

A model of random walk with varying transition probabilities

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Outline

1. Motivation
2. Model description
3. Model application

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Random walk

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A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O .

[Karl Pearson: *The problem of the random walk*. (1905)]

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Motivation

- Failure of a machine
 - repair after failure
 - preventive maintenance
- Occurrence of a disease
 - cure of the disease
 - prevention (i.e. lifestyle change)
- Development of sports match
 - goal scored, point achieved
 - period won

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Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ "Success punished"

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

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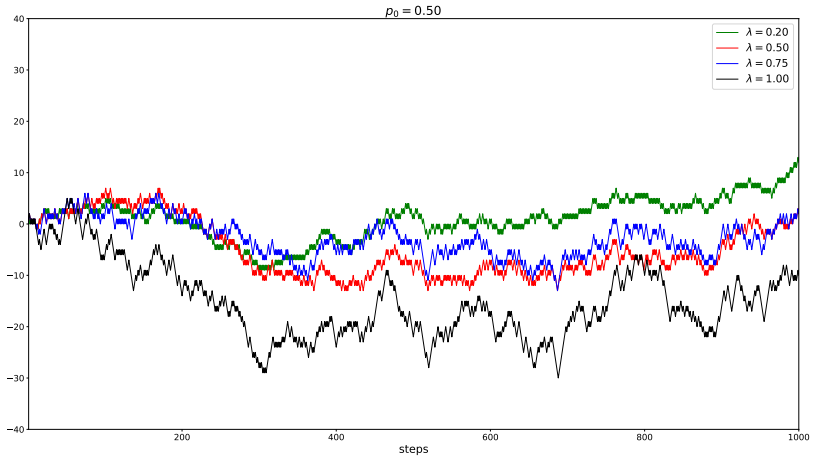
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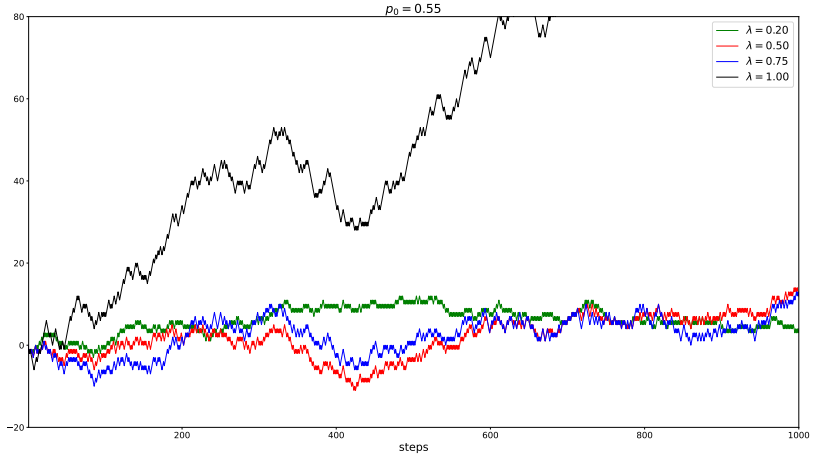
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Example - RW development



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Walk steps properties

$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1),$$

$$\lim_{t \rightarrow +\infty} EX_t = 0.$$

$$\text{Var } X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2,$$

$$\lim_{t \rightarrow +\infty} \text{Var } X_t = 1.$$

Walk steps properties

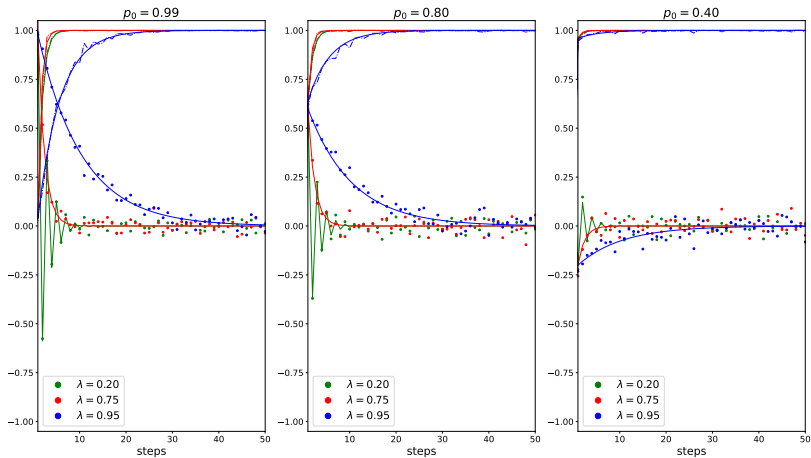
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Example - RW steps



Walk probabilities properties

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2},$$

$$\lim_{t \rightarrow +\infty} EP_t = \frac{1}{2}.$$

$$\text{Var } P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda) (3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_0, \lambda)^2,$$

$$\lim_{t \rightarrow +\infty} \text{Var } P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}.$$

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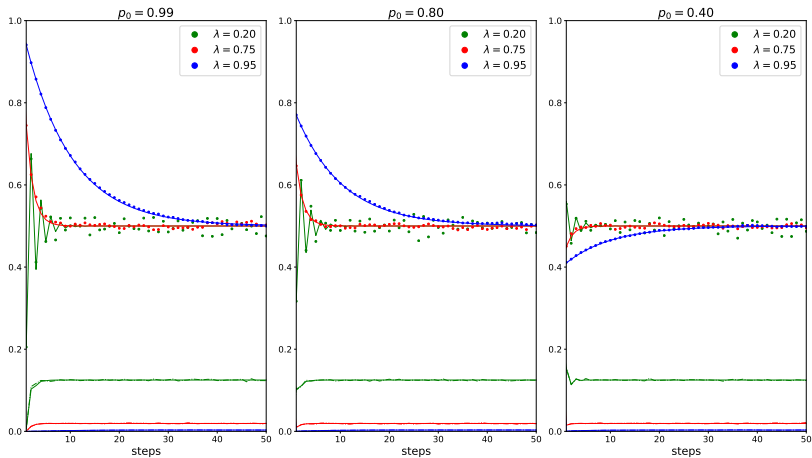
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Example - RW probabilities



Walk position properties

$$ES_t = S_0 + (2p_0 - 1) \frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)},$$

$$\lim_{t \rightarrow +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}.$$

$$\text{Var } S_t = t + 4 \sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda),$$

$$\lim_{t \rightarrow +\infty} \text{Var } S_t = c_1(p_0, \lambda)t + c_2(p_0, \lambda).$$

Walk position properties

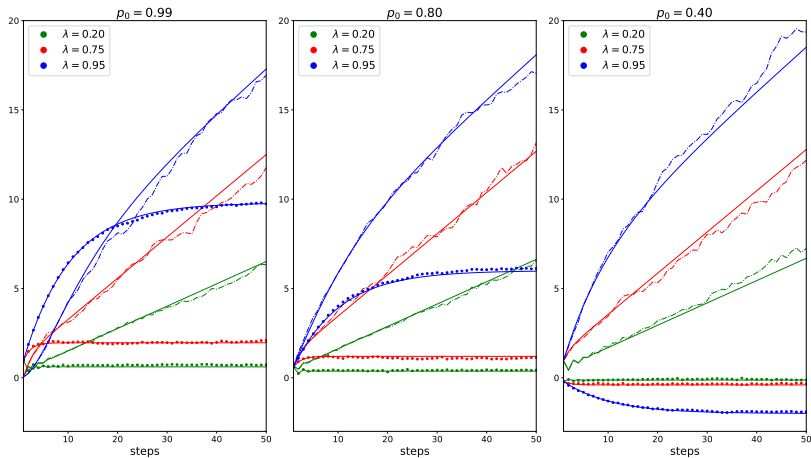
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Example - RW position



Success rewarding model

$$EX_t = 2p_0 - 1,$$

$$\text{Var } X_t = 4p_0(1 - p_0),$$

$$EP_t = p_0,$$

$$\text{Var } P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1 - \lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2,$$

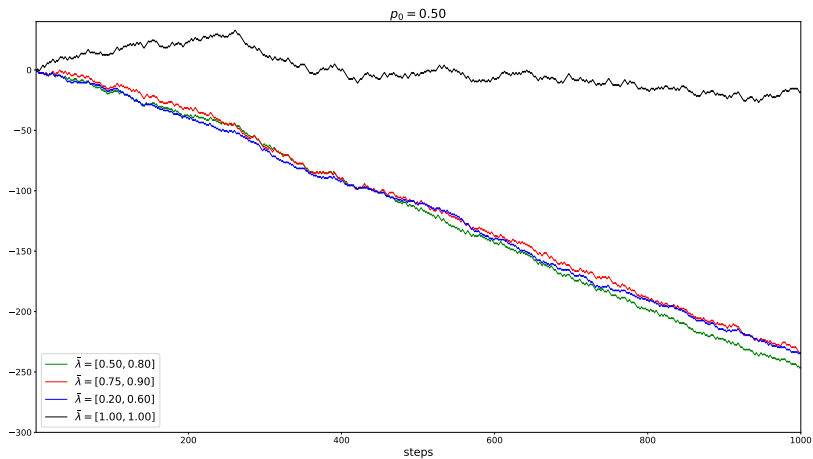
$$ES_t = S_0 + t(2p_0 - 1),$$

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Two-parameter model

- Two parameters λ , each affecting one direction of the walk.
- Again two variants - success punishing and success rewarding
- $\frac{1}{2}[(1 + X_i)\lambda_0 P_{i-1} + (1 - X_i)(1 - \lambda_1(1 - P_{i-1}))]$
 - > Two parameter success punishing model
- $\frac{1}{2}[(1 - X_i)\lambda_0 P_{i-1} + (1 + X_i)(1 - \lambda_1(1 - P_{i-1}))]$
 - > Two parameter success rewarding model

Example - two-parameter model



Model fitting

- Find $\vec{\lambda}$ with known p_0 , model type
- Find p_0 with known $\vec{\lambda}$, model type
- Find $p_0, \vec{\lambda}$ with known model type
 - Using maximal likelihood estimate & numerical optimization
- Find model type without any prior knowledge
 - Using Akaike information criterion & numerical optimization

Real life application

- Success rewarding model well suited for modelling tennis matches
- Model trained on 2009–2018 men Grand Slam tournaments
- Model applied on 2019 US Open
 - Live betting against bookmaker
 - 0.52 units total necessary bankroll
 - 2.24 units total profit → 430% ROI
 - Only 128 bets placed

Summary

- A specific model of a random walk with memory
- Model properties derived
- Application shows big future potential of the model
- Possible applications in a set of real life scenarios