Discrete random walks with memory: Models and applications

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Outline

1. Prepare mathematical model
2. Describe its properties
3. Apply it on data

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Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O.

[Karl Pearson: The problem of the random walk. (1905)]

Where is the *"Drunken sailor"*?

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Let$\{X_k\}_{k=1}^{\infty}$be a sequence of discrete random variables. For each positive integer n, let S_n denote the sum $X_1 + X_2 + \cdots + X_n$, with $S_0 = 0$. The sequence $\{S_n\}_{n=1}^{\infty}$ is called a random walk. If the common range of the X_k's is $\mathbb{R}_m$, then $\{S_n\}$ is a random walk in $\mathbb{R}_m$.

If for $\forall k$; $X_k \sim B(p = \frac{1}{2})$, the walk is called the standard random walk.

Random walk properties

- Discrete random process
- $n-$dimensional, on a matrix, graph, finite or infinite set
- Self avoiding, reinforced
- Brownian motion, polymer creation, games simulation, sports simulation

Random walk with memory

- Based on standard random walk (Bernoulli distribution with $p = 0.5$, discrete time).
- Constant total step size:

$$l_i^+ + l_i^- = 2 \ \forall i \in \mathbb{N}.$$

- At the beginning the step sizes are equal ($l_1^+ = l_1^- = 1$) and further for $t > 1$ evolve using a memory parameter $\lambda \in (0, 1)$:

$$X_{t-1} = 1 \rightarrow \begin{cases} l_t^+ = \lambda l_{t-1}^+ \\ l_t^- = 2 - \lambda l_{t-1}^+ \end{cases} \quad X_{t-1} = -1 \rightarrow \begin{cases} l_t^+ = 2 - \lambda l_{t-1}^- \\ l_t^- = \lambda l_{t-1}^- \end{cases}$$

- Loïc Turban. *On a random walk with memory and its relation with markovian processes.* Journal of Physics A: Mathematical and Theoretical (2010).

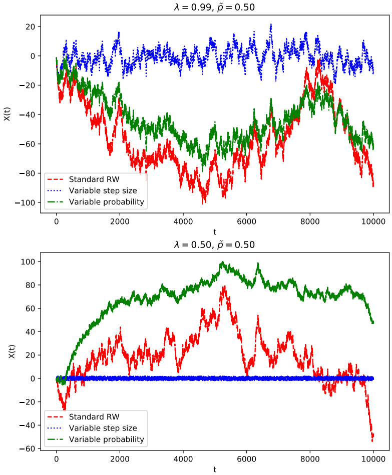
Random walk with varying transition probability

- Based on standard random walk (Bernoulli distribution with $p = 0.5$, discrete time).
- Step size remains constant, transition probability changes
- First step realized according to starting probability p_0 which then for $t > 1$ evolve using a memory parameter $\lambda \in (0, 1)$:

$$X_t = 1 \rightarrow p_t = \lambda p_{t-1}$$

$$X_t = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

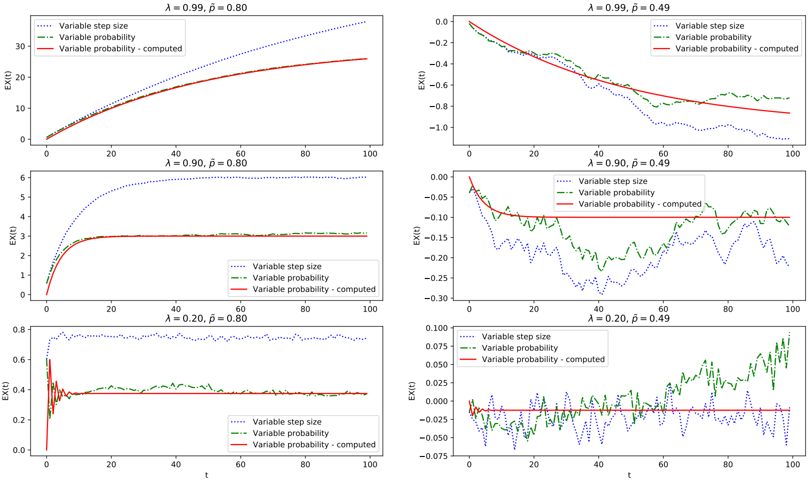
Example - RW evolution



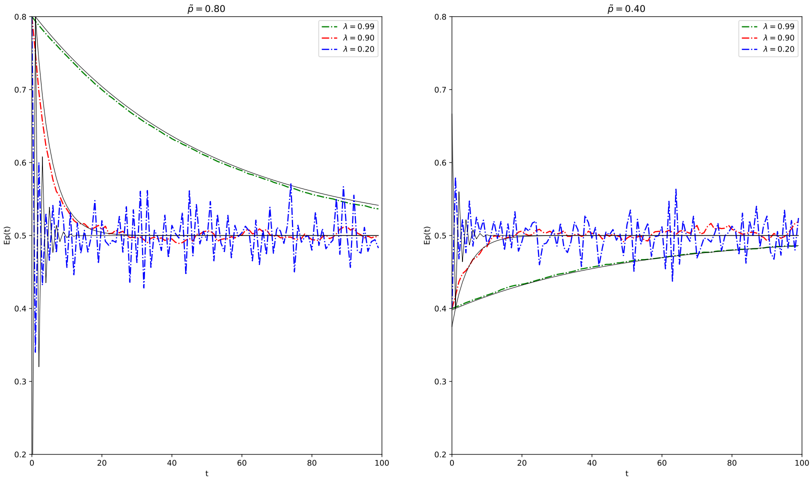
Random walk with varying transition probability - properties

- $P_t = \lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1-X_t)$
- $P_t = p_0 \lambda^t + \frac{1}{2}(1-\lambda)\sum_{i=1}^{t} \lambda^{t-i}(1-X_i)$
- $EP_t = (2\lambda - 1)^t p_0 + \frac{1-(2\lambda-1)^t}{2}$
- $ES_t = S_0 + (2p_0 - 1)\frac{1-(2\lambda-1)^t}{2(1-\lambda)}$

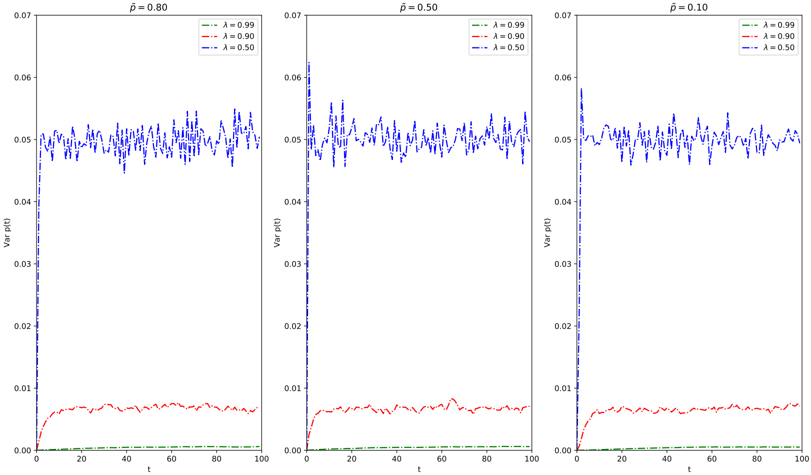
Example - Expected position of the walker



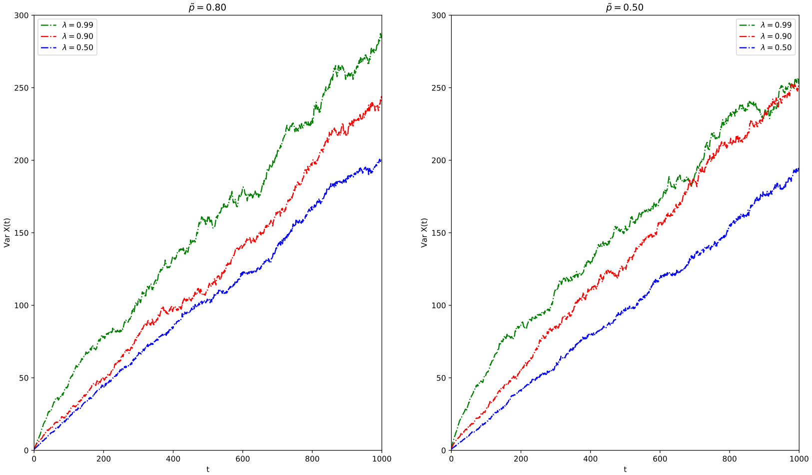
Example - Expected transition probability



Example - Transition probability variance



Example - Walker's position variance



Alternative definitions

- Formulas to obtain next transition probability P_i:
  - $\frac{1}{2}[(1+X_i)\lambda P_{i-1} + (1-X_i)(1-\lambda(1-P_{i-1}))]$
    - Success punished
  - $\frac{1}{2}[(1-X_i)\lambda P_{i-1} + (1+X_i)(1-\lambda(1-P_{i-1}))]$
    - Success rewarded
  - $\frac{1}{2}[(1+X_i)\lambda_0 P_{i-1} + (1-X_i)(1-\lambda_1(1-P_{i-1}))]$
    - Success punished with multiple λ
  - $\frac{1}{2}[(1-X_i)\lambda_0 P_{i-1} + (1+X_i)(1-\lambda_1(1-P_{i-1}))]$
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Data generation

- Using Python & Numpy package
- Different parameters:
  - $\lambda \in \{0.01, 0.1, 0.25, 0.5, 0.6, 0.8, 0.9, 0.99\}$
  - $\bar{\lambda} = \{[0.1, 0.2], [0.2, 0.5], [0.5, 0.8], [0.6, 0.9], [0.1, 0.9], [0.5, 0.99], [0.8, 0.01], [0.99, 0.9]\}$
  - $p_0 = \{0.01, 0.1, 0.25, 0.5, 0.8, 0.9, 0.99\}$
  - $n = \{2, 3, 4, 5, 10, 50, 100, 1000\}$
  - 100 walks for each combination

Fitting parameters on generated data

- Find $\bar{\lambda}$ with known p_0, model type
- Find p_0 with known $\bar{\lambda}$, model type
- Find p_0, $\bar{\lambda}$ with known model type
- Find p_0, $\bar{\lambda}$ and model type

Parameter fitting evaluation

- ?????

Real life applications

- Reliability analysis, medical data analysis
- Sport modelling
  - Tennis with model "Success rewarded"
  - Applied for live betting on US Open against Tipsport bookmaker
  - Source code and paper available at https://github.com/tomaskourim/mathsport2019

Results

- New models of Random walk described
- Properties derived
- Strenght of the models shown using simulated data
- Possible real life applications shown
- Source code and paper available at https://github.com/tomaskourim/amistat2019

Next steps????

- Model implementation
  - λ optimization
  - p_1 optimization
- Model improvement
  - Other versions of random walk with memory
  - Combination with other approaches
- Model testing
  - Model evaluation granularity
  - Performance on a larger dataset
  - Betting module for more bookmakers
  - Application of the model to *best-of-three* matches
- Application in other domains

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Thank you.

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