

A model of random walk with varying transition probabilities

Tomáš Kouřim ¹ Petr Volf ²

¹Faculty of Nuclear Sciences and Physical Engineering, CTU Prague

²Institute of Information Theory and Automation, CAS CR Prague

SMTDA 2020

Outline

1. Motivation
2. Model description
3. Model application

Outline

1. Motivation
2. Model description
3. Model application

Outline

1. Motivation
2. Model description
3. Model application

Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O .

[Karl Pearson: The problem of the random walk.(1905)]

Where is the *drunken sailor*?

Random walk

Definition

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O .

[Karl Pearson: The problem of the random walk. (1905)]

Where is the *drunken sailor*?

Motivation

- Failure of a machine
 - repair after failure
 - preventive maintenance
- Occurrence of a disease
 - cure of the disease
 - prevention (i.e. lifestyle change)
- Development of sports match
 - goal scored, point achieved
 - period won

Motivation

- Failure of a machine
 - repair after failure
 - preventive maintenance
- Occurrence of a disease
 - cure of the disease
 - prevention (i.e. lifestyle change)
- Development of sports match
 - goal scored, point achieved
 - period won

Motivation

- Failure of a machine
 - repair after failure
 - preventive maintenance
- Occurrence of a disease
 - cure of the disease
 - prevention (i.e. lifestyle change)
- Development of sports match
 - goal scored, point achieved
 - period won

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ "Success punished"

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

→ "Success rewarded"

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ “Success punished”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

→ “Success rewarded”

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ “Success punished”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

→ “Success rewarded”

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

→ “Success punished”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

→ “Success rewarded”

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

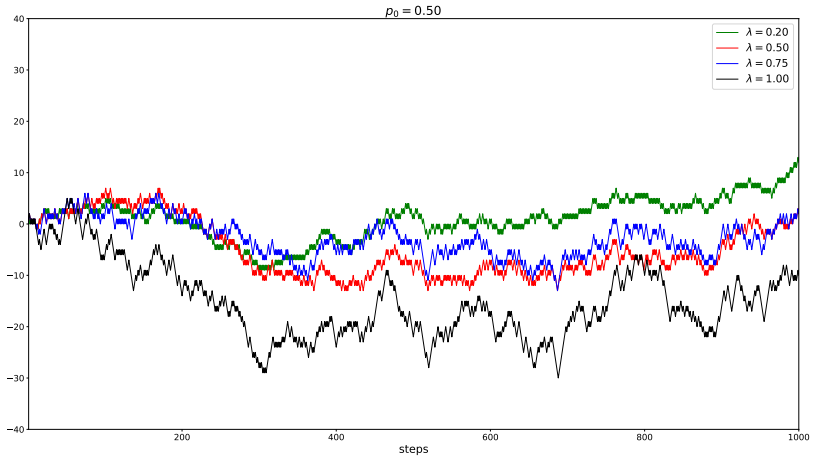
$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

-> "Success punished"

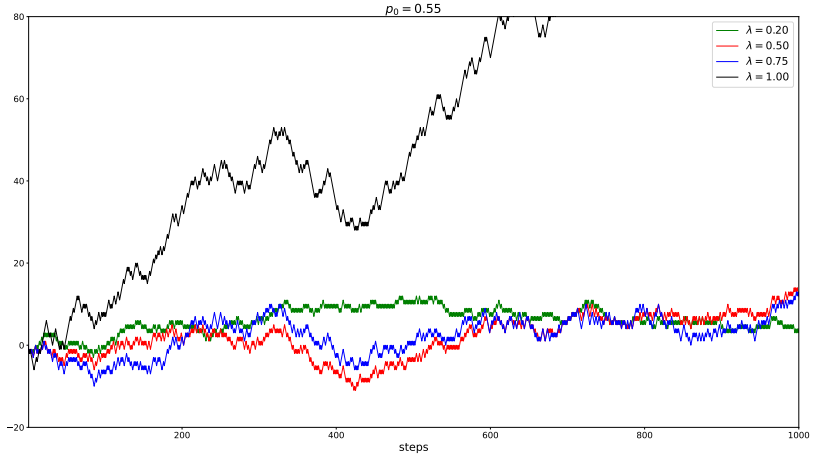
$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda p_{t-1}$$

-> "Success rewarded"

Example - RW development



Example - RW development



Walk steps properties

$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1)$$

$$\lim_{t \rightarrow +\infty} EX_t = 0$$

$$\text{Var } X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2$$

$$\lim_{t \rightarrow +\infty} \text{Var } X_t = 1$$

Walk steps properties

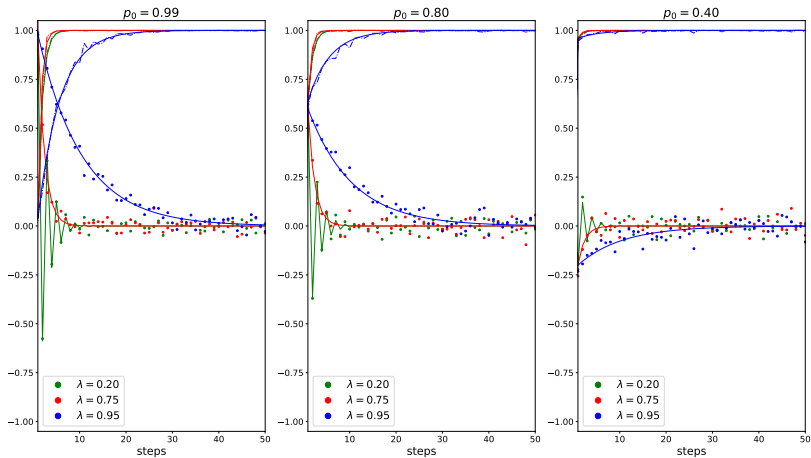
$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1)$$

$$\lim_{t \rightarrow +\infty} EX_t = 0$$

$$\text{Var } X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2$$

$$\lim_{t \rightarrow +\infty} \text{Var } X_t = 1$$

Example - RW steps



Walk probabilities properties

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}$$

$$\lim_{t \rightarrow +\infty} EP_t = \frac{1}{2}$$

$$Var P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda) (3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_0, \lambda)^2$$

$$\lim_{t \rightarrow +\infty} Var P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}$$

Walk probabilities properties

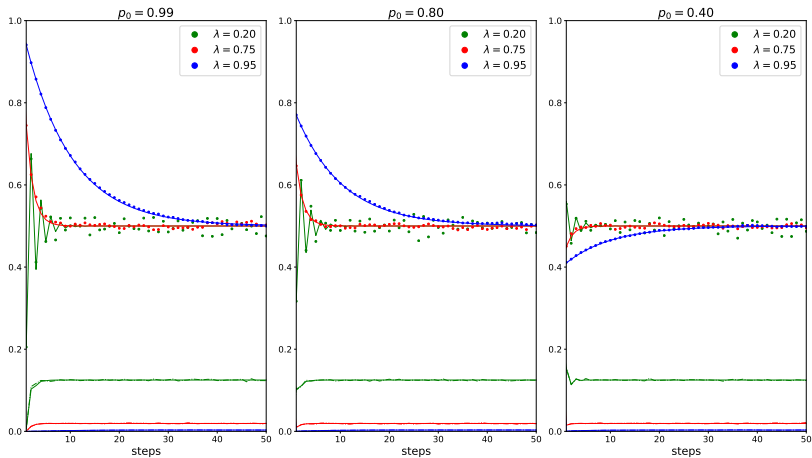
$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}$$

$$\lim_{t \rightarrow +\infty} EP_t = \frac{1}{2}$$

$$\text{Var } P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda) (3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_0, \lambda)^2$$

$$\lim_{t \rightarrow +\infty} \text{Var } P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}$$

Example - RW probabilities



Walk position properties

$$ES_t = S_0 + (2p_0 - 1) \frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)}$$

$$\lim_{t \rightarrow +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}$$

$$\text{Var } S_t = t + 4 \sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda)$$

$$\lim_{t \rightarrow +\infty} \text{Var } S_t = c_1(p_0, \lambda)t + c_2(p_0, \lambda)$$

Walk position properties

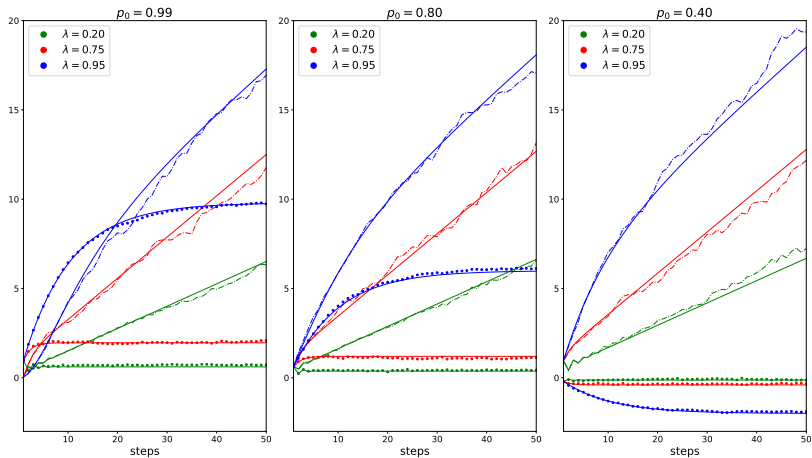
$$ES_t = S_0 + (2p_0 - 1) \frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)}$$

$$\lim_{t \rightarrow +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}$$

$$\text{Var } S_t = t + 4 \sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda)$$

$$\lim_{t \rightarrow +\infty} \text{Var } S_t = c_1(p_0, \lambda)t + c_2(p_0, \lambda)$$

Example - RW position



Success rewarding model

$$EX_t = 2p_0 - 1$$

$$\text{Var } X_t = 4p_0(1 - p_0)$$

$$EP_t = p_0$$

$$\text{Var } P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1 - \lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2$$

$$ES_t = S_0 + t(2p_0 - 1)$$

$$\text{Var } S_t = 4p_0(1 - p_0)t^2 + a(p_0, \lambda)t - a(p_0, \lambda) \frac{1 - (2\lambda - \lambda^2)^t}{(1 - \lambda)^2}$$

Two-parameter model

- Two λ parameters each affecting one direction of the walk
- Again two variants – success punishing and success rewarding

$$X_{t-1} = 1 \rightarrow p_t = \lambda_0 p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda_1(1 - p_{t-1})$$

→ “Two-parameter success punishing model”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda_0(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda_1 p_{t-1}$$

→ “Two-parameter success rewarding model”

Two-parameter model

- Two λ parameters each affecting one direction of the walk
- Again two variants – success punishing and success rewarding

$$X_{t-1} = 1 \rightarrow p_t = \lambda_0 p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda_1(1 - p_{t-1})$$

→ “Two-parameter success punishing model”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda_0(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda_1 p_{t-1}$$

→ “Two-parameter success rewarding model”

Two-parameter model

- Two λ parameters each affecting one direction of the walk
- Again two variants – success punishing and success rewarding

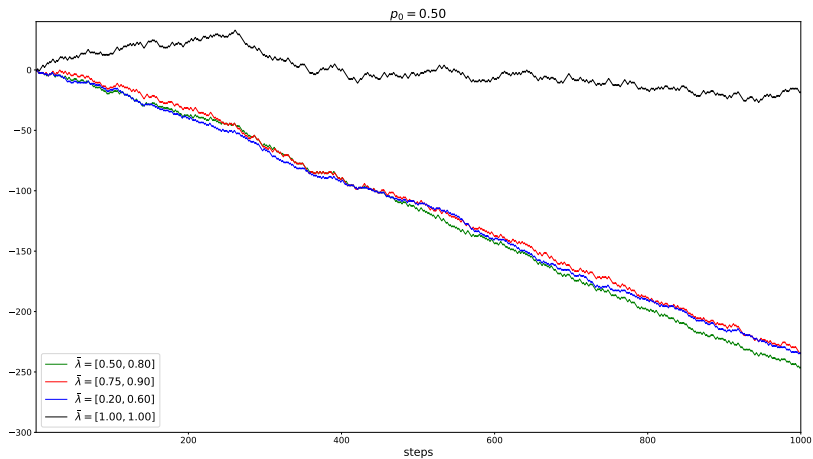
$$X_{t-1} = 1 \rightarrow p_t = \lambda_0 p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda_1(1 - p_{t-1})$$

→ “Two-parameter success punishing model”

$$X_{t-1} = 1 \rightarrow p_t = 1 - \lambda_0(1 - p_{t-1}) \quad X_{t-1} = -1 \rightarrow p_t = \lambda_1 p_{t-1}$$

→ “Two-parameter success rewarding model”

Example - two-parameter success punishing model



Model fitting

1. Find $\vec{\lambda}$ with known p_0 , model type
2. Find p_0 with known $\vec{\lambda}$, model type
3. Find $p_0, \vec{\lambda}$ with known model type
4. Find model type without any prior knowledge

Model fitting

1. Find $\vec{\lambda}$ with known p_0 , model type
2. Find p_0 with known $\vec{\lambda}$, model type
3. Find $p_0, \vec{\lambda}$ with known model type
 - Using maximal likelihood estimate & numerical optimization
4. Find model type without any prior knowledge

Model fitting

1. Find $\vec{\lambda}$ with known p_0 , model type
2. Find p_0 with known $\vec{\lambda}$, model type
3. Find $p_0, \vec{\lambda}$ with known model type
 - Using maximal likelihood estimate & numerical optimization
4. Find model type without any prior knowledge
 - Using Akaike information criterion & numerical optimization

Real life application

- Success rewarding model well suited for modelling tennis matches
- Model trained on 2009–2018 men Grand Slam tournaments
- Model applied on 2019 US Open
 - Live betting against bookmaker
 - 0.52 units total necessary bankroll
 - 2.24 units total profit → 430% ROI
 - Only 128 bets placed

Real life application

- Success rewarding model well suited for modelling tennis matches
- Model trained on 2009–2018 men Grand Slam tournaments
- Model applied on 2019 US Open
 - Live betting against bookmaker
 - 0.52 units total necessary bankroll
 - 2.24 units total profit → 430% ROI
 - Only 128 bets placed

Real life application

- Success rewarding model well suited for modelling tennis matches
- Model trained on 2009–2018 men Grand Slam tournaments
- Model applied on 2019 US Open
 - Live betting against bookmaker
 - 0.52 units total necessary bankroll
 - 2.24 units total profit → 430% ROI
 - Only 128 bets placed

Summary

- A specific model of a random walk with memory
- Model properties derived
- Application shows big future potential of the model
- Possible applications in a set of real life scenarios

Thank you.

tom@skourim.com