moments, the goal of moment matching then becomes that of matching the second moment and, possibly, the fourth moment.

Using Quasi-Random Sequences

A quasi-random sequence (also called a *low-discrepancy* sequence) is a sequence of representative samples from a probability distribution.[19] Descriptions of the use of quasi-random sequences appear in Brotherton-Ratcliffe, and Press *et al*.[20] Quasi-random sequences can have the desirable property that they lead to the standard error of an estimate being proportional to $1/M$ rather than $1/\sqrt{M}$, where M is the sample size.

Quasi-random sampling is similar to stratified sampling. The objective is to sample representative values for the underlying variables. In stratified sampling, it is assumed that we know in advance how many samples will be taken. A quasi-random sampling procedure is more flexible. The samples are taken in such a way that we are always "filling in" gaps between existing samples. At each stage of the simulation, the sampled points are roughly evenly spaced throughout the probability space.

Figure 21.14 shows points generated in two dimensions using a procedure by Sobol'.[21] It can be seen that successive points do tend to fill in the gaps left by previous points.

21.8 FINITE DIFFERENCE METHODS

Finite difference methods value a derivative by solving the differential equation that the derivative satisfies. The differential equation is converted into a set of difference equations, and the difference equations are solved iteratively.

To illustrate the approach, we consider how it might be used to value an American put option on a stock paying a dividend yield of q. The differential equation that the option must satisfy is, from equation (17.6),

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{21.21}$$

Suppose that the life of the option is T. We divide this into N equally spaced intervals of length $\Delta t = T/N$. A total of $N+1$ times are therefore considered

$$0,\ \Delta t,\ 2\,\Delta t,\ \ldots,\ T$$

Suppose that $S_{\max}$ is a stock price sufficiently high that, when it is reached, the put has virtually no value. We define $\Delta S = S_{\max}/M$ and consider a total of $M+1$ equally spaced stock prices:

$$0,\ \Delta S,\ 2\,\Delta S,\ \ldots,\ S_{\max}$$

The level $S_{\max}$ is chosen so that one of these is the current stock price.

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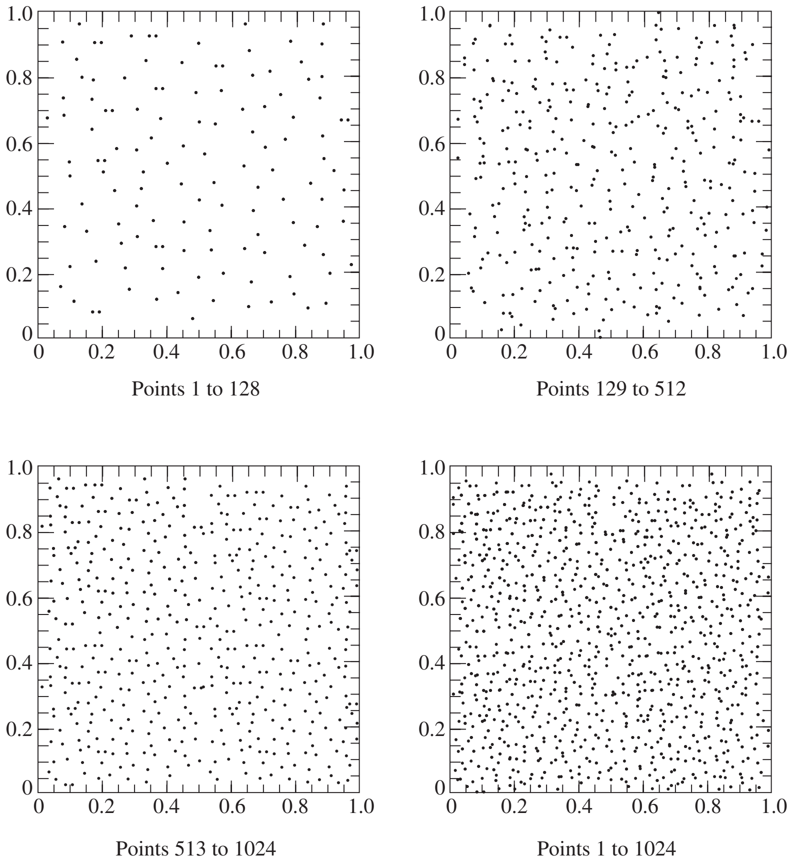
[19] The term *quasi-random* is a misnomer. A quasi-random sequence is totally deterministic.

[20] See R. Brotherton-Ratcliffe, "Monte Carlo Motoring," *Risk*, December 1994: 53–58; W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn. Cambridge University Press, 1992.

[21] See I. M. Sobol', *USSR Computational Mathematics and Mathematical Physics*, 7, 4 (1967): 86–112. A description of Sobol's procedure is in W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn. Cambridge University Press, 1992.

Figure 21.14 First 1,024 points of a Sobol' sequence.



The time points and stock price points define a grid consisting of a total of $(M + 1)(N + 1)$ points, as shown in Figure 21.15. We define the (i, j) point on the grid as the point that corresponds to time $i\ \Delta t$ and stock price $j\ \Delta S$. We will use the variable $f_{i,j}$ to denote the value of the option at the (i, j) point.

Implicit Finite Difference Method

For an interior point (i, j) on the grid, $\partial f/\partial S$ can be approximated as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S} \tag{21.22}$$

or as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \tag{21.23}$$

Equation (21.22) is known as the *forward difference approximation*; equation (21.23) is known as the *backward difference approximation*. We use a more symmetrical approximation by averaging the two:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta S} \tag{21.24}$$

For $\partial f/\partial t$, we will use a forward difference approximation so that the value at time $i\,\Delta t$ is related to the value at time $(i+1)\,\Delta t$:

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{21.25}$$

Consider next $\partial^2 f/\partial S^2$. The backward difference approximation for $\partial f/\partial S$ at the (i, j) point is given by equation (21.23). The backward difference at the $(i, j+1)$ point is

$$\frac{f_{i,j+1} - f_{i,j}}{\Delta S}$$

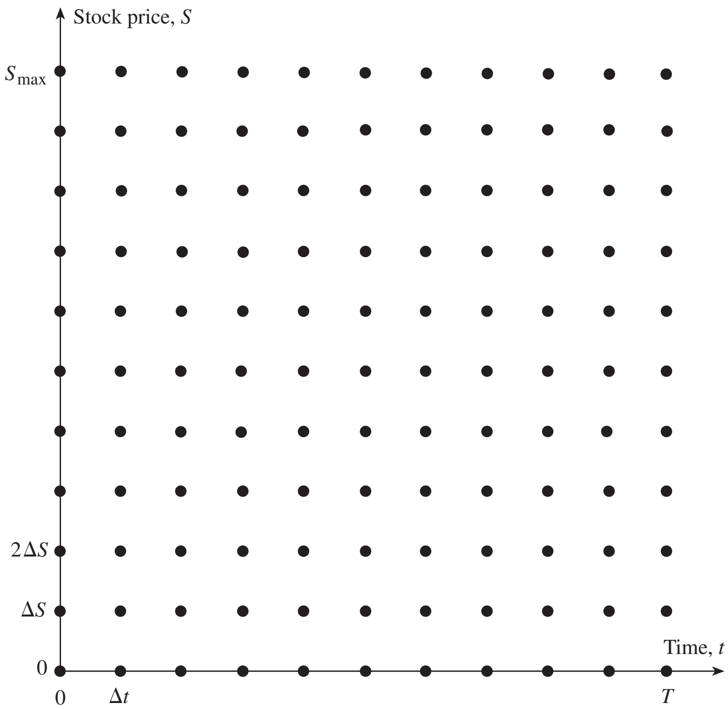
Hence a finite difference approximation for $\partial^2 f/\partial S^2$ at the (i, j) point is

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S}\right) \Big/ \Delta S$$

or

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} \tag{21.26}$$

Figure 21.15 Grid for finite difference approach.



Substituting equations (21.24), (21.25), and (21.26) into the differential equation (21.21) and noting that $S = j\,\Delta S$ gives

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q)j\,\Delta S\frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta S} + \tfrac{1}{2}\sigma^2 j^2\,\Delta S^2\frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} = rf_{i,j}$$

for $j = 1, 2, \ldots, M - 1$ and $i = 0, 1, \ldots, N - 1$. Rearranging terms, we obtain

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{21.27}$$

where

$$a_j = \tfrac{1}{2}(r - q)j\,\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t$$
$$b_j = 1 + \sigma^2 j^2 \Delta t + r\,\Delta t$$
$$c_j = -\tfrac{1}{2}(r - q)j\,\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t$$

The value of the put at time T is $\max(K - S_T, 0)$, where S_T is the stock price at time T. Hence,

$$f_{N,j} = \max(K - j\,\Delta S, 0), \quad j = 0, 1, \ldots, M \tag{21.28}$$

The value of the put option when the stock price is zero is K. Hence,

$$f_{i,0} = K, \quad i = 0, 1, \ldots, N \tag{21.29}$$

We assume that the put option is worth zero when $S = S_{\max}$, so that

$$f_{i,M} = 0, \quad i = 0, 1, \ldots, N \tag{21.30}$$

Equations (21.28), (21.29), and (21.30) define the value of the put option along the three edges of the grid in Figure 21.15, where $S = 0$, $S = S_{\max}$, and $t = T$. It remains to use equation (21.27) to arrive at the value of f at all other points. First the points corresponding to time $T - \Delta t$ are tackled. Equation (21.27) with $i = N - 1$ gives

$$a_j f_{N-1,j-1} + b_j f_{N-1,j} + c_j f_{N-1,j+1} = f_{N,j} \tag{21.31}$$

for $j = 1, 2, \ldots, M - 1$. The right-hand sides of these equations are known from equation (21.28). Furthermore, from equations (21.29) and (21.30),

$$f_{N-1,0} = K \tag{21.32}$$

$$f_{N-1,M} = 0 \tag{21.33}$$

Equations (21.31) are therefore $M - 1$ simultaneous equations that can be solved for the $M - 1$ unknowns: $f_{N-1,1}, f_{N-1,2}, \ldots, f_{N-1,M-1}$.[22] After this has been done, each value

[22] This does not involve inverting a matrix. The $j = 1$ equation in (21.31) can be used to express $f_{N-1,2}$ in terms of $f_{N-1,1}$; the $j = 2$ equation, when combined with the $j = 1$ equation, can be used to express $f_{N-1,3}$ in terms of $f_{N-1,1}$; and so on. The $j = M - 2$ equation, together with earlier equations, enables $f_{N-1,M-1}$ to be expressed in terms of $f_{N-1,1}$. The final $j = M - 1$ equation can then be solved for $f_{N-1,1}$, which can then be used to determine the other $f_{N-1,j}$.

of $f_{N-1,j}$ is compared with $K - j\,\Delta S$. If $f_{N-1,j} < K - j\,\Delta S$, early exercise at time $T - \Delta t$ is optimal and $f_{N-1,j}$ is set equal to $K - j\,\Delta S$. The nodes corresponding to time $T - 2\,\Delta t$ are handled in a similar way, and so on. Eventually, $f_{0,1}, f_{0,2}, f_{0,3}, \ldots, f_{0,M-1}$ are obtained. One of these is the option price of interest.

The control variate technique can be used in conjunction with finite difference methods. The same grid is used to value an option similar to the one under consideration but for which an analytic valuation is available. Equation (21.20) is then used.

Example 21.10

Table 21.4 shows the result of using the implicit finite difference method as just described for pricing the American put option in Example 21.1. Values of 20, 10, and 5 were chosen for M, N, and ΔS, respectively. Thus, the option price is evaluated at \$5 stock price intervals between \$0 and \$100 and at half-month time intervals throughout the life of the option. The option price given by the grid is \$4.07. The same grid gives the price of the corresponding European option as \$3.91. The true European price given by the Black–Scholes–Merton formula is \$4.08. The control variate estimate of the American price is therefore

$$4.07 + (4.08 - 3.91) = \$4.24$$

Explicit Finite Difference Method

The implicit finite difference method has the advantage of being very robust. It always converges to the solution of the differential equation as ΔS and Δt approach zero.[23] One of the disadvantages of the implicit finite difference method is that $M - 1$ simultaneous equations have to be solved in order to calculate the $f_{i,j}$ from the $f_{i+1,j}$. The method can be simplified if the values of $\partial f/\partial S$ and $\partial^2 f/\partial S^2$ at point (i, j) on the grid are assumed to be the same as at point $(i + 1, j)$. Equations (21.24) and (21.26) then become

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\,\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2}$$

The difference equation is

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q)j\,\Delta S\,\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} + \tfrac{1}{2}\sigma^2 j^2 \Delta S^2\,\frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2} = r f_{i,j}$$

or

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1} \qquad \textbf{(21.34)}$$

[23] A general rule in finite difference methods is that ΔS should be kept proportional to $\sqrt{\Delta t}$ as they approach zero.

Table 21.4 Grid to value American option in Example 21.1 using implicit finite difference methods.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	0.05	0.04	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00
85	0.09	0.07	0.05	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00
80	0.16	0.12	0.09	0.07	0.04	0.03	0.02	0.01	0.00	0.00	0.00
75	0.27	0.22	0.17	0.13	0.09	0.06	0.03	0.02	0.01	0.00	0.00
70	0.47	0.39	0.32	0.25	0.18	0.13	0.08	0.04	0.02	0.00	0.00
65	0.82	0.71	0.60	0.49	0.38	0.28	0.19	0.11	0.05	0.02	0.00
60	1.42	1.27	1.11	0.95	0.78	0.62	0.45	0.30	0.16	0.05	0.00
55	2.43	2.24	2.05	1.83	1.61	1.36	1.09	0.81	0.51	0.22	0.00
50	4.07	3.88	3.67	3.45	3.19	2.91	2.57	2.17	1.66	0.99	0.00
45	6.58	6.44	6.29	6.13	5.96	5.77	5.57	5.36	5.17	5.02	5.00
40	10.15	10.10	10.05	10.01	10.00	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

where

$$a_j^* = \frac{1}{1 + r\,\Delta t}\left(-\tfrac{1}{2}(r - q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2\,\Delta t\right)$$

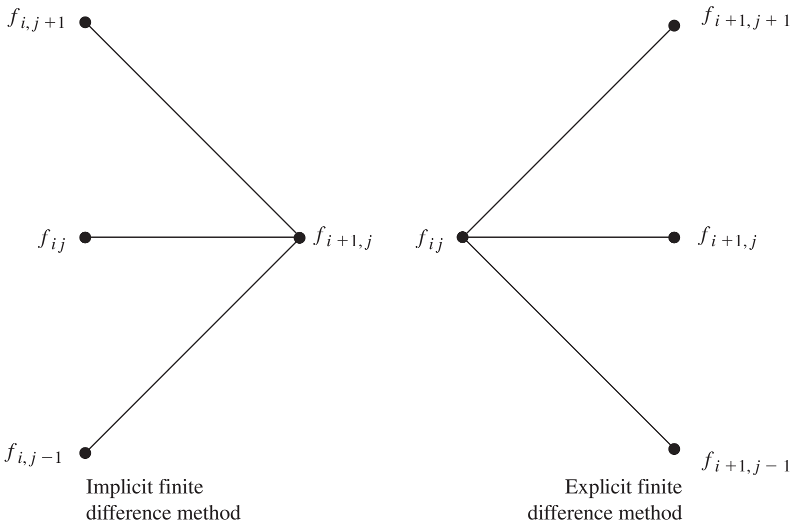
$$b_j^* = \frac{1}{1 + r\,\Delta t}\left(1 - \sigma^2 j^2\,\Delta t\right)$$

$$c_j^* = \frac{1}{1 + r\,\Delta t}\left(\tfrac{1}{2}(r - q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2\,\Delta t\right)$$

This creates what is known as the *explicit finite difference method*.[24] Figure 21.16 shows the difference between the implicit and explicit methods. The implicit method leads to equation (21.27), which gives a relationship between three different values of the option at time $i\,\Delta t$ (i.e., $f_{i,j-1}$, $f_{i,j}$, and $f_{i,j+1}$) and one value of the option at time $(i + 1)\,\Delta t$ (i.e., $f_{i+1,j}$). The explicit method leads to equation (21.34), which gives a relationship

[24] We also obtain the explicit finite difference method if we use the backward difference approximation instead of the forward difference approximation for $\partial f/\partial t$.

Figure 21.16 Difference between implicit and explicit finite difference methods.



between one value of the option at time $i\,\Delta t$ (i.e., $f_{i,j}$) and three different values of the option at time $(i+1)\,\Delta t$ (i.e., $f_{i+1,j-1}, f_{i+1,j}, f_{i+1,j+1}$).

Example 21.11

Table 21.5 shows the result of using the explicit version of the finite difference method for pricing the American put option described in Example 21.1. As in Example 21.10, values of 20, 10, and 5 were chosen for M, N, and ΔS, respectively. The option price given by the grid is \$4.26.[25]

Change of Variable

When geometric Brownian motion is used for the underlying asset price, it is computationally more efficient to use finite difference methods with ln S rather than S as the underlying variable. Define $Z = \ln S$. Equation (21.21) becomes

$$\frac{\partial f}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial Z} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial Z^2} = rf$$

The grid then evaluates the derivative for equally spaced values of Z rather than for equally spaced values of S. The difference equation for the implicit method becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q - \sigma^2/2)\,\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta Z} + \tfrac{1}{2}\sigma^2\,\frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta Z^2} = r\,f_{i,j}$$

or

$$\alpha_j\, f_{i,j-1} + \beta_j\, f_{i,j} + \gamma_j\, f_{i,j+1} = f_{i+1,j} \qquad \textbf{(21.35)}$$

[25] The negative numbers and other inconsistencies in the top left-hand part of the grid will be explained later.

Table 21.5 Grid to value American option in Example 21.1 using explicit finite difference methods.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	−0.11	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
85	0.28	−0.05	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	−0.13	0.20	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
75	0.46	0.06	0.20	0.04	0.06	0.00	0.00	0.00	0.00	0.00	0.00
70	0.32	0.46	0.23	0.25	0.10	0.09	0.00	0.00	0.00	0.00	0.00
65	0.91	0.68	0.63	0.44	0.37	0.21	0.14	0.00	0.00	0.00	0.00
60	1.48	1.37	1.17	1.02	0.81	0.65	0.42	0.27	0.00	0.00	0.00
55	2.59	2.39	2.21	1.99	1.77	1.50	1.24	0.90	0.59	0.00	0.00
50	4.26	4.08	3.89	3.68	3.44	3.18	2.87	2.53	2.07	1.56	0.00
45	6.76	6.61	6.47	6.31	6.15	5.96	5.75	5.50	5.24	5.00	5.00
40	10.28	10.20	10.13	10.06	10.01	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

where

$$\alpha_j = \frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) - \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

$$\beta_j = 1 + \frac{\Delta t}{\Delta Z^2}\sigma^2 + r\,\Delta t$$

$$\gamma_j = -\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) - \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

The difference equation for the explicit method becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q - \sigma^2/2)\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta Z} + \tfrac{1}{2}\sigma^2\frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta Z^2} = rf_{i,j}$$

or

$$\alpha_j^* f_{i+1,j-1} + \beta_j^* f_{i+1,j} + \gamma_j^* f_{i+1,j+1} = f_{i,j} \tag{21.36}$$

where

$$\alpha_j^* = \frac{1}{1 + r\Delta t}\left[-\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2\right] \tag{21.37}$$

$$\beta_j^* = \frac{1}{1 + r\Delta t}\left(1 - \frac{\Delta t}{\Delta Z^2}\sigma^2\right) \tag{21.38}$$

$$\gamma_j^* = \frac{1}{1 + r\Delta t}\left[\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2\right] \tag{21.39}$$

The change of variable approach has the property that α_j, β_j, and γ_j as well as α_j^*, β_j^*, and γ_j^* are independent of j. In most cases, a good choice for ΔZ is $\sigma\sqrt{3\Delta t}$.

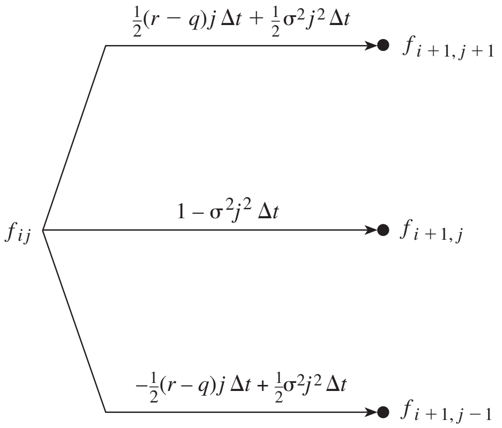
Relation to Trinomial Tree Approaches

The explicit finite difference method is equivalent to the trinomial tree approach.[26] In the expressions for a_j^*, b_j^*, and c_j^* in equation (21.34), we can interpret terms as follows:

$-\frac{1}{2}(r - q)j\,\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$:	Probability of stock price decreasing from $j\Delta S$ to $(j - 1)\Delta S$ in time Δt.
$1 - \sigma^2 j^2 \Delta t$:	Probability of stock price remaining unchanged at $j\Delta S$ in time Δt.
$\frac{1}{2}(r - q)j\,\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$:	Probability of stock price increasing from $j\Delta S$ to $(j + 1)\Delta S$ in time Δt.

This interpretation is illustrated in Figure 21.17. The three probabilities sum to unity. They give the expected increase in the stock price in time Δt as $(r - q)j\,\Delta S\,\Delta t = (r - q)S\,\Delta t$. This is the expected increase in a risk-neutral world. For small values

Figure 21.17 Interpretation of explicit finite difference method as a trinomial tree.



[26] It can also be shown that the implicit finite difference method is equivalent to a multinomial tree approach where there are $M + 1$ branches emanating from each node.

of Δt, they also give the variance of the change in the stock price in time Δt as $\sigma^2 j^2 \Delta S^2 \Delta t = \sigma^2 S^2 \Delta t$. This corresponds to the stochastic process followed by S. The value of f at time $i\,\Delta t$ is calculated as the expected value of f at time $(i+1)\,\Delta t$ in a risk-neutral world discounted at the risk-free rate.

For the explicit version of the finite difference method to work well, the three "probabilities"

$$-\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t,$$
$$1-\sigma^2 j^2 \Delta t$$
$$\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t$$

should all be positive. In Example 21.11, $1-\sigma^2 j^2 \Delta t$ is negative when $j \geq 13$ (i.e., when $S \geq 65$). This explains the negative option prices and other inconsistencies in the top left-hand part of Table 21.5. This example illustrates the main problem associated with the explicit finite difference method. Because the probabilities in the associated tree may be negative, it does not necessarily produce results that converge to the solution of the differential equation.[27]

When the change-of-variable approach is used (see equations (21.36) to (21.39)), the probability that $Z = \ln S$ will decrease by ΔZ, stay the same, and increase by ΔZ are

$$-\frac{\Delta t}{2\Delta Z}(r-q-\sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2$$
$$1-\frac{\Delta t}{\Delta Z^2}\sigma^2$$
$$\frac{\Delta t}{2\Delta Z}(r-q-\sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

respectively. These movements in Z correspond to the stock price changing from S to $Se^{-\Delta Z}$, S, and $Se^{\Delta Z}$, respectively. If we set $\Delta Z = \sigma\sqrt{3\Delta t}$, then the tree and the probabilities are identical to those for the trinomial tree approach discussed in Section 21.4.

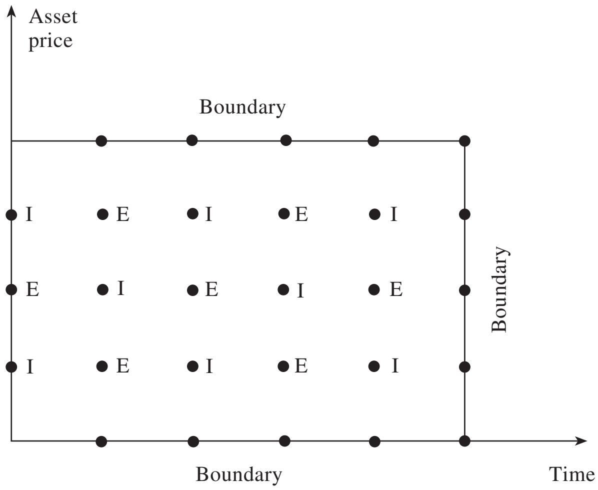
Other Finite Difference Methods

Researchers have proposed other finite difference methods which are in many circumstances more computationally efficient than either the pure explicit or pure implicit method.

In what is known as the *hopscotch method*, we alternate between the explicit and implicit calculations as we move from node to node. This is illustrated in Figure 21.18. At each time, we first do all the calculations at the "explicit nodes" (E) in the usual way. The "implicit nodes" (I) can then be handled without solving a set of simultaneous equations because the values at the adjacent nodes have already been calculated.

[27] J. C. Hull and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25 (March 1990): 87–100, show how this problem can be overcome. In the situation considered here it is sufficient to construct the grid in ln S rather than S to ensure convergence.

Figure 21.18 The hopscotch method. I indicates node at which implicit calculations are done; E indicates node at which explicit calculations are done.



In the *Crank–Nicolson* method, the estimate of

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

is set equal to an average of that given by the implicit and the explicit methods.

Applications of Finite Difference Methods

Finite difference methods can be used for the same types of derivative pricing problems as tree approaches. They can handle American-style as well as European-style derivatives but cannot easily be used in situations where the payoff from a derivative depends on the past history of the underlying variable. Finite difference methods can, at the expense of a considerable increase in computer time, be used when there are several state variables. The grid in Figure 21.15 then becomes multidimensional.

The method for calculating Greek letters is similar to that used for trees. Delta, gamma, and theta can be calculated directly from the $f_{i,j}$ values on the grid. For vega, it is necessary to make a small change to volatility and recalculate the value of the derivative using the same grid.

SUMMARY

We have presented three different numerical procedures for valuing derivatives when no analytic solution is available. These involve the use of trees, Monte Carlo simulation, and finite difference methods.

Binomial trees assume that, in each short interval of time Δt, a stock price either moves up by a multiplicative amount u or down by a multiplicative amount d. The sizes of u

and d and their associated probabilities are chosen so that the change in the stock price has the correct mean and standard deviation in a risk-neutral world. Derivative prices are calculated by starting at the end of the tree and working backward. For an American option, the value at a node is the greater of (a) the value if it is exercised immediately and (b) the discounted expected value if it is held for a further period of time Δt.

Monte Carlo simulation involves using random numbers to sample many different paths that the variables underlying the derivative could follow in a risk-neutral world. For each path, the payoff is calculated and discounted at the risk-free interest rate. The arithmetic average of the discounted payoffs is the estimated value of the derivative.

Finite difference methods solve the underlying differential equation by converting it to a difference equation. They are similar to tree approaches in that the computations work back from the end of the life of the derivative to the beginning. The explicit finite difference method is functionally the same as using a trinomial tree. The implicit finite difference method is more complicated but has the advantage that the user does not have to take any special precautions to ensure convergence.

In practice, the method that is chosen is likely to depend on the characteristics of the derivative being evaluated and the accuracy required. Monte Carlo simulation works forward from the beginning to the end of the life of a derivative. It can be used for European-style derivatives and can cope with a great deal of complexity as far as the payoffs are concerned. It becomes relatively more efficient as the number of underlying variables increases. Tree approaches and finite difference methods work from the end of the life of a security to the beginning and can accommodate American-style as well as European-style derivatives. However, they are difficult to apply when the payoffs depend on the past history of the state variables as well as on their current values. Also, they are liable to become computationally very time consuming when three or more variables are involved.

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