A model of random walk with varying transition probabilities

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Abstract

This paper considers a model of the one-dimensional discrete time random walk in which the position of the walker is controlled by a varying transition probabilities. Namely, these probabilities depend on the last walker moves. Hence, transition probabilities evolve in time, they depend on the walk history, thus making the walk a non-Markovian stochastic process. The paper follows on the recent work of the authors. Two basic version of the model are introduced, some their properties are recalled and new theoretical results derived. Then, more complex variants of models are presented. Development of walks themselves as well as the properties of connected sequences of transition probabilities are illustrated also with the aid of simulation. Application to tennis results prediction is mentioned (described in short?), too.

1 Introduction

Modelling sporting events and reliability analysis are two seemingly very different areas of mathematics, yet they offer many similarities. The key factor in both domains is a certain probability measure determining the development of observed phenomenon. This probability measure is called "hazard rate" in reliability analysis and simply "probability of scoring" in modelling of sport events. It is altered either continuously with time according to some underlying probability function or suddenly as a reaction to an event. Breakage or repair of a given machine in reliability analysis or scoring (goal, point, basket etc.) in sports modelling. The most prominent tool in reliability analysis is the Cox's model ZDROJ expressing the changes of a hazard rate with a base hazard rate and a covariate vector. This model, operating with continuous time, is often used for sports modelling as well. Many types of sport such as tennis or volleyball, however, are played in a strictly discrete matter with steps divided by individual *points*, *games* or *sets*. Other real life situations with similar properties can be found as well, including the recurrence of diseases or recidivism in crime. Thus authors thus recently presented a novel discrete time random process model - discrete random walk with varying probabilities - which aims to serve as a discrete alternative to the standard Cox's model ZDROJ. In this paper selected properties of the model are presented and possible real life implementations of the model are discussed.

The rest of the paper is organized as follows. Next section.... The last section concludes this work.

2 Theoretical properties

The original motivation of the model comes from modelling of sport events where the probability of a success (i.e. scoring a goal, achieving a point etc.) is at the center of interest. After each occurrence of such success this probability either decreases or increases, and thus two basic model alternatives exist - *success punishing* and *success rewarding*. The basic version of the model has 2 parameters, starting success probability p_0 and a memory coefficient λ affecting the severity of probability change after a success. Formally the walk is defined as follows:

Definition 1. Let $p_0, \lambda \in (0, 1)$ be constant parameters, $\{P_n\}_{n=0}^{\infty}$ and $\{X_n\}_{n=1}^{\infty}$ sequences of discrete random variables with $P_0 = p_0$ and for $t \geq 1$

$$P(X_t = 1|P_{t-1} = p_{t-1}) = p_{t-1}, \;\; P(X_t = -1|P_{t-1} = p_{t-1}) = 1 - p_{t-1},$$

and *(success punishing)*

$$P_t = \lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1-X_t) \tag{1}$$

or *(success rewarding)*

$$P_t = \lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1+X_t).$$

The sequence $\{S_n\}_{n=0}^{\infty}$, $S_N = S_0 + \sum_{i=1}^{N} X_i$ for $n \in \mathbb{N}$, with $S_0 \in \mathbb{R}$ some given starting position, is called a *random walk with varying probabilities*, with $\{X_n\}_{n=1}^{\infty}$ being the steps of the walker and $\{P_n\}_{n=0}^{\infty}$ transition probabilities. Depending on the chosen formula to calculate P_i the walk type is either *success punishing* REF or *success rewarding* REF.

The model was first introduced in [1] and a more thorough description was provided in [3]. More properties were then presented in [4] and a practical implementation of the model in modelling tennis matches was presented in [2]. Next let us recall the results of the previous work.

2.1 Success punishing model

The basic properties of the *success punishing* version of the walk are presented in this section. Previous results are presented as a set of expressions only, the reader is kindly asked to see referred papers for full proves of those expressions.

For the expected value and variance of the step size for the $t \geq 1$ iteration of the walk X_t it holds [4]

$$EX_t = (2\lambda - 1)^{t-1}(2p_0 - 1),$$

$$Var\, X_t = 1 - (2\lambda - 1)^{2(t-1)}(2p_0 - 1)^2.$$

For the expected value and variance of the transition probability or the $t \geq 1$ iteration of the walk P_t it holds [1, 4]

$$EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}, \tag{2}$$

$$Var\, P_t = (3\lambda^2 - 2\lambda)^t p_0^2 + \sum_{i=0}^{t-1} K(i)(3\lambda^2 - 2\lambda)^{t-1-i} - k(t)^2,$$

where

$$k(t) = EP_t = (2\lambda - 1)^t p_0 + \frac{1 - (2\lambda - 1)^t}{2}$$

and

$$K(t) = k(t) \cdot (-3\lambda^2 + 4\lambda - 1) + (1 - \lambda)^2.$$

Finally, the expected position of the walker S_t after $t \geq 1$ iterations can be expressed as [1]

$$ES_t = S_0 + (2p_0 - 1)\frac{1 - (2\lambda - 1)^t}{2(1 - \lambda)}. \tag{3}$$

The last formula missing is the one expressing the variance of the position of the walker $Var\, S_t$. Before it is presented, let us first prove a support proposition.

Proposition 2. *For all $t \geq 1$*

$$E(P_t S_t) = (2\lambda - 1)^t p_0 S_0 + \sum_{i=0}^{t-1}(2\lambda - 1)^i q(t - 1 - i; p_0, S_0, \lambda), \tag{4}$$

where

$$q(j; p_0, S_0, \lambda) = (1 - \lambda)s(j; p_0, S_0, \lambda) + 2\lambda\pi(j; p_0, \lambda) + (1 - 2\lambda)p(j; p_0, \lambda) + \lambda - 1$$

and $p(j; p_0, \lambda) = EP_j$ given by (2), $s(j; p_0, S_0, \lambda) = ES_j$ given by (3) and $\pi(j; p_0, \lambda) = EP_j^2$ given as

$$EP_t^2 = (3\lambda^2 - 2\lambda)^t p_0^2 + \frac{1 - (3\lambda^2 - 2\lambda)^t}{3\lambda + 1}\frac{(\lambda + 1)}{2} - (p_0 - \frac{1}{2})(3\lambda - 1)(\lambda - 1)M(t - 1; \lambda), \tag{5}$$

where

$$M(t; \lambda) = \sum_{i=0}^{t-1}(3\lambda^2 - 2\lambda)^{t-1-i}(2\lambda - 1)^i.$$

Proof. The formula for EP_t^2 follows from the proof of Proposition 2.5 in [4]. To prove (4) let us start with expressing the value of $E(P_tS_t)$ from the knowledge of past steps as

$$E(P_tS_t) = E[E(P_tS_t|P_{t-1})] = E[E((\lambda P_{t-1}+\frac{1}{2}(1-\lambda)(1-X_t))(S_{t-1}+X_t)|P_{t-1})] =$$

$$= E[E(\lambda P_{t-1}S_{t-1}+\frac{1-\lambda}{2}S_{t-1}-\frac{1-\lambda}{2}X_tS_{t-1}+\lambda X_tP_{t-1}+\frac{1-\lambda}{2}X_t-\frac{1-\lambda}{2}X_t^2|P_{t-1})]$$

and using $E(X_t|P_{t-1}) = 2P_{t-1} - 1$ and $EX_t^2 = 1$ finally

$$E(P_tS_t) = (2\lambda-1)E(P_{t-1}S_{t-1})+(1-\lambda)ES_{t-1}+2\lambda EP_{t-1}^2+(2\lambda-1)EP_{t-1}+\lambda-1. \tag{6}$$

Further we will continue using mathematical induction. For $t = 1$ using the definition of the walk it holds that

$$E(P_1S_1) = p_0(p_0\lambda(S_0+1)) + (1-p_0)[(1-(1-p_0)\lambda)(S_0-1)] =$$

$$= (2\lambda-1)p_0S_0 + (1-\lambda)S_0 + 2\lambda p_0^2 - (2\lambda-1)p_0 + \lambda - 1.$$

When substituting $t = 1$ into (4) we obtain

$$E(P_1S_1) = (2\lambda-1)p_0S_0 + \sum_{i=0}^{0}(2\lambda-1)^i q(0-i;p_0,S_0,\lambda) =$$

$$= (2\lambda-1)p_0S_0 + (1-\lambda)s(0;p_0,\lambda) + 2\lambda\pi(0;p_0,\lambda) + (1-2\lambda)p(0;p_0,\lambda) + \lambda - 1$$

and finally using (2), (3) and (5)

$$E(P_1S_1) = (2\lambda-1)p_0S_0 + (1-\lambda)S_0 + 2\lambda p_0^2 + (1-2\lambda)p_0 + \lambda - 1.$$

Equation (4) thus holds $t = 1$. Now for the induction step $t \to t+1$ we get by substituting(4) into (6)

$$E(P_{t+1}S_{t+1}) = (2\lambda-1)E(P_tS_t) + (1-\lambda)ES_t + 2\lambda EP_t^2 + (2\lambda-1)EP_t + \lambda - 1 =$$

$$(2\lambda-1)[(2\lambda-1)^t p_0S_0+\sum_{i=0}^{t-1}(2\lambda-1)^i q(t-1-i)]+(1-\lambda)s(t)+2\lambda\pi(t)+(2\lambda-1)p(t)+\lambda-1 =$$

$$(2\lambda-1)^{t+1}p_0S_0 + \sum_{i=0}^{t}(2\lambda-1)^i q(t-i).$$

□

Theorem 3. *For all* $t \geq 1$

$$Var\, S_t = t + 4\sum_{i=0}^{t-1}\sigma(i;p_0,0,\lambda) - a(t;p_0,\lambda),$$

with $\sigma(i;p_0,S_0,\lambda) = E(P_tS_t)$ *given by (4) and*

$$a(t;p_0,\lambda) = (2p_0-1)\sum_{i=0}^{t-1}\frac{1-(2\lambda-1)^i}{1-\lambda} + (2p_0-1)^2\frac{(1-(2\lambda-1)^t)^2}{4(1-\lambda)^2}.$$

Proof. As clearly the value S_0 does not affect the variance, let us from now assume $S_0 = 0$. From the definition of variance

$$Var\, S_t = ES_t^2 - E^2 S_t \tag{7}$$

and (3) follows that to prove the proposition it is enough to prove that

$$ES_t^2 = t + 4\sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - (2p_0 - 1)\sum_{i=0}^{t-1} \frac{1-(2\lambda-1)^i}{1-\lambda}. \tag{8}$$

First of all, let us express ES_t^2 from the knowledge of past walk development. From the definition of the expected value and the definition of the walk (Definition 1) it follows

$$ES_t^2 = E[E(S_t^2|P_{t-1})] = E[E((S_{t-1}+X_t)^2|P_{t-1})] = E(S_{t-1}^2+2(2P_{t-1}-1)S_{t-1}+1) =$$

$$= ES_{t-1}^2 + 4E(P_{t-1}S_{t-1}) - 2ES_{t-1} + 1, \tag{9}$$

where the fact that $EX_t^2 = 1$ was used. The theorem will be once again proved using mathematical induction. For $t = 1$ we get using the definition of the walk (Definition 1)

$$ES_1^2 = p_0(S_0+1)^2 + (1-p_0)(S_0-1)^2 = 1.$$

Substituting $t = 1$ into (8) we obtain

$$ES_1^2 = 1 + 4\sigma(0; p_0, 0, \lambda) - (2p_0 - 1)\frac{1-(2\lambda-1)^0}{1-\lambda} = 1$$

and (8) thus holds for $t = 1$. Now for the induction step $t \to t+1$ we get by substituting (8) into (9)

$$ES_{t+1}^2 = ES_t^2 + 4E(P_tS_t) - 2ES_t + 1 =$$

$$= t + 4\sum_{i=0}^{t-1} \sigma(i; p_0, 0, \lambda) - (2p_0 - 1)\sum_{i=0}^{t-1} \frac{1-(2\lambda-1)^i}{1-\lambda} +$$

$$+4\sigma(t; p_0, 0, \lambda) - 2(2p_0 - 1)\frac{1-(2\lambda-1)^t}{2(1-\lambda)} + 1 =$$

$$(t+1) + 4\sum_{i=0}^{t} \sigma(i; p_0, 0, \lambda) - (2p_0 - 1)\sum_{i=0}^{t} \frac{1-(2\lambda-1)^i}{1-\lambda},$$

which proves (8). Substituting (8) and (3) into (7) then proves the theorem. □

2.2 Success rewarding model

Similar formulas can be derived for the *success rewarding* model. Once again for previous results only the formulas are presented with proofs in the referred literature. For the sake of clarity the set of expressions is presented in the same manner as in the previous section.

For the expected value and variance of the step size for the $t \geq 1$ iteration of the walk X_t it holds [4]

$$EX_t = 2p_0 - 1,$$
$$Var\, X_t = 4p_0(1 - p_0).$$

For the expected value and variance of the transition probability or the $t \geq 1$ iteration of the walk P_t it holds [4]

$$EP_t = p_0, \tag{10}$$

$$Var\, P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1-\lambda)^2 \sum_{i=0}^{t-1} (2\lambda - \lambda^2)^i - p_0^2.$$

As the sum in the formula equals $\frac{1-(2\lambda-\lambda^2)^t}{1-(2\lambda-\lambda^2)}$, it can be further simplified as

$$Var\, P_t = (2\lambda - \lambda^2)^t p_0^2 + p_0(1-\lambda)^2 \frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2} - p_0^2 =$$

$$= p_0[(2\lambda - \lambda^2)^t(p_0 - 1) + 1] - p_0^2,$$
$$Var\, P_t = p_0[(1 - p_0)(1 - (2\lambda - \lambda^2)^t)].$$

Finally, the expected position of the walker S_t after $t \geq 1$ iterations can be expressed as [4]

$$ES_t = S_0 + t(2p_0 - 1). \tag{11}$$

To prove a formula allowing to compute the variance of the position of the walker, let us again start with a support proposition.

Proposition 4. *For all* $t \geq 1$

$$E(P_tS_t) = p_0S_0 + p_0t + 2\lambda p_0(p_0 - 1)\frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2}. \tag{12}$$

Proof. We will once again start with expressing $E(P_tS_t)$ from the knowledge of the past step.

$$E(P_tS_t) = E(E(P_{t-1}S_{t-1}|P_{t-1}) = E[E((\lambda P_{t-1} + \frac{1}{2}(1-\lambda)(1+X_t))(S_{t-1}+X_t)|P_{t-1})] =$$

$$= E[E(\lambda P_{t-1}S_{t-1} + \frac{1-\lambda}{2}S_{t-1} + \frac{1-\lambda}{2}X_tS_{t-1} + \lambda X_tP_{t-1} + \frac{1-\lambda}{2}X_t + \frac{1-\lambda}{2}X_t^2|P_{t-1})]$$

and using $E(X_t|P_{t-1}) = 2P_{t-1} - 1$ and $EX_t^2 = 1$ finally

$$E(P_tS_t) = E(P_{t-1}S_{t-1}) + 2\lambda EP_{t-1}^2 - (2\lambda - 1)EP_{t-1}. \tag{13}$$

Further we will continue using mathematical induction. For $t = 1$ using the definition of the walk it holds that

$$E(P_1S_1) = p_0(1-(1-p_0)\lambda)(S_0+1) + (1-p_0)\lambda p_0(S_0-1) =$$

$$= p_0S_0 + 2\lambda p_0^2 - (2\lambda-1)p_0.$$

When substituting$t = 1$ into (12) we obtain

$$E(P_1S_1) = p_0S_0 + p_0 + 2\lambda p_0(p_0-1)\frac{1-(2\lambda-\lambda^2)^0}{(1-\lambda)^2} =$$

$$= p_0S_0 + p_0 + 2\lambda p_0(p_0-1)$$

and finally

$$E(P_1S_1) = p_0S_0 + 2\lambda p_0^2 - (2\lambda-1)p_0.$$

Equation (12) thus holds $t = 1$. Now for the induction step $t \to t+1$ we get by substituting(12) into (13)

$$E(P_{t+1}S_{t+1}) = E(P_tS_t) + 2\lambda EP_t^2 - (2\lambda-1)EP_t$$

and further using

$$EP_t^2 = p_0((2\lambda-\lambda^2)^t(p_0-1)+1),$$

which follows from the proof of Proposition 3.7 in [4], and (10)

$$E(P_{t+1}S_{t+1}) = p_0S_0 + p_0t + 2\lambda p_0(p_0-1)\frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2} + 2\lambda p_0((2\lambda-\lambda^2)^t(p_0-1)+1) - (2\lambda-1)p_0 =$$

$$= p_0S_0 + p_0(t+1) + 2\lambda p_0(p_0-1)[\frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2} + (2\lambda-\lambda^2)^t] =$$

$$= p_0S_0 + p_0(t+1) + 2\lambda p_0(p_0-1)\frac{1-(2\lambda-\lambda^2)^{t+1}}{(1-\lambda)^2}.$$

□

Theorem 5. *For all $t \geq 1$ holds*

$$Var\, S_t = 4p_0(1-p_0)t^2 + a(p_0,\lambda)t - a(p_0,\lambda)\frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2},$$

where

$$a(p_0,\lambda) = \frac{8p_0(1-p_0)}{(1-\lambda)^2}.$$

Proof. As clearly the value S_0 does not affect the variance, let us from now assume $S_0 = 0$. From the definition of variance and (11) follows that to prove the theorem it is enough to prove that

$$ES_t^2 = t^2 + a(p_0,\lambda)t - a(p_0,\lambda)\frac{1-(2\lambda-\lambda^2)^t}{(1-\lambda)^2}. \tag{14}$$

First of all let us recall that formula (9) holds for the *success rewarding* type of the model as well. The theorem will be once again proved using mathematical induction. For $t = 1$ the definition of the walk yields the same result as in the proof of Theorem (3). Substituting $t = 1$ into (14) we obtain

$$ES_1^2 = 1 + a(p_0, \lambda)t - a(p_0, \lambda) = 1$$

and (14) thus holds for $t = 1$. Now for the induction step $t \to t + 1$ we get by substituting (14), (12) and (11) into (9)

$$ES_{t+1}^2 = ES_t^2 + 4E(P_t S_t) - 2ES_t + 1 =$$

$$= t^2 + a(p_0, \lambda)t - a(p_0, \lambda)\frac{1 - (2\lambda - \lambda^2)^t}{(1 - \lambda)^2} + 4(p_0 t + 2\lambda p_0(p_0 - 1)\frac{1 - (2\lambda - \lambda^2)^t}{(1 - \lambda)^2}) - 2t(2p_0 - 1) + 1 =$$

$$= (t + 1)^2 + a(p_0, \lambda)(t + 1) - a(p_0, \lambda)\frac{1 - (2\lambda - \lambda^2)^{t+1}}{(1 - \lambda)^2}.$$

Substituting (14) and (11) into the definition of variance then proves the theorem. □

3 Model application

The model is especially well suited for simulation of random processes where a single or just a few events significantly affect the process's future development. An example such a process can be found in sports modelling. The authors recently presented a study where the model was used for modelling the men tennis Grand Slam matches with results suggesting the model might provide precious insight when modelling tennis. Here is a brief summary of the modelling approach.

The *success rewarding* version of the model was selected as the historical results show that the development of a tennis match follows such pattern. The p_0 was found using input bookmaker odds from *Pinnacle Sports*, an industry leading bookmaker. The appropriate λ parameter was then found from historical data using the maximal likelihood estimate. The model was tested on a database consisting of 4255 tennis matches that took place between 2009 and 2018 and the results suggest that such a model could be used for *in-play* odds prediction. For full details of the model derivation and testing, see the original paper [2].

The model was implemented and tested in real life setup where it actively bet *in-play* against Tipsport, the biggest bookmaker in the Czech Republic. The test was set up in the following manner.

An automated betting and odds scraping tool developed using the Python programming language and Selenium framework running on a remote server (Digital Ocean) was developed and deployed. The tool operates with Tipsport's website and scrapes it for both pre-match as well as in-play odds. It was set up to continuously observe Tipsport's odds offerings for 2019 men tennis US Open, especially the set winning odds, and store the odds together with some general information about the match, such as the respective players, starting time etc., into a database (Postgresql database). To test the model, Tipsport's starting odds were used to obtain parameter p_0 and optimized λ trained on the 2018 tennis season was chosen as the second necessary model parameter in a same way se in [2]. Every match was observed individually and after each finished set next set winning probabilities were computed using the presented model. Whenever the actual set winning odds offered by Tipsport for one of the players a_i were higher than the probability implied set winning odds for that player computed by the model, i.e. $a_i > \frac{1}{p}$, a bet was made. The amount bet was computed as $p_i U$, where U is a base bankroll-dependent betting unit. This amount was further rounded with precision CZK 1 (due to betting limitations of Tipsport). Overall, 128 bets were made (and 3 additional bets that were cancelled due to one of the players forfeiting the match because of injury) with the total amount $59.85U$ bet. The expected number of wins among these bets was 59.85 whereas the actual number of wins was 57. The minimal account balance over the entire US Open was $-0.52U$ and the final balance $2.24U$, creating a theoretical return on investment (ROI) of 430% within only 2 weeks, which is an outstanding performance.

This study just briefly shows the posibilities of the presented model. The testing dataset, consisting of only 128 bets, is too small to provide a strong evidence favoring the model over bookmaker's ods, but the results are more than convincing. A more conclusive test on a bigger dataset will be subject of further study.

4 Conclusion

A specific model of a random walk with varying probabilities was considered in this paper. New properties of the model were derived and proved in Section XXX, providing another insight into the model behaivor. A model application from previous study was recalled and the model was tested on a new, real life dataset with convincing results.

References

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