Folha 4

1

a) $y = \sin x - 1 + e^{-\sin x}$

$y' = \cos x + (-\sin x)' e^{-\sin x}$
$= \cos x - \cos x\, e^{-\sin x}$

$y' + y \cos x = \cos x - \cos x\, e^{-\sin x} + (\sin x - 1 + e^{-\sin x}) \cos x$

$= \cancel{\cos x} - \cancel{\cos x\, e^{-\sin x}} + \cos x \sin x - \cancel{\cos x} + \cancel{\cos x\, e^{-\sin x}}$
$= \cos x \sin x$
$= \frac{1}{2} \cdot 2 \cos x \sin x = \frac{1}{2} \sin(2x)$

b)

$z = \cos x$

$z'' = ((\cos x)')' = (-\sin x)' = -\cos x$

$z'' + z = -\cos x + \cos x = 0$ ✓

c) $y = \cos^2 x$

$y'' = ((\cos^2 x)')' = (-2\cos x \sin x)' = (-\sin(2x))' = -2\cos(2x)$

$y'' + y = -2\cos(2x) + \cos^2 x \neq 0$ ✗

d)

$y = Cx - C^2, \ C \in \mathbb{R}$

$y' = C \ , \ C \in \mathbb{R}$

$(y')^2 - x \cdot y' + y = C^2 - x \cdot C + Cx - C^2 = 0$

2

a) $y = Cx$

1 constante, logo equação diferencial de 1º grau

$y' = C$

$y'x = Cx$

$y'x = y \Leftrightarrow y'x - y = 0$

b) $y = Ax + B$

2 constantes $\Rightarrow$ equação diferencial de 2º grau

$y' = (Ax+B)' = A$

$y'' = 0$

c) $y = e^{Cx}, C \in \mathbb{R}$

1 constante $\Rightarrow$ eq. diferencial de 1º grau

$y' = (e^{Cx})' = Ce^{Cx}$

$y' = Cy \Leftrightarrow \frac{y'}{y} = C$

$\ln y = \ln(e^{Cx}) = Cx = \frac{y'}{y} x$

$\ln y = \frac{y'}{y} x \Leftrightarrow y \ln y = y'x \Leftrightarrow y'x - y\ln y = 0$

3.

$y = A\sin(x+B)$

$y' = A\cos(x+B)$

$y'' = -A\sin(x+B)$

$y''' = -A\cos(x+B)$

$y''' - y' = -A\cos(x+B) + A\cos(x+B) = 0$

$y''' - y' = 0$

4.

a) $y'' - \sin x = 0$

$y'' = \sin x$

$y' = \int \sin x \, dx = -\cos x + C, C \in \mathbb{R}$

$y' = -\cos x + C, C \in \mathbb{R}$

$y = \int (-\cos x + C) dx = -\int \cos x \, dx + \int C \, dx$

$= -\sin x + Cx + C_2, \; C, C_2 \in \mathbb{R}$

b) $\varphi(x) = 2x - \sin x$

Solução geral $-\sin x + Cx + C_2$

$\varphi(0) = -\sin 0 + 0 + C_2 \Leftrightarrow C_2 = \varphi(0) \Leftrightarrow C_2 = 0$

$\varphi'(0) = 1 \Leftrightarrow -\cos 0 + C = 1 \Leftrightarrow -1 + C = 1 \Leftrightarrow C = 2$

$\varphi(x) = -\sin x + 2x + 0$

$= 2x - \sin x$

5

a) $y' - \dfrac{1}{(1+x^2)\,\text{arctg}\,x} = 0$

$y' = \dfrac{1}{(1+x^2)\,\text{arctg}\,x}$

$\int y' \, dx = \int \dfrac{1}{(1+x^2)\,\text{arctg}\,x} dx$

$y = \int (\text{arctg}\,x)' \times \dfrac{1}{\text{arctg}\,x} dx$

$y = \ln|\text{arctg}(x)|$

b) $y' - \sqrt{1-x^2} = 0 \Leftrightarrow y' = \sqrt{1-x^2} \Leftrightarrow \int y' \, dx = \int \sqrt{1-x^2} \, dx$

Considerando $x = \sin t$) $\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t$

$t = \arcsin x$

$dx = (\sin t)' dt = \cos t \, dt$

$$y = \int \cos t \cos t \, dt = \int \cos^2 t \, dt = \int \frac{1+\cos(2t)}{2} \, dt = \int \frac{1}{2} dt + \frac{1}{2}\int \cos(2t) \, dt$$

$$= \frac{t}{2} + \frac{1}{2} \times \frac{1}{2} \int 2 \cdot \cos(2t) \, dt = \frac{t}{2} + \frac{1}{4} \sin(2t) + C, \ C \in \mathbb{R}$$

$$= \frac{\arcsin x}{2} + \frac{2 \sin(\arcsin x) \cos(\arcsin x)}{4}$$

$$= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}$$

c) $y' - \frac{x^4+x^2+1}{x^2+1} = 0 \Leftrightarrow y' - \left(x^2 + \frac{1}{x^2+1}\right) = 0 \Leftrightarrow y' = x^2 + \frac{1}{x^2+1}$

$$\begin{array}{r|l} x^4 + x^2 + 1 & x^2+1 \\ \hline -x^4 - x^2 \phantom{+1} & x^2 \\ 1 & \end{array}$$

$\Leftrightarrow \int y' dx = \int \left(x^2 + \frac{1}{x^2+1}\right) dx$

$\Leftrightarrow y = \int x^2 dx + \int \frac{1}{x^2+1} dx$

$\Leftrightarrow y = \frac{x^3}{3} + \operatorname{arctg}(x) + C, \ C \in \mathbb{R}$

6

a)

$x + yy' = 0$

$yy' = -x$

$\int yy' dx = \int -x \, dx$

$\int y \, dy = -\int x \, dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + C, \ C \in \mathbb{R}$

$\frac{y^2}{2} + \frac{x^2}{2} = C, \ C \in \mathbb{R}$

$y^2 + x^2 = C_2, \ C_2 \in \mathbb{R} \quad (C_2 = 2C)$

b) $xy' - y = 0 \Leftrightarrow xy' = y \Leftrightarrow \frac{1}{y} y' = \frac{1}{x}$

$\int \frac{1}{y} y' dx = \int \frac{1}{x} dx \Leftrightarrow \ln|y| = \ln|x| + C, \ C \in \mathbb{R}$

$\Leftrightarrow |y| = e^{\ln|x| + C}, \ C \in \mathbb{R}$

$\Leftrightarrow |y| = |x| \cdot e^C, \ C \in \mathbb{R}$

$\Leftrightarrow y = x \cdot C_2, \ C' \in \mathbb{R}$

c) $(t^2 - xt^2)\frac{dx}{dt} + x^2 = -tx^2$

$(t^2 - xt^2)\, x' = -tx^2 - x^2$

$t^2(1-x)\, x' = x^2(-1-t)$

$\frac{(1-x)x'}{x^2} = -\frac{1+t}{t^2}$

$\int \frac{1-x}{x^2} dx = -\int \frac{1+t}{t^2} dt \Leftrightarrow \int \frac{1}{x^2} dx - \int \frac{1}{x} dx = -\left(\int \frac{1}{t^2} dt + \int \frac{1}{t} dt\right.$

$$\frac{x^{-1}}{-1} - \ln|x| = -\left(\frac{t^{-1}}{-1} + \ln|t|\right) + C,\ C \in \mathbb{R}$$

$$-x^{-1} - \ln|x| = t^{-1} - \ln|t| + C,\ C \in \mathbb{R}$$

$$\ln|x| - \ln|t| = -\frac{1}{x} - \frac{1}{t} + C$$

$$\ln\left|\frac{x}{t}\right| = -\frac{1}{x} - \frac{1}{t} + C$$

$$\left|\frac{x}{t}\right| = e^{-\frac{1}{x} - \frac{1}{t} + C}$$

$$\frac{x}{t} = e^{-\frac{1}{x} - \frac{1}{t}} \cdot C,\ C \in \mathbb{R}$$

d) $(x^2-1)y' + 2xy^2 = 0$

$$(x^2-1)y' = -2xy^2$$

$$\frac{1}{y^2}y' = \frac{-2x}{x^2-1}$$

$$\int \frac{1}{y^2}\,dy = \int \frac{-2x}{x^2-1}\,dx$$

$$\frac{y^{-1}}{-1} = -\int \frac{(x^2-1)'}{x^2-1}\,dx$$

$$-y^{-1} = -\ln(x^2-1) + C$$

$$y^{-1} = \ln|x^2-1| + C_2$$

$$y = \frac{1}{\ln|x^2-1| + C}$$

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a) $xy' + y = y^2$

$$xy' = y^2 - y$$

$$\frac{1}{y^2-y}y' = \frac{1}{x}$$

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{x}\,dx \qquad \frac{1}{y(y-1)} = \frac{A^{(y-1)}}{y} + \frac{B^{(y)}}{y-1} = -\frac{1}{y} + \frac{1}{y-1}$$

$$\int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = \int \frac{1}{x}\,dx \qquad \begin{cases} A+B=0 \\ -A=1 \end{cases} \Leftrightarrow \begin{cases} B=1 \\ A=-1 \end{cases}$$

$$-\ln|y| + \ln|y-1| = \ln|x| + C$$

$$e^{-\ln|y| + \ln|y-1|} = e^{\ln|x| + C},\ C \in \mathbb{R}$$

$$\frac{1}{|y|} * |y-1| = |x| \cdot C_2,\ C_2 \in \mathbb{R}$$

$$\left|\frac{y-1}{y}\right| = |x| \cdot C_2,\ C_2 \in \mathbb{R}$$

$$\frac{y-1}{y} = C_3 \cdot x,\ C_3 \in \mathbb{R}$$

$$\to 1 - \frac{1}{y} = C_3 \cdot x,\ C_3 \in \mathbb{R}$$

$$\frac{1}{y} = 1 - C_3x,\ C_3 \in \mathbb{R}$$

$$y = \frac{1}{1-C_3x},\ C_3 \in \mathbb{R}$$

$$\frac{1}{2} = \frac{1}{1-C_3 \cdot 1} \Leftrightarrow 1 - C_3 = 2 \Leftrightarrow C_3 = -1$$

b) $xy + x + y'\sqrt{4+x^2} = 0$

$y'\sqrt{4+x^2} = -x(1+y)$

$\frac{1}{1+y} y' = -\frac{x}{\sqrt{4+x^2}}$

$\int \frac{1}{1+y} dy = -\int \frac{x}{\sqrt{4+x^2}} dx$

$\ln|1+y| = -\frac{1}{2}\int \frac{2x}{\sqrt{4+x^2}} dx = -\frac{1}{2}\int \frac{(4+x^2)'}{\sqrt{4+x^2}} dx = -\frac{1}{2}\int (4+x^2)'(4+x^2)^{-\frac{1}{2}} dx$

$= -\frac{1}{2} \times \frac{(4+x^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{4+x^2} + C$

$\ln|1+y| = -\sqrt{4+x^2} + C, \; C \in \mathbb{R}$

$|1+y| = e^{-\sqrt{4+x^2}} \cdot e^C, C \in \mathbb{R}$

$1+y = C_2 \cdot e^{-\sqrt{4+x^2}}, C_2 \in \mathbb{R}$

$y = C_2 \cdot e^{-\sqrt{4+x^2}} - 1, \; C_2 \in \mathbb{R}$

$y(0) = 1$

$C_2 \cdot e^{-\sqrt{4+0^2}} - 1 = 1$

$C_2 e^{-\sqrt{4}} = 2$

$C_2 = 2e^2$ $\qquad y = 2e^2 \cdot e^{-\sqrt{4+x^2}} - 1$

c) $(1+x^3)y' = x^2 y$

$\frac{1}{y} y' = \frac{x^2}{1+x^3}$

$\int \frac{1}{y} dy = \int \frac{x^2}{1+x^3} dx$

$\ln|y| = \frac{1}{3}\ln|1+x^3| + C$

$y = e^{\frac{1}{3}\ln|1+x^3| + C}$

$2 = e^{\frac{1}{3}\ln|2| + C}$

$\ln 2 = \frac{1}{3}\ln|2| + C$

$3 = C$

8

a) $(x^2+y^2)y' = xy$

$\Leftrightarrow y' = \frac{xy}{x^2+y^2}$

Considerando $f(x,y) = y'$

$f(\lambda x, \lambda y) = \frac{\lambda x \lambda y}{(\lambda x)^2 + (\lambda y)^2} = \frac{\lambda^2 xy}{\lambda^2 x^2 + \lambda^2 y^2} = \frac{\lambda^2 xy}{\lambda^2(x^2+y^2)} = \frac{x\cdot}{x^2+\cdot}$

É um equação diferencial homogénea

$z = \frac{y}{x} \Leftrightarrow y = xz$

$y' = \frac{xy}{x^2+y^2} \Leftrightarrow \frac{x \cdot xz}{x^2+(xz)^2} = \frac{x^2 z}{x^2+x^2z^2} \Leftrightarrow \frac{x^2 z}{x^2(1+z^2)} = \frac{z}{1+z^2}$

$y' = (xz)' = z + xz'$

$z + xz' = \frac{z}{1+z^2}$

$xz' = \frac{z}{1+z^2} - z$

$xz' = \frac{z - z^2 - z^3}{1+z^2}$

$\frac{1+z}{z-z^2-z^3} z' = \frac{1}{x}$

$\frac{xy}{x^2+y^2} = \frac{x \cdot x \cdot z}{x^2+(x^2 \cdot z^2)} = \frac{x^2 z}{x^2(1+z^2)} = \frac{z}{1+z^2}$

$z = \frac{y}{x} \Leftrightarrow y = xz$

$y' = (xz)' = z + xz'$

$z + xz = \frac{z}{1+z^2}$

$xz = \frac{z}{1+z^2} - z \Leftrightarrow xz = \frac{z - z - z^3}{1+z^2}$

$\Leftrightarrow xz = -\frac{z^3}{1+z^2} \Leftrightarrow \frac{1+z^2}{z^3} z' = -\frac{1}{x}$

$\Leftrightarrow \int \frac{1}{z^3} dz + \int \frac{1}{z} dz = -\int \frac{1}{x} dx$

$\Leftrightarrow \frac{z^{-2}}{-2} + \ln|z| = -\ln|x| + C, \ C \in \mathbb{R}$

$\Leftrightarrow \frac{\left(\frac{y}{x}\right)^{-2}}{-2} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C, \ C \in \mathbb{R}$

$\Leftrightarrow -\frac{x^2}{2y^2} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C$

$\Leftrightarrow \ln|y| - \cancel{\ln|x|} - \frac{x^2}{2y^2} = \cancel{-\ln|x|} + C$

$\Leftrightarrow \ln|y| - \frac{x^2}{2y^2} = C, \ C \in \mathbb{R}$

b) $y'\left(1 - \ln\frac{y}{x}\right) = \frac{y}{x} \Leftrightarrow y' = \frac{\frac{y}{x}}{1-\ln\left(\frac{y}{x}\right)}$

$f(x,y) = y'$

$f(\lambda x, \lambda y) = \frac{\frac{\lambda y}{\lambda x}}{1 - \ln\left(\frac{\lambda y}{\lambda x}\right)} = \frac{\frac{y}{x}}{1-\ln\left(\frac{y}{x}\right)} = f(x,y)$

$z = \frac{y}{x} \Leftrightarrow y = zx$

$y' = z + xz'$

$z + xz' = \frac{z}{1+\ln(z)}$

$xz' = \frac{z - z - z\ln(z)}{1+\ln(z)}$

$xz' = -\frac{z\ln(z)}{1+\ln(z)}$

$\frac{1+\ln(z)}{z\ln(z)} z' = \cancel{x} - \frac{1}{x}$

$\int \left(\frac{1+\ln(z)}{z\ln(z)}\right) dz = \int -\frac{1}{x} dx$

$\int \frac{1}{z\ln z} dz + \int \frac{\ln(z)}{z\ln(z)} dz = -\int \frac{1}{x} dx$

$\int \frac{(\ln z)'}{\ln z} dz + \int \frac{1}{z} dz = -\ln|x| + C, C \in \mathbb{R}$

$\ln|\ln(z)| + \ln(z) = -\ln|x| + C, C \in \mathbb{R}$

$\ln|z\ln(z)| = \ln\left|\frac{1}{x}\right| + C, C \in \mathbb{R}$

$z\ln(z) = \frac{1}{x} \cdot C, \quad C \in \mathbb{R}$

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a) $f(x,y) = y' = \frac{y}{x}(1 + \ln|y| - \ln(x))$

$= \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right)$

$f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x}\left(1 + \ln\left(\frac{\lambda y}{\lambda x}\right)\right)$

$= \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right) = f(x,y)$

b) $z = \frac{y}{x} \Leftrightarrow y = zx$

$y' = z + z'x$

$z + z'x = z(1 + \ln(z))$

$z'x = z(1 + \ln(z)) - z$

$z'x = z\ln(z)$

$\frac{1}{z\ln(z)} z' = -\frac{1}{x}$

$\int \frac{1}{z\ln(z)} dz = -\int \frac{1}{x} dx \Leftrightarrow \int \frac{(\ln z)'}{\ln(z)} dz = -\ln|x| + C, C \in \mathbb{R}$

$\Leftrightarrow \ln|\ln(z)| = -\ln|x| + C$

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a) $(2x+\sin y)dx + x\cos y\, dy = 0$

$$\begin{cases} \frac{dF}{dx} = 2x + \sin y \\ \frac{dF}{dy} = x\cos y \end{cases}$$

$F = \int (2x+\sin y)dx = x^2 + x\sin y + \varphi(y)$

$\frac{dF}{dy} = x\cos y + \varphi'(y) = x\cos y$

$\Rightarrow x\cos y + \varphi'(y) = x\cos y$

$\varphi'(y) = 0$

$\int \varphi'(y)dy = \int 0\, dy$

$\varphi'(y) = C,\ C \in \mathbb{R}$

$F = x^2 + x\sin y + C, C\in\mathbb{R}$

b) $(2xy - x - e^y)dx = (xe^y + y - x^2)dy$

$(2xy - x - e^y)dx - (xe^y + y - x^2)dy = 0$

$(2xy - x - e^y)dx + (x^2 - xe^y - y)dy = 0$

$\frac{dF}{dx} = 2xy - x - e$

$F = \int (2xy - x - e^y)dx = x^2y - \frac{x^2}{2} - xe^y + \varphi(y)$

$\frac{dF}{dy} = x^2 - xe^y + \varphi'(y)$

$x^2 - xe^y + \varphi'(y) = x^2 - xe^y - y$

$\varphi'(y) = -y$

$\varphi(y) = \int -y\, dy$

$\varphi(y) = -\frac{y^2}{5} \cancel{dx} + C,\ C\in\mathbb{R}$

$F(x) = x^2y - \frac{x^2}{2} - \frac{y^2}{2} - xe^y$

c) $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$

$\frac{dF}{dx} = \frac{y}{x} + 6x$

$F = \int \left(\frac{y}{x} + 6x\right)dx = y\ln|x| + 3x^2 + \varphi(y)$

$\frac{dF}{dy} = \ln x + \varphi'(y) = \ln x - 2$

$\Rightarrow \ln x + \varphi'(y) = \ln x - 2 \Leftrightarrow \varphi'(y) = -2 \Leftrightarrow \varphi(y) = -2y + C,\ C\in\mathbb{R}$

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$$e^x \sec y - tg\, y + y' = 0$$

$$\Leftrightarrow e^x \sec y - tg\, y + \frac{dy}{dx} = 0$$

$$\Leftrightarrow (e^x \sec y - tg\, y)\,dx + dy = 0$$

$$\underbrace{e^{\beta x}\cos y(e^x \sec y - tg\, y)}_{M(x,y)}\,dx + \underbrace{e^{\beta x}\cos y}_{N(x,y)}\,dy = 0 \Leftrightarrow$$

$$\Leftrightarrow (e^{(\beta+1)x} - e^{\beta x}\sin y)\,dx + e^{\beta x}\cos y\,dy = 0$$

$$\frac{dM}{dy} = -e^{\beta x}\cos y \qquad \frac{dN}{dx} = \beta e^{\beta x}\cos y$$

$$-e^{\beta x}\cos y = \beta e^{\beta x}\cos y \Leftrightarrow \beta = -1$$

$$M(x,y) = x - e^{-x}\sin y \qquad F = \int M\,dx = \int x - e^{-x}\sin y\,dx = \frac{x^2}{2} + e^{-x}\sin y + \varphi$$
$$N(x,y) = e^{-x}\cos y$$

$$\frac{dF}{dy} = e^{-x}\cos y + \varphi'(y) = N(x,y)$$

$$\cancel{e^{-x}\cos y} + \varphi'(y) = \cancel{e^{-x}\cos y}$$

$$\varphi'(y) = 0 \Leftrightarrow \varphi(y) = c,\ c \in \mathbb{R}$$

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a) $\underbrace{y}_{M(x,y)}dx + \underbrace{(y^2 - x)}_{N(x,y)}dy = 0$

$$\frac{dM}{dy} = 1 \qquad \frac{dN}{dx} = -1$$

$$\frac{dM}{dy} \neq \frac{dN}{dx}$$

Considerando o fator $y^{-2}$

$$y^{-2}\cdot y\,dx + y^{-2}(y^2 - x)\,dy = 0$$

$$\underbrace{y^{-1}}_{M(x,y)}dx + \underbrace{(1 - xy^{-2})}_{N(x,y)}dy = 0$$

$$\frac{dM}{dy} = -1y^{-2} = -\frac{1}{y^2}$$

$$\frac{dN}{dx} = -\frac{1}{y^2}$$

$$F = \int M\,dx = \int y^{-1}dx = y^{-1}x + \varphi(y)$$

$$\frac{dF}{dy} = \frac{d\left(y^{-1}x + \varphi(y)\right)}{dy} = -xy^{-2} + \varphi(y)$$

$\frac{\partial F}{\partial y} = N(x,y)$

$\varphi'(y) - y^{-2}x = 1 - xy^{-2}$

$\varphi'(y) = 1$

$\varphi(y) = y + C, \ C \in \mathbb{R}$

$F = y^{-1}x + y + C$

$= x + y^2 + Cy$

b)

$(2y - x^3)dx + x\,dy = 0$

$M(x,y) = 2y - x^3 \qquad N(x,y) = x$

$\frac{dM}{dy} = 2 \quad \neq \quad \frac{dN}{dx} = 1$

Considerando o fator integrante: $x$

$x \cdot M(x,y) = 2yx - x^4 \quad | \quad x \cdot N(x,y) = x^2$

$\frac{dM}{dy} = 2x \quad = \quad \frac{dN}{dx} = 2x$

$F = \int M dx = \int (2yx - x^4)\,dx = yx^2 - \frac{x^5}{5} + \varphi(y)$

$\frac{dF}{dy} = x^2 + \varphi'(y)$

$N = \frac{dF}{dy} \Leftrightarrow x^2 = x^2 + \varphi'(y) \Leftrightarrow \varphi'(y) = 0 \Leftrightarrow \varphi(y) = C, \ C \in \mathbb{R}$

$F = yx^2 - \frac{x^5}{5} + C, \ C \in \mathbb{R}$

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a) $y' + 2y = \cos x$ EDO de 1ª Ordem

$p(x) = 2$

$q(x) = \cos x$

$e^{\int p(x)dx} = e^{2x}$

$e^{2x}y' + 2e^{2x}y = e^{2x}\cos x$

$(e^{2x}y)' = e^{2x}\cos x$

$e^{2x}y = \int e^{2x}\cos x\,dx$

$$\int e^{2x}\cos x\,dx = \sin x\, e^{2x} - 2\int e^{2x}\sin x\,dx = \circledast$$

$$f(x)=e^{2x} \quad g'(x)=\cos x \Leftrightarrow \begin{cases} f'(x)=2e^{2x} \\ g(x)=\sin x \end{cases}$$

$$f'(x)=\sin x \quad g(x)=e^{2x} \Leftrightarrow \begin{cases} f(x)=-\cos x \\ g(x)=2e^{2x} \end{cases}$$

$$\circledast = \sin x\, e^{2x} - 2\left(-\cos x\, e^{2x} + 2\int e^{2x}\cos x\,dx\right)$$

$$= \sin x\, e^{2x} + 2\cos x\, e^{2x} - 4\int e^{2x}\cos x\,dx$$

$$I = P - 4I \Leftrightarrow I = \frac{P}{5}$$

$$\int e^{2x}\cos x\,dx = \frac{\sin x\, e^{2x} + 2\cos x\, e^{2x}}{5} + C,\ C\in\mathbb{R}$$

$$e^{2x}y = \frac{\sin x\, e^{2x} + 2\cos x\, e^{2x}}{5} + C,\ C\in\mathbb{R}$$

$$y = \frac{\sin x + 2\cos x}{5} + \frac{C}{e^{2x}},\ C\in\mathbb{R}$$

b)

$$x^3y' - y - 1 = 0$$

$$x^3y' - y = 1$$

$$y' - x^{-3}y = x^{-3}$$

$$P(x) = -x^{-3}$$
$$Q(x) = x^{-3}$$

$$e^{\int P(x)dx} = e^{-\int x^{-3}dx} = e^{\frac{x^{-2}}{2}}$$

$$e^{\frac{x^{-2}}{2}}y' - x^{-3}\cdot e^{\frac{x^{-2}}{2}}y = e^{\frac{x^{-2}}{2}}\cdot x^{-3}$$

$$\left(e^{\frac{x^{-2}}{2}}y\right)' = \int e^{\frac{x^{-2}}{2}}x^{-3}\,dx$$

$$\int e^{\frac{x^{-2}}{2}}x^{-3}\,dx = -\int -x^{-3}e^{\frac{x^{-2}}{2}}\,dx = -\int \left(\frac{x^{-2}}{2}\right)' e^{\frac{x^{-2}}{2}}\,dx$$

$$= -e^{\frac{x^{-2}}{2}} + C,\ C\in\mathbb{R}$$

$$e^{\frac{x^{-2}}{2}}y = -e^{\frac{x^{-2}}{2}} + C,\ C\in\mathbb{R}$$

$$y = -1 + \frac{C}{e^{\frac{x^{-2}}{2}}},\ C\in\mathbb{R}$$

c) $\frac{1}{x}y' - \frac{1}{x^2+1}y = \frac{\sqrt{x^2+1}}{x}$, $x \neq 0$

$y' - \frac{x}{x^2+1}y = \sqrt{x^2+1}$, $x \neq 0$

$P(x) = -\frac{x}{x^2+1}$

$Q(x) = \sqrt{x^2+1}$

$\int P(x)dx = \int -\frac{x}{x^2+1}dx = -\frac{1}{2}\int \frac{2x}{x^2+1}dx = -\frac{1}{2}\ln|x^2+1| + C = \ln(x^2+1)^{-\frac{1}{2}} + C$

Considerando $C=0$, $\int P(x)dx = \ln(\sqrt{x^2+1})^{-1}$

$e^{\int p(x)dx} = e^{\ln(\sqrt{x^2+1})^{-1}} = (\sqrt{x^2+1})^{-1} = \frac{1}{\sqrt{x^2+1}}$

$\frac{1}{\sqrt{x^2+1}}y' - \frac{x}{(x^2+1)(\sqrt{x^2+1})}y = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$

$\Leftrightarrow \left(\frac{1}{\sqrt{x^2+1}}y\right)' = 1$

$\frac{1}{\sqrt{x^2+1}}y = x + C$, $C \in \mathbb{R}$

$y = (x+C)(\sqrt{x^2+1})$, $C \in \mathbb{R}$

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$x^2y' + 2xy = 1$

$(x^2y)' = 1$

$\int (x^2y)'dx = \int 1\,dx$

$x^2y = x + C$, $C \in \mathbb{R}$

$y = \frac{x+C}{x^2}$, $C \in \mathbb{R}$

$\lim_{x\to+\infty} y = \lim_{x\to+\infty} \left(\frac{x+C}{x^2}\right) = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0$

15

a) $xy' + y = y^2\ln x$ $\qquad z = y^{1-2} = y^{-1}$

$xy^{-2}y' + y^{-2}y = \ln x$ $\qquad z' = (y^{-1})' = -(y^{-2})y'$

$\Leftrightarrow xz' + z = \ln x$

$\Leftrightarrow z' + \frac{1}{x}z = \frac{1}{x}\ln x$

$P(x) = \frac{1}{x}$ $\qquad \int P(x)dx = \int \frac{1}{x}dx = \ln|x| + C$, $C \in \mathbb{R}$

Considerando $C=0$, $\ln|x| + 0 = \ln|x|$, $x > 0$
$= \ln x$

$e^{\int p(x)dx} = e^{\ln x} = x$

$xz' + z = \ln x \Leftrightarrow (xz)' = \ln x \Leftrightarrow xz = \int \ln x\,dx$

$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C, C \in \mathbb{R}$

$\begin{cases} f'(x) = 1 \\ g(x) = \ln(x) \end{cases} \Leftrightarrow \begin{cases} f(x) = x \\ g'(x) = \frac{1}{x} \end{cases}$

$xz = x \ln x - x + C$

$z = \ln x - \cancel{1} + \frac{C}{x}$

$y^{-1} = \frac{x \ln x - x + C}{x}$

$y = \frac{x}{x \ln x - x + C}, x \in \mathbb{R}$

Para $y = 0$

$xy' + y = y^2 \ln x$

$x0 + 0 = 0 \ln x$

$0 = 0$

$y = 0$ sol particular

15

b) $y' - \frac{y}{2x} = 5x^2 y^5$

$y^{-5} y' - \frac{y^{-4}}{2x} = 5x^2$

$z = y^{-4}$

$z' = (y^{-4})' = -4y^{-5} y'$

$\frac{z'}{-4} - \frac{z}{2x} = 5x^2 \Leftrightarrow z' + \frac{2z}{x} = -20x^2$

$P(x) = \frac{2}{x}$

$\int P(x) dx = 2 \ln|x| + C, C \in \mathbb{R}$

$= \ln x^2 + C,$

$e^{\int P(x) dx} = e^{\ln x^2 + 0} = x^2$

$x^2 z' + \frac{2x^2 z}{x} = -20x^2 \cdot x^2$

$x^2 z' + 2xz = -20x^4$

$(x^2 z)' = -20x^4$

$x^2 z = \int -20x^4 dx$

$x^2 z = -20\frac{x^5}{5} + C, C \in \mathbb{R}$

$x^2 z = -4x^5 + C, C \in \mathbb{R}$

$z = \frac{-4x^5 + C}{x^2}, C \in \mathbb{R}$

$y^{-4} = \frac{-4x^5 + C}{x^2}, C \in \mathbb{R}$

$y^4 = \frac{x^2}{-4x^5 + C}, C \in \mathbb{R}$

$y = \sqrt[4]{\frac{x^2}{-4x^5 + C}}, C \in \mathbb{R}$

Para $y = 0$

$y' = 0$

$0 - 0 = 0$

$0 = 0$ ~~Sola~~ $y = 0$ é sol particular

16

a) $y' - \frac{2y}{x} = x^3$ Equação linear completa

Eq. homogénea associada:

$y' - \frac{2y}{x} = 0$ Eq de variáveis separáveis

$y' = \frac{2y}{x}$

$\frac{1}{y} y' = \frac{2}{x}$

$\int \frac{1}{y} dy = \int \frac{2}{x} dx$

$\ln|y| = 2\ln|x| + C, C \in \mathbb{R}$

$|y| = e^{\ln x^2 + C}, C \in \mathbb{R}$

$|y| = x^2 \cdot e^C, C \in \mathbb{R}$

$|y| = x^2 \cdot C_1, C_1 = e^C, C_1 \in \mathbb{R}$

$y = x^2 \cdot C_2, C_2 = \pm C_1, C_2 \in \mathbb{R}$

Solução particular (Método de variação das constantes

$C_2'(x) \cdot x^2 = x^3$

$C_2'(x) = x$

$C_2(x) = \frac{x^2}{2}$

$y_p = \frac{x^2}{2}$

$y = x^2 \times C_2 + \frac{x^2}{2} \cdot x^2, C_2 \in \mathbb{R}$

$= x^2 \cdot C_2 + \frac{x^4}{2}, C_2 \in \mathbb{R}$

b) $y' \sin x + y \cos x = \sin^2 x$

Considerando a EDO Homo associada

$y' \sin x + y \cos x = 0$

$y' \sin x = -y \cos x$ EDO de variáveis separáveis

$y' \frac{1}{y} = -\frac{\cos x}{\sin x}$

$\int \frac{1}{y} dy = -\int \frac{\cos x}{\sin x} dx$

$\ln|y| = -\int \frac{(\sin x)'}{\sin x} dx$

$\ln|y| = -\ln|\sin x| + C, C \in \mathbb{R}$

$|y| = e^{\ln (\sin x)^{-1}} \cdot e^C \quad C \in \mathbb{R}$

$|y| = \frac{1}{\sin x} \cdot C_2, C_2 = e^C, C_2 \in \mathbb{R}$

16

a) $y' - \frac{2y}{x} = x^3$ Equação linear completa

Eq. homogénea associada:

$y' - \frac{2y}{x} = 0$ Eq de variáveis separáveis

$y' = \frac{2y}{x}$

$\frac{1}{y} y' = \frac{2}{x}$

$\int \frac{1}{y} dy = \int \frac{2}{x} dx$

$\ln|y| = 2\ln|x| + C, C \in \mathbb{R}$

$|y| = e^{\ln x^2 + C}, C \in \mathbb{R}$

$|y| = x^2 \cdot e^C, C \in \mathbb{R}$

$|y| = x^2 \cdot C_1, C_1 = e^C, C_1 \in \mathbb{R}$

$y = x^2 \cdot C_2, C_2 = \pm C_1, C_2 \in \mathbb{R}$

Solução particular (Método de variação das constantes)

$C_2'(x) \cdot x^2 = x^3$

$C_2'(x) = x$

$C_2(x) = \frac{x^2}{2}$

$y_p = \frac{x^2}{2}$

$y = x^2 \times C_2 + \frac{x^2}{2} x^2, C_2 \in \mathbb{R}$

$= x^2 \cdot C_2 + \frac{x^4}{2}, C_2 \in \mathbb{R}$

b) $y' \sin x + y \cos x = \sin^2 x$

Considerando a EDO Homo associada

$y' \sin x + y \cos x = 0$

$y' \sin x = -y \cos x$ EDO de variáveis separáveis

$y' \frac{1}{y} = -\frac{\cos x}{\sin x}$

$\int \frac{1}{y} dy = -\int \frac{\cos x}{\sin x} dx$

$\ln|y| = -\int \frac{(\sin x)'}{\sin x} dx$

$\ln|y| = -\ln|\sin x| + C, C \in \mathbb{R}$

$|y| = e^{\ln (\sin x)^{-1}} \cdot e^C, C \in \mathbb{R}$

$|y| = \frac{1}{\sin x} \cdot C_2, C_2 = e^C, C_2 \in \mathbb{R}$

$y = \frac{1}{\sin x} \cdot C_3, \quad C_3 = \pm C_2, \quad C_3 \in \mathbb{R}$

$C_3'(x) \frac{1}{\sin x} = \frac{\sin^2 x}{\sin x}$

$C_3'(x) = \sin^2 x$

$C_3(x) = \int \sin^2 x \, dx$

$\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos(2x)}{2} dx$

$= \frac{x}{2} - \frac{1}{4} \sin(2x)$

$C_3(x) = \frac{x}{2} - \frac{1}{4} \sin(2x)$

$y = \frac{1}{\sin x} C_3 + \left(\frac{x}{2} - \frac{1}{4} \sin(2x)\right) \frac{1}{\sin x}$

$= \left(C_3 + \frac{x}{2} - \frac{1}{4} \sin(2x)\right) \frac{1}{\sin x}$

$= \frac{C_3}{\sin x} + \frac{x}{2 \sin x} - \frac{2 \sin(x) \cos(x)}{4 \sin x}$

$= C_3 \operatorname{cosec}(x) + \frac{x}{2} \operatorname{cosec}(x) - \frac{\cos(x)}{2}, \quad C_3 \in \mathbb{R}$

c)

$y' - \frac{x}{x^2+1} y = \sqrt{x^2+1}$

$y_h \Rightarrow \quad y' - \frac{x}{x^2+1} y = 0$

$y' = \frac{x}{x^2+1} y$

$\frac{1}{y} y' = \frac{x}{x^2+1}$

$\int \frac{1}{y} dy = \int \frac{x}{x^2+1} dx$

$\ln|y| = \frac{1}{2} \int \frac{2x}{x^2+1} dx$

$\ln|y| = \frac{1}{2} \ln|x^2+1| + C, \quad C \in \mathbb{R}$

$\ln|y| = \ln \sqrt{x^2+1} + C, \quad C \in \mathbb{R}$

$y = e^{\ln \sqrt{x^2+1} + C}, \quad C \in \mathbb{R}$

$y = \sqrt{x^2+1} \cdot C_2, \quad C_2 = \pm e^C, \quad C_2 \in \mathbb{R}$

$C_2'(x) \sqrt{x^2+1} = \sqrt{x^2+1}$

$C_2'(x) = 1$

$C_2(x) = x$

$y = \sqrt{x^2+1} \cdot C_2 + x \cancel{C_2} \sqrt{x^2+1}$

$= (x + C) \sqrt{x^2+1}$

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a) $y'+y=\sin x$

EDO homogénea associada: $y'+y=0$

Eq. característica: $r+1=0$
$r=-1$

raiz simples logo $SFS=\{e^{-x}\}$

$y_h=C_1 \cdot e^{-x}$

$y_p$: $C_1'(x)\cdot e^{-x}=\sin x$
$C_1'(x)=\sin e^{x}$
$C_1(x)=\int \sin e^{x}$

$\int \sin x\, e^{x}dx = e^{x}\cos x - \int e^{x}\cos x\, dx = ⊛$

$f'(x)=\sin x$, $g(x)=e^{x}$ $\Leftrightarrow$ $\begin{cases} f(x)=\cos x \\ g'(x)=e^{x}\end{cases}$

$f'(x)=\cos x$, $g(x)=e^{x}$ $\Leftrightarrow$ $\begin{cases} f(x)=\sin x \\ g'(x)=e^{x}\end{cases}$

$⊛ = e^{x}\cos x-(e^{x}\sin x-\int e^{x}\sin x\, dx)$

$=e^{x}\cos x-e^{x}\sin x - e^{x}\sin x$

$I=P-I \Leftrightarrow 2I=P \Leftrightarrow I=\frac{P}{2}$

$\int \sin x\, e^{x}dx=\frac{e^{x}\cos x-e^{x}\sin x}{2}=\frac{e^{x}(\cos x-\sin x)}{2}$

$C_1(x)=\frac{e^{x}(\cos x-\sin x)}{2}$

$\therefore y=C_1e^{-x}+\frac{e^{x}(\cos x-\sin x)}{2}\cdot e^{-x}$

$=C_1e^{-x}+\frac{\cos x-\sin x}{2}$, $C_1\in\mathbb{R}$

b) $y''-y+2\cos x=0$

$y''-y=-2\cos x$

EDO homogénea: $y''-y=0$

Eq característica: $r^2-1=0$
$r=\pm1$

$SFS=\{e^{x}, e^{-x}\}$

$y_h=C_1e^{x}+C_2e^{-x}$

$y_p$: $\begin{cases} C_1'(x)e^{x}+C_2'(x)e^{-x}=0 \\ C_1'(x)e^{x}-C_2'(x)e^{-x}=-2\cos x\end{cases}$

$$(eq_1+eq_2)\ \begin{cases} 2C_1'(x)\,e^{x} = -2\cos x \\ \text{—} \end{cases} \Leftrightarrow \begin{cases} C_1'(x) = -\cos x\, e^{-x} \\ \text{—} \end{cases} \Leftrightarrow \circledast_3$$

$$C_1(x) = \int(-\cos x\, e^{-x})\,dx = \circledast^2$$

$$\int -\cos x\, e^{-x}\,dx = -e^{-x}\sin x - \int e^{-x}\sin x\,dx = \circledast$$

$$\begin{matrix} f'(x) = \cos x \\ g(x) = -e^{-x} \end{matrix} \Leftrightarrow \begin{cases} f(x) = \sin x \\ g'(x) = e^{-x} \end{cases}$$

$$\begin{matrix} f'(x) = \sin x \\ g(x) = e^{-x} \end{matrix} \Leftrightarrow \begin{cases} f(x) = -\cos x \\ g'(x) = -e^{-x} \end{cases}$$

$$\circledast = -e^{-x}\sin x - \left(-\cos x\, e^{-x} + \int -\cos x\, e^{-x}\,dx\right)$$

$$= -e^{-x}\sin x + \cos x\, e^{-x} - \int -\cos x\, e^{-x}\,dx$$

$$I = P - I$$

$$I = \frac{P}{2}$$

$$\int -\cos x\, e^{-x}\,dx = \frac{e^{-x}(\cos x - \sin x)}{2}$$

$$\circledast^2\quad C_1(x) = \frac{e^{-x}(\cos x - \sin x)}{2}$$

$$\circledast_3 = \begin{cases} C_1'(x)\,e^{x} + C_2' e^{-x} = 0 \\ C_1'(x) = -\cos x\, e^{-x} \end{cases} \Leftrightarrow \begin{cases} C_2'(x) = -C_1'(x)\,e^{x}\cdot e^{+x} \\ \text{—} \end{cases}$$

$$\Leftrightarrow \begin{cases} C_2'(x) = -\cos x\, e^{-x} e^{x} e^{x} \\ \text{—} \end{cases} \Leftrightarrow \begin{cases} C_2'(x) = -\cos x\, e^{x} \end{cases}$$

$$C_2(x) = \int -\cos e^{x}\,dx \underset{\substack{\uparrow \\ \text{Symbolab}}}{=} -\frac{e^{x}(\sin(x) + \cos(x))}{2}$$

$$C_1(x) = \frac{e^{-x}(\cos x - \sin x)}{2}$$

$$y = C_1 e^{x} + C_2 e^{-x} + \frac{e^{-x}(\cos x - \sin x)e^{x}}{2} + \frac{e^{x}(\cos x + \sin x)e^{-x}}{2}$$

$$= C_1 e^{x} + C_2 e^{-x} + \frac{\cos x - \sin x}{2} + \frac{\cos x + \sin x}{2}$$

$$= C_1 e^{x} + C_2 e^{-x} + \cos x$$

2c)

$$y'' + y' = 2y + 3 - 6x$$

$$y'' + y' - 2y = 3 - 6x$$

$$y_h \Rightarrow y'' + y' - 2y = 0$$

$$\text{Eq. Carac} \Rightarrow r^2 + r - 2 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1+8}}{2} \Leftrightarrow r = 1 \vee r = -2$$

$$SFS = \{e^{x}, e^{-2x}\}$$

$$y_h = C_1 e^{x} + C_2 e^{-2x},\ C_1, C_2 \in \mathbb{R}$$

$$\begin{cases} C_1'(x)e^{x}+C_2'(x)e^{-2x}=0 \\ C_1'(x)e^{x}-2C_2'(x)e^{-2x}=3-6x \end{cases} \underset{2eq-1eq}{\Leftrightarrow} \begin{cases} C_1'(x)e^{x}=-C_2'(x)e^{-2x} \\ -3C_2'(x)e^{-2x}=3-6x \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1'(x)=-C_2'(x)e^{-3x} \\ C_2'(x)=(2x-1)e^{2x} \end{cases} \Leftrightarrow \begin{cases} C_1'(x)=(1-2x)e^{-x} \\ C_2'(x)=(2x-1)e^{2x} \end{cases}$$

$$C_1(x)=\int(1-2x)e^{-x}dx=-e^{-x}(1-2x)-\int 2e^{-x}dx=\circledast$$

$$\begin{matrix} f'(x)=e^{-x} \\ g(x)=(1-2x) \end{matrix} \Leftrightarrow \begin{cases} f(x)=-e^{-x} \\ g'(x)=-2 \end{cases}$$

$$\circledast=e^{-x}(2x-1)+2\int -e^{-x}dx=e^{-x}(2x-1)+2e^{-x}$$

$$C_2(x)=\int(2x-1)e^{2x}dx=\frac{1}{2}e^{2x}(2x-1)-\frac{1}{2}\int 2e^{2x}dx=\circledast$$

$$\begin{matrix} f'(x)=e^{2x} \\ g(x)=2x-1 \end{matrix} \Leftrightarrow \begin{cases} f(x)=\frac{1}{2}e^{2x} \\ g'(x)=2 \end{cases}$$

$$\circledast=\frac{1}{2}e^{2x}(2x-1)-\frac{1}{2}e^{2x}=\frac{1}{2}e^{2x}(2x-1-1)=e^{2x}(x-1)$$

$$C_1(x)=e^{-x}(2x-1)+2e^{-x}$$

$$C_2(x)=e^{2x}(x-1)$$

$$y_p=\left(e^{-x}(2x-1)+2e^{-x}\right)e^{x}+\left(e^{2x}(x-1)\cdot e^{-2x}\right)$$
$$=2x-1+2+x-1=3x$$

$$y=C_1e^{x}+C_2e^{-2x}+3x,\quad C_1,C_2\in\mathbb{R}$$

d) $y''-4y'+4y=xe^{2x}$

$y''-4y'+4y=0$

$r^2-4r+4=0 \Leftrightarrow (r-2)^2=0 \Leftrightarrow r=2$
raiz dupla

$SFS=\{e^{2x},xe^{2x}\}$

$y_h=C_1e^{2x}+C_2xe^{2x},\quad C_1,C_2\in\mathbb{R}$

$$\begin{cases} C_1'(x)e^{2x}+C_2'(x)xe^{2x}=0 \\ 2C_1'(x)e^{2x}+C_2'(x)e^{2x}+2C_2'(x)xe^{2x}=xe^{2x} \end{cases} \underset{eq_2-2eq_1}{\Leftrightarrow} \begin{cases} C_1'(x)=-C_2'(x)x \\ C_2'(x)=x \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1'(x)=-x^2 \\ C_2'(x)=x \end{cases} \Leftrightarrow \begin{cases} C_1(x)=-\frac{x^3}{3} \\ C_2(x)=\frac{x^2}{2} \end{cases}$$

$$y_p=-\frac{x^3}{3}\cdot e^{2x}+\frac{x^2}{2}xe^{2x}=x^3e^{2x}\left(-\frac{1}{3}+\frac{1}{2}\right)=x^3e^{2x}\cdot\frac{1}{6}$$

$$y=C_1e^{2x}+C_2xe^{2x}+\frac{1}{6}x^3e^{2x}$$

e)

$y'' + y' = e^{-x}$

Eq caract: $r^2 + r = 0$

$r(r+1) = 0$

$r = 0 \vee r = -1$

$SFS = \{1, e^{-x}\}$

$y_h = C_1 + C_2 e^{-x}$

$y_p: \begin{cases} C_1'(x) + C_2'(x) e^{-x} = 0 \\ C_1'(x) \cdot 1 - C_2'(x) e^{-x} = e^{-x} \end{cases} \Leftrightarrow \begin{cases} C_1'(x) = -e^{-x} \\ C_2'(x) = -1 \end{cases} \Leftrightarrow \begin{cases} C_1(x) = e^{-x} \\ C_2(x) = -x \end{cases}$

$y_p = e^{-x} - xe^{-x} = e^{-x}(1-x)$

$y = C_1 + C_2 e^{-x} + e^{-x}(1-x) = C_1 + e^{-x}(1 + C_2 - x) = C_1 + e^{-x}(C_3 - x)$

f)

$y'' + 4y = tg(2x)$

Eq caract $r^2 + 4 = 0$

$r = \pm\sqrt{-4}$

$r = \pm 2i$

$SFS = \{e^0 \cos(2x), e^0 \sin(2x)\}$

$= \{\cos(2x), \sin(2x)\}$

$y_h = C_1 \cos(2x) + C_2 \cdot \sin(2x)$

$y_p: \begin{cases} C_1'(x) \cos(2x) + C_2' \sin(2x) = 0 \\ -2C_1'(x) \sin(2x) + 2C_2'(x) \cos(2x) = tg(2x) \end{cases}$

$\Leftrightarrow \begin{cases} C_1'(x) = -C_2' \, tg(2x) \\ 2C_2' \, tg(2x) \sin(2x) + 2C_2' \cos(2x) = tg(2x) \end{cases}$

$\Leftrightarrow \begin{cases} \text{---} \\ 4C_2' \cos 2x = tg\, 2x \end{cases} \Leftrightarrow \begin{cases} C_1'(x) = -\dfrac{tg^2(2x)}{4\cos(2x)} \\ C_2' = \dfrac{tg\, 2x}{4\cos 2x} \end{cases}$

$C_1'(x) = \int -\dfrac{tg^2(2x)}{4\cos(2x)}\, dx = -\dfrac{1}{4} \int \dfrac{\sin^2(2x)}{\cos^3(2x)}\, dx$

$= \dfrac{1}{8} \int \dfrac{-2\sin^2(2x)}{\cos^3(2x)}\, dx = -\dfrac{1}{8} \int \dfrac{(2x)' \sin^2(2x)}{\cos^3(2x)}\, dx$

$= -\dfrac{1}{8} \int \dfrac{\sin^2(y)}{\cos^3(y)}\, dy = -\dfrac{1}{8} \int \dfrac{1-\cos^2(y)}{\cos^3(y)}\, dx \ldots$

...

do later

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$y' + y\cos x = \cos x$

$y' = \cos x\,(1-y)$

$\frac{1}{1-y}\, y' = \cos x$

$\int \frac{1}{1-y}\, dy = \int \cos x\, dx$

$-\ln|1-y| = \sin x + C,\ C \in \mathbb{R}$

$\ln|1-y| = -\sin x + C_2,\ C_2 = -C_1,\ C_2 \in \mathbb{R}$

$1-y = e^{-\sin x + C_2},\ C_2 \in \mathbb{R}$

$-y = e^{-\sin x} \cdot C_3 - 1,\ C_3 = e^{C_2},\ C_3 \in \mathbb{R}$

$y = 1 - e^{-\sin x} \cdot C_3$

$y(0) = 2$

$1 - e^{-\sin 0} \cdot C_3 = 2$

$e^0 C_3 = 1-2$

$C_3 = -1$

$y = 1 + e^{-\sin x}$

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a) $(1+x^2)y' + 4xy = 0$

$(1+x^2)y' = -4xy$

$\frac{1}{y}\, y' = -\frac{4x}{1+x^2}$

$\int \frac{1}{y}\, dy = \int -\frac{4x}{1+x^2}\, dx$

$\ln|y| = -2\ln|1+x^2| + C,\ C \in \mathbb{R}$

$|y| = e^{\ln(1+x^2)^{-2} + C},\ C \in \mathbb{R}$

$y = \frac{1}{(1+x^2)^2} \cdot C$

b) $y'' + y + 2\sin x = 0$

$y'' + y = -2\sin x$

Eq. carac.: $r^2 + 1 = 0$

$r = \pm i$

$SFS = \{\cos(x), \sin(x)\}$

$y_h = C_1 \cdot \cos(x) + C_2 \cdot \sin(x)$

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0 \\ -C_1'(x)\sin x + C_2'(x)\cos x = -2\sin x \end{cases}$$

$$\begin{cases} C_1' \cos x = -C_2' \sin x \\ \text{—} \end{cases} \Leftrightarrow \begin{cases} C_1' = -C_2' \frac{\sin x}{\cos x} \\ -C_2' \frac{\sin x}{\cos x} \cdot \sin x + C_2' \cos x = -2\sin x \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{—} \\ \frac{C_2' \sin^2 x + C_2' \cos^2 x}{\cos x} = -2\sin x \end{cases} \Leftrightarrow \begin{cases} \text{—} \\ C_2' = -2\sin x \cos x \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1' = 2\sin^2 x \\ C_2' = -\sin(2x) \end{cases}$$

$$C_1 = \int 2\sin^2 x \, dx = \cancel{2} \int \frac{1-\cos(2x)}{\cancel{2}} dx \Leftrightarrow \int 1 dx - \int \cos(2x) dx = x - \frac{1}{2}\sin(2x)$$

$$C_2 = \int -\sin(2x) dx = -\frac{1}{2} \int 2\sin(2x) dx = \frac{1}{2}\cos(2x)$$

$$y_p = \left(x - \frac{1}{2}\sin(2x)\right)\cos(x) + \frac{1}{2}\cos(2x)\sin(x)$$

$$= x\cos x - \frac{1}{2}\sin x \cos^2 x + \frac{1}{2}\cos^2 x \sin x - \frac{1}{2}\sin^3 x$$

$$= x\cos x - \frac{1}{2}\sin x \cos^2 x - \frac{1}{2}\sin^3 x$$

$$= x\cos x - \frac{1}{2}\sin x(\cos^2 x + \sin^2 x) = x\cos x - \frac{1}{2}\sin x$$

$$y = C_1 \cos x + C_2 \sin x + x\cos x - \frac{1}{2}\sin x$$

$$= C_1 \cos x + \left(C_2 - \frac{1}{2}\right)\sin x + x\cos x, \quad C_1, C_2 \in \mathbb{R}$$

$$= C_1 \cos x + C_3 \sin x + x\cos x$$

c) $(1+x^2)y' - y = 0$

$$(1+x^2)y' = y$$

$$\frac{1}{y} y' = \frac{1}{1+x^2}$$

$$\int \frac{1}{y} dy = \int \frac{1}{1+x^2} dx$$

$$\ln|y| = \operatorname{arctg} x + C$$

$$y = e^{\operatorname{arctg} x} \cdot C, \quad C \in \mathbb{R}$$

d) $y''' + 4y' = \cos x$

Eq carat: $r^3 + 4r = 0$

$r(r^2+4) = 0$

$r = 0 \vee r = \pm 2i$

$$\{1, \cos(2x), \sin(2x)\}$$

$$y_h = C_1 + C_2 \cos(2x) + C_3 \sin(2x)$$

d)

$y''' + 4y' = \cos x$

Eq característica: $r^3 + 4r = 0$
$r(r^2+4) = 0$
$r = 0 \vee r = \pm 2i$

$b(x) = \cos x = P_0(x) \cdot e^{0x} \cdot \cos(1x)$

Sabe-se que: $y_p = x^k \cdot e^{\alpha x} \cdot (P(x)\cos(\beta x) + Q(x)\sin(\beta x))$

↳ $k$ é a multiplicidade de $0 + 1 \cdot i$ como solução ~~de~~ caract.

$\cancel{0+}i$ não é solução, logo $k = 0$

$y_p = P(x)\cos(x) + Q(x)\sin(x)$

O grau de $P$ e $Q$ é igual ao grau de $P(x)$ em $b(x)$, logo $0$: $P = A$ e $Q = B$

$y_p = A\cos(x) + B\sin(x)$

$(y_p)''' + 4y_p' = \cos x$

$y_p' = (A\cos(x) + B\sin(x))' = -A\sin(x) + B\cos(x)$

$y_p'' = -A\cos(x) - B\sin(x)$

$y_p''' = A\sin(x) - B\cos(x)$

$A\sin(x) - B\cos(x) + 4(-A\sin(x) + B\cos(x)) = \cos x$

$A\sin x - B\cos x - 4A\sin(x) + 4B\cos(x) = \cos x$

$3B\cos(x) - 3A\sin(x) = \cos x$

$\begin{cases} 3B = 1 \\ -3A = 0 \end{cases} \Leftrightarrow \begin{cases} B = \frac{1}{3} \\ A = 0 \end{cases}$

$y_p = \frac{1}{3}\sin x$

$y = 1 + C_1 \cdot \cos(2x) + C_2 \cdot \sin(2x) + \frac{1}{3}\sin x$

e) $y' - 3x^2 y = x^2$

$y' = x^2 + 3x^2 y$
$y' = x^2(1+3y)$

$\frac{1}{1+3y} y' = x^2 \Leftrightarrow \int \frac{1}{1+3y} dy = \int x^2 dx \Leftrightarrow \frac{1}{3}\int \frac{3}{1+3y} dy = \int x^2 dx$

$\Leftrightarrow \frac{1}{3}\ln|1+3y| = \frac{x^3}{3} + C, C \in \mathbb{R}$

$\Leftrightarrow \ln|1+3y| = x^3 + 3C, C \in \mathbb{R} \Leftrightarrow |1+3y| = e^{x^3 + C_2}, C_2 = 3C, C_2 \in \mathbb{R}$

$|1+3y| = e^{x^3} \cdot C_3, \quad C_3 = e^{C_2}, \quad C_3 \in \mathbb{R}$

$1+3y = C_4 e^{x^3}, \quad C_4 = \pm C_3, \quad C_4 \in \mathbb{R}$

$y = \frac{1}{3} C_4 e^{x^3} - \frac{1}{3}, \quad C_4 \in \mathbb{R}$

$y = C_5 e^{x^3} - \frac{1}{3}, \quad C_5 \in \mathbb{R}$

f)

$y''' - 3y' + 2y = 12e^x$

Eq. carac: $r^3 - 3r + 2 = 0$

Considerando $r=1$

$1^3 - 3 \cdot 1 + 2 = 3 - 3 = 0$

1 é raiz de $r^3 - 3r + 2$

| | 1 | 0 | -3 | 2 |
|---|---|---|---|---|
| 1 | | 1 | 1 | -2 |
| | 1 | 1 | -2 | 0 = R |

$r^3 - 3r + 2 = (r-1)(r^2 + r - 2)$

$r^2 + r - 2 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1+8}}{2} \Leftrightarrow r = \frac{-1 \pm 3}{2}$

$\Leftrightarrow r = -2 \vee r = 1$

$(r-1)^2 (r+2)$

$SFS = \{e^x, xe^x, e^{2x}\}$

$y_h = C_1 e^x + C_2 xe^x + C_3 e^{2x}$

$b(x) = 12e^x = P_0(x) \cdot e^{1x} \cdot \cos(0x)$

$P(x) = 12 \rightarrow$ polinómio grau 0

$\alpha = 1$

$\beta = 0$

$1 + 0i = 1 \rightarrow$ raiz dupla

$y_p = x^2 \cdot A \cdot e^x \cdot \cos(0) + x^2 B \cdot e^x \cdot \sin(0)$
$= Ax^2 e^x$

$y' = (Ax^2 e^x)' = 2xAe^x + Ax^2 e^x$

$y'' = 2Ae^x + 2xAe^x + 2xAe^x + Ax^2e^x = 2Ae^x + 4xAe^x + Ax^2e^x$

$y''' = 2Ae^x + 4Ae^x + 4xAe^x + 2xAe^x + Ax^2e^x$
$= 6Ae^x + 6xAe^x + Ax^2e^x$

$y''' - 3y' + 2y = 12e^x$

$6Ae^x + 6xAe^x + Ax^2e^x - 3(2xAe^x + Ax^2e^x) + 2(Ax^2e^x) = 12e^x$

$6Ae^x + \cancel{6xAe^x} + Ax^2e^x - \cancel{6xAe^x} - 3Ax^2e^x + 2Ax^2e^x = 12e^x$

$6Ae^x = 12e^x$

$6A = 12$

$A = 2$

$y_p = 2x^2e^x$

$y = C_1e^x + C_2xe^x + C_3e^{-2x} + 2x^2e^x$

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$z = y'$

$y'' = z'$

$xz' - z = 3x^2$

$z' - \frac{1}{x}z = 3x$

$P(x) = -\frac{1}{x}$

$Q(x) = 3x$

$\int P(x)\,dx = -\ln|x| = \ln\left|\frac{1}{x}\right|$

$e^{\int P(x)\,dx} = e^{\ln|1/x|} = \frac{1}{x}$

$z' - \frac{1}{x}z = 3x$

$\frac{1}{x}z' - \frac{1}{x^2}z = 3$

$\left(\frac{1}{x}z\right)' = 3$

$\frac{1}{x}z = 3x + C$

$z = 3x^2 + Cx$

$y' = 3x^2 + Cx$

$y = x^3 + \frac{Cx^2}{2} + C_2, \quad C, C_2 \in \mathbb{R}$

$= x^3 + C_3x^2 + C_2, \quad C_3 = \frac{C}{2}, \quad C_3, C_2 \in \mathbb{R}$

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a) $y = C_1x + C_2e^x$

$y' = C_1 + C_2e^x$

$y'' = 0 + C_2e^x = C_2e^x$

$(1-x)C_2e^x + x(C_1 + C_2e^x) - (C_1x + C_2e^x) =$

$= \cancel{C_2e^x} - \cancel{xC_2e^x} + \cancel{C_1x} + \cancel{C_2xe^x} - \cancel{C_1x} - \cancel{C_2e^x} = 0$

Logo $\{x, e^x\}$ é SFS

b) $y = C_1 x + C_2 e^x$

c)

Se $y = \beta x^2$

$y' = 2\beta x$
$y'' = 2\beta$

$(1-x)\cdot 2\beta + x(2\beta x) - \beta x^2$

$= 2\beta - 2\beta x + 2\beta x^2 - \beta x^2$

$= \beta x^2 - 2\beta x + 2\beta$ : ⊛

Se $\beta = 1$

⊛ $= x^2 - 2x + 2$

$y = x^2$ é solução ~~de~~ particular da EDO