

A Model of a Random Walk with Varying Transition Probabilities

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Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O .

[Karl Pearson: *The problem of the random walk*. (1905)]

Where is the *drunken sailor*?

Random walk

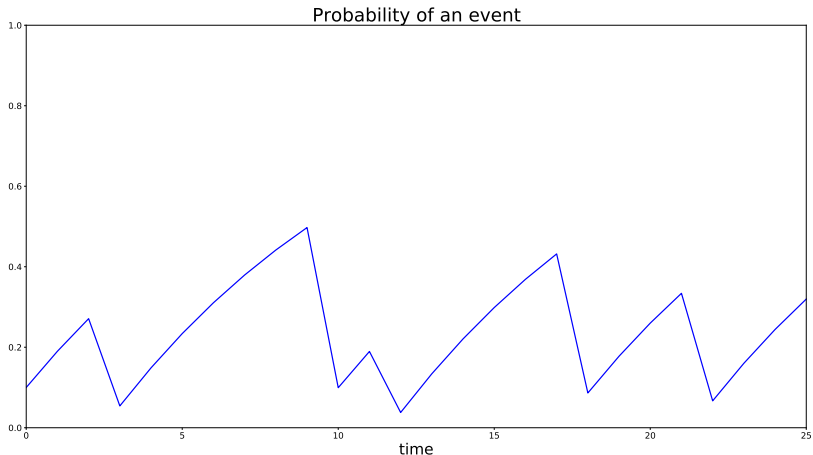
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Motivation - Random process with varying probability



Motivation

- Failure of a machine
 - repair after failure
 - preventive maintenance
- Occurrence of a disease
 - cure of the disease
 - prevention (i.e. lifestyle change)
- Development of sports match
 - goal scored, point achieved
 - period won

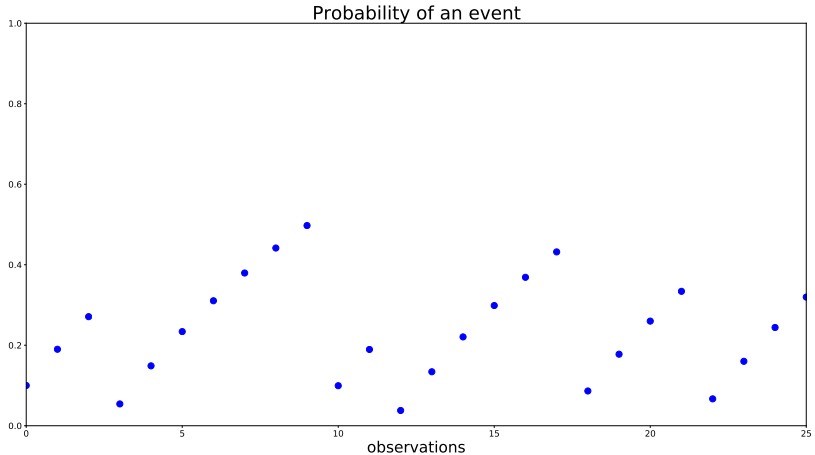
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Random walk with varying probabilities

- Random walk with memory based the on standard Bernoulli random walk
- Given starting probability p_0
- $X_t \in \{-1, 1\}$ with $X_t \sim \text{Bernoulli}(p_{t-1})$
- Memory coefficient $\lambda \in (0, 1)$ affecting the development of probabilities p_t as

$$X_t = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_t = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ "Success punishing"

$$X_t = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_t = -1 \rightarrow p_t = \lambda p_{t-1}$$

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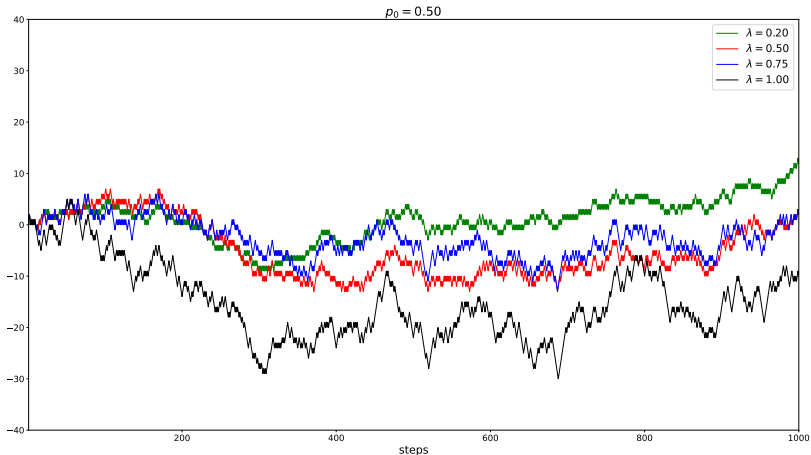
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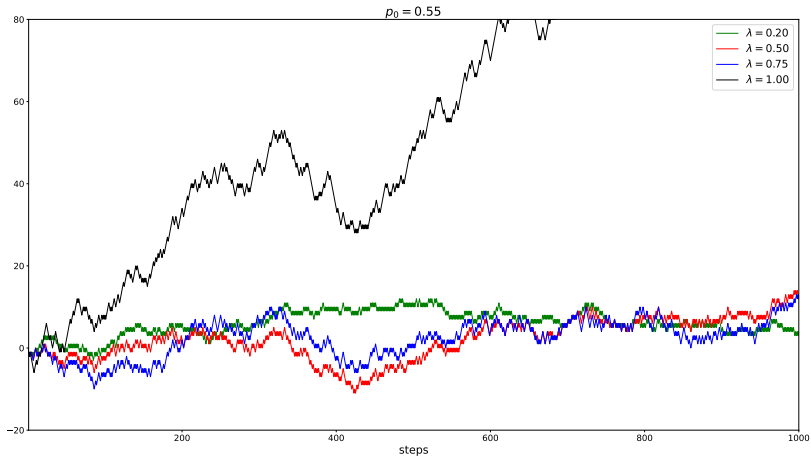
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Example - RW development



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Walk steps properties

$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1)$$

$$\lim_{t \rightarrow +\infty} EX_t = 0$$

$$\text{Var } X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2$$

$$\lim_{t \rightarrow +\infty} \text{Var } X_t = 1$$

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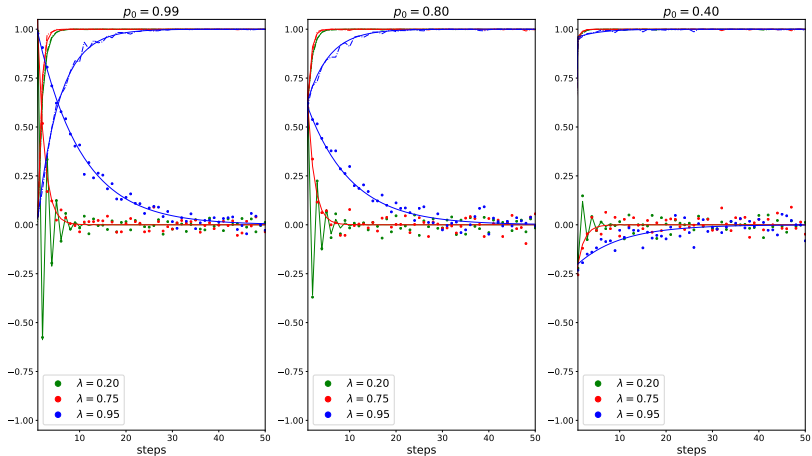
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Example - RW steps



Walk probabilities properties

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}$$

$$\lim_{t \rightarrow +\infty} EP_t = \frac{1}{2}$$

$$Var P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda) (3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_0, \lambda)^2$$

$$\lim_{t \rightarrow +\infty} Var P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}$$

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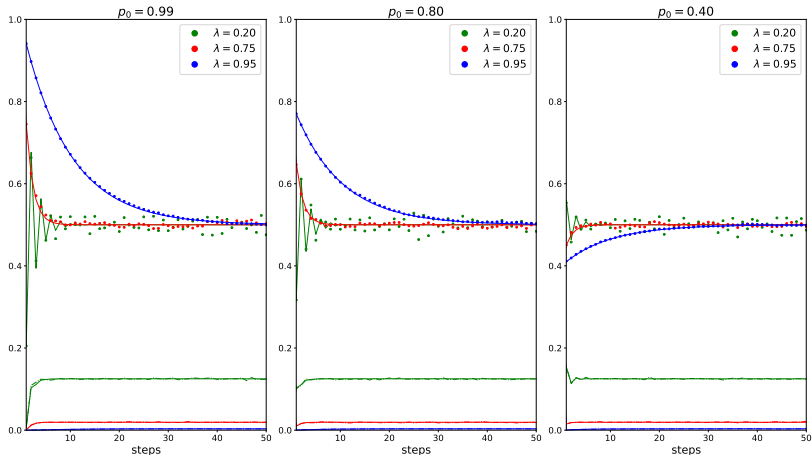
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Walk position properties

$$ES_t = S_0 + (2p_0 - 1) \frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)}$$

$$\lim_{t \rightarrow +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}$$

$$Var S_t = t + 4 \sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda)$$

$$\lim_{t \rightarrow +\infty} Var S_t = c_1(p_0, \lambda)t + c_2(p_0, \lambda)$$

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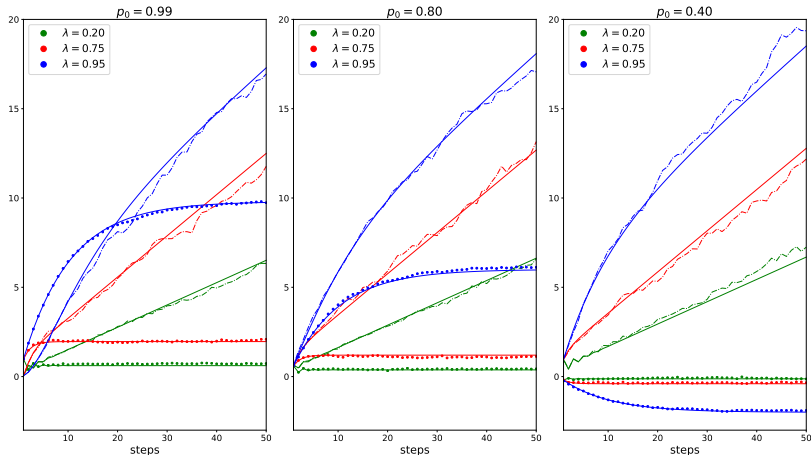
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Example - RW position



Success rewarding model

$$EX_t = 2p_0 - 1$$

$$\text{Var } X_t = 4p_0(1 - p_0)$$

$$EP_t = p_0$$

$$\text{Var } P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1 - \lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2$$

$$ES_t = S_0 + t(2p_0 - 1)$$

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More complex models

- Two memory coefficients λ each affecting one direction of the walk
- Again two variants – success punishing and success rewarding

$$X_{t-1} = 1 \rightarrow p_t = \lambda_0 p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda_1(1 - p_{t-1})$$

→ “Two-parameter success punishing model”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda_0(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda_1 p_{t-1}$$

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- M steps, $\lambda(t)$, n -dimensional walk

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1. Find p_0
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Real life application

- Results of tennis matches from February till May 2021 together with real life odds (both pre-match and in-play) provided by a bookmaker
- Data divided into training (February - April) and testing (May) datasets
- p_0 estimated using the first set winning odds provided by the bookmaker
- Single lambda success rewarding model selected as best fit using AIC and training data

Real life application

- Simulated in-play betting on set winner
 - Bet if $p \geq 1.2 \frac{1}{odds}$
 - Three betting strategies tested
 - ROI 98-148% within just one month of betting
 - Only 65 bets placed

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Summary

- A specific model of a random walk with memory
- Model properties derived
- Possible applications in a set of real life scenarios
- Initial results show big potential of the model

Thank you.

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