Homotopy Algebras in Classical (Quantum) Field Theory

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$$\begin{tikzcd}
\cdots \arrow[r, "d_{k-2}"] & C^{k-1} \arrow[r, "d_{k-1}"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h_{k-1}"'] & C^{k} \arrow[r, "d_{k}"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h_{k}"'] & C^{k+1} \arrow[r, "d_{k+1}"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h_{k+1}"'] & \cdots \arrow[dl, "h_{k+2}"'] \\
\cdots \arrow[r, "d'_{k-2}"] & D^{k-1} \arrow[r, "d'_{k-1}"] & D^{k} \arrow[r, "d'_{k}"] & D^{k+1} \arrow[r, "d'_{k+1}"] & \cdots
\end{tikzcd}$$

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Field Theory

Classical Field Theory

$$\{\Phi_A\} := \mathcal{F} = \Gamma(M, \mathbb{F}_M)$$

$$S : \mathcal{F} \to \mathbb{R}$$

$$S[\Phi] = \int_M \mathcal{L}(\Phi, \partial\Phi)$$

$$\delta S = 0 \to \text{ EoM}$$

Quantum Field Theory

$$Z[J] = \frac{1}{Z[0]} \int_{\mathcal{F}} \mathcal{D}\Phi e^{\frac{i}{\hbar}(S[\Phi] + \int_M J\Phi)}$$

$$S_{if} = \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_n} Z[J]\Big|_{J=0}$$

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Motivation and Summary

- L_∞-algebras are homotopy generalisations of Lie algebras.
- Classical Field Theories correspond to L_∞-algebras.
- Scattering amplitudes are encoded in minimal models.
- BV formalism provides the bridge
- Equivalences are quasi-isomorphisms of L_∞-algebras.

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Lie algebras

A vector space $\mathfrak{g}$

Antisymmetric bilinear bracket

$$[-,-] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$

Jacobi Identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

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From Lie algebras to L_∞-algebras

Jacobi identity "up to homotopy"

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=d[X,Y,Z]$$

$d^2=0$ is called differential

If $X = Y + dZ$ then "X and Y are homotopic"

in L_∞-algebras the Jacobi identity is allowed to hold (only) up to higher coherent homotopies

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Notation

- $X, Y, Z \to l_1, l_2, l_3, \ldots$

- $[l_1, l_2] \to \mu_2(l_1, l_2)$

- $[l_1, [l_2, l_3]] + [l_3, [l_1, l_2]] + [l_2, [l_3, l_1]] = 0$

$$\to \mu_2(l_1, \mu_2(l_2, l_3)) + \mu_2(l_3, \mu_2(l_1, l_2)) + \mu_2(l_2, \mu_2(l_3, l_1)) = 0$$

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Graded vector space

$$L = \bigoplus_{k\in\mathbb{Z}} L_k = \cdots \bigoplus L_{-1} \bigoplus L_0 \bigoplus L_1 \cdots$$

Graded totally antisymmetric multilinear "brackets"

$$\mu_i : \wedge^i L \to L \text{ of degree } |\mu_i| = 2 - i$$

Satisfying homotopy Jacobi identities

$$\sum_{k=1}^{i} (-1)^{i-k} \sum_{\sigma\in Sh(k;i)} \chi(\sigma, l_1, \ldots, l_i)\mu_{i-k+1}(\mu_k(l_{\sigma(1)}, \ldots, l_{\sigma(k)}), \ldots, l_{\sigma(i)}) = 0$$

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Examples

- $\mu_1(\mu_1(l)) = 0 \implies \mu_1$ is a differential, turning L into a complex

$$\cdots \xrightarrow{\mu_1} L_{-1} \xrightarrow{\mu_1} L_0 \xrightarrow{\mu_1} L_1 \xrightarrow{\mu_1} \cdots$$

$$H^{\bullet}_{\mu_1}(L) = \ker(\mu_1)/\mathrm{im}(\mu_1) \text{ cohomology groups}$$

$$[l] = \{l' \in \ker(\mu_1) : l' = l + \mu_1(\alpha)\}$$

$$\cdots \xrightarrow{0} H^{-1} \xrightarrow{0} H^{0} \xrightarrow{0} H^{1} \xrightarrow{0} \cdots$$

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Examples

- $\mu_1(\mu_2(l_1, l_2)) = \mu_2(\mu_1(l_1), l_2) \pm \mu_2(l_1, \mu_1(l_2))$

μ_1 **is a derivation respect to** μ_2

- $\mu_2(l_1, \mu_2(l_2, l_3)) \pm \mu_2(l_3, \mu_2(l_1, l_2)) \pm \mu_2(l_2, \mu_2(l_3, l_1))$

$$= \mu_3(\mu_1(l_1), l_2, l_3) \pm \mu_3(l_1, \mu_1(l_2), l_3) \pm \mu_3(l_1, l_2, \mu_1(l_3))$$

$$+\mu_1(\mu_3(l_1, l_2, l_3))$$

μ_3 **provides homotopy for** μ_2

$\vdots$

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Dual picture of L_∞-algebras

Higher brackets in a basis $\{\tau_\alpha\}$: $\mu_i(\tau_{\beta_1}, \ldots, \tau_{\beta_i}) = f^\alpha_{\beta_1 \ldots \beta_i} \tau_\alpha$

"Shifted" dual space $L[1]^*$ with dual basis ξ^α, i.e $\xi^\alpha(s\tau_\beta) = \delta^\alpha_\beta$

Functions on L[1] are polynomials in the generators ξ^α

F.O.D.O: $Q := \sum_{i=1}^{\infty} f^\alpha_{\beta_1 \ldots \beta_i} \xi^{\beta_1} \ldots \xi^{\beta_i} \frac{\partial}{\partial \xi^\alpha}$

$Q^2 = 0 \iff$ Homotopy Jacobi identities

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Q is a differential (HVF), and $(C^{\infty}(L[1]), Q)$ is a differential-graded algebra (dga).

L_{∞}-algebras are dual to dga!

The homological vector field encodes all the higher brackets!

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(classical) BV formalism in a nutshell

The most general approach to the quantisation of gauge theories

- Resolve the **quotient space of classical observables**
  - Introduce ghost fields to resolve gauge redundancy (BRST)
  - Introduce anti-fields to resolve EoM
- Structure of BV space:
  - $\mathcal{F}_{BV} = \mathfrak{g}[1] \oplus \mathcal{F} \oplus \mathfrak{g}^*[-2] \oplus \mathcal{F}^*[-1] = T^*[-1](\mathcal{F} \oplus \mathfrak{g}[1])$
  - $S_{BV}[c, \Phi, c^*, \Phi^*]$ s.t. $S_{BV}[0, \Phi, 0, 0] = S[\Phi]$
  - (F, G) for $F, G \in C^\infty(\mathcal{F}_{BV}), |(-,-)| = 1$

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BV formalism and L_∞-algebras

BV (classical) transformations $Q_{BV} = (-, S_{BV})$

$Q_{BV}^2 = 0 \iff (S_{BV}, S_{BV}) = 0$: "Classical Master Equation"

$C^\infty(\mathcal{F}_{BV})$ is now a dga due to Q_{BV}

$\implies \mathcal{F}_{BV}[-1]$ is an L_∞-algebra

with higher Brackets given by $Q_{BV} \iff S_{BV}$

This L_∞-algebra comes with $\langle -, - \rangle$ due to the antibracket

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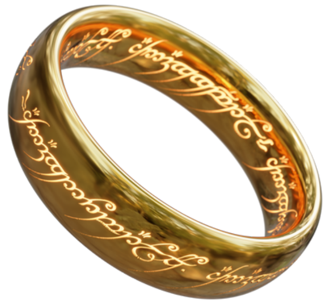
L_∞-algebras and Classical Field Theory

This homotopy Lie-algebra encodes everything ther is to know about a classical theory!

$$\underbrace{\begin{array}{c}\text{gauge}\\ \text{symmetries}\end{array}}_{\ldots,\,L_{-1},\,L_0} \longrightarrow \underbrace{\begin{array}{c}\text{Classical}\\ \text{Fields}\end{array}}_{L_1} \longrightarrow \underbrace{\begin{array}{c}\text{Equations}\\ \text{of Motion}\end{array}}_{L_2} \longrightarrow \underbrace{\begin{array}{c}\text{Noether}\\ \text{Identities}\end{array}}_{L_3,\,L_4,\,\ldots}$$

$$S = \sum_{i\geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a,\ldots,a)\rangle =: S_{MC}[a]$$

$$S_{BV} = S_{MC}[c + a + a^* + c^*]$$



All Field Theories are hMC!

Why should I care?

Homotopy algebraic technology in Physics

- Equivalences are (quasi)-isomorphisms
- Factorisation
  - Colour-stripping
- Strictification theorem
  - Colour-Kinematics duality
  - Rendering a field theory cubic
- Homotopy Transfer and Minimal Model
  - Tree-level Scattering Amplitudes
  - Berends-Giele recursion relations

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Further applications

- "Double copy prescription" $\text{gravity} = \text{gauge} \ \otimes \text{gauge}$

- Generalisation to $loop - level$ Scattering-Amplitudes

more . . .