Stochastic differential equations

With insights to the Langevin equation

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Overview

1. Stochastic differential equation

2. The Langevin equation

Comparison

ODE form:

- $m\frac{d\mathbf{v}}{dt} = \mathbf{F}$
- Deterministic, our output value is fully determined by the parameter values with no randomness.
- Equation of motion for many common things, ie. cars.

SDE form (Langevin Equation):

- $m\frac{d\mathbf{v}}{dt} = -\lambda\mathbf{v} + \eta(t)$
- The force acting on a particle is the sum of a viscous force proportional to the particle's velocity, $-\lambda v$ and a noise term $\eta(t)$
- Stochastic, our model has some randomness in the term $\eta(t)$.
- Brownian motion for particles suspended in a medium.

Relevant methods

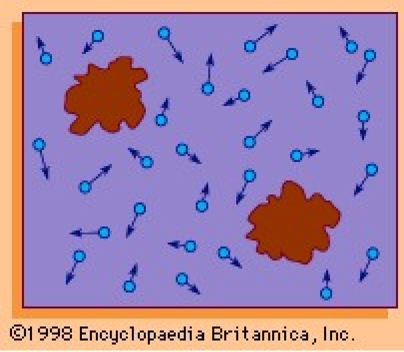
- Analytical:
  - SDE: Ito calculus gives us the integral form $m\mathbf{v} = \int^{t}(-\lambda\mathbf{v} + \eta(t))dt$
  - PDE alternative: Focker Planck equations
    - Treat the time evolution of the full probability distribution (Similar to Schrödinger equation)
    - Inconvenient: Not easy to solve for 2 and 3 dims.
- Numerical: Euler/Runge-Kutta integration with random component
  - Keeps the dynamical treatment of the problem
  - Simple to implement in our times
  - Repeat the simulation many times to obtain a distribution

Langevin equation

- First stochastic differential equation.
- Developed to describe the Brownian motion, ie the motion of particles in a medium.
- Nowadays extended to many other fields with a generic form.
  - Thermal noise
  - Stock market
  - Particle in a fluid
- Two sets of variables: **macroscopic** (slow) and **microscopic** (fast/stochastic)
  - Fast: Local thermodynamic equilibrium in a liquid settles within a few collisions.
  - Slow: Parts like mass and energy takes longer to relax to equilibrium.

The equation

Intuitive derivation:



Take newton's classical particle in a fluid equation and add some stochastic noise.

$$m\frac{d\mathbf{v}}{dt} + \lambda\mathbf{v} = 0 \quad \Longrightarrow \quad m\frac{d\mathbf{v}}{dt} + \lambda\mathbf{v} = \eta(t) \tag{1}$$

Coments on η

- step of a random walk $\overbrace{\Longrightarrow}^{CLT}$ Normal uncorrelated distribution
- gaussian with $< \eta(t)\eta(t') >= 2\lambda k_B T\delta(t - t')$
- approximation. In a microscopic reality gas' molecules speeds are of course correlated at the collision time but we look at slow property (speed of the particle).
- $\bar{\eta} = 0$

Generic form

Mathematical derivation starting from Zwanzig operator (separate slow and fast variables)

$$\frac{dA_i}{dt} = k_B T \sum_j [A_i, A_j] \frac{d\mathcal{H}}{dA_j} - \sum_j \lambda_{i,j}(A) \frac{d\mathcal{H}}{dA_j} + \sum_j \frac{d\lambda_{i,j}(A)}{dA_j} + \eta_i(t)$$

Example simulation

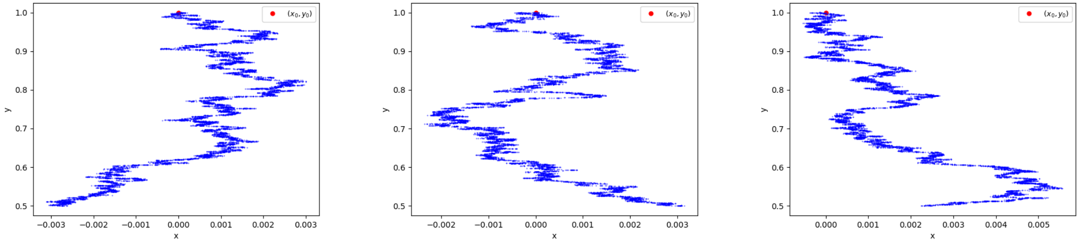


Figure: Different examples of one particle random walk (migration to XZ-plane located at y=0.5)

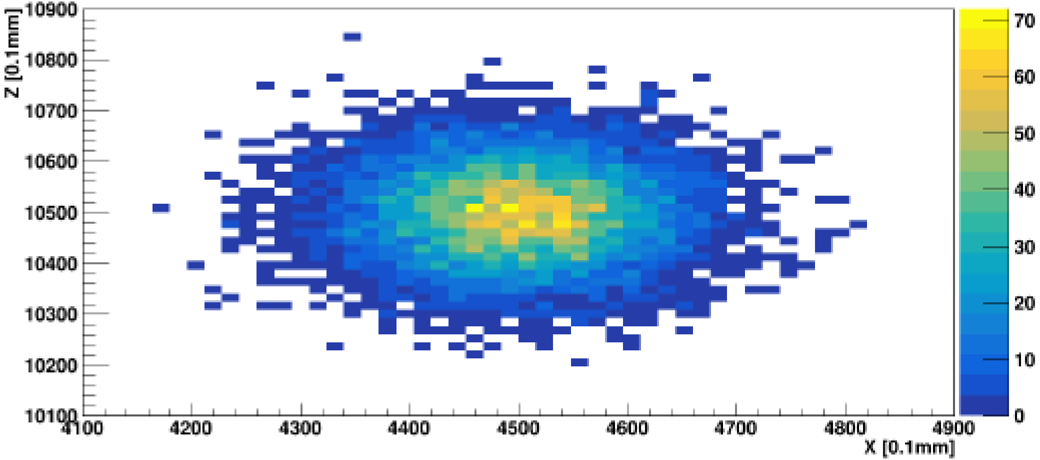


Figure: Probability distribution (2D gaussian)

References

- https://www.probabilitycourse.com/chapter11
- F. B. Tomás *Simulations in GLAD-TPC* Universidade de Santiago de Compostela, 2020
- Coffey, W. T. and Kalmykov, Yu P. and Waldron, J. T. *The Langevin Equation. With Applications in Physics, Chemistry and Electrical Engineering* (pp. 1-25) World Scientific Publishing Co. Pte. Ltd., Singapore, 1996.
- https://en.wikipedia.org/wiki/Langevin$_e$*quation*

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