Discrete random walks with memory: Models and applications

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Outline

1. Prepare mathematical model
2. Describe its properties
3. Apply it on data

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Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O.

[Karl Pearson: The problem of the random walk. (1905)]

Where is the *"Drunken sailor"*?

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Let$\{X_k\}_{k=1}^{\infty}$be a sequence of independent, identically distributed discrete random variables. For each positive integer n, let S_n denote the sum $X_1 + X_2 + \cdots + X_n$, with $S_0 = 0$. The sequence $\{S_n\}_{n=1}^{\infty}$ is called a random walk. If the common range of the X_k's is $\mathbb{R}_m$, then $\{S_n\}$ is a random walk in $\mathbb{R}_m$.

For $X_k \sim B(p = \frac{1}{2})$ it is called the standard random walk.

Random walk properties

- Discrete random process
- $n-$dimensional, on a matrix, graph, finite or infinite set
- Self avoiding, reinforced
- Brownian motion, polymer creation, games simulation, sports simulation

Random walk with memory

- Based on standard random walk (Bernoulli distribution with $p = 0.5$, discrete time).
- Constant total step size:

$$l_i^+ + l_i^- = 2 \ \forall i \in \mathbb{N}.$$

- At the beginning the step sizes are equal ($l_1^+ = l_1^- = 1$) and further for $t > 1$ evolve using a memory parameter $\lambda \in (0, 1)$:

$$X_{t-1} = 1 \rightarrow \begin{cases} l_t^+ = \lambda l_{t-1}^+ \\ l_t^- = 2 - \lambda l_{t-1}^+ \end{cases} \quad X_{t-1} = -1 \rightarrow \begin{cases} l_t^+ = 2 - \lambda l_{t-1}^- \\ l_t^- = \lambda l_{t-1}^- \end{cases}$$

- Loïc Turban. *On a random walk with memory and its relation with markovian processes.* Journal of Physics A: Mathematical and Theoretical (2010).

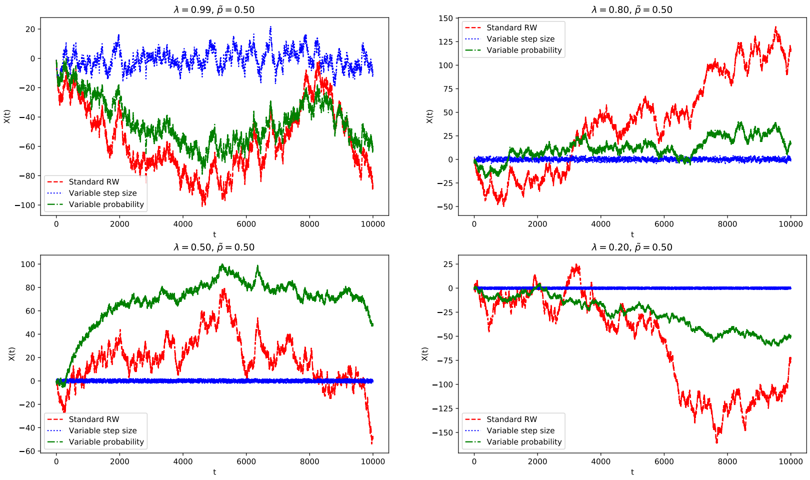
Random walk with varying transition probability

- Based on standard random walk (Bernoulli distribution with $p = 0.5$, discrete time).
- Step size remains constant, transition probability changes
- First step realized according to starting probability p_0 which then for $t > 1$ evolve using a memory parameter $\lambda \in (0, 1)$:

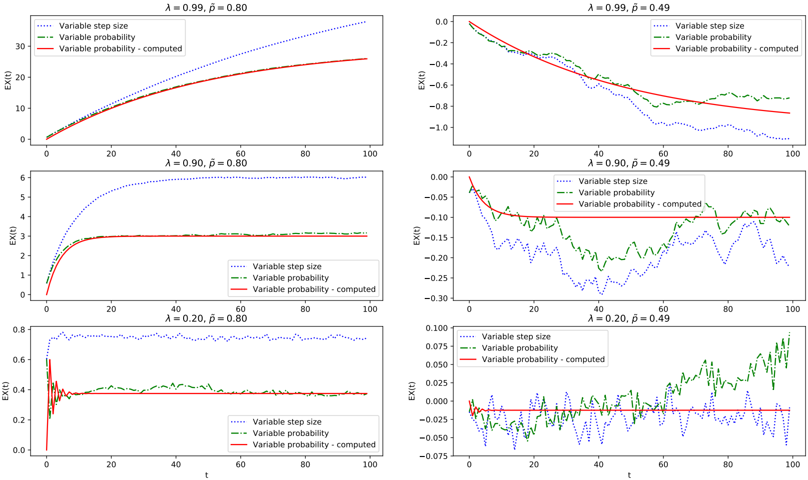
$$X_t = 1 \rightarrow p_t = \lambda p_{t-1}$$

$$X_t = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

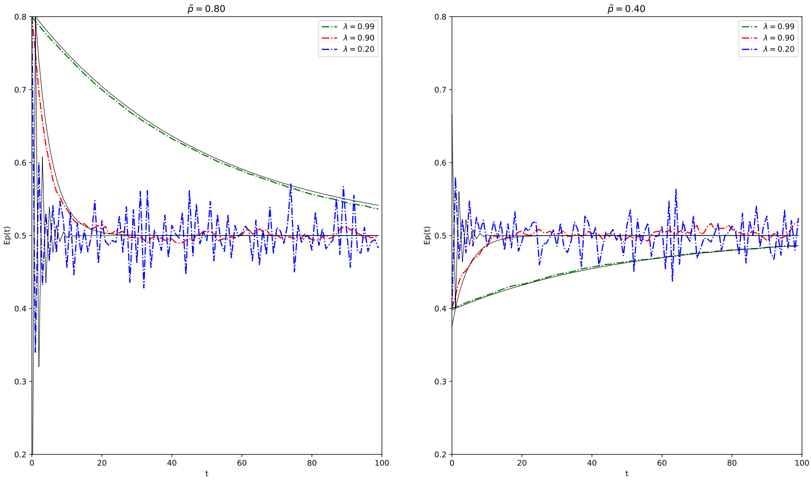
Example - RW evolution



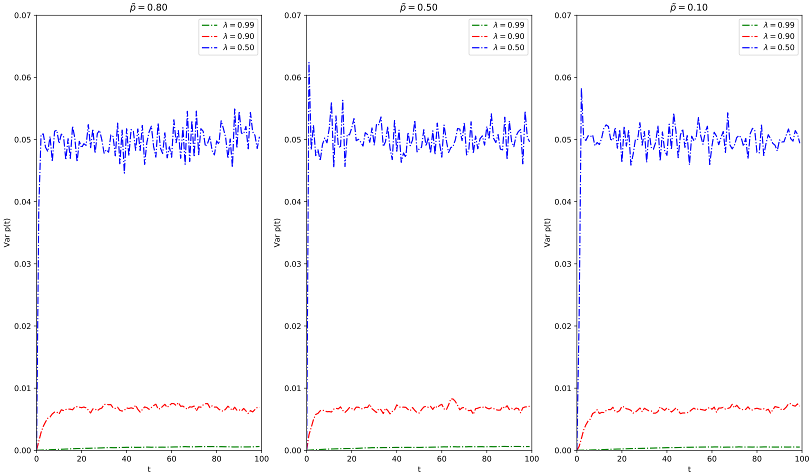
Example - Expected position of the walker



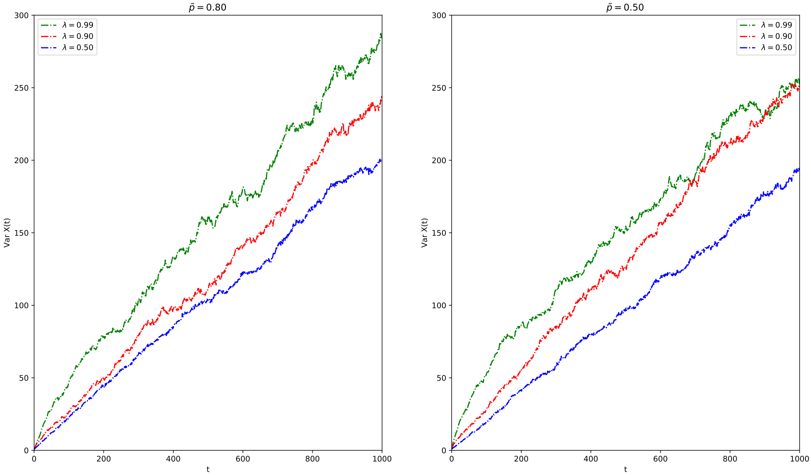
Example - Expected transition probability



Example - Transition probability variance



Example - Walker's position variance



Alternative definitions

- Formulas to obtain next transition probability P_i:
  - $\frac{1}{2}[(1+X_i)\lambda P_{i-1} + (1-X_i)(1-\lambda(1-P_{i-1}))]$
    - Success punished
  - $\frac{1}{2}[(1-X_i)\lambda P_{i-1} + (1+X_i)(1-\lambda(1-P_{i-1}))]$
    - Success rewarded
  - $\frac{1}{2}[(1+X_i)\lambda_0 P_{i-1} + (1-X_i)(1-\lambda_1(1-P_{i-1}))]$
    - Success punished with multiple λ
  - $\frac{1}{2}[(1-X_i)\lambda_0 P_{i-1} + (1+X_i)(1-\lambda_1(1-P_{i-1}))]$
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  - $\frac{1}{2}[(1+X_i)\lambda P_{i-1}+(1-X_i)(1-\lambda(1-P_{i-1}))]$
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Data generation

- Ways to generate data, different conditions, number of repetitions
- popis jaka data generuju

Fitting on generated data

- Which types are detectable and predictable from the data
- Error rates
- Grafy?

Real life examples

- Kratce popsat co jsem delal do Aten
- Zminit, ze jsem desne vydelal na US Open

Results

- Zajimavy nastroj s possible implementations
- link na github kam neco nahraju

Next steps

- Model implementation
  - λ optimization
  - p_1 optimization
- Model improvement
  - Other versions of random walk with memory
  - Combination with other approaches
- Model testing
  - Model evaluation granularity
  - Performance on a larger dataset
  - Bbetting module for more bookmakers
  - Application of the model to *best-of-three* matches
- Application in other domains

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Thank you.

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