A model of random walk with varying transition probabilities

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May 15, 2020

Outline

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1. Motivation
2. Model description
3. Model application

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Random walk

Definition

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Motivation

- Failure of a machine
  - repair after failure
  - preventive maintenance
- Occurrence of a disease
  - cure of the disease
  - prevention (i.e. lifestyle change)
- Development of sports match
  - goal scored, point achieved
  - period won

Random walk with varying probabilities

- Random walk with memory
- Memory coefficient $\lambda \in (0, 1)$ affecting the transition probabilities
- First step of the walk X_1 depends on an initial transition probability p_0
- Further steps depend on a transition probability p_t evolving as

$$X_{t-1} = 1 \rightarrow p_t = \lambda p_{t-1} \quad X_{t-1} = -1 \rightarrow p_t = 1 - \lambda(1 - p_{t-1})$$

–> "Success punished"

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Formal definition

Definition

Let $p_0, \lambda \in (0, 1)$ be constant parameters, $\{P_n\}_{n=0}^{\infty}$ and $\{X_n\}_{n=1}^{\infty}$ sequences of discrete random variables with $P_0 = p_0$ and for $t \geq 1$

$$P(X_t = 1 | P_{t-1} = p_{t-1}) = p_{t-1}, \;\; P(X_t = -1 | P_{t-1} = p_{t-1}) = 1 - p_{t-1},$$

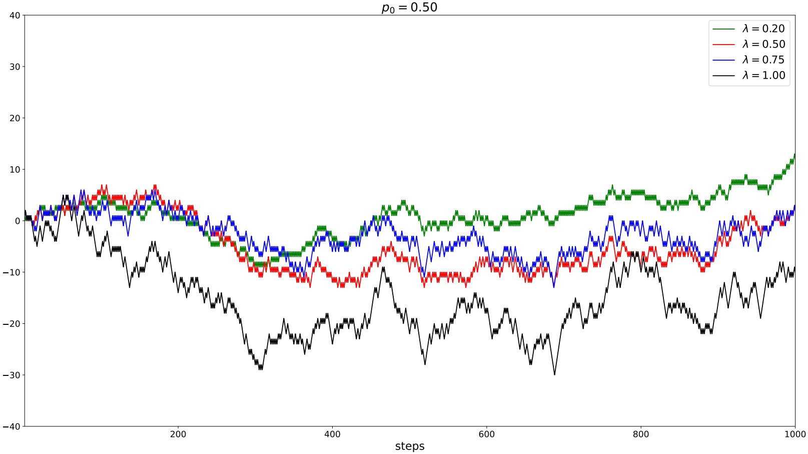
and (*success punishing*)

$$P_t = \lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1 - X_t)$$

or (*success rewarding*)

$$P_t = \lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1 + X_t).$$

Example - RW development



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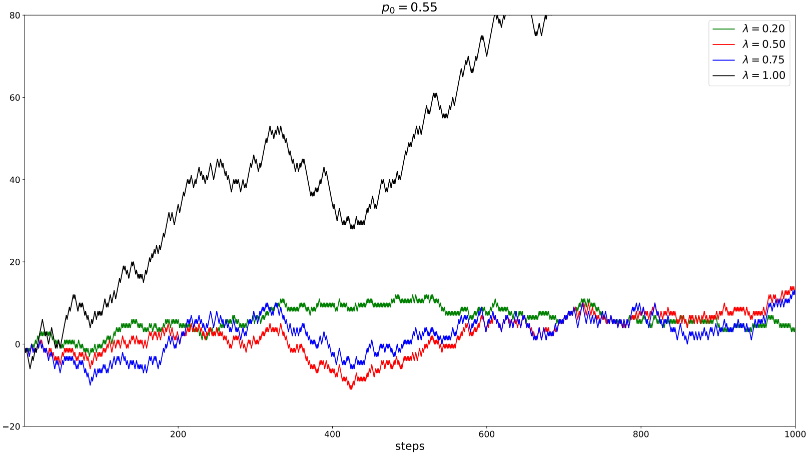


Walk steps properties

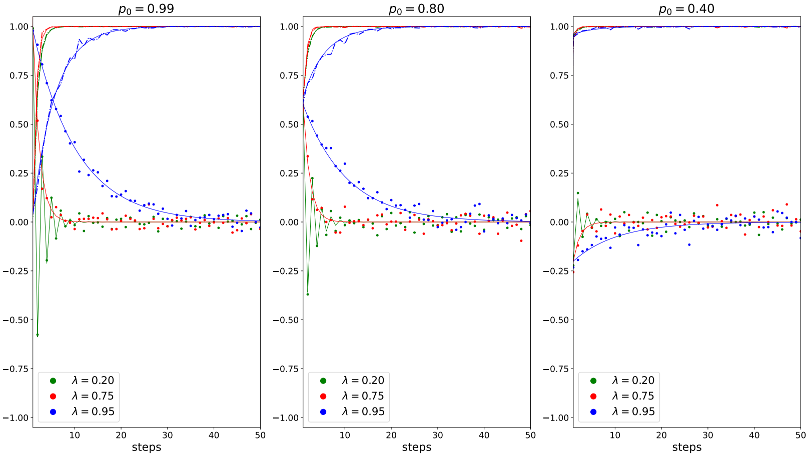
$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1),$$

$$\lim_{t \to +\infty} EX_t = 0.$$

$$Var\, X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2,$$

$$\lim_{t \to +\infty} Var\, X_t = 1.$$

Example - RW steps



Walk probabilities properties

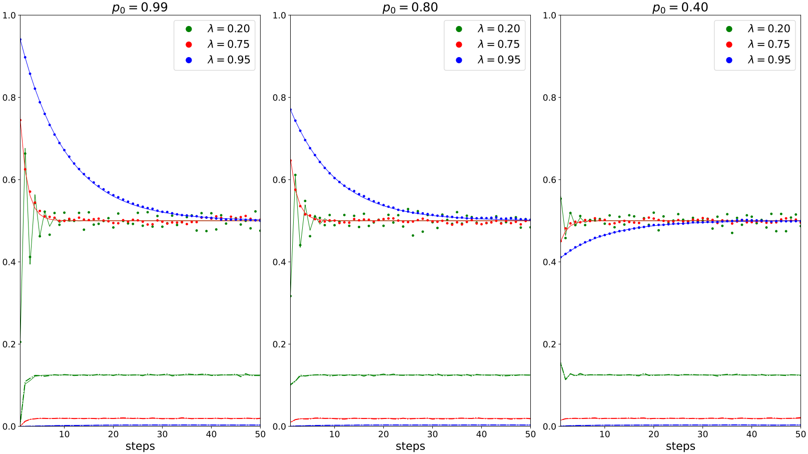
$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2},$$

$$\lim_{t \to +\infty} EP_t = \frac{1}{2}.$$

$$Var\,P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i; p_0, \lambda)(3\lambda^2 - 2\lambda)^{t-1-i} - k(t; p_o, \lambda)^2,$$

$$\lim_{t \to +\infty} Var\,P_t = \frac{\frac{1}{2}(1 - \lambda^2)}{-3\lambda^2 + 2\lambda + 1} - \frac{1}{4}.$$

Example - RW probabilities



Walk position properties

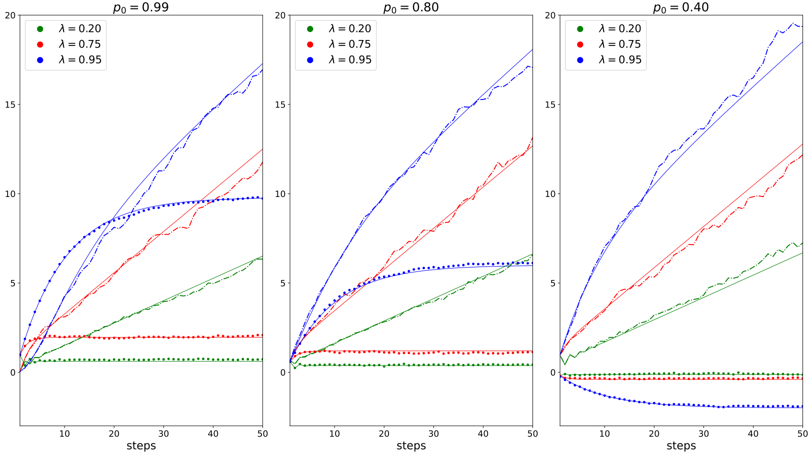
$$ES_t = S_0 + (2p_0 - 1)\frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)},$$

$$\lim_{t \to +\infty} ES_t = S_0 + \frac{(2p_0 - 1)}{2(1 - \lambda)}.$$

$$Var\, S_t = t + 4\sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - a(t; p_0, \lambda),$$

$$\lim_{t \to +\infty} Var\, S_t = c_1(p_0, \lambda)t + c_2(p_0, \lambda).$$

Example - RW position



Success rewarding model

$$EX_t = 2p_0 - 1,$$

$$Var\, X_t = 4p_0(1 - p_0),$$

$$EP_t = p_0,$$

$$Var\, P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1 - \lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2,$$

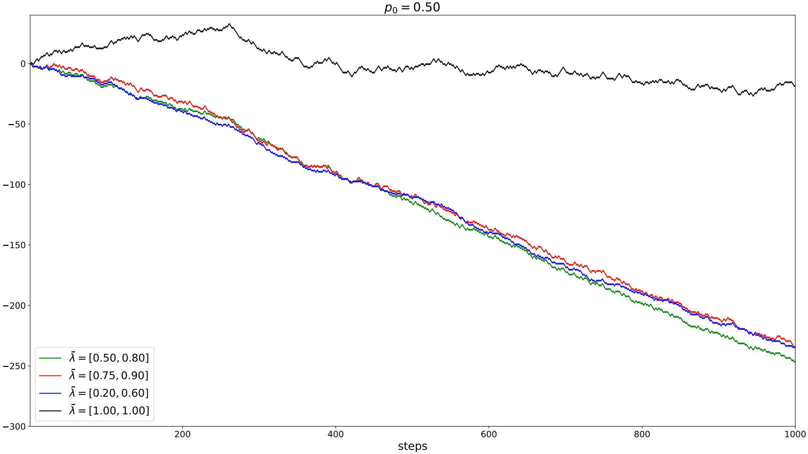
$$ES_t = S_0 + t(2p_0 - 1),$$

$$Var\, S_t = 4p_0(1 - p_0)t^2 + a(p_0, \lambda)t - a(p_0, \lambda)\frac{1 - (2\lambda - \lambda^2)^t}{(1 - \lambda)^2}.$$

Two-parameter model

- Two parameters λ, each affecting one direction of the walk.
- Again two variants - success punishing and success rewarding
- $\frac{1}{2}[(1+X_i)\lambda_0 P_{i-1} + (1-X_i)(1-\lambda_1(1-P_{i-1}))]$
  - Two parameter success punishing model
- $\frac{1}{2}[(1-X_i)\lambda_0 P_{i-1} + (1+X_i)(1-\lambda_1(1-P_{i-1}))]$
  - Two parameter success rewarding model

Example - two-parameter model



Model fitting

- Find $\overrightarrow{\lambda}$ with known p_0, model type
- Find p_0 with known $\overrightarrow{\lambda}$, model type
- Find p_0, $\overrightarrow{\lambda}$ with known model type
  - Using maximal likelihood estimate & numerical optimization
- Find model type without any prior knowledge
  - Using Akaike information criterion & numerical optimization

Real life application

- Success rewarding model well suited for modelling tennis matches
- Model trained on 2009–2018 men Grand Slam tournaments
- Model applied on 2019 US Open
  - Live betting against bookmaker
  - 0.52 units total necessary bankroll
  - 2.24 units total profit $\rightarrow$ 430% ROI
  - Only 128 bets placed

Summary

- A specific model of a random walk with memory
- Model properties derived
- Application shows big future potential of the model
- Possible applications in a set of real life scenarios