;

    for(i = 1;i <= n;i++)
      for(j = 1;j <= n;j++)
        r += 1;
    return r;
  }
  ```

- ```
  int f4(int n)
  {
    int i,r = 0;

    for(i = 1;i <= n;i++)
      r += i;
    return r;
  }
  ```

- ```
  int f5(int n)
  {
    int i,j,r = 0;

    for(i = 1;i <= n;i++)
      for(j = i;j <= n;j++)
        r += 1;
    return r;
  }
  ```

- ```
  int f6(int n)
  {
    int i,j,r = 0;

    for(i = 1;i <= n;i++)
      for(j = 1;j <= i;j++)
        r += j;
    return r;
  }
  ```

Paper and pencil exercises (part 1b, solutions)

Let $r(n)$ be the value returned by the function and let $t(n)$ be the corresponding number of iterations of the body of the inner loop. Then,

- for the `f1` function,

$$t_1(n) = r_1(n) = \sum_{i=1}^{n} 1 = n.$$

  In this case we have $t_1(n) = \Theta(n)$.

- for the `f2` function,

$$t_2(n) = r_2(n) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{i} 1\Big) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

  In this case we have $t_2(n) = \Theta(n^2)$.

- for the `f3` function,

$$t_3(n) = r_3(n) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} 1\Big) = \sum_{i=1}^{n} n = n\sum_{i=1}^{n} 1 = n^2.$$

  Since $t_3(n) = r_3(n)$ we have $t_3(n) = \Theta(n^2)$.

- for the `f4` function,

$$r_4(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

  and $t_4(n) = r_1(n)$.

- for the `f5` function,

$$\begin{aligned} t_5(n) = r_5(n) &= \sum_{i=1}^{n}\Big(\sum_{j=i}^{n} 1\Big) = \sum_{i=1}^{n}(n-i+1) \\ &= (n+1)\sum_{i=1}^{n} 1 - \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \end{aligned}$$

  In this case we have $t_5(n) = \Theta(n^2)$.

- for the `f6` function,

$$\begin{aligned} r_6(n) &= \sum_{i=1}^{n}\Big(\sum_{j=1}^{i} j\Big) = \sum_{i=1}^{n} \frac{i(i+1)}{2} \\ &= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{12}(2n+1+3) = \frac{n(n+1)(n+2)}{6} \end{aligned}$$

  and $t_6(n) = r_2(n)$. Note that $r_6(n) = \Theta(n^3)$.

Paper and pencil exercises (part 2a, some problems)

Each one of the statements

- S1: $f(n) = O\big(g(n)\big)$,
- S2: $f(n) = \Omega\big(g(n)\big)$,
- S3: $f(n) = \Theta\big(g(n)\big)$,

can be either true of false for each of the following pairs of functions. Determine which ones are true and explain why.

$f(n)$	$g(n)$
$\log n^2$	$\log n + 2$
$\sqrt{n}$	$\log n^2$
$n\sqrt{n}$	n^2
$n \log n$	n
$n \log n$	$10n \log n + n$
$n \log n$	$n^2 + n \log n$

List the functions given below from lowest to highest order. Mark the functions with the same order.

20	$\log n$	$\log \log n$	1.000001^n	2^n	$0.1n \log n$
n	3^n	$\frac{n}{\log n}$	$n \log n + 100$	2^{n+10}	$n + \frac{100}{n}$
$n!$	$n + 10^9$	$n^2 + n\sqrt{n} \log n$	$\log^2 n$	n^2	$\log n + 10 \log \log n$

Paper and pencil exercises (part 2b, solutions)

$f(n)$	$g(n)$	comparison to perform	true statements
$\log n^2$	$\log n + 2$	$\log n$ compared to $\log n$, tie	S1, S2, and S3
$\sqrt{n}$	$\log n^2$	$n^{1/2}$ compared to $\log n$, $f(n)$ wins	S2
$n\sqrt{n}$	n^2	$n^{3/2}$ compared to n^2, $f(n)$ loses	S1
$n \log n$	n	$n \log n$ compared to n, $f(n)$ wins	S2
$n \log n$	$10n \log n + n$	$n \log n$ compared to $n \log n$, tie	S1, S2, and S3
$n \log n$	$n^2 + n \log n$	$n \log n$ compared to n^2, $f(n)$ loses	S1

rank	function(s)
1	20
2	$\log \log n$
3	$\log n$, $\log n + 10 \log \log n$
4	$\log^2 n$
5	$\frac{n}{\log n}$
6	n, $n + \frac{100}{n}$, $n + 10^9$
7	$0.1n \log n$, $n \log n + 100$
8	n^2, $n^2 + n\sqrt{n} \log n$
9	1.000001^n
10	2^n, 2^{n+10}
11	3^n
12	$n!$

Extra problems

Give a formula for the value returned by each of the following functions (if applicable), and give their running time in Big Theta notation. Write a program to confirm your results (you can find these functions in the file `extra_functions.c`, stored in the archive `P03.tgz`).

- ```
  int g1(int n)
  {
    int i,j,r = 0;

    for(i = 0;i <= n;i++)
      for(j = i;j >= 0;j--)
        r += i - j;
    return r;
  }
  ```

- ```
  int g2(int n)
  {
    int i,j,r = 0;

    for(i = 0;i < 2 * n;i += 2)
      for(j = i;j <= 2 * n;j += 2)
        r += j;
    return r;
  }
  ```

- ```
  void g3(int n,int *a)
  {
    for(int i = 1;i <= n;i++)
      for(int j = i;j <= n;j += i)
        a[j] = i;
  }
  ```
  ```

Computational complexity of three algorithms

Given a sequence of n distinct integers $a_1, a_2, \ldots, a_n$, our task is to determine all pairs (a_i, a_j) such that $a_i + a_j$ is equal to a given v. The program `find_pairs.c` solves this problem in three different ways.
Study, compile, and run the program. What can we say about the time and space complexities of each of the three algorithms coded in the program? (In the space complexity do not take into account the space needed to store the algorithm's input.) Which of the three algorithms has a better computational complexity? Which one uses less extra space?

Auxiliary information:

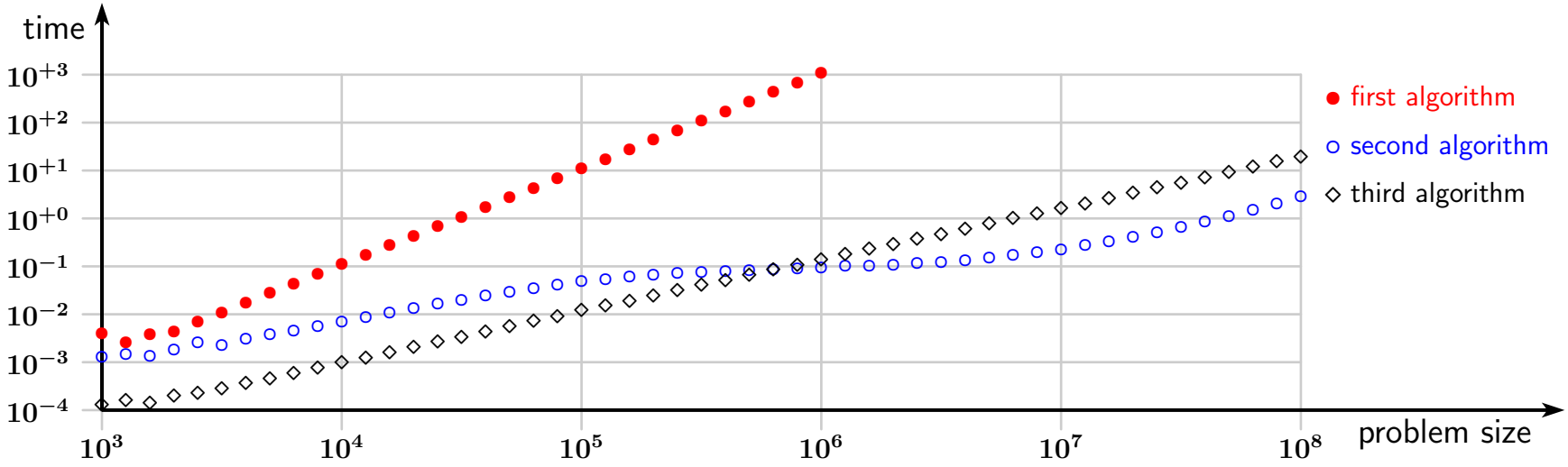
- In the three algorithms (a_i, a_j) is considered to be the same as (a_j, a_i), so only one of the two is printed. It is also assumed that in a pair $i \neq j$.
- The second algorithm requires non-negative integers.
- The `qsort` function sorts an array using a user supplied comparison function. [Study how this is done!] It has an average case complexity of $O(n \log n)$ and a worst case complexity of $O(n^2)$. It can be replaced by another sorting routine that has a worst case complexity of $O(n \log n)$. So, in this problem assume that sorting can be done in $O(n \log n)$.
- The `calloc` function allocates a memory region with a size (number of bytes) that is the product of its two arguments, fills that region with zeros, and returns a pointer to its starting location.
- The `malloc` function allocates a memory region with a size that is given in its only argument and returns a pointer to its starting location. The initial contents of the memory are arbitrary.
- The `free` function deallocates a memory region previously allocated by either the `calloc` or `malloc` functions.

Computational complexity of three algorithms (solution)

The first algorithm uses two nested `for` loops to iterate over all pairs of indices of the input array. Is has a time complexity of $O(n^2)$ and a space complexity of $O(1)$. The algorithm can be modified to work with real numbers.

For a sum value of `v` (an unsigned integer), the second algorithm uses a temporary array with `v+1` elements to record the unsigned integers that appear in the input array. It then computes `v-a[i]` for all elements of the input array and checks if the difference is marked or not in the temporary array. It has a time complexity of $O(n)$ and a space complexity of $O(v)$. The algorithm cannot be modified to work with real numbers.

The third algorithm sorts the input array and then makes a single pass (one index going up and another index going down) through the array. It has a time complexity of $O(n \log n)$, due to the sorting routine, and a space complexity of $O(n)$, to store the sorted array. The algorithm can be modified to work with real numbers.



Computational complexity examples
— TP.04 —

Summary:

- Classes of problems
- Examples

Recommended bibliography for this lecture:

- **Analysis of Algorithms**, Jeffrey J. McConnell, second edition, Jones and Bertlett Publishers, 2008.
- **The Algorithm Design Manual**, chapter 2, Steven S. Skiena, second edition, Springer, 2008.
- **Introduction to Algorithms**, chapters 1 and 3, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, third edition, The MIT press, 2009.
- **Análise da Complexidade de Algoritmos**, chapter 1, António Adrego da Rocha, FCA.

Classes of problems

In computer science problems are subdivided into classes:

- Problems that can be solved in polynomial time, i.e., for which there exits an $O(n^k)$ algorithm, are considered the tractable problems. They are said to belong to the class P (the P stands for **p**olynomial-time).
- There exist also some problems whose solution can be verified in polynomial time. Consider for example the problem of factoring an integer: finding its factors is believed to be a difficult problem, but verifying the factorization can be done using multiplications and primality tests, operations that are known to be in P. (Primality testing was only proved to be in P in 2002!) Problems of this type are said to belong to the class NP (NP stands for **n**ondeterministic **p**olynomial-time). It is not known if P=NP.
- There exists an important subset of NP problems that are equivalent to a certain "prototype" problem (the so-called boolean satisfiability problem). Finding an efficient solution to one such problem automatically provides an efficient solution to the rest of them. These are said to belong to the class NP-complete. Many meaningful problems are known to belong to this class.

Examples (part 1, $O(n)$ algorithms)

Examples of $O(n)$ algorithms:

- sum of the elements of a vector

```
double vector_sum(unsigned int n,double a[n])
{
  unsigned int i;
  double r;

  r = a[0];
  for(i = 1;i < n;i++)
    r += a[i];
  return r;
}
```

- inner product of two vectors

```
double vector_inner_product(unsigned int n,
                            double a[n],
                            double b[n])
{
  unsigned int i;
  double r;

  r = a[0] * b[0];
  for(i = 1;i < n;i++)
    r += a[i] * b[i];
  return r;
}
```

- sum of two vectors:

```
void vector_addition(unsigned int n,
                     double a[n],
                     double b[n],
                     double r[n])
{ // r = a + b
  unsigned int i;

  for(i = 0;i < n;i++)
    r[i] = a[i] + b[i];
}
```

- find the first index `i` for which `a[i]` is equal to `x`

```
unsigned int find_index(unsigned int n,int a[n],int x)
{ // returns n when x is not found
  unsigned int i;

  for(i = 0;i < n && a[i] != x;i++)
    ;
  return i;
}
```

- recursive computation of $n!$

```
double factorial(unsigned int n)
{
  return (n < 2) ? 1.0 : n * factorial(n - 1);
}
```

Examples (part 2, $O(n^2)$ algorithms)

Examples of $O(n^2)$ algorithms:

- multiplication of multi-precision integers (one base 10 digit per array entry)

```
void multiplication(unsigned int n,
                    unsigned int a[n],
                    unsigned int b[n],
                    unsigned int r[2 * n])
{ // r = a * b
  unsigned int i,j;

  assert(sizeof(int) >= 4 && n <= 47721858);
  for(i = 0;i < 2 * n;i++)
    r[i] = 0;
  for(i = 0;i < n:i++)
    if(a[i] != 0)
      for(j = 0;j < n;j++)
        r[i + j] += a[i] * b[j];
  for(i = j = 0;i < 2 * n;i++)
  {
    j += r[i];
    r[i] = j % 10;
    j /= 10;
  }
  assert(j == 0);
}
```

[**Homework:** Why 47721858?]

- sum of two matrices

```
void matrix_addition(unsigned int n,
                     double A[n][n],
                     double B[n][n],
                     double R[n][n])
{ // R = A + B
  unsigned int i,j;

  for(i = 0;i < n;i++)
    for(j = 0;j < n;j++)
      R[i][j] = A[i][j] + B[i][j];
}
```

- insertion sort

```
void insertion_sort(unsigned int n,double a[n])
{
  unsigned int i,j;
  double d;

  for(i = 1;i < n;i++)
  {
    d = a[i];
    for(j = i;j > 0 && d < a[j - 1];j--)
      a[j] = a[j - 1];
    a[j] = d;
  }
}
```

[**Homework:** What are the best and worst cases?]

Examples (part 3, improved multiplication)

The multiplication of two integers, A and B, each with with $2n$ bits, can be done as follows.

- Split A into two halves, A_1 and A_0, each with n bits, so that $A = A_1 2^n + A_0$.
- Likewise, split B into two halves, B_1 and B_0, again each with n bits, so that $B = B_1 2^n + B_0$.
- In the standard multiplication method the product of A and B is computed with the formula $(A_1 B_1)2^{2n} + (A_1 B_0 + A_0 B_1)2^n + (A_0 B_0)$. This requires 4 multiplications of numbers with half the number of bits, so using this formula recursively gives rise to an $O(n^2)$ algorithm to compute the product.
- It is possible to eliminate one multiplication by doing more additions and subtractions, by taking advantage of the identity $A_1 B_0 + A_0 B_1 = (A_1 + A_0)(B_1 + B_0) - A_1 B_1 - A_0 B_0$. Its right hand side requires only one extra multiplication (the products $A_1 B_1$ and $A_0 B_0$ can be reused).
- Thus, if $T(n)$ is the time required to multiply two n-bits integers, we have $T(2n) \leqslant 3T(n) + \alpha n$, where the αn term captures the time required to do additions, subtractions, and house-keeping tasks. It follows (we omit a few details here), that $T(n) = O(n^{\log_2 3})$. Note that $\log_2 3 \approx 1.58496$ is considerably smaller than 2, so this simple method is a substantial improvement over the original method. [**Homework:** Compare n^2 with $2n^{\log_2 3}$ for $n = 10^k$, $k = 1, 2, 3, 4, 5, 6$.]
- Using more advanced methods it is possible to multiply two integers much faster than this. The best known methods do the job in $O(n \log n \log\log n)$ steps and use FFTs (Fast Fourier Transforms), or, what is similar but uses only integers, NTTs (Number Theoretical Transforms).

Examples (part 4, $O(n^3)$ algorithms)

Examples of $O(n^3)$ algorithms:

- multiplication of two matrices

```
void matrix_matrix_product(unsigned int n,
                           double A[n][n],
                           double B[n][n],
                           double R[n][n])
{ // R = A * B
  unsigned int i,j,k;

  for(i = 0;i < n;i++)
    for(j = 0;j < n;j++)
    {
      R[i][j] = 0.0;
      for(k = 0;k < n;k++)
        R[i][j] += A[i][k] * B[k][j];
    }
}
```

- matrix inversion (not much different from the next algorithm).

- determinant of a matrix

```
double matrix_determinant(unsigned int n,
                          double A[n][n])
{ // A is modified
  unsigned int i,j,k;
  double r,t;

  r = 1.0;
  for(i = 0;i < n;i++)
  {
    // find the biggest element (the pivot)
    j = i;
    for(k = i + 1;k < n;k++)
      if(fabs(A[k][i]) > fabs(A[j][i]))
        j = k;
    // exchange lines (if necessary)
    if(j != i)
      for(r = -r,k = i;k < n;k++)
      {
        t = A[i][k];
        A[i][k] = A[j][k];
        A[j][k] = t;
      }
    // Gauss-Jordan elimination
    for(r *= A[i][i],j = i + 1;j < n;j++)
      for(k = i + 1;k < n;k++)
        A[j][k] -= (A[j][i] / A[i][i]) * A[i][k];
  }
  return r;
}
```

Examples (part 5, improved matrix multiplication)

The matrix multiplication of a $(2n) \times (2n)$ matrix can be split as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

This yields 8 multiplications and 4 additions of $n \times n$ matrices. Doing this recursively gives rise to a $O(n^3)$ algorithm. Since the computational complexity of a matrix addition is smaller than that of a matrix multiplication, the number of additions is irrelevant in theory (but not in practice).

Strassen found a way to compute

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

that only requires 7 multiplications (and 18 additions). Winograd later reduced the number of additions to 15. The product is given by

$$\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & W + V + (A_{11} + A_{12} - A_{21} - A_{22})B_{22} \\ W + U + A_{22}(B_{21} + B_{12} - B_{11} - B_{22}) & W + U + V \end{bmatrix}$$

where $U = (A_{21} - A_{11})(B_{12} - B_{22})$, $V = (A_{21} + A_{22})(B_{12} - B_{11})$, and $W = A_{11}B_{11} + (A_{21} + A_{22} - A_{11})(B_{11} + B_{22} - B_{12})$. This gives rise to a $O(n^{\log_2 7}) \approx O(n^{2.807})$ algorithm to multiply matrices. With considerable effort, it is possible to reduce the exponent to a number closer to 2. The current record is an exponent of only 2.376. Note, however, that all known methods with an exponent smaller that 2.7 are so complex, and with horrendous proportionality constants, that they are **useless** in practice!

Examples (part 6, exponential complexity)

The following algorithms have exponential complexity:

- computing Fibonacci numbers (in a dumb way)

```
double F(unsigned int n)
{
  return (n < 2) ? (double)n : F(n - 1) + F(n - 2);
}
```

Work for the practice class: what is the computational complexity of this function? How about the computational complexity of this "improved" way of computing Fibonacci numbers?

```
double Fi(unsigned int n)
{
  static double Ft[10] = { 0,1,1,2,3,5,8,13,21,34 };

  return (n < 10) ? Ft[n] : Fi(n - 1) + Fi(n - 2);
}
```

- print all possible sums of the elements of an array (generate all subsets)

```
void print_all_sums(unsigned int n,double a[n])
{ // 1 <= n <= 60
  unsigned long mask;
  unsigned int i,j;
  double s;

  assert(n > 0 && n <= 8 * (int)sizeof(long) - 1);
  mask = 0; // a bit set to 0 means that the
            // corresponding a[i] will not
            // contribute to the sum
  do
  {
    // do sum
    s = 0.0;
    for(i = j = 0;i < n;i++)
      if(((mask >> i) & 1ul) != 0ul)
      {
        s += a[i];
        printf("%sa[%u]",(j == 0) ? "" : "+",i);
        j = 1; // next time print a + sign
      }
    printf("%s = %.3f\n",(j == 0) ? "empty" : "",s);
    // next subset (discrete binary counter)
    mask++;
  }
  while(mask < (1ul << n));
}
```

Examples (part 7, worst than exponential complexity)

When called with m equal to 0, the following recursive function prints all permutations of the integers a_0, a_1, ..., a_{n-1}, assumed to be distinct and to be stored in the array a[]:

```
void print_all_permutations(unsigned int n,unsigned int m,int a[])  // int a[] is a synonym of int *a
{
  unsigned int i;

  if(m < n - 1)
  { // not yet at the end
    for(i = m;i < n;i++)
    {
#define swap(i,j)  do { int t = a[i]; a[i] = a[j]; a[j] = t; } while(0)
      swap(i,m);                          // exchange a[i] with a[m]
      print_all_permutations(n,m + 1,a); // recurse
      swap(i,m);                          // undo the exchange of a[i] with a[m]
#undef swap
    }
  }
  else
  { // visit permutation
    for(i = 0;i < n;i++)
      printf("%s%d",(i == 0) ? "" : " ",a[i]);
    printf("\n");
  }
}
```

(When called with $m > 0$ this function does not change the values of $a_0, \ldots, a_{m-1}$.) Note that although inside the function the contents of the a[] array are reordered, when it returns they fall back to their original values. The computational complexity of this function is $O(n \times n!)$, because the block that prints each permutation is visited $n!$ times.

Examples (part 8, algorithms with "small" computational complexity)

We conclude this tour of examples with two algorithms that have small computational complexity:

- computation of x^y using a mathematical formula (y is a real number)

```
double power_dd(double x,double y)
{
  return exp(y * log(x));
}
```

  This is an $O(1)$ algorithm. It works only when $x > 0$.

- computation of x^n using recursion (n is an integer)

```
double power_di(double x,int n)
{
  if(n < 0)  return power_di(1.0 / x,-n);
  if(n == 0) return 1.0;
  double t = power_di(x * x,n / 2); // the integer division discards a fractional part
  return (n % 2 == 0) ? t : x * t;  // take care of the discarded fractional part
}
```

  This is an $O(\log n)$ algorithm. With the exception of $x = 0$ and $n < 0$, it works for any x. (By convention $0^0 = 1$; 0 raised to a negative power is illegal.)

- computation of $a^b \bmod c$ (u32 is `unsigned int` and u64 is `unsigned long long`)

```
u32 power_mod(u32 a,u32 b,u32 c)
{
# define mult_mod(x,y,m)  (u32)(((u64)(x) * (u64)(y)) % (u64)(m))  // (x*y) mod m
  if(b == 0) return 1u;
  u32 t = power_mod(mult_mod(a,a,c),b / 2,c);
  return (b % 2 == 0) ? t : mult_mod(a,t,c);
# undef mult_mod
}
```

Computational complexity examples (exercises)

— P.04 —

Summary:

- Formal and empirical computational complexity of several algorithms
- The traveling salesman problem (**theme of the first written report**)

Formal and empirical computational complexity of several algorithms

Extract the file `examples.c` from the archive `P04.tgz`. This file contains the code of all functions described in the TP.04 lecture. Redoing the formal analysis done in the TP lecture, determine the computational complexity of each one of the functions. Confirm your results by adding code to each of the functions to count (and print at the end) the number of times that the body of the innermost loop is executed.

Note that in C it is possible to write code like this (in C++ that is not possible):

```
for(n = 1;n <= 10;n++)
{
  double A[n][n]; // inside a function, non static arrays do NOT need to have a size defined at compile time!
  int i,j;

  for(i = 0;i < n;i++)
    for(j = 0;j < n;j++)
      A[i][j] = (double)rand() / (double)RAND_MAX; // one way to get uniformly distributed pseudo-random numbers
  //
  // put more stuff here, such as a call to an O(n^2) or an O(n^3) function
  //
}
```

For each interesting case (say, one for each computational complexity), make a log-log graph of the number of times the inner loop was executed versus the problem size.

The traveling salesman problem (part 1)

Modify the code of the `tsp_v1()` function, located in the `tsp.c` file (get it from the `P04.tgz` archive), so that instead of printing each permutation, the modified function solves the traveling salesman problem. Do some research to see what this problem is all about.

Things to do (the more you do correctly, the better your grade will be):

- [Mandatory] At the start of the `main()` function in the file `tsp.c` put your student number in the line `n_mec = 0;` (for groups with two or more students, present results for each student number).
- [Mandatory] Compute the length of the best (shortest) tour for $3, 4, \ldots, 15$ cities.
- Compute the length of the worst (longest) tour for $3, 4, \ldots, 15$ cities.
- Measure the time it takes to solve the problem for the first k cities, for $2 \leqslant k \leqslant 15$.
- Make an histogram of the length of all tours for 12 and 15 cities.
- Compute of the shortest and longest tour found using say, one million (the more the better) random permutations of the order the cities are visited; in the report, try to compare the results of this probabilistic algorithm with what the histogram of the lengths or the tours suggests should happen.
- For the final report, make separate plots of the best and worst tours, just to see what they look like. (There is code to produce SVG files; use it!)
- At the start of the `main()` function in the file `tsp.c` change the line `special = 0;` to `special = 1;` and redo all computations. For this case, the inter-city distances are asymmetric.

The traveling salesman problem (part 2)

The first written report (about the traveling salesman problem) must have:

- A title page (front page) with the name of the course, the name of the report, date, the names of the students, and an estimate of the percentage each student contributed to the project (the sum of percentages should be 100%).
- A short introduction describing the problem; the source of material adapted from the internet must be properly cited.
- A small description of the method used to find the solutions.
- A description of the solutions found; this should include a graph of the execution time of the program as a function of the number of cities, likewise for the lengths of the shortest and longest tours, and figures of the solutions.
- Comments or attempts at explanations of the results found (this can be placed near where the results are presented in the report).
- An appendix with all the code (use a small font).

The traveling salesman problem (part 3)

Test data for `n_mec = 0;` and `special = 0;`

n	minLength	minPath	maxLength	maxPath
3	1160	0, 1, 2	1160	0, 1, 2
4	1179	0, 2, 1, 3	1417	0, 1, 2, 3
5	1400	0, 2, 1, 4, 3	2232	0, 1, 3, 2, 4
6	1409	0, 2, 1, 4, 3, 5	2385	0, 1, 5, 4, 2, 3
7	1437	0, 2, 5, 3, 6, 1, 4	2664	0, 3, 1, 5, 4, 2, 6
8	1440	0, 2, 7, 5, 3, 6, 1, 4	3022	0, 3, 2, 4, 5, 1, 7, 6

Test data for `n_mec = 0;` and `special = 1;`

n	minLength	minPath	maxLength	maxPath
3	996	0, 1, 2	1324	0, 2, 1
4	1036	0, 1, 3, 2	1529	0, 3, 2, 1
5	1213	0, 1, 4, 3, 2	2364	0, 3, 1, 2, 4
6	1250	0, 1, 4, 3, 5, 2	2691	0, 3, 2, 4, 5, 1
7	1305	0, 5, 6, 1, 4, 3, 2	2970	0, 6, 3, 2, 4, 5, 1
8	1307	0, 5, 3, 6, 1, 4, 7, 2	3325	0, 3, 7, 6, 2, 4, 5, 1

The traveling salesman problem (part 4)

The following code can be used to generate a random permutation of the integers $0, 1, \ldots, n-1$:

```
#include <math.h>
#include <assert.h>
#include <stdlib.h>

void rand_perm(int n,int a[])
{
  int i,j,k;

  for(i = 0;i < n;i++)
    a[i] = i;
  for(i = n - 1;i > 0;i--)
  {
    j = (int)floor((double)(i + 1) * (double)rand() / (1.0 + (double)RAND_MAX)); // range 0..i
    assert(j >= 0 && j <= i);
    k = a[i];
    a[i] = a[j];
    a[j] = k;
  }
}
```

Elementary data structures (part 1)
— TP.05 —

Summary:

- Data containers
- Arrays (and circular buffers)
- Linked lists (singly- and doubly-linked)
- Stacks
- Queues
- Deques

Recommended bibliography for this lecture:

- **Analysis of Algorithms**, Jeffrey J. McConnell, second edition, Jones and Bertlett Publishers, 2008.
- **Introduction to Algorithms**, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, third edition, The MIT press, 2009.
- **Algorithms**, Robert Sedgewick and Kevin Wayne, fourth edition, Addison Wesley, 2011
- **Estruturas de Dados e Algoritmos em C**, António Adrego da Rocha, terceira edição, FCA.

Data containers

In most programs it is necessary to store and manipulate information. This information may be stored and manipulated directly by the programmer, or it may be encapsulated in a so-called data container. A data container is a data structure that specifies how the information is organized and stored, together with a standardized interface to access and modify the information (thus, a class).

There are many types of data containers, differing on the services they provide and on the computational complexity of these services (the computational complexity depends on how the information is internally organized in the data container). The choice of data container should be done according to the answers to the following questions:

- Does the insertion and retrieval of information obey some rules? (Say, do we always remove the newest, oldest, largest, or smallest item of information?)
- Do we need efficient random access to the information?
- Do we need efficient sequential access to the information?
- Do we need to efficiently insert information in a random location?
- Do we need to efficiently delete information from a random location?
- Do we need to efficiently search for information?
- Is the information single valued, say, an integer, or is it multi-valued, say, a (key,value) pair?

Arrays (part 1, computational complexity)

An array is one data structure that can be used to implement a data container. Assuming that the items of information are to be stored consecutively in memory (in array elements) it follows that

- it is necessary to specify the size of the array before it is used; if later on it turns out that that size is too small, it will be necessary to resize the array (that is an $O(n)$ operation, but, if done rarely, its amortized cost is low)
- given a position (an index) random access to information is fast: $O(1)$
- sequential access is also fast
- inserting information at one end of the used part of the array (assuming it is not full) is fast: $O(1)$
- inserting information at an arbitrary location, opening space for it, is slow: $O(n)$
- replacing information at an arbitrary location is fast: $O(1)$
- deleting information at one end is fast: $O(1)$
- deleting information at an arbitrary location, closing the space it would otherwise leave behind, is slow: $O(n)$; without closing the space, it is fast: $O(1)$
- if the information is stored in the array in random order, then searching for information is slow: $O(n)$
- if the information is stored in the array in sorted order, then searching for information is fast: $O(\log n)$

Arrays (part 2, implementation of a circular buffer)

A circular buffer is an array in which index arithmetic is done modulo the size of the array (for an array with n elements, index n is the same as index 0). Besides supporting the usual array operations, by keeping track of the indices of the first and last data items stored in it, a circular buffer also supports $O(1)$ insertion and deletion of data at either end. Thus a circular buffer is a very efficient way of implementing a queue and a deque (see next slides).

Circular buffers are usually used in device drivers to implement efficiently a queue with a given maximum size, because it does not require any dynamic memory operations (allocation and deallocation of memory).

The following code illustrates one possible way to increment and decrement an index in a circular buffer (of floating point numbers):

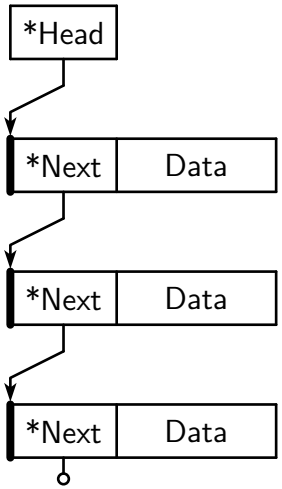
```
class circular_buffer
{
  private:
    int max_size; // the maximum size of the circular buffer
    double *data; // array allocated in the constructor
  public:
    circular_buffer(int max_size = 100)  { this->max_size = max_size; data = new double[max_size]; }
    ~circular_buffer(void)               { delete[] data;                                           }
  private:
    int increment_index(int i)  { return (i + 1 <  max_size) ? i + 1 : 0;            }
    int decrement_index(int i)  { return (i - 1 >= 0       ) ? i - 1 : max_size - 1; }
    //
    // data[i] accesses the number stored in position i
    // data[increment_index(i)], accesses the number stored in the position after position i
    // data[decrement_index(i)], accesses the number stored in the position before position i
    //
    // ... (rest of the code for the circular_buffer class)
    //
}
```

Linked lists (part 1, overview)

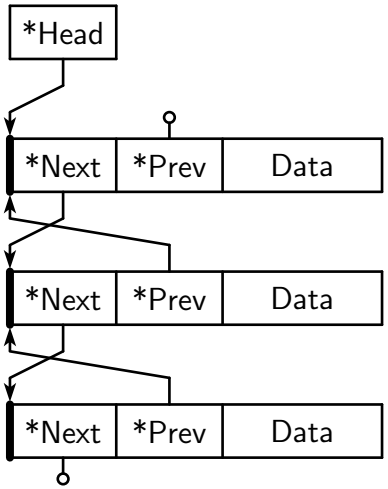
A linked list is a dynamic data structure, because it can grow as much as needed. In order to be able to do this, each node of information contains

- the information itself
- in a singly-linked list, a pointer to the next node of information
- in a doubly-linked list, a pointer to the next node and another to the previous node of information

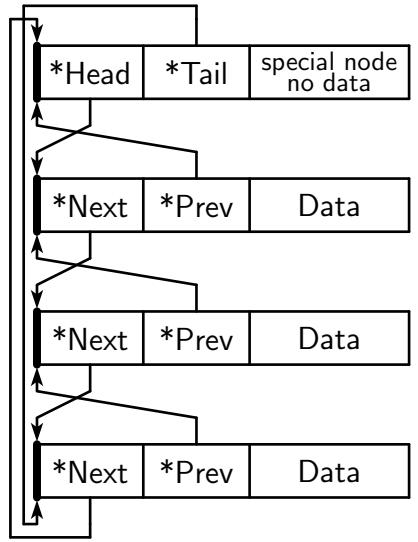
In a singly linked-list, the nodes of information are linked as follows (a small circle represents a `nullptr` pointer, and a `*` before a field name means that it is a pointer):



In a doubly linked-list, the nodes of information are linked as follows (note that when inserting or removing information it may be necessary to deal with `nullptr` pointers):



Using an extra special node to hold the head, in the `Next` field, and the tail, in the `Prev` field, of a doubly linked-list makes its implementation far simpler (no `nullptr` pointers):



Linked lists (part 2, computational complexity)

Because of the way information is organized in a linked list,

- it is necessary to keep track of the first node (the head) of the list
- it is possible to keep track of the last node (the tail) of the list
- given a position (an index), random access to information is slow: $O(n)$
- forward sequential access (from head to tail) is fast; backward sequential access (from tail to head) is slow for singly-linked lists and fast for doubly-linked lists if the tail is known
- inserting information at the head of the list is fast: $O(1)$
- if the tail of the list is known, inserting information at the end is fast: $O(1)$
- if the tail of the list is not known, inserting information at the end is slow: $O(n)$
- inserting information after a node is fast: $O(1)$
- deleting information at the head of the list is fast: $O(1)$
- on a singly-linked list, deleting a node of information is slow: $O(n)$
- on a doubly-linked list, deleting a node of information is fast: $O(1)$
- searching for information is slow, even if the data is stored in sorted order: $O(n)$

Linked lists (part 3, operations)

The following operations are usually supported in a linked list implementation:

- creation of the linked list
- destruction of the linked list
- insertion of a new node of information (before the head of the list, after the tail of the list, or after a given node of information)
- deletion of a node of information (of the head of the list, of the tail of the list, or of a given node of information)
- given a node of information, determine its next node of information
- given a node of information, determine its previous node of information

Given the class

```
class list_node
{
  private:
    list_node *next;
    // ... (rest of the code for the list_node class)
}
```

the following code finds the tail of a singly-linked list:

```
list_node *tail = head;
if(tail != nullptr)
  while(tail->next != nullptr)
    tail = tail->next;
```

Stacks

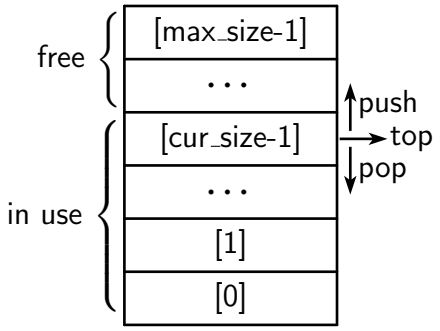
A stack, associated with a usage policy of First In Last Out (FILO), or, what is the same, Last In First Out (LIFO), is a data container that supports the following operations:

- creation of the stack
- destruction of the stack
- add a new element to the top of the stack, called **push**
- remove the element at the top of the stack, called **pop**
- take a look at the top element of the stack, called **top**
- determine the current size of the stack

It is possible to implement a stack using

- an array (in this case the stack has a maximum size, specified when the stack is created),
- a linked list (keeping the top of the stack at the head of the list), or
- a deque (see next slides).

The simplest implementation uses an array and two integers: `max_size`, which is the maximum size of the stack, and `cur_size`, which is the current size of the stack. In this case, the index of the top of the stack is `cur_size-1`, the stack is empty when `cur_size==0` and it is full when `cur_size==max_size`. The following figure illustrates how the information is organized when a stack is implemented using an array.



[**Homework:** implement a stack using a linked list.]

Queues

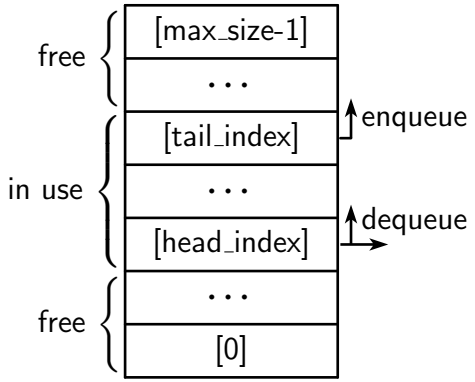
A queue, associated with a usage policy of First In First Out (FIFO), or, what is the same, Last In Last Out (LILO), is a data container that supports the following operations:

- creation of the queue
- destruction of the queue
- add a new element to the back (tail) of the queue, called **enqueue**
- remove the element at the front (head) of the queue, called **dequeue**
- determine the current size of the queue

It is possible to implement a queue using

- a circular buffer (in this case the queue has a maximum size, specified when the queue is created),
- a linked list (preferably one that keeps track of the tail, to make the **enqueue** operation efficient), or
- a deque (see next slides).

The simplest implementation uses a circular buffer and four integers: `max_size`, which is the maximum size of the queue, `cur_size`, which is the current size of the queue, `head_index`, which is the index of the head of the queue, and `tail_index`, which is the index of the tail of the queue.



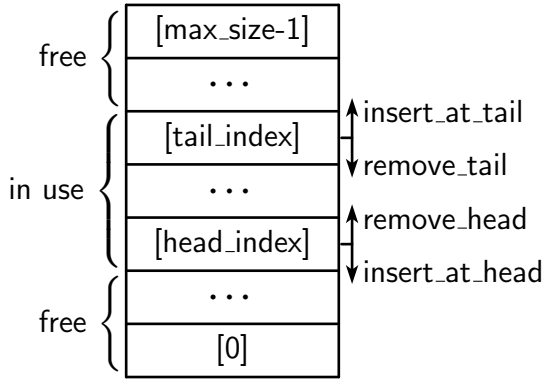
In an empty queue `cur_size==0` and the tail index has to be one more than the head index (modulo `max_size`).

Deques

A deque is a double-ended queue. It is similar to a queue, but it is possible to insert and remove elements at both ends of the queue. It supports the following operations:

- creation of the deque
- destruction of the deque
- insert a new element at the front (head) or at the back (tail) of the deque
- remove the element at the front (head) or at back (tail) the of the deque
- determine the current size of the deque

It can be implemented using either a circular buffer or a doubly-linked list (given that it may be necessary to move in either direction, using a singly-linked list to implement a deque is inefficient).



An insertion at either end forces the size of the deque to increase. That determines the direction of movement of the head or of the tail index.

Elementary data structures (part 1, exercises)

— P.05 —

Summary:

- Stacks
- Singly-linked lists
- Queues
- Deques
- Doubly-linked lists

Stacks

Extract the files `aStack.h` and `aStack_demo.cpp` from the archive `P05.tgz`. Study the generic implementation of a stack (file `aStack.h`). The purpose of the program `aStack_demo.cpp` is to verify if the parentheses of each of its text arguments are balanced. When called as follows (warning: copying and pasting may not work properly on the following line; if it does not work the accute accent is the culprit)

```
./aStack_demo 'abc' 'a(b)' 'a(b' 'a)b' 'a(b(c)(d((ef)g))h)i'
```

it should produce the output

```
abc
  good
a(b)
  '(' at position 1 and matching ')' at position 3
  good
a(b
  unmatched '(' at position 1
  bad
a)b
  unmatched ')' at position 1
  bad
a(b(c)(d((ef)g))h)i
  '(' at position 3 and matching ')' at position 5
  '(' at position 9 and matching ')' at position 12
  '(' at position 8 and matching ')' at position 14
  '(' at position 6 and matching ')' at position 15
  '(' at position 1 and matching ')' at position 17
  good
```

The code in `aStack_demo.cpp` is incomplete. Complete it using a stack.

Modify the `aStack.h` class so that the stack can grow as much as needed. (Hint: write a private member function that resizes the stack, and start with a stack with a maximum size of, say, 100.)

Singly-linked Lists

Extract the files `sList.h` and `sList_test.cpp` from the archive `P05.tgz`. The file `sList.h` implements a generic singly-linked list. Study it. Study also the file `sList_test.cpp`, that tests the correctness of the implementation in `sList.h`.

Queues

Extract the files `lQueue.h` and `lQueue_demo.cpp` for the archive `P05.tgz`. The file `lQueue.h` contains a skeleton of an implementation of a generic queue based on a singly-linked list. Complete the implementation and write code to test it.

Deque

Implement a generic deque (double-ended queue) using an array. On a deque, insertion and deletion can occur at both ends of the queue.

Doubly-linked lists

Work to be done at home: using the code in `sList.h` as starting point, implement a doubly-linked list. Hints:

- the `move()` member function can be improved, but that is not strictly necessary,
- the various `insert` and `remove` member functions have to be modified.
- the computational complexity of some of these functions may change!

Elementary data structures (part 2)
— TP.06 —

Summary:

- Heaps
- Priority queues
- Binary trees
- Tries
- Hash tables

Recommended bibliography for this lecture:

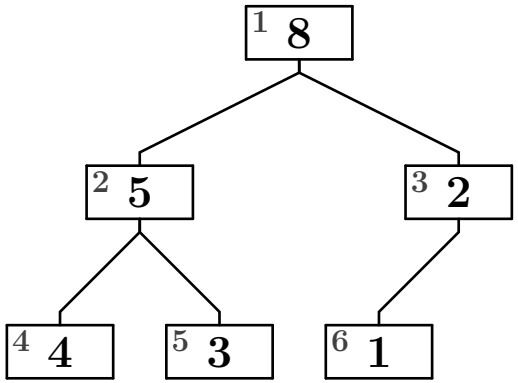
- **Analysis of Algorithms**, Jeffrey J. McConnell, second edition, Jones and Bertlett Publishers, 2008.
- **Introduction to Algorithms**, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, third edition, The MIT press, 2009.
- **Algorithms**, Robert Sedgewick and Kevin Wayne, fourth edition, Addison Wesley, 2011
- **Estruturas de Dados e Algoritmos em C**, António Adrego da Rocha, terceira edição, FCA.

Heaps (part 1, properties)

A heap is an array with internal structure. The internal structure of the array takes the form of an order relation (the heap property) that has to be maintained between the data stored in certain index positions. To be more precise, in a **binary max-heap** with n data items, stored in indices 1, 2, $\ldots$, n, the information stored in index i, with $2 \leqslant i \leqslant n$, cannot be larger than the information stored in index $\lfloor i/2 \rfloor$; $\lfloor x \rfloor$ is the largest integer that is not larger than x (i.e., the `floor` function). For example, the data stored in the following array (left hand side) satisfies the binary max-heap property

index	value
0	-
1	8
2	5
3	2
4	4
5	3
6	1

Implicit binary tree organization



because $8 \geqslant 5$ (index 1 versus index 2), $8 \geqslant 2$ (index 1 versus index 3), $5 \geqslant 4$ (index 2 versus index 4), $5 \geqslant 3$ (index 2 versus index 5), and $2 \geqslant 1$ (index 3 versus index 6). The max-heap property can be easily checked using, for example, the following code

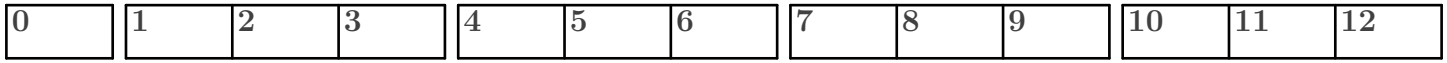
```
for(int i = 2;i <= n;i++)
    assert(heap[i / 2] >= heap[i]);
```

We may also have a binary min-heap, and even multi-way heaps. The starting index is usually either 1 or 0.

Heaps (part 2, m-way heaps)

For each index of a heap we define its parent index, and its children indices. In a max-heap the data stored in a given position cannot be larger that the data stored in the parent position, and it cannot be smaller that the data stored in each one of its children positions. This implicitly defines a tree organization for the data, as shown on the right hand side of the example given in the previous slide. The first index of the array used by the heap is the root index, as it corresponds to the root of the implicit tree.

In a m-way heap with a root index of 0, the parent of index $i > 0$ is $\lfloor \frac{i-1}{m} \rfloor$ and its children are $mi+1, \ldots, mi+m$. For $m = 3$ the array is implicitly subdivided in the following way:



In a m-way heap with a root index of 1, the parent of index $i > 1$ is $\lfloor \frac{i+m-2}{m} \rfloor$ and its children are $mi - m + 2, \ldots, mi + 1$. Note that for $m = 2$ the formulas are particularly simple. For $m = 3$ the array is implicitly subdivided in the following way:

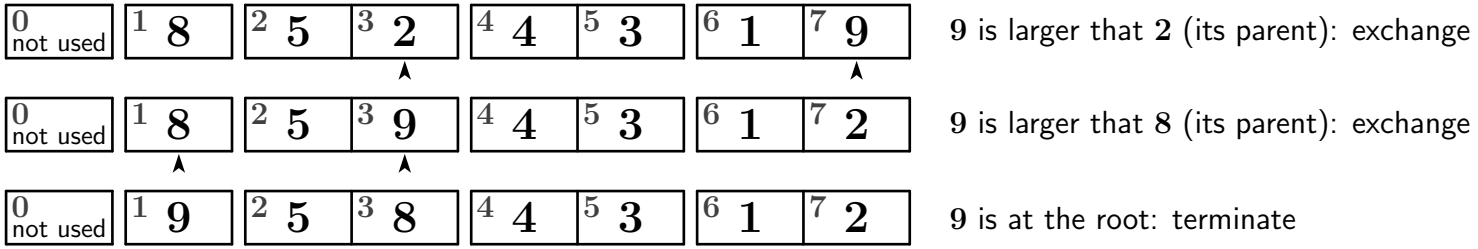
The path to the root of a node with index i is the sequence of indices i, $\mathbf{parent}(i)$, $\mathbf{parent}\big(\mathbf{parent}(i)\big)$, and so on, that stops at the root index. When $m = 2$ and the root index is 1, the path to the root of i is i, $\lfloor \frac{i}{2} \rfloor$, $\lfloor \frac{i}{2^2} \rfloor$, $\lfloor \frac{i}{2^3} \rfloor$, $\ldots$, $\lfloor \frac{i}{2^k} \rfloor$, where $k = \lfloor \log_2 i \rfloor$; $\log_2 x$ is the base-2 logarithm of x. In particular, for a binary heap of size n, the longest path to the root has length $1 + \lfloor \log_2 n \rfloor$, which is $O(\log n)$.

Heaps (part 3, insertion)

A max-heap supports at least the following operations (a min-heap is similar, with the word "largest" replaced by the word "smallest"):

- creation and destruction of the heap
- inspection of the largest data item (the root of a max-heap holds the largest data value)
- insertion of a data item
- removal of a data item

To insert a data item `v` on a heap with size `n` the first step is to place it at the end of the heap. The heap property is then enforced on the path to the root of the new node, by exchanging the data of a node with that of its parent whenever the heap property is not satisfied (there is no need to change data elsewhere). Thus, insertion is an $O(\log n)$ operation. The following example illustrate what happens when 9 is inserted on the heap $8, 5, 2, 4, 3, 1$:

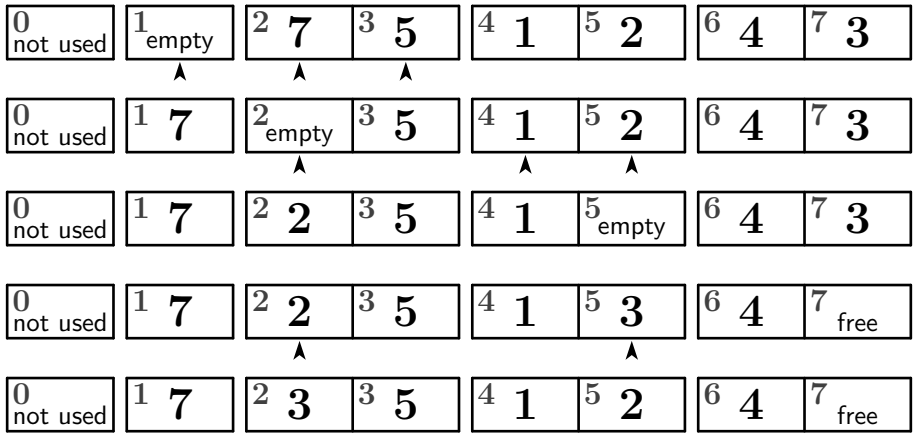


In C or C++, all this can be done as follows ($m = 2$ and the root index is 1):

```
for(i = ++n;i > 1 && heap[i / 2] < v;i /= 2)
  heap[i] = heap[i / 2];
heap[i] = v;
```

Heaps (part 4, removal)

To remove a data item, we replace it by the largest of its children, and then we do the same to fill the empty slot vacated by each child that was moved towards the root, until that cannot be done any more. This procedure leaves an empty slot in the heap. If it is not the last slot, we move the last slot to the empty slot and then enforce the heap property on the path to the root starting at that slot. In the worst case about $2 \log_2 n$ operations need to be done — $O(\log n)$ — to remove a data item. The following example illustrates what happens when the root (9) is removed from the heap $9, 7, 5, 1, 2, 4, 3$:



In C or C++, removal of the data at position `pos` can be done as follows ($m = 2$ and the root index is 1):

```
for(i = pos;2 * i <= n;heap[i] = heap[j],i = j)
  j = (2 * i + 1 <= n && heap[2 * i + 1] > heap[2 * i]) ? 2 * i + 1 : 2 * i; // select largest child
for(;i > 1 && heap[i / 2] < heap[n];i /= 2)
  heap[i] = heap[i / 2];
heap[i] = heap[n--];
```

[**Homework:** what happens in the second `for` loop when i is equal to n?]

Priority queues

A priority queue is a data container that supports the following operations:

- creation and destruction of the priority queue
- inspection of the largest data item (**peek**)
- insertion of a data item (**enqueue**)
- removal of the largest data item (**dequeue**)

It can be implemented easily using a max-heap. Conceptually, a priority queue is similar to an ordinary queue, its only difference being that instead of removing the **oldest** element it is the **largest** that gets removed. (In terms of implementation, a priority queue and an ordinary queue are quite different.)

Instead of inspecting and removing the largest data item, a priority queue can allow the inspection and removal of its smallest data item. This variant of the priority queue is, obviously, implemented using a min-heap.

Binary trees (part 1, overview)

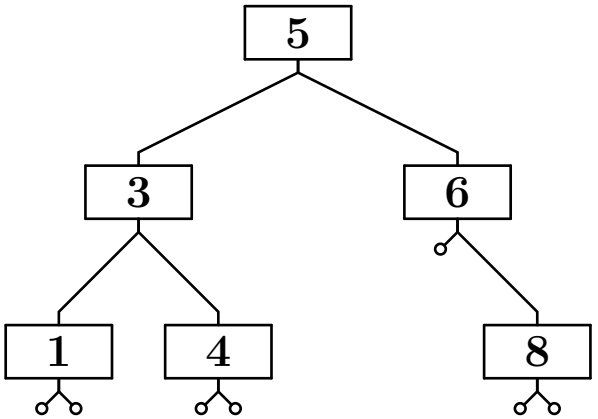
A binary tree is a dynamic data structure composed of nodes. Each node of information contains

- the information itself (the data item)
- a pointer to the node on the "left" side (the left branch); in an ordered binary tree the data items stored on this side are all of them smaller that the data item stored on the node
- a pointer to the node on the "right" side (the right branch); in an ordered binary tree the data items stored on this side are all of them larger that the data item stored on the node
- optionally, a pointer to the parent node

A null pointer (`NULL` in C and `nullptr` in C++) indicates the nonexistence of a node. The **root** node is the node where the tree begins. It is the only node without a parent node. A **leaf** node is a node whose left and right node pointers are null.

The **height** of a tree is the number of levels it has. Each time a left or a right pointer is followed the level increases by 1. It is usual to consider that the root is at level 0. We may also talk about the height of the left or right branches of a node: that it the height of the tree that has as root the left or right node of the node. It is not necessary for a tree to have its leaves all at the same level (but that is usually desired).

Visually, a tree is usually depicted upside-down, i.e., with its root on top:



Binary trees (part 2, node contents)

In these slides each node of the **ordered binary tree** will be implemented (in C) as follows:

```
typedef struct tree_node
{
  struct tree_node *left;   // pointer to the left branch (a sub-tree)
  struct tree_node *right;  // pointer to the right branch (a sub-tree)
  struct tree_node *parent; // optional
  int data;                 // the data item (we use an int here, but it can be anything)
}
tree_node;
```

Each node may also keep information about

- the height of the subtree having that node as root,
- the difference of the heights of its left and right branches (the balance)
- the level of the node, or
- other useful information that may be need by a particular program.

In the following slides we present C code snippets of functions that do useful things to a tree. **Study them carefully.** Most of them take the form of a recursive function, because trees are recursive structures: the left and right branches of a node are also trees!

In some of these functions we pass a pointer to the location in memory where the pointer to the root of the tree is stored (a pointer to a pointer). Things are done in this way because it may be necessary to change the root of the tree!

Binary trees (part 3, insertion)

The following non-recursive function creates a new node and inserts it in the tree at the appropriate location:

```
tree_node *new_tree_node(int data,tree_node *parent); // sets left and right to NULL

void insert_non_recursive(tree_node **link,int data)
{
  tree_node *parent = NULL;
  while(*link != NULL)
  {
    parent = *link;
    link = (data <= (*link)->data) ? &((*link)->left) : &((*link)->right); // select branch
  }
  *link = new_tree_node(data,parent);
}
```

This can also be done recursively as follows:

```
void insert_recursive(tree_node **link,tree_node *parent,int data)
{
  if(*link == NULL)
    *link = new_tree_node(data,parent);
  else if(data <= (*link)->data)
    insert_recursive(&((*link)->left),*link,data);
  else
    insert_recursive(&((*link)->right),*link,data);
}
```

These functions are used as follows:

```
tree_node *root = NULL;

insert_nonrecursive(&root,4);
insert_recursive(&root,NULL,7);
```

What is the height of an initially empty tree after insertion of $1, 2, \ldots, 31$? What about the heigth of an initially empty tree after insertion of $f(1), f(2), \ldots, f(31)$, where the function $f(n)$ reverses the order of the least significant 5 bits of its argument?

Binary trees (part 4, search in an ordered and in an unordered binary tree)

For an **ordered** binary tree, the following non-recursive and recursive functions return one node of the tree for which its data field is equal to a given value (or NULL if there is no such node):

```
tree_node *search_non_recursive(tree_node *link,int data)
{
  while(link != NULL && data != link->data)
    link = (data < link->data) ? link->left : link->right;
  return link;
}

tree_node *search_recursive(tree_node *link,int data)
{
  if(link == NULL || link->data == data)
    return link;
  if(data < link->data)
    return search_recursive(link->left,data);
  else
    return search_recursive(link->right,data);
}
```

These functions are used as follows:

```
tree_node *root,*n;
int data;

n = search_non_recursive(root,data);
n = search_recursive(root,data);
```

For an **unordered** binary tree, the following recursive function returns one node of the tree for which its data field is equal to a given value (or NULL if there is no such node):

```
tree_node *search_recursive(tree_node *link,int data)
{
  node *n;

  if(link == NULL || link->data == data)
    return link;
  // try the left branch
  if((n = search_recursive(link->left,data)) != NULL)
    return n;
  // not found in the left branch, try the right branch
  return search_recursive(link->right,data);
}
```

This function is used as follows:

```
tree_node *root,*n;
int data;

n = search_recursive(root,data);
```

Note that for an unordered binary tree it may be necessary to search the entire tree, which is an order $O(n)$ operation, while in an ordered binary tree it is necessary to examine at most $h + 1$ nodes, where h is the maximum height of any node of the tree.

Binary trees (part 5, traversal)

The following non-recursive and recursive functions traverse the entire tree in different orders:

```
void visit(tree_node *n)
{
  printf("%d\n",n->data);
}

void traverse_breadth_first(tree_node *link)
{
  queue *q = new_queue();

  enqueue(q,link);
  while(is_empty(q) == 0)
  {
    link = dequeue(q);
    if(link != NULL)
    {
      visit(link);
      enqueue(q,link->left);
      enqueue(q,link->right);
  }
  free_queue(q);
}
```

```
void traverse_depth_first_recursive(tree_node *link)
{
  if(link != NULL)
  {
    visit(link);
    traverse_depth_first_recursive(link->left);
    traverse_depth_first_recursive(link->right);
  }
}

void traverse_in_order_recursive(tree_node *link)
{
  if(link != NULL)
  {
    traverse_in_order_recursive(link->left);
    visit(link);
    traverse_in_order_recursive(link->right);
  }
}
```

They are used as follows:

```
tree_node *root;

traverse_breadth_first(root);
traverse_depth_first_recursive(root);
traverse_in_order_recursive(root);
```

[**Homework:** do the `traverse_depth_first` function in a non-recursive way.]

Binary trees (part 6a, misc)

Does the purpose of each of the following functions match its name?

```
// use count_nodes(root) to count the nodes of an entire tree

int count_nodes(tree_node *link)
{
  return (link == NULL) ? 0 : count_nodes(link->left) + 1 + count_nodes(link->right);
}


// use count_leaves(root) to count the leaves (terminal nodes) of an entire tree

int count_leaves(tree_node *link)
{
  if(link == NULL)
    return 0;
  if(link->left == NULL && link->right == NULL)
    return 1;
  return count_leaves(link->left) + count_leaves(link->right);
}


// use check_node(root,NULL,INT_MIN,INT_MAX) to check an entire ordered binary tree

void check_node(tree_node *link,tree_node *parent,int min_bound,int max_bound)
{
  if(link != NULL)
  {
    assert(min_bound <= link->data && link->data <= max_bound && link->parent == parent);
    check_node(link->left,link,min_bound,link->data);
    check_node(link->right,link,link->data,max_bound);
  }
}
```

Binary trees (part 6b, misc)

Does the purpose of each of the following functions match its name?

```
// use set_level_nodes(root,0) to set the level of each node of an entire tree

void set_level(tree_node *link,int level)
{
  if(link != NULL)
  {
    link->level = level;
    set_level(link->left,level + 1);
    set_level(link->right,level + 1);
  }
}


// use set_height(root) to set the height of each node of an entire tree

int set_height(tree_node *link)
{
  int left_height,right_height;

  if(link == NULL)
    return 0;
  left_height = set_height(link->left);
  right_height = set_height(link->right);
  link->height = (left_height >= right_height) ? 1 + left_height : 1 + right_height;
  return link->height;
}
```

Binary trees (part 7a, balancing)

A binary tree is said to be **perfect** if all its leaves are at the same level. A perfect binary tree with height h has $2^h - 1$ nodes: 2^{h-1} leaves and $2^{h-1} - 1$ internal nodes. Abusing the concept of perfect binary tree, we will say that a binary tree is also perfect if some of the leaves at the last level of the tree are missing (but all other levels are full). Such a tree has a height as small as possible. That is desirable because many operations that modify a binary tree do a number of elementary operations that is proportional to the height of the tree. A perfect binary tree with n nodes has a height equal to $\lceil \log_2(n+1) \rceil$, which is $O(\log n)$.

A binary tree is said to be **balanced** if for each of its nodes the height of its left branch does not differ by more than 1 from the height of its right branch. A balanced binary tree of height h has at least G_h nodes, where $G_0 = 0$, $G_1 = 1$ and, for $n > 1$, $G_n = G_{n-1} + 1 + G_{n-2}$, and it has at most $2^h - 1$ nodes. It can be verified that $G_n = F_{n+2} - 1$, where F_n is the n-th Fibonacci number. It follows that the height of a balanced binary tree is $\Theta(\log n)$. It is possible, using simple $O(\log n)$ operations, to keep a binary tree balanced after the insertion or removal of a node. Searching is also an $O(\log n)$ operation in a balanced binary tree.

Binary trees (part 7b, how to keep an ordered binary tree balanced)

Explain here how to balance a binary tree (to be done, maybe, in the 2019/2020 school year).

Tries

A trie is essentially an m-way tree (each node of the tree has m children.) An index is usually used to select the appropriate children. For example, storing and searching for a telephone number can be done very efficiently using a trie (here m will be 10). The most significant digit of the telephone number selects the child of the root of the tree that must be followed, the second most significant digit selects the child of that node, and so on. Only the nodes that are needed are actually allocated.

Hash tables (part 1, overview)

A hash table is a data container that supports (at least) the following operations:

- creation and destruction of the hash table
- insertion of a data item composed of two parts: the key and its corresponding value
- search for a data item with a given key
- removal of a data item given its key

In an array, information is accessed given its index. In a hash table, information is accessed given its key. In a properly dimensioned hash table, the insertion, removal and search operations have very good expected computational complexity: $O(1)$. This is better that the computational complexity of the same operations in a balanced binary tree, which is $O(\log n)$, where n is the number of data items stored in the data container.

A hash table is usually the data container used to implement an associative array, a symbol table, or a dictionary. (Other data containers may also be used for this, but the hash table is usually more efficient.) The key may not be an integer; it may be, for example, a string.

In the insert operation, if a data item with the same key already exists in the hash table, the insert operation should either fail or it should replace the corresponding value (and no new data item is created). The programmer has to decide which is best for a given application.

An hash table is usually implemented using an array. It may be an array of data items (keys and respective values), if **open addressing** (explained later) is used, or it may be an array of pointers to the heads of (doubly-)linked lists of data items, if **chaining** is used. Instead of a pointer to the head of a linked list, it is also possible to use a pointer to the root of a binary tree, or even a pointer to another hash table.

Hash tables (part 2a, hash functions)

Let s be the size of the array used to implement the hash table. An hash function h maps each possible key k to an integer i in the range $0, \ldots, s-1$. This integer will then be used to access the array. This conversion is necessary because the key itself may not be an integer, and even if it is an integer, its value may be too small or too large.

If the number of keys, n, is larger than the size of the array, s, then it is inevitable that two (or more) keys map, via the hash function, to the same index. Even when n is smaller than s, it is possible, if the hash function is not chosen with extreme care, for these so-called **collisions** to happen. Indeed, due to the birthday paradox, if the hash function spreads the indices in an uniform way, then there is at least a 50% chance of a collision when $n \geqslant (1+\sqrt{1+8s\log 2})/2$.

A good hash function should attempt to avoid too many collisions. There are many ways to attempt to do this. One of them it to treat the key, or rather, its memory representation, as a possibly very large integer, and to choose as hash function the remainder of the division of this large integer by the array size. When the key is a string this gives rise to code such as:

```
unsigned int hash_function(const char *str,unsigned int s)
{ // for 32-bit unsigned integers, s should be smaller that 16777216u
  unsigned int h;

  for(h = 0u;*str != '\0';str++)
    h = (256u * h + (0xFFu & (unsigned int)*str)) % s;
  return h;
}
```

It turns out that hash functions of this form are better (less collisions) when s is a prime number. (Furthermore, the order of 256 in the multiplicative group of remainders co-prime to s should be large; in particular, s should not be an even number.)

Hash tables (part 2b, more hash functions)

The hash function presented in the previous slide is reasonably good but it is slow, because it requires a remainder operation in each iteration of the for loop, and it works better when s is a prime number. Furthermore, for a 32-bit `unsigned int` data type, due to a possible integer overflow, it should not be used when `s` is larger than or equal to $2^{32-8} = 16777216$.

One way to solve these problems is to get rid of all but the last of the remainder operations, as done in the following variant of the hash function of the previous slide:

```
unsigned int hash_function(const char *str,unsigned int s)
{
  unsigned int h;

  for(h = 0u;*str != '\0';str++)
    h = 157u * h + (0xFFu & (unsigned int)*str); // arithmetic overflow may occur here (just ignore it!)
  return h % s; // due to the unsigned int data type, it is guaranteed that 0 <= h % s < s
}
```

(The multiplication factor 157 was chosen is an almost arbitrary way.) Note that return values smaller that $2^{32} \bmod s$ (for a 32-bit data type) will be slightly more probable than those larger than or equal to $2^{32} \bmod s$ (and, of course, smaller that s). This defect does not cause any significant problem when s is many times smaller (say, 1000 times smaller) that the largest possible unsigned integer. Of course, this potential problem can be almost eliminated if 64-bit integers are used (`unsigned long long` data type).

Hash tables (part 2c, even more hash functions)

Other possible hash functions are based on so-called cyclic redundancy checksums (CRC), or on so-called message digest signatures (such as MD5), or on secure hash algorithms (such as SHA-1). The following hash function is based on a 32-bit cyclic redundancy checksum.

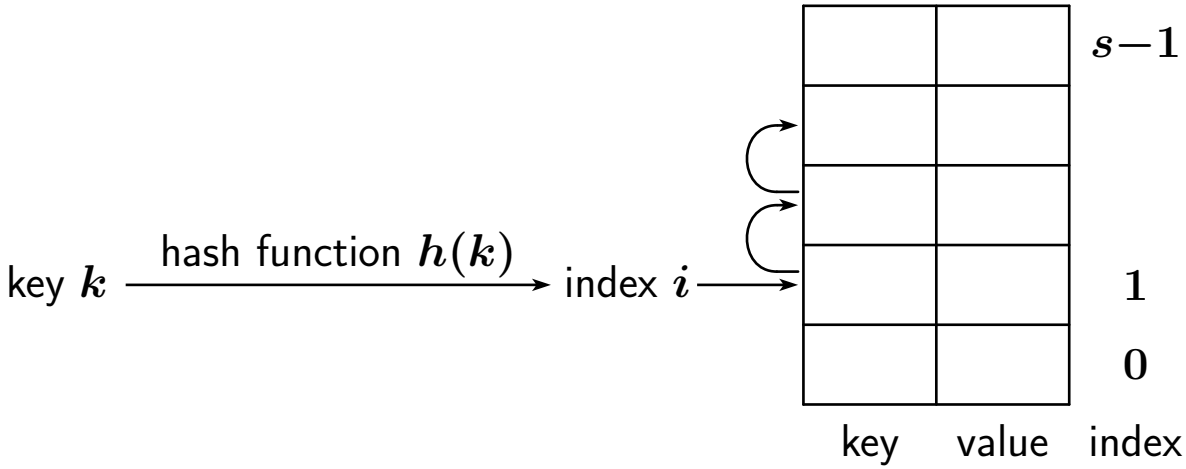
```
unsigned int hash_function(const char *str,unsigned int s)
{
  static unsigned int table[256];
  unsigned int crc,i,j;

  if(table[1] == 0u) // do we need to initialize the table[] array?
    for(i = 0u;i < 256u;i++)
      for(table[i] = i,j = 0u;j < 8u;j++)
        if(table[i] & 1u)
          table[i] = (table[i] >> 1) ^ 0xAED00022u; // "magic" constant
        else
          table[i] >>= 1;
  crc = 0xAED02016u; // initial value (chosen arbitrarily)
  while(*str != '\0')
    crc = (crc >> 8) ^ table[crc & 0xFFu] ^ ((unsigned int)*str++ << 24);
  return crc % s;
}
```

For the curious, the "magic" constant encodes the coefficients (bits) of a primitive polynomial in the finite field $GF(2^{32})$. In this case the polynomial is $x^{32} + x^{30} + x^{26} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^2 + 1$. [**Homework:** In the function given above replace the 32-bit CRC by a 64-bit CRC; use a "magic" constant of `0xAED0AED0AED0011Full`.]

Hash tables (part 3a, open addressing)

When a hash table uses open addressing the key and respective value are stored directly in the array:



This implies that $n \leqslant s$, and that it is necessary to resolve collisions by looking at other positions of the array when position i, with $i = h(k)$, does not contain the correct key. One possibility is to try $(i+1) \bmod s$, $(i+2) \bmod s$, and so on, until either the desired key is found or an empty array position is found. Instead of trying consecutive positions, it is also possible to try positions further apart, with jumps of j between positions: $(i+j) \bmod s$, $(i+2j) \bmod s$, and so on. For this to work is is necessary and sufficient that $\gcd(j, s) = 1$, which is ensured if $0 < j < s$ and if s is a prime number. Instead of using a fixed j, in **double hashing** each key uses its own j, computed by another hash function.

Open addressing has several major disadvantages:

- the hash table cannot have more that s keys
- when the hash table is nearly full and there are collisions the worst search time can be quite large
- it is difficult to remove keys from the hash table

Hash tables (part 3b, open addressing)

The following C code exemplifies one way to perform a key search in a hash table that uses open addressing (compare to similar code that used separate chaining, presented in part 4b):

```
typedef struct hash_data
{
  char key[10]; // empty when key[0] = '\0'
  int value;
}
hash_data;

#define hash_size 1009u

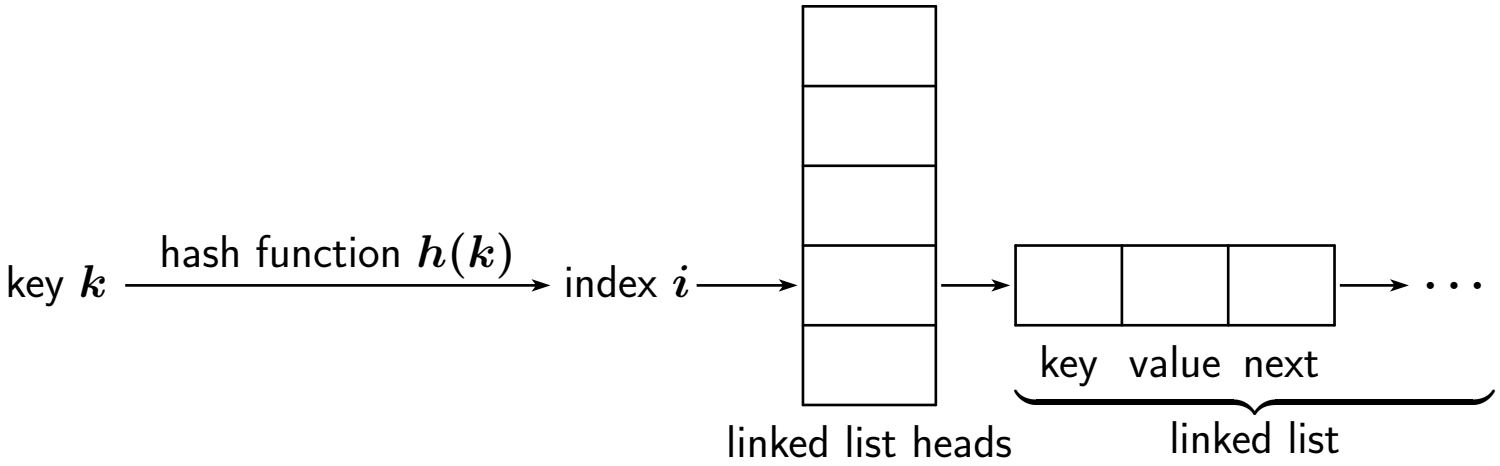
hash_data hash_table[hash_size];

hash_data *find_data(const char *key)
{
  unsigned int idx;

  idx = hash_function(key,hash_size);
  while(hash_table[idx].key[0] != '\0' && strcmp(key,hash_table[idx].key) != 0)
    idx = (idx + 1u) % hash_size; // try the next array position
  return (hash_table[idx].key[0] == '\0') ? NULL : &hash_table[idx];
}
```

Hash tables (part 4a, separate chaining)

When the hash table uses chaining, sometimes also called separate chaining, the array stores pointers to the heads of linked lists. For each key the hash function specifies in which list we will operate (search, insert or remove). Of course, it is possible to replace the linked lists by a more sophisticated data structure, such as a binary search tree (or even another hash table!), that provides the same operations but with lower computational complexity.



When separate chaining is being used, the main purpose of the hash function is to distribute the keys as evenly an possible among the s positions of the array. A good spread implies a small number of collisions, and so a search operation will be fast. (If linked lists are being used, the worst search time is proportional to the length of the longest linked list.)

Because linked lists are dynamic data structures, a hash table that uses them can store more keys that the size of the array. The average search time will be $\max(1, n/s)$ if all keys are equally probable. So, the performance of a hash table implemented with separate chaining will degrade gracefully if its array is under-dimensioned.

Hash tables (part 4b, separate chaining)

The following C code exemplifies one way to perform a key search in a hash table that uses separate chaining (compare to similar code that used open addressing):

```
typedef struct hash_data
{
  struct hash_data *next;
  char key[10];
  int value;
}
hash_data;

#define hash_size 1009u

hash_data *hash_table[hash_size];

hash_data *find_data(const char *key)
{
  unsigned int idx;
  hash_data *hd;

  idx = hash_function(key,hash_size);
  hd = hash_table[idx];
  while(hd != NULL && strcmp(key,hd->key) != 0)
    hd = hd->next;
  return hd;
}
```

Hash tables (part 4c, separate chaining)

The following C code exemplifies how to allocate a new hash_data structure:

```
#include <stdio.h>
#include <stdlib.h>

hash_data *new_hash_data(void)
{
  hash_data *hd = (hash_data *)malloc(sizeof(hash_data));
  if(hd == NULL)
  {
    fprintf(stderr,"Out of memory\n");
    exit(1);
  }
  return hd;
}
```

The following C code exemplifies how to visit all nodes of a hash table:

```
void visit_all(void)
{
  unsigned int i;
  hash_data *hd;

  for(i = 0u;i < hash_size;i++)
    for(hd = hash_table[i];hd != NULL;hd = hd->next)
      visit(hd);
}
```

Hash tables (part 5, perfect hash functions)

A perfect hash function is an hash function that does not generate any collisions. Perfect hash functions are desired when the set of keys is fixed and known (for example, the reserved keywords of a programming language). In a perfect hash function it is usually desired that $s = n$ (to save space). As there are no collisions, the hash table is best implemented using open addressing.

Although perfect hash functions are rare (recall the birthday paradox), it is not difficult to construct a perfect hash function $h(k)$, that maps k to one of the integers $0, 1, \ldots, s-1$. Mathematically this can be expressed in a more compact way by $h : k \mapsto \{i\}_{i=0}^{s-1}$. The main idea is to use two distinct hash functions $h_1 : k \mapsto \{i\}_{i=0}^{s-1}$ and $h_2 : k \mapsto \{i\}_{i=0}^{t-1}$ and to use an auxiliary table T of size t. Indeed, when t is not too small ($t \approx s/2$ works well), and assuming that there are no key pairs that give rise to simultaneous collisions in the two hash functions, is it usually possible to construct the table in such a way that

$$h(k) = \big(h_1(k) + T[h_2(k)]\big) \bmod s$$

is a perfect hash function. Note that for a key k without a colision in $h_2(k)$ it is possible to assign to $h(k)$ any desired value.

Let K_i be the set of keys for which h_2 evaluates to i; mathematically we have $K_i = \{\ k\ :\ h_2(k) = i\ \}$. The table T can be constructed using a greedy approach by considering each K_i in turn, starting with the largest set and ending with the smallest set (the size of a set is its number of elements). For each i one tries $T[i] = 0$, $T[i] = 1$, and so on, until either s is reached (a failure!) or all the values of $h(k)$, for $k \in K_i$, do not collide with the values of $h(k)$ for the k belonging to the sets already dealt with. Sets with 1 or 0 elements, done at the end, do not pose any problem! In practice, this greedy approach works surprising well, if one is not too ambitious in the choice of t.

First written report work

— P.06 —

Summary:

- The traveling salesman problem.