Ficha 5

1

a) $f(t) = 2\sin(3t) + t - 5e^{-t}$

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{2\sin(3t) + t - 5e^{-t}\}(s)$

$= 2\mathcal{L}\{\sin(3t)\}(s) + \mathcal{L}\{t\}(s) - 5\mathcal{L}\{e^{-t}\}(s)$

$= 2 \cdot \underset{(s>0)}{\frac{3}{s^2-3}} + \underset{(s>0)}{\frac{1!}{s^2}} - 5\underset{(s>-1)}{\frac{1}{s+1}} = \frac{6}{s^2-3} + \frac{1}{s^2} - \frac{5}{s+1}, \; s>0$

$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

b) $f(t) = e^{2t}\cos(5t)$

Considerando $g(t) = \cos(5t)$

Então $f(t) = e^{2t} g(t)$

$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s-2) = \underset{(s-2)>0}{\frac{(s-2)}{(s-2)^2+5^2}} = \frac{s-2}{s^2-4s+29}, \; s>2$

c) $f(t) = \underbrace{t}_{g(t)} \cdot e^{3t} = e^{3t} g(t)$

$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s-3) = \mathcal{L}\{t\}(s-3) = \underset{(s-3>0)}{\frac{1!}{(s-3)^2}} = \frac{1}{(s-3)^2}, \; s>3$

d) $f(t) = \pi - 5e^{-t}\underbrace{t^{10}}_{g(t)}$

$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\pi - 5e^{-t}g(t)\}(s) = \pi\mathcal{L}\{1\}(s) - 5\mathcal{L}\{e^{-t}g(t)\}(s)$

$= \pi \cdot \frac{0!}{t}_{t>0} - 5 \cdot \mathcal{L}\{t^{10}\}(s+1) = \frac{\pi}{t} - 5 \cdot \frac{10!}{(s+1)^{11}}, s>0$

e)

$f(t) = (3t-1)\sin t = 3t\sin t - \sin t$

$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{3t\sin t - \sin t\}(s) = 3\mathcal{L}\{t\overset{g(t)}{\widetilde{\sin t}}\}(s) - \mathcal{L}\{\sin t\}(s)$

$= 3\cdot(-1)^1 \cdot F'(s) - \frac{1}{s^2+1}_{s>0}$

$= -3\cdot\left(\frac{-2s}{(s^2+1)^2}\right) - \frac{1}{s^2+1}$

$= \frac{6s}{(s^2+1)^2} - \frac{1}{s^2+1}, s>0$

C.A.

$F'(s) = ?$

$F(s) = \mathcal{L}\{\sin t\}(s)$

$= \frac{1}{s^2+1}, s>0$

$F'(s) = \left(\frac{1}{s^2+1}\right)' = -\frac{2s}{(s^2+1)^2}$

f) $f(t) = (1 - H_\pi(t))\sin t$

$= \sin t - H_\pi(t)\sin t$

$= \sin t + H_\pi(t)\sin(t-\pi)$

$F(s) = \mathcal{L}\{\sin t + H_\pi(t)\sin(t-\pi)\}(s)$

$= \mathcal{L}\{\sin t\}(s) + e^{-\pi s} \cdot \mathcal{L}\{\sin t\}(s), s$

$= \frac{1}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}, s>0$

g) $f(t) = (t-2)^2 e^{2(t-2)} H_2(t)$

Considerando $g(t) = t^2 e^{2t}$

Então temos $f(t) = H_2(t)\cdot g(t-2)$

$F(s) = \mathcal{L}\{H_2(t)\cdot g(t-2)\}(s) =$

$= e^{-2s}\cdot \mathcal{L}\{g(t)\}(s)$

$= e^{-2s}\cdot \frac{2!}{(s-2)^3}, s>2$

C.A $\Rightarrow \mathcal{L}(g(t))(s) = ?$

$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{t^2 e^{2t}\}(s)$

$\mathcal{L}\{t^2\}(s-2) = \frac{2!}{(s-2)^3}, s>2$

2

a) $F(s) = \frac{2s}{s^2-9} = 2\cdot\frac{s}{s^2-9} = 2\cdot\frac{s}{s^2-3^2} = 2\cdot\mathcal{L}\{\cosh(3t)\}(s)$

$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{2\cdot\mathcal{L}\{\cosh(3t)\}(s)\}(t) = 2\cosh(3t)$

b) $F(s) = \frac{4}{s^7} = 4\cdot\frac{1}{s^7} = \frac{4}{6!}\cdot\frac{6!}{s^7} = \frac{4}{6!}\cdot\frac{6!}{s^{6+1}} = \frac{4}{6!}\cdot\mathcal{L}\{t^6\}(s)$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{4}{6!}\cdot\mathcal{L}\{t^6\}(s)\} = \frac{4}{6!}\cdot t^6$

c)

$$F(s) = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2} = \frac{1!}{(s+3)^{1+1}} = \frac{1!}{(s-(-3))^{1+1}} = \mathcal{L}\{e^{-3t} \cdot t\}(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{-3t}t$$

d)

$$F(s) = \frac{1}{s^2+s-2} = \frac{1}{(s-1)(s+2)} = \frac{1}{3} \times \frac{1}{s-1} - \frac{1}{3} \times \frac{1}{s+2} = \frac{1}{3}\mathcal{L}\{e^t\}(s) - \frac{1}{3}\mathcal{L}\{e^{-2t}\}(s)$$

$$= ⊛$$

$\cancel{s^2+2}$

$$s^2+s-2=0 \Leftrightarrow \cancel{(s+1)(s+2)=0} \qquad \frac{1}{(s-1)(s+2)} = \frac{\overset{(s+2)}{A}}{s-1} + \frac{\overset{(s-1)}{B}}{s+2}$$

$$s = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$\begin{cases} B+A=0 \\ -B+2A=1 \end{cases} \Leftrightarrow \begin{cases} B=-\frac{1}{3} \\ A=\frac{1}{3} \end{cases}$$

$$s = \frac{-1 \pm 3}{2}$$

$$s=1 \vee s=-2$$

$$⊛ = \mathcal{L}\left\{\frac{1}{3}e^t - \frac{1}{3}e^{-2t}\right\}(s)$$

$$f(t) = \frac{1}{3}\left(e^t - e^{-2t}\right)$$

e)

$$F(s) = \frac{1}{s^2+4s+6} = \frac{1}{s^2+4s+4+2} = \frac{1}{(s+2)^2+2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}$$

$$= \frac{\sqrt{2}}{2} \cdot \mathcal{L}\{e^{-2t} \cdot \sin(\sqrt{2}t)\}(s)$$

$$= \mathcal{L}\left\{\frac{\sqrt{2}}{2} \cdot e^{-2t} \cdot \sin(\sqrt{2}t)\right\}(s)$$

$$\mathcal{L}^{-1} = \frac{\sqrt{2}}{2}e^{-2t} \cdot \sin(\sqrt{2}t)$$

f)

$$F(s) = \frac{3s-1}{s^2-4s+13} = \frac{3s-1}{s^2-4s+4+9} = \frac{3s-1}{(s-2)^2+9} = \frac{3(s-2+2)-1}{(s-2)^2+9}$$

$$= \frac{3(s-2)+6-1}{(s-2)^2+9} = 3\frac{(s-2)}{(s-2)^2+3^2} + 5 \cdot \frac{1}{(s-2)^2+3^2}$$

$$= 3\frac{(s-2)}{(s-2)^2+3^2} + \frac{5}{3} \times \frac{3}{(s-2)^2+3^2} = 3\mathcal{L}\{e^{2t} \cdot \cos(3t)\}(s) + \frac{5}{3}\mathcal{L}\{e^{2t}\sin(3t)\}(s)$$

$$= \mathcal{L}\left\{e^{2t}\left(3\cos(3t) + \frac{5}{3}\sin(3t)\right)\right\}(s)$$

$$\mathcal{L}^{-1}(F(s)) = \left(3\cos(3t) + \frac{5}{3}\sin(3t)\right)e^{2t}$$

g)

$$F(s) = \frac{4s+e^{-s}}{s^2+s-2} = \frac{4s}{s^2+s-2} + \frac{e^{-s}}{s^2+s-2} = \frac{4s}{(s-1)(s+2)} + \frac{e^{-s}}{(s-1)(s+2)} = ⊛$$

$$\frac{4s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{8}{3} \times \frac{1}{s-1} + \frac{4}{3} \times \frac{1}{s+2}$$

$$\begin{cases} A+B=4 \\ -A+2B=0 \end{cases} \Leftrightarrow \begin{cases} A=\frac{8}{3} \\ B=\frac{4}{3} \end{cases}$$

$$⊛ = \frac{8}{3}\times\left(\frac{1}{s-1}\right)+\frac{4}{3}\times\left(\frac{1}{s+2}\right)+e^{-s}\left(\frac{1}{(s-1)(s+2)}\right)$$

$$= \frac{8}{3}\,\mathcal{L}\{e^{1t}\}(s)+\frac{4}{3}\times\mathcal{L}\{e^{-2t}\}+e^{-s}\,\mathcal{L}\{\tfrac{1}{3}(e^{t}-e^{-2t})\}(s)$$

$$= \mathcal{L}\{\tfrac{8}{3}e^{t}+\tfrac{4}{3}e^{-2t}+\tfrac{1}{3}(e^{t-1}-e^{-2(t-1)})\,H_1(t)\}(s)$$

$$f(t) = \frac{8}{3}e^{t}+\frac{4}{3}e^{-2t}+\frac{1}{3}H_1(t)\cdot(e^{t-1}-e^{-2t+2})$$

b)

$$F(s) = \frac{s}{(s^2+4)^2} = \frac{1}{2}\times\frac{2s}{(s^2+4)^2} = -\frac{1}{2}\times\frac{-2s}{(s^2+4)^2} = -\frac{1}{2}\,\frac{-(s^2+4)'}{(s^2+4)^2} = -\frac{1}{2}\times\left(\frac{1}{(s^2+4)}\right)'$$

$$= (-1)^1\times\frac{1}{2}\times\left(\frac{1}{s^2+4}\right)' = \frac{1}{2}t\times\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)(s) = \frac{1}{2}\times t\times\frac{1}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)(s)$$

$$= \frac{1}{4}t\sin(2t)$$

3

$$F(s) = \int_0^{+\infty} e^{-st} f(t)$$

a)

$$\int_0^{+\infty} t^{10}e^{-2t}\,dt = F(2) \text{ si } F(s) = \int_0^{+\infty} t^{10}e^{-2t}\,dt = \mathcal{L}\{t^{10}\}(s) = \frac{10!}{s^{11}}$$

$$F(2) = \mathcal{L}\{t^{10}\}(2) = \frac{10!}{2^{11}}$$

b) $\int_0^{+\infty} e^{-3t} t \sin t \, dt = \mathcal{L}\{t\sin t\}(3) = \frac{2\times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$

$\mathcal{L}\{t\sin t\}(s) = (-1)^1 \times \left(\frac{1}{s^2+1}\right)' = -1 \times \frac{-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$

4

$f(0)=2$

$f'(t)+2f(t)=e^t$

$\mathcal{L}\{f'(t)+2f(t)\}(s)=\mathcal{L}\{e^t\}(s)$

$\mathcal{L}\{f'(t)\}(s)+2\mathcal{L}\{f(t)\}(s)=\frac{1}{s-1}, \quad s>1$

$s^1 \times \mathcal{L}\{f(t)\}(s) - s^0 \times f(0) + 2\mathcal{L}\{f(t)\}(s) = \frac{1}{s-1}$

$\mathcal{L}\{f(t)\}(s)(s+2) = \frac{1}{s-1}+2$

$\mathcal{L}\{f(t)\}(s) = \frac{1}{(s-1)(s+2)} + \frac{2}{s+2}$

$= \frac{1}{3} \times \left(\frac{1}{s-1} - \frac{1}{s+2} + \frac{6}{s+2}\right) = \frac{1}{3}\left(\frac{1}{s-1}+\frac{5}{s+2}\right)$

$\frac{1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{1}{3}\times\frac{1}{s-1} - \frac{1}{3}\times\frac{1}{s+2}$

$\begin{cases} A+B=0 \\ 2A-B=1 \end{cases} \Leftrightarrow \begin{cases} B=-\frac{1}{3} \\ A=\frac{1}{3} \end{cases}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-1}+\frac{5}{s+2}\right)\right\}(s)$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(s) + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(s)$

$= \frac{1}{3}e^t + \frac{5}{3}e^{-2t}$

5

a) $\mathcal{L}\{(t-2+e^{-2t})\cos(4t)\}(s) = \mathcal{L}\{t\cos(4t)\}(s) - 2\mathcal{L}\{\cos(4t)\} + \mathcal{L}\{e^{-2t}\cos(4t)\}(s)$

$= (-1)^1\left(\frac{s}{s^2+4^2}\right)' - 2\frac{s}{s^2+4^2} + \frac{(s+2)}{(s+2)^2+4^2}, \quad s>0$

$= -\left(\frac{(s^2+4^2)-s(2s)}{(s^2+16)^2}\right) + \frac{-2s}{s^2+16} + \frac{s+2}{s^2+2s+20}, \quad s>0$

$= -\left(\frac{s^2+16-2s^2}{(s^2+16)^2}\right) - \frac{2s}{s^2+16} + \frac{s+2}{s^2+2s+20}, \quad s>0$

$= \frac{s^2-16}{(s^2+16)^2} - \frac{2s}{s^2+16} + \frac{s+2}{s^2+2s+20}$

b)

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-4s+6}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-4s+4+2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s-2)^2+2}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-2+2)-1}{(s-2)^2+2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{2(s-2)+3}{(s-2)^2+2}\right\}(t)$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+(\sqrt{2})^2}\right\}(t) + \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-2)^2+(\sqrt{2})^2}\right\}(t)$$

$$= 2e^{2t}\cdot\cos(\sqrt{2}\,t) + \frac{3}{\sqrt{2}}e^{2t}\cdot\sin(\sqrt{2}t)$$

c)

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+2s+5)}\right\}(t) =$$

$$\frac{2s}{(s-1)(s^2+2s+5)} = \frac{A^{(s^2+2s+5)}}{s-1} + \frac{Bs^{(s-1)}+C^{(s-1)}}{s^2+2s+5} = \frac{1}{4}\times\frac{1}{s-1} + \frac{\cancel{1}}{4}\left(\frac{s-s}{s^2+2s+5}\right)$$

$$\begin{cases} A+B=0 \\ 2A-B+C=\cancel{0}\,2 \\ 5A-C=0 \end{cases} \Leftrightarrow \begin{cases} \text{—} \\ 3A+C=2 \\ 5A-C=0 \end{cases} \Leftrightarrow \begin{cases} B=-A \\ C=2-3A \\ 8A=2 \end{cases}$$

$$\Leftrightarrow \begin{cases} B=-\frac{1}{4} \\ C=\frac{5}{4} \\ A=\frac{1}{4} \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+2s+5)}\right\}(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1-1-5}{s^2+2s+1+4}\right\}(t)$$

$$= \frac{1}{4}e^t - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}(s) + \frac{1}{4}\times 3\,\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}(t)$$

$$= \frac{1}{4}e^t - \frac{1}{4}e^t\cos(2t) + \frac{3}{4}e^t\sin(2t)$$

$$= \frac{1}{4}e^t\left(1-\cancel{1}\cos(2t)+3\sin(2t)\right)$$

6

$t^m * t^n$

$$\mathcal{L}\{t^m * t^n\}(s) = \mathcal{L}\{t^m\}(s)\times\mathcal{L}\{t^n\}(s)$$

$$= \frac{m!}{s^{m+1}}\times\frac{n!}{s^{n+1}} = \frac{m!\,n!}{s^{m+1+n+1}} = \frac{m!\,n!}{s^{m+n+2}}$$

$$\mathcal{L}^{-1}\left\{\frac{m!\,n!}{s^{m+n+2}}\right\}(s) = m!\,n!\times\mathcal{L}^{-1}\left\{\frac{1}{s^{n+m+2}}\right\}(s) = \frac{m!\,n!}{(m+n+1)!}\times\mathcal{L}^{-1}\left\{\frac{(m+n+1)!}{s^{n+m+2}}\right\}(s)$$

$$= \frac{m!\,n!}{(n+m+1)!}\times t^{m+n+1}$$

7

$$y'(t) = 1-\sin t - \int_0^t y(\tau)\,d\tau$$

$$\mathcal{L}\{y'(t)\}(s) = \mathcal{L}\{t^0\} - \mathcal{L}\{\sin t\} - \mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\}$$

$$s\,\mathcal{L}\{f(t)\}(s) - 0 = \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$s\,\mathcal{L}\{f(t)\} + \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{1}{s^2+1}$$

$$\mathcal{L}\{f(t)\}\left(s + \frac{1}{s}\right) = \frac{1}{s} - \frac{1}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s\left(s+\frac{1}{s}\right)} - \frac{1}{(s^2+1)\left(s+\frac{1}{s}\right)}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} - \frac{1}{s^3+s+s^2+\frac{1}{s}} = \frac{1}{s^2+1} - \frac{s}{s^4+2s^2+1}$$

$$= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \Leftrightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right\}(t)$$

$$\int \frac{s}{(s^2+1)^2}\,ds = \frac{1}{2}\int \frac{2s}{(s^2+1)^2}\,ds = \frac{1}{2}\int \frac{(s^2+1)'}{(s^2+1)^2}\,ds$$

$$= -\frac{1}{2} \times \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) - \frac{1}{2}\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+1}\right)'\right\}(t)$$

$$= \sin(t) - \frac{1}{2}t\sin(t)$$

8

a) $3x' - x = \cos t$

$$3\mathcal{L}\{x'\}(s) - \mathcal{L}\{x\} = \mathcal{L}\{\cos t\}$$

$$3\left(s\,\mathcal{L}\{x\}(s) + 1 - \mathcal{L}\{x\}\right) = \frac{s}{s^2+1}$$

$$s\,\mathcal{L}\{x\}(s) - \mathcal{L}\{x\} = \frac{s}{3(s^2+1)} - 3$$

$$\mathcal{L}\{x\}(s)(s-1) = \frac{1}{3} \times \frac{s}{s^2+1} - 3$$

$$\mathcal{L}\{x\} = \frac{1}{3} \times \frac{s}{(s^2+1)(s-1)} - \frac{3}{s-1}$$

$$x = \mathcal{L}^{-1}\left\{\frac{1}{3} \times \frac{s}{(s^2+1)(s-1)} - \frac{3}{s-1}\right\}(s)$$

$$\frac{s}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1}^{(s-1)} + \frac{C}{s-1}^{(s^2+1)} = \frac{1}{2} \times \frac{1-s}{s^2+1} + \frac{1}{2} \times \frac{1}{s-1}$$

$$\begin{cases} A + C = 0 \\ -A + B = 1 \\ -B + C = 0 \end{cases} \Leftrightarrow \begin{cases} A = -C \\ C + B = 1 \\ C - B = 0 \end{cases} \Leftrightarrow \begin{cases} A = -\frac{1}{2} \\ C = \frac{1}{2} \\ B = +\frac{1}{2} \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{2} \times \frac{1}{s^2+1} - \frac{1}{2} \times \frac{s}{s^2+1} + \frac{1}{2} \times \frac{1}{s-1}\right) - \frac{3}{s-1}\right\}(s)$$

$$= \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{9}{s-1}\right\}(t)$$

a)

$$3x' - x = \cos t$$

$$\mathcal{L}\{3x'\}(s) - \mathcal{L}\{x\}(s) = \mathcal{L}\{\cos t\}(s)$$

$$3\mathcal{L}\{x'\}(s) - \mathcal{L}\{x\}(s) = \mathcal{L}\{\cos t\}(s)$$

$$3(s\mathcal{L}\{x\}(s) + 1) - \mathcal{L}\{x\}(s) = \frac{s}{s^2+1}, s>0$$

$$3s\mathcal{L}\{x\}(s) + 3 - \mathcal{L}\{x\}(s) = \frac{s}{s^2+1}, s>0$$

$$\mathcal{L}\{x\}(s)(3s-1) = \frac{s}{s^2+1} - 3, \ s>0$$

$$\mathcal{L}\{x\}(s) = \frac{s}{(s^2+1)(3s-1)} - \frac{3}{3s-1} \ s>0$$

$$x = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(3s-1)} - \frac{3}{3s-1}\right\}, s>0$$

$$\frac{s}{(s^2+1)(3s-1)} = \frac{A}{3s-1}^{(s^2+1)} + \frac{B+Cs}{s^2+1}^{(3s-1)} = \frac{3}{10} \times \frac{1}{3s-1} + \frac{1}{10} \times \frac{3-s}{s^2+1}$$

$$\begin{cases} A + 3C = 0 \\ 3B - C = 1 \\ A - B = 0 \end{cases} \Leftrightarrow \begin{cases} \text{—} \\ 3B - C = 1 \\ -B - 3C = 0 \end{cases} \Leftrightarrow \begin{cases} -9C - C = 1 \\ B = -3C \end{cases} \Leftrightarrow \begin{cases} A = \frac{3}{10} \\ C = -\frac{1}{10} \\ B = \frac{3}{10} \end{cases}$$

$$x = \mathcal{L}^{-1}\left\{\frac{3}{10} \times \frac{1}{3s-1} + \frac{1}{10} \times \frac{3}{s^2+1} - \frac{1}{10} \times \frac{s}{s^2+1} - \frac{3}{3s-1}\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{-\frac{9}{10} \times \frac{3}{3s-1}\right\} + \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= -\frac{9}{10} \times \frac{1}{\frac{1}{3}} \times \mathcal{L}^{-1}\left\{\frac{1}{\frac{s}{\frac{1}{3}} - 1}\right\}(t) + \frac{3}{10}\sin(t) - \frac{1}{10}\cos(t)$$

$$= -\frac{9}{10} e^{\frac{t}{3}} + \frac{3}{10}\sin t - \frac{1}{10}\cos t$$

b)

$$y'' + 36y = 0 \qquad y(0) = -1 \qquad y'(0) = -2$$

$$\mathcal{L}\{y'' + 36y\}(s) = 0$$

$$\mathcal{L}\{y''\}(s) + 36\mathcal{L}\{y\}(s) = 0$$

$$s^2\mathcal{L}\{y\} + 2s + 1 + 36\mathcal{L}\{y\}(s) = 0$$

$$\mathcal{L}\{y\}(s^2+36) = -(2s+1)$$

$$\mathcal{L}\{y\} = -2 \times \frac{s}{s^2+36} - 1 \times \frac{1}{s^2+36}$$

$$y = \mathcal{L}^{-1}\left\{-2 \times \frac{s}{s^2+36} - \frac{1}{6} \times \frac{6}{s^2+36}\right\}(s) = -2\cos(6t) - \frac{1}{6}\sin(6t)$$

c)

$y'' + 2y' + 3y = 3t \qquad y(0) = 0 \quad y'(0) = 1$

$\mathcal{L}\{y'' + 2y' + 3y\}(s) = \mathcal{L}\{3t\}(s)$

$\mathcal{L}\{y''\} + 2\{y'\} + 3\mathcal{L}\{y\} = 3\mathcal{L}\{t\}(s)$

$s^2\mathcal{L}\{y\} - s + 2s\mathcal{L}\{y\} + 3\mathcal{L}\{y\} = \frac{3}{s^2}$

$\mathcal{L}\{y\}(s^2 + 2s + 3) = \frac{3}{s^2} - s$

$\mathcal{L}\{y\} = \frac{3}{s^2(s^2+2s+3)} - \frac{s}{s^2+2s+3} = \frac{3-s^3}{s^2(s^2+2s+3)}$

$\frac{3-s^3}{s^2(s^2+2s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C+Ds}{s^2+2s+3} = -\frac{2}{3} \times \frac{1}{s} + \frac{1}{s^2} + \frac{1}{3} \times \frac{2-s}{s^2+2s+3}$

$\begin{cases} A + D = -1 \\ 2A + B + C = 0 \\ 3A + 2B = 0 \\ 3B = 3 \end{cases} \Leftrightarrow \begin{cases} D = -\frac{1}{3} \\ C = \frac{2}{3} \\ A = -\frac{2}{3} \\ B = 1 \end{cases}$

$y = \mathcal{L}^{-1}\left\{\frac{3-s^3}{s^2(s^2+2s+3)}\right\} = -\frac{2}{3} \times \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^{1+1}}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s+1-1-2}{(s+1)^2+2}\right\}$

$= -\frac{2}{3} \times t^0 + t^1 - \frac{1}{3}\left(\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\} - \sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}\right)$

$= -\frac{2}{3} + t - \frac{1}{3} e^{-t}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{3} e^{-t} \sin(\sqrt{2}t)$

d) $y''' + 2y'' + y' = x \qquad y''(0) = 1 \quad y'(0) = 0 \quad y(0) = 0$

$\mathcal{L}\{y''' + 2y'' + y'\}(s) = \mathcal{L}\{x\}(s)$

$\mathcal{L}\{y'''\}(s) + 2\{y''\} + \mathcal{L}\{y'\} = \frac{1}{s^2}$

$s^3\mathcal{L}\{y\}(s) - 1 + 2s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} = \frac{1}{s^2}$

$\mathcal{L}\{y\}(s^3 + 2s^2 + s) = \frac{1}{s^2} + 1$

$\mathcal{L}\{y\} = \frac{1+s^2}{s^2(s^3+2s^2+s)} = \frac{s^2+1}{(s^3)(s+1)^2}$

$\frac{s^4+1}{s^3(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{(s+1)^2} = \frac{4}{s} - \frac{2}{s^2} + \frac{1}{s^3} - \frac{4}{s+1} - \frac{2}{(s+1)^2}$

$\begin{cases} A + D = 0 \\ 2A + B + D + E = 0 \\ A + 2B + C = 1 \\ B + 2C = 0 \\ C = 1 \end{cases} \Leftrightarrow \begin{cases} D = -4 \\ E = -2 \\ A = 4 \\ B = -2 \\ C = 1 \end{cases}$

$$y = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{2}{s^2} + \frac{1}{s^3} - \frac{4}{s+1} - \frac{2}{(s+1)^2}\right\}(x)$$

$$= 4 - 2x + \frac{1}{2}x^2 - 4e^{-x}\cdot x^0 - 2e^{-x}\cdot x$$

$$= 4 - 2x + \frac{1}{2}x^2 - 4e^{-x} - 2e^{-x}x$$

e)

$$y'' + y' = \frac{e^{-t}}{2}$$

$$\mathcal{L}\{y'' + y'\} = \mathcal{L}\{\frac{e^{-t}}{2}\}$$

$$s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} = \frac{1}{2} \times \frac{1}{s+1}$$

$$\mathcal{L}\{y\} = \frac{1}{2} \times \frac{1}{(s+1)(s^2+s)} = \frac{1}{2} \times \frac{1}{s(s+1)^2} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$\begin{cases} A+B=0 \\ 2A+B+C=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} B=-1 \\ C=-1 \\ A=1 \end{cases}$$

$$y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= \frac{1}{2}t - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-t}t$$

9

$$y'' + y = t^2 + 1$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t^2+1\}$$

$$s^2\mathcal{L}\{y\} - 2\pi s - \pi^2 + \mathcal{L}\{y\} = \mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$

$$\mathcal{L}\{y\}(s^2+1) = \frac{2}{s^3} + \frac{1}{s} + 2\pi s + \pi^2$$

$$\mathcal{L}\{y\} = \frac{2}{s^3(s^2+1)} + \frac{1}{s(s^2+1)} + \frac{2\pi s + \pi^2}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{2 + s^2 + 2\pi s^4 + \pi^2 s^3}{s^3(s^2+1)}$$

$$\frac{2 + s^2 + s^3\pi^2 + 2\pi s^4}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D + Es}{s^2+1} = -\frac{1}{s} + \frac{2}{s^3} + \frac{\pi^2 + 2\pi s + s}{s^2+1}$$

$$\begin{cases} A+E=2\pi \\ B+D=\pi^2 \\ A+C=1 \\ B=0 \\ C=2 \end{cases} \Leftrightarrow \begin{cases} E=2\pi+1 \\ D=\pi^2 \\ A=-1 \\ B=0 \\ C=2 \end{cases}$$