Advanced Algorithms. Home Exam 2023/24

The Problem

The practical background of this problem is the placement of machines in a factory hall, in such a way that closely related machines get placed close to each other. However, we work straight away with an abstract formulation: In the plane with x, y-coordinate system, grid points are points (x, y) with integer coordinates x and y. The Manhattan distance between any two points (x, y) and (x', y') is defined as $|x - x'| + |y - y'|$.
We are given a complete graph (clique) with n nodes which are denoted by the indices $1, 2, \ldots, n$. Every edge, that is, every unordered pair of distinct nodes $\{i, j\}$, has a given positive weight w_{ij}. Note that $w_{ij} = w_{ji}$.
The problem is to map the n nodes to n distinct grid points, so as to minimize the weighted sum of all Manhattan distances between the nodes, where the sum is taken over all $\binom{n}{2}$ pairs of nodes. More formally, when every node i is mapped to a grid point (x_i, y_i), the weighted sum of distances is $\sum_{i<j} w_{ij} \cdot (|x_i - x_j| + |y_i - y_j|)$. Note that a solution must decide on the set S of the n grid points used, and on the actual mapping of the n nodes to S. The problem is known to be NP-complete.
The exam subject is to develop and analyze polynomial-time approximation algorithms for this problem.
In more detail, you may consider some of these aspects:
Algorithms may use an appropriately chosen fixed set S of grid points, or they may choose S adaptively, depending on the given weights.
Approximation ratios can be based on lower bounds on any solution, and on upper bounds on the distances in S. More refined bounds could also take the numbers of pairs of points in S with different distances into account.
Remember that approximation ratios do not have to be constant; also ratios being mildly growing functions of n could be of interest. But try to make them as good as you can.

The mapping of nodes onto the chosen grid points could use greedy rules (heavy edges first?) and/or branching rules, to some extent. But it could also be randomized, or a combination thereof.
In a randomized mapping, can you use linearity of expectation to calculate the expected weighted sum of distances, and expected approximation ratios? Does randomization have benefits for this problem, compared to deterministic rules? (Maybe yes, maybe not. This is really an open-ended discussion question.)
The aspects above are only suggestions. You may develop more ideas, and creatively work in further directions.

Submission

Mail your final report to `ptr@chalmers.se` as PDF attachment (no other formats please!). Given the complexity of the task and the need to rewrite drafts, handwriting is discouraged.
Write your name and study programme on the title page. The final submission deadline is announced on the course web page. Do not wait until the last minute, but submit when you are done. See also the Instructions below.
Quality is more important than quantity. However, as a rule of thumb, your report should have at least 4 pages of text (with usual font size, spacing, and margins), plus the title page and possible references.

Criteria for a Good Report

- Correctness: There are no major factual mistakes and invalid calculations. The final "product" does not have to be perfect or contain brilliant results, but what you write must be sound.
- Depth: You provide some solid, substantial results that are fully worked out, not only some trivial observations or vague heuristic guesses.
- Clarity: All notations are well defined. Algorithms, as well as proofs of their properties, are well described. One can follow your arguments step by step without extra efforts. (See also the general grading criteria on the course web page.)
- Negative statements (that something is probably not possible to do) are motivated by good reasons.

Instructions

- You have some freedom to choose your specific working directions, as long as you stick to the given problem.
- Only a certain selection of concepts and tools from the course are suitable for this problem while most others are probably not, and it is part of the task to figure out what fits here – which would also be the typical situation in practice. (We have already had assignments that covered the main course contents. The exam does *not* have the ambition to cover everything again.)
- Do not hesitate to discuss, and to send questions or drafts. But take availability times into account; see the course web page. Do not expect feedback on the drafts that goes much into technical details; the purpose is only to provide some guidance and check whether you are on a meaningful track.
- You can submit arbitrarily many drafts, at any time. Only the last version submitted before the deadline will count for your grade. In fact, **drafts are strongly encouraged**. The worst strategy would be to start thinking only a few days before the deadline – this would most likely result in a very poor report, because the details may be more tricky than anticipated at first glance.

- Utmost academic honesty is expected. The words about cheating (see the course web page) apply also here. In particular, you must cite all literature you have used, acknowledge all sources of help, and always describe the contents in your own way. Also mark very clearly in the text what is taken from other sources and is not your own thoughts.