MCMC Assignment 1

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Assignment 1

In this assignment, we will show, that if we want to generate from some continuous random variable $X \sim f$, we can just generate uniformly from the region

$$C_f = \{(u,v) : 0 \leq u \leq \sqrt{f(\frac{u}{v})}\}$$

.

If we denote $Z = \frac{U}{V}$, distribution of Z is

$$g_Z(z) = \frac{f(z)}{K}.$$

So if we are able to generate uniformly (U, V) from the set C_f, the ratio of the generated uniform variables has the desired distribution up to normalising constant K, which is mostly area of the set C_f.

Proof of the distribution g(z):

We will proof () with the help of theorem of transformation of random vectors. Let us denote

$$Z = \frac{Y}{X} \quad U = X,$$

Jacobian of this transformation is u. So based on the transformation of the random vectors theorem, we have

$$g_{(Z,U)}(z,u) = \frac{u}{K}\mathbb{I}\{(u,z) : 0 \leq u \leq f(z)\},$$

where

$$K = \int \int_0^{\sqrt{f(z)}} ududz = \frac{1}{2} \int f(z)dz.$$

Next, the marginal distribution of Z is

$$g_Z(z) = \int_0^{\sqrt{f(z)}} \frac{u}{K} du = \frac{\frac{1}{2} f(z)}{K}.$$

If we plug-in K into equation above, we have

$$g_Z(z) = \frac{f(z)}{\int f(z)dz}$$

Next, in the process of generating from the some continuous distribution f, we will choose the accept-reject method. The algorithm is following:

1. Generate $U \sim U(0, b)$, $V \sim U(c, d)$,where b, c, d are boundaries based on the set C_f (which is the tricky part). One of the option to choose the boundaries is:

$$b = \sup \sqrt{f(z)}, \quad c = u \inf \sqrt{f(z)}, \quad d = u \inf \sqrt{f(z)}$$

2. Set $Z = \frac{U}{V}$ if $u \leq \sqrt{f(z)}$, otherwise, go back to the step 1.

Steps above describes, how we can generate from the general distribution f. Right now, we will show, how we can generate from the $X \sim Cauchy$. Let us denote $f(x) = \frac{1}{1+x^2}$ and the set C_f for our case is following:

$$C_f = \{(u, v) : 0 \leq u \leq \sqrt{\frac{1}{1 + (\frac{v}{u}}})^2\},$$

Right now, we need to choose the boundaries for both u and v, for generating uniformly from the set C_f. But in this case, it is quite easy, so with some little changes in C_f, we have

$$C_f = \{(u, v) : u^2 + v^2 \leq 1, u \geq 0, v \in [-1, 1]\}.$$

So for the successful generating from the Cauchy distribution, we just generate uniformly from the half circle with radius 1. The algorithm is following

1. Generate $U \sim U(0, 1)$, $V \sim U(-1, 1)$
2. Set $Z = \frac{V}{U}$ if $u^2 + v^2 \leq 1$, otherwise go to the step 1.

The code for this algorithm can be found in GitHub repository created by me for purposes of solving the assignments for the MCMC course.