Discrete random walks with memory: Models and applications

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Abstract

The contribution focuses on non-Markov random processes, i.e. processes with memory, and is especially concerned with random processes where either one or just a small number of events significantly affects its future development. Reliability analysis, medical studies or sport statistics provide many real-life examples of such processes. This paper focuses on statistical event-history analysis with discretized time, where the role of the hazard rate can be substituted by different variants of transition probabilities. In the simplest case the Bernoulli scheme is used.

The main concern of the paper is the formulation of models describing the dependence of transition probabilities on the process development, as well as on exogenous factors – covariates. Such an impact can be incorporated explicitly and transition probabilities modulated using a few parameters reflecting the current state of the walk as well as the information about the past path. In more complicated cases, as well as in the presence of exogenous covariates, the changes of probabilities are modeled via a regression model, for instance the logistic one. The behavior of proposed random walks is studied both theoretically and with the aid of simulations. Finally, the approach is illustrated on several real data examples.

1 Introduction

Random wak is a well described mathematicall object first introduced by K.Pearson in 1905 [3]. Since then many variations of a random walk have been derived and described and it was applied on many reali life problems (zdroje). Yet there are still new posibilities and options how to alter and improve the classical random walk and present yet another new model representing different real life events. One of such modifications is the random walk with varying step size indroduced in 2010 by Turban [4] which served as an inspiration to the random walk with varying probabilities introduced by Kouřim [1, 2]. It was later shown that such a random walk can be sucesfully applied to predict *in-play* odds and used for actual betting against one of the comercial bookmakers (zdroj Ateny, work in progress). In this paper, the theoretical properties of the model are described and furher examined and the results are tested on generated data. The rest of the paper is organized as follows....

2 Random walk with varying probabilities

The random walk with varying probabilities is based on a standard Bernoulli (zdroj) random walk with some starting transition probability p_0. This probability is then altered after each step of the walk using a coefficient λ so that the repetition of the same step becomes less probable. Formally, it can be defined [2]

Definition 1. Let $\{X_n\}_{n=1}^{\infty}$ and $\{P_n\}_{n=1}^{\infty}$ be sequences of discrete random variables, and $p_0 \in [0,\,1]$ and $\lambda \in (0,\,1)$ constant parameters, such that the first random variable X_1 is given by

$$P(X_1 = 1) = p_0$$

$$P(X_1 = -1) = 1 - p_0.$$

Further

$$P_1 = \lambda p_0 + \frac{1}{2}(1-\lambda)(1-X_1) \tag{1}$$

and for $i \geq 2$

$$P(X_i = 1|P_{i-1} = p_{i-1}) = p_{i-1}$$

$$P(X_i = -1|P_{i-1} = p_{i-1}) = 1 - p_{i-1}$$

$$P_i = \lambda P_{i-1} + \frac{1}{2}(1-\lambda)(1-X_i). \tag{2}$$

The sequence $\{S_n\}_{n=0}^{\infty}$, $S_N = S_0 + \sum_{i=1}^{N} X_i$ for $n \in \mathbb{N}$, with $S_0 \in \mathbb{R}$ some given starting position, is called a *random walk with varying probabilities*, with $\{X_n\}_{n=1}^{\infty}$ being the steps of the walker and $\{P_n\}_{n=1}^{\infty}$ transition probabilities.

From [2], it can be further derived that at each step $t+k,\ t,\,k > 0$ the value of a transition probability P_{t+k} can be computed from the knowledge of transition probability P_t and the realization of the walk $X_{t+1},\,\ldots,\,X_{t+k}$ using formula

$$P_{t+k} = P_t \lambda^{t+k} + \frac{1}{2}(1-\lambda)\sum_{i=t+1}^{t+k} \lambda^{t+k-i}(1-X_i).$$

2.1 Properties

Basic properties of the random walk with varying are described in [2], namely that

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1-(2\lambda-1)^t}{2}$$

and

$$ES_t = S_0 + (2p_0 - 1)\frac{1-(2\lambda-1)^t}{2(1-\lambda)}$$

for $\forall t \geq 1$. This further yields $EP_t \to \frac{1}{2}$ and $ES_t \to S_0 + \frac{2p_0-1}{2(1-\lambda)}$ for $t \to +\infty$.

Now to describe the variance of the transition probability, let us proove the following propositions.

Proposition 1. *For $\forall t \geq 1$, it holds that*

$$Var(P_t) = TODO$$

Proof. TODO □

Proposition 2. *For $\forall t \geq 1$, it holds that*

$$E(Var(P_t)) = TODO$$

Proof. TODO

And similarly for the variance of the position of the walker, following statements can be proved. □

Proposition 3. *For $\forall t \geq 1$, it holds that*

$$Var(S_t) = TODO$$

Proof. TODO □

Proposition 4. *For $\forall t \geq 1$, it holds that*

$$E(Var(S_t)) = TODO$$

Proof. TODO □

3 Random walk with varying transition probability - alternatives

Pripomenuti alternativnich definic z tezi

vytvorit nove simulace a obrazky - pozor na floating point zaokrouhlovani

3.1 Basic properties

odvozeni expected value prsti a pozice - to uz je nova vec, to v tezich nebylo

3.2 More properties

odvozeni novych veci pro alternativni definice RW

-> stredni hodnoty rozptulu prsti a pozice

-> ocekavane hodnoty po t krocich, kdyz uz jsem v pozici t

4 Simulations

Nasimulovani ruznych druhu prochazky s ruznymi parametry. Nasledne pokus o zpetne odhaleni druhu prochazky a jejich paramteru. Budu to delat zrejme podle MLE, vyhodnocovat asi podle te nejvetsi verohodnosti, pripadne podle goodness-of-fit

5 Results

Zhodnoceni prace, ze je to vlastne skvele, ze neco takoveho je, a na co vsechno by se to vlastne dalo pouzit.

References

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