A Model of a Random Walk with Varying Transition Probabilities

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Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O.

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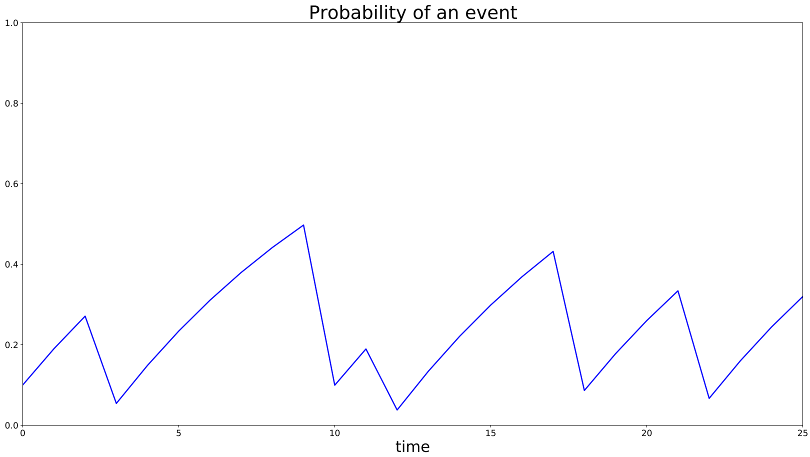

Motivation

- Failure of a machine
  - repair after failure
  - preventive maintenance
- Occurrence of a disease
  - cure of the disease
  - prevention (i.e. lifestyle change)
- Development of sports match
  - goal scored, point achieved
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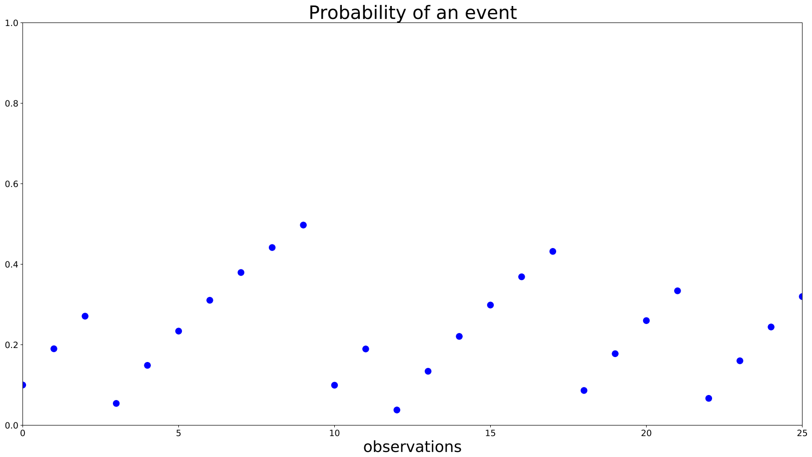

Random walk with varying probabilities

- Random walk with memory based the on standard Bernoulli random walk
- Given starting probability p_0
- $X_t \in \{-1, 1\}$ with $X_t \sim \mathrm{Bernoulli}(p_{t-1})$
- Memory coefficient $\lambda \in (0, 1)$ affecting the development of probabilities p_t as

$$X_t = 1 \to p_t = \lambda p_{t-1} \qquad X_t = -1 \to p_t = 1 - \lambda(1 - p_{t-1})$$

$\Rightarrow$ "Success punishing"

$$X_t = 1 \to p_t = 1 - \lambda(1 - p_{t-1}) \qquad X_t = -1 \to p_t = \lambda p_{t-1}$$

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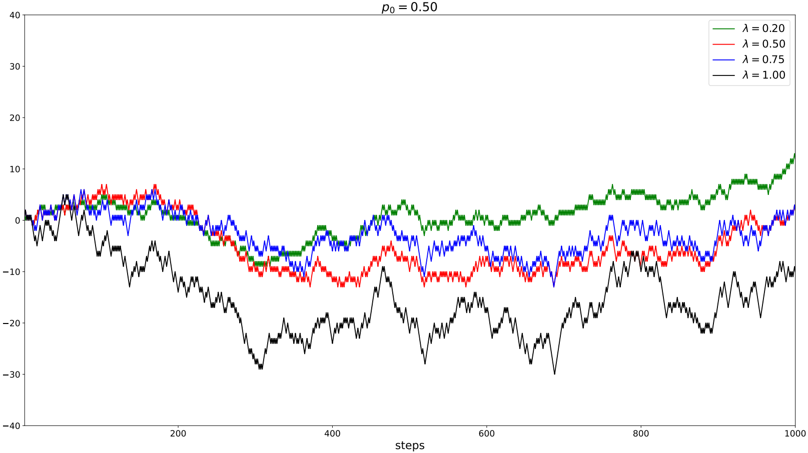
Example - RW development



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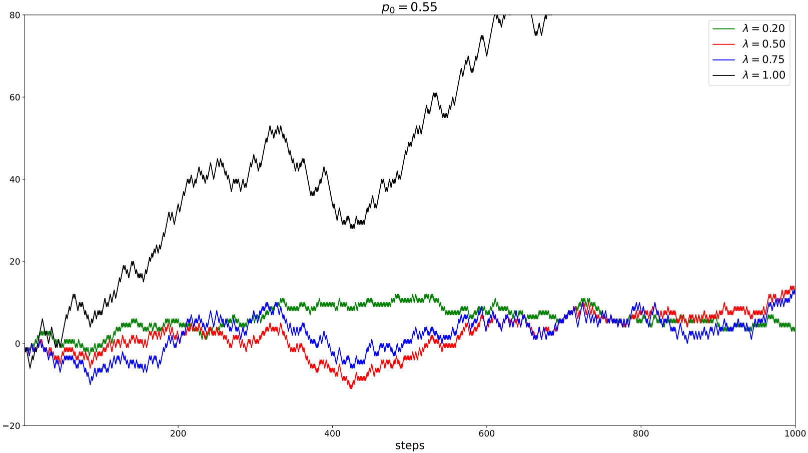

Walk steps properties

$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1)$$

$$\lim_{t \to +\infty} EX_t = 0$$

$$Var\, X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2$$

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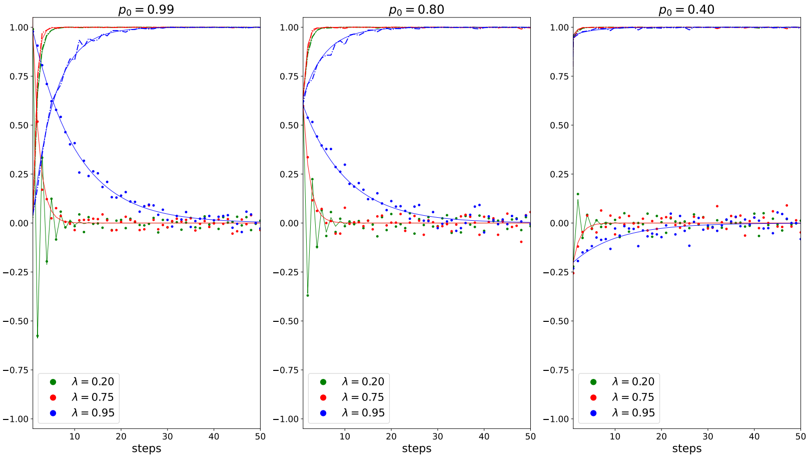

Walk probabilities properties

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}$$

$$\lim_{t \to +\infty} EP_t = \frac{1}{2}$$

$$Var\ P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda)(3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_o, \lambda)^2$$

$$\lim_{t \to +\infty} Var\ P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}$$

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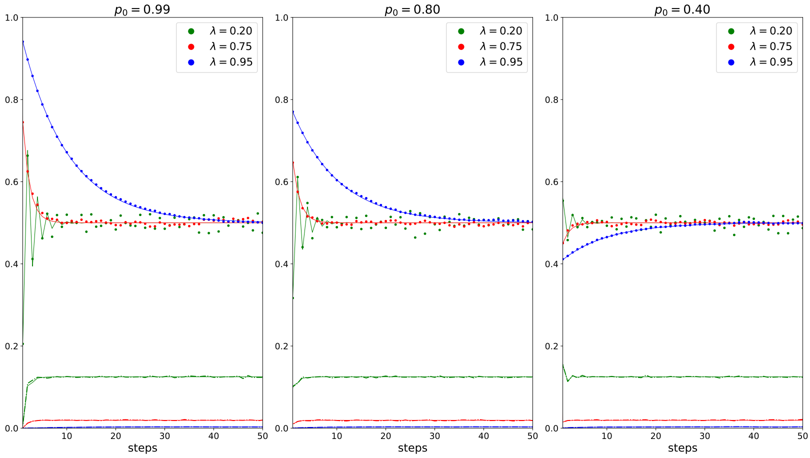

Walk position properties

$$ES_t = S_0 + (2p_0 - 1)\frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)}$$

$$\lim_{t \to +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}$$

$$Var\, S_t = t + 4\sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda)$$

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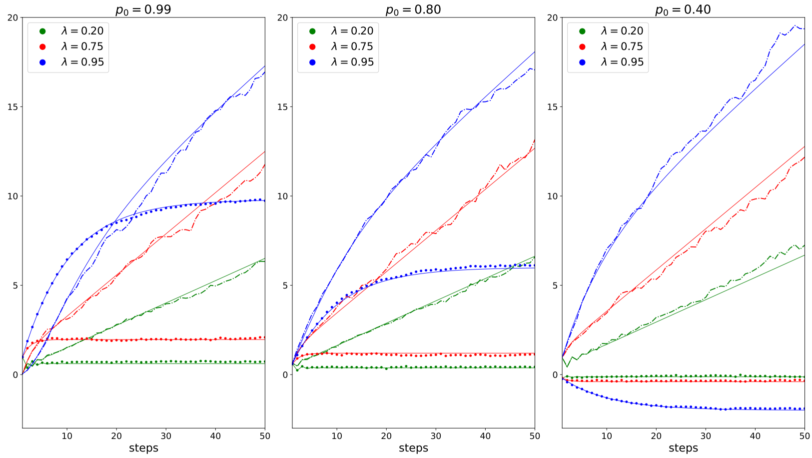

Success rewarding model

$$EX_t = 2p_0 - 1$$

$$Var\,X_t = 4p_0(1 - p_0)$$

$$EP_t = p_0$$

$$Var\,P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1-\lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2$$

$$ES_t = S_0 + t(2p_0 - 1)$$

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More complex models

- Two memory coefficients λ each affecting one direction of the walk
- Again two variants – success punishing and success rewarding

$$X_{t-1} = 1 \rightarrow p_t = \lambda_0 p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda_1(1 - p_{t-1})$$

–> "Two-parameter success punishing model"

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- M steps, $\lambda(t)$, n-dimensional walk

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Model fitting

1. Find p_0
2. Find λ parameter(s)
3. Choose optimal model type
   - Akaike Information Criterion (AIC) & numerical optimization

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Real life application

- Results of tennis matches from February till May 2021 together with real life odds (both pre-match and in-play) provided by a bookmaker
- Data divided into training (February - April) and testing (May) datasets
- p_0 estimated using the first set winning odds provided by the bookmaker
- Single lambda success rewarding model selected as best fit using AIC and training data

Real life application

- Simulated in-play betting on set winner
  - Bet if $p \geq \frac{1.2}{\frac{1}{odds}}$
  - Three betting strategies tested
  - ROI 98-148% withing just one month of betting
  - Only 65 bets placed

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Summary

- A specific model of a random walk with memory
- Model properties derived
- Possible applications in a set of real life scenarios
- Initial results show big potential of the model

Thank you.

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