Understand the Assumptions of OLS Regression

Understand and empirically test OLS assumptions. Dealing with breaches of OLS assmptions.

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Agenda

- **Understand the Assumptions of OLS Regression**
  - Linearity, independence, homoscedasticity, normality of residuals.
- **Testing Assumptions in Stata**
  - Nonlinearity: `rvfplot`, `lowess`
  - Heteroscedasticity: `hettest`, `imtest`
  - Distribution of errors: `swilk`, `qnorm`
- **Addressing Violations of Assumptions**
  - Transformations (quadratic, log, square root).
  - Robust or clustered standard errors.

Gauss–Markov Assumptions

The Ordinary Least-Squares (OLS) Estimator

- The least-squares method is attributed to Karl Friedrich Gauss (1821).
- The Gauss–Markov theorem states that OLS provides the Best Linear Unbiased Estimator (BLUE) under specific conditions.

Gauss–Markov Assumptions

1. **Zero Conditional Mean of Errors**
    - $E(\varepsilon|X) = 0$ ensures no omitted variable bias.
2. **Homoscedasticity (Constant Variance of Errors)**
    - Variance of errors should not depend on X.
3. **No Autocorrelation**
    - Errors should not be correlated across observations.

Additional Assumptions for OLS

4. **Correct Model Specification**
   - Omitted variables or incorrect functional form can bias results.
5. **Absence of Multicollinearity**
   - Explanatory variables should not be highly correlated.
6. **Normally Distributed Residuals**
   - Important for inference, particularly for small samples.

Next Steps:

- Investigate influential cases and potential outliers.
- Use robust methods if assumptions are violated.

Correct Model

Two Parts:

- **Model Specification:** Ensuring the correct functional form and inclusion of relevant variables.
- **Residuals Assumptions:** Ensuring homoscedasticity, normality, and independence of residuals.

Model Specification

All X-variables are Relevant, and None Irrelevant

You should not include X-variables that you have no theoretical or logical reason to include (Mehmetoglu and Jakobsen, 2022).

Testing Model Improvements in Stata:

- Run the restricted (smaller) model.
- Run the unrestricted (larger) model with additional variables.
- Use an **F-test** to check significance of added variables.

```
*Run the restricted model
reg Y X1 X2 X3
*Run the unrestricted model
reg Y X1 X2 X3 X4 X5
*Conduct an F-test to check if X_4 and X_5 jointly improve the model
test X4 X5
```

Detecting Model Misspecification with `linktest` in Stata

Purpose of `linktest`:

- Checks if the regression model is correctly specified.
- Identifies whether the wrong functional form is used.
- Detects omitted variables.

How It Works:

- Runs a regression including:
  - _hat: the predicted values of Y.
  - _hatsq: the squared predicted values.
- If _hatsq is significant, the model is misspecified.

Interpreting Results:

- If _hatsq is NOT significant: The model is correctly specified.
- If _hatsq is significant: The model has omitted variables or incorrect functional form.

```
*Estimate the factors that are associated with the trust in the legal
    system (0  10  )
use ESSGBdiagnostics.dta, clear
quietly regress trstlgl age woman political_interest religious
linktest
```

Interpreting Output:

- If _hatsq is significant ($p < 0.05$), the model is misspecified.
- Consider adding relevant variables or transforming predictors.

However, passing a diagnostic test like `linktest` (or `ovtest`) not mean that we have specified the best possible model, either statistically or substantively.

Linearity Assumption in Regression

Definition:

- A one-unit increase in X_i results in a constant change in Y, holding other variables constant.
- The effect of X on Y does not depend on the level of X.

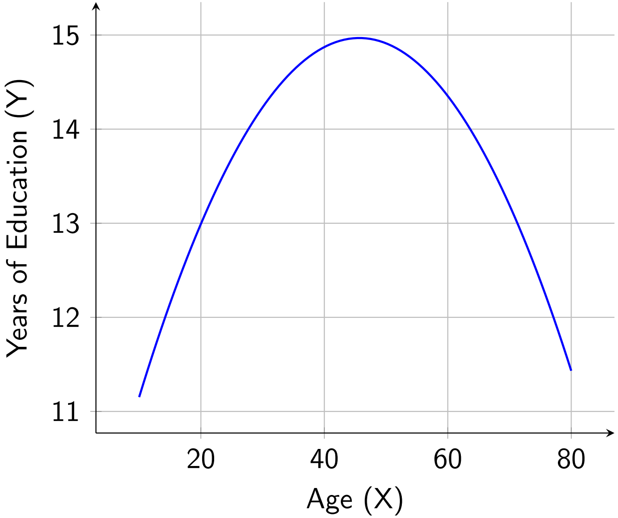
When Linearity is Violated:

- In real-world data, the impact of X on Y often changes at different levels of X.
- Example:
  - Moving from $X = 3$ to $X = 4$ increases Y.
  - Moving from $X = 45$ to $X = 46$ decreases Y.
- This indicates a nonlinear relationship.

Consequences of Misspecification:

- Incorrect slope estimates lead to misleading interpretations.
- Standard errors may be biased, affecting hypothesis testing.
- Model predictions may not accurately reflect reality.

Nonlinear Relationship: Quadratic Curve



Equation: $Y = 8.712 + 0.274X - 0.003X^2$

Equation: $Y = 8.712 + 0.274X - 0.003X^2$
Interpretation:

- Initially, Y (education) increases with X (age).
- A turning point occurs where education peaks.
- After a certain age, the negative X^2 term dominates, causing Y to decline.

Finding the Maximum/Minimum of $Y = 8.712 + 0.274X - 0.003X^2$

Equation:

$$Y = 8.712 + 0.274X - 0.003X^2$$

Step 1: Compute the First Derivative

$$\frac{dY}{dX} = 0.274 - 0.006X$$

To find the critical point, set the derivative equal to zero:

$$0.274 - 0.006X = 0$$

Step 2: Solve for X

$$X = \frac{0.274}{0.006} = 45.67$$

This means the function reaches a turning point at approximately $X = 45.67$ years.

Step 3: Second Derivative Test

$$\frac{d^2Y}{dX^2} = -0.006$$

Since the second derivative is negative, the function has a maximum at $X = 45.67$.

Interpretation of the Maximum Point

Key Findings:

- Education (Y) increases with age (X) until $X = 45.67$.
- After this age, the negative effect of X^2 dominates, causing education levels to decline.
- The maximum value of Y occurs around age 46.

Practical Implication:

- The quadratic term in regression helps capture nonlinearity in relationships.
- Ignoring nonlinear effects could lead to misleading interpretations.

Detecting and Addressing Nonlinearity

How to Detect Nonlinearity:

- Scatterplots: Visualize Y vs. X to check for patterns.
- Residual Plots: Look for systematic patterns in residuals.
- Ramsey's RESET Test (`ovtest` in Stata) detects omitted nonlinear terms.

Solutions to Nonlinearity:

- Polynomial Regression: Include X^2 or X^3 terms.
- Log Transformations: Use log-linear or log-log models - we will talk about them later :).
- Nonparametric Methods: Consider splines or kernel regressions (advanced, not covered in this course).

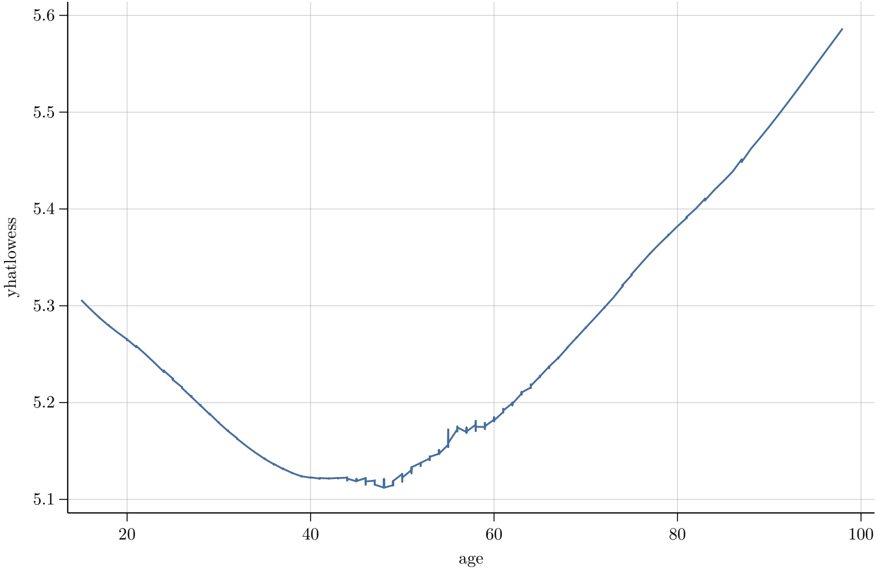
```
*Detect potential nonlinearity
scatter Y X       *Check for nonlinearity
ovtest            *Ramseys RESET test
gen X2 = X^2      *Create squared term
reg Y X X2        *Fit polynomial model
 \end{verbatim}
```

```
. *Modelling curvilinearity when there is an interaction
. use ESS5GBdiagnostics.dta, clear
. regress trstlgl age woman political_interest religious

      Source |       SS           df       MS      Number of obs   =     1,902
-------------+----------------------------------   F(4, 1897)      =     25.18
       Model |  532.142701         4  133.035675   Prob > F        =    0.0000
    Residual |  10023.8426     1,897  5.28404986   R-squared       =    0.0504
-------------+----------------------------------   Adj R-squared   =    0.0484
       Total |  10555.9853     1,901  5.55285917   Root MSE        =    2.2987

------------------------------------------------------------------------------------
           trstlgl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------------+----------------------------------------------------------------
               age |  -.0045918   .0028796    -1.59   0.111    -.0102393    .0010557
             woman |  -.3239065    .108797    -2.98   0.003    -.5372809   -.1105321
political_interest |   .4534565   .0581223     7.80   0.000     .3394662    .5674468
         religious |   .0892736   .0217054     4.11   0.000     .0467046    .1318426
             _cons |   4.172909   .2254221    18.51   0.000     3.730808    4.615011
------------------------------------------------------------------------------------
```

```
*Curvilinearity
lowess trstlgl age, nograph gen(yhatlowess)  // predicts value of
   regression
line yhatlowess age, sort   // graph bivariate relationship
```


5.6
5.5
5.4
5.3
5.2
5.1
yhatlowess
20
40
60
80
100
age

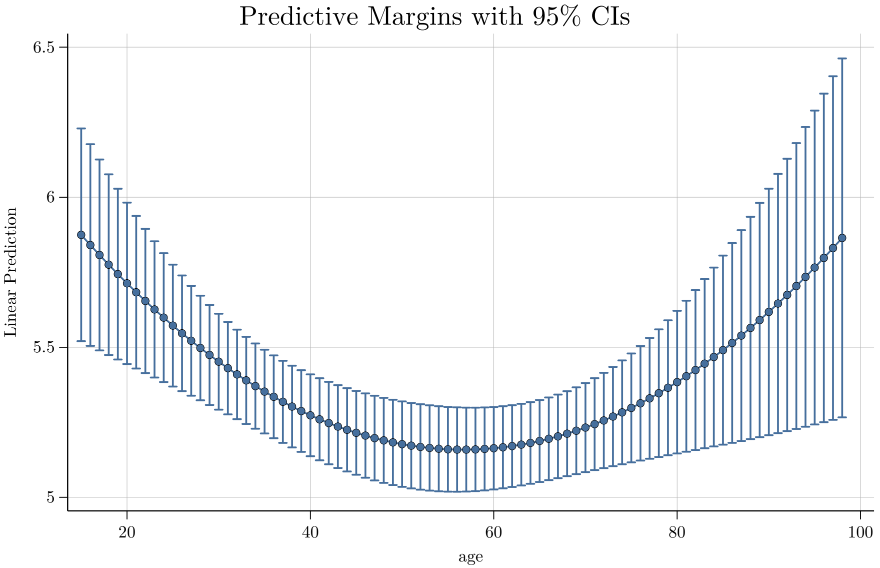
```
*Regression including squared term
regress trstlgl c.age##c.age woman political_interest religious

margins, at(age=(15(1)98))
marginsplot
```

```
. *Regression including squared term
. regress trstlgl c.age##c.age woman political_interest religious

      Source |       SS           df       MS      Number of obs   =     1,902
-------------+----------------------------------   F(5, 1896)      =     21.84
       Model |   574.94511         5  114.989022   Prob > F        =    0.0000
    Residual |  9981.04017     1,896  5.26426169   R-squared       =    0.0545
-------------+----------------------------------   Adj R-squared   =    0.0520
       Total |  10555.9853     1,901  5.55285917   Root MSE        =    2.2944

------------------------------------------------------------------------------------
           trstlgl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------------+----------------------------------------------------------------
               age |  -.0467456   .0150601    -3.10   0.002    -.0762818   -.0172095
                   |
       c.age#c.age |   .0004126   .0001447     2.85   0.004     .0001288    .0006964
                   |
             woman |  -.3257212    .108595    -3.00   0.003    -.5386995    -.112743
political_interest |   .4723812   .0583918     8.09   0.000     .3578623       .5869
         religious |   .0864412   .0216875     3.99   0.000     .0439074    .1289751
             _cons |   5.074485   .3880667    13.08   0.000     4.313403    5.835568
------------------------------------------------------------------------------------
```


Predictive Margins with 95% CIs
Linear Prediction
6.5
6
5.5
5
20
40
60
80
100
age

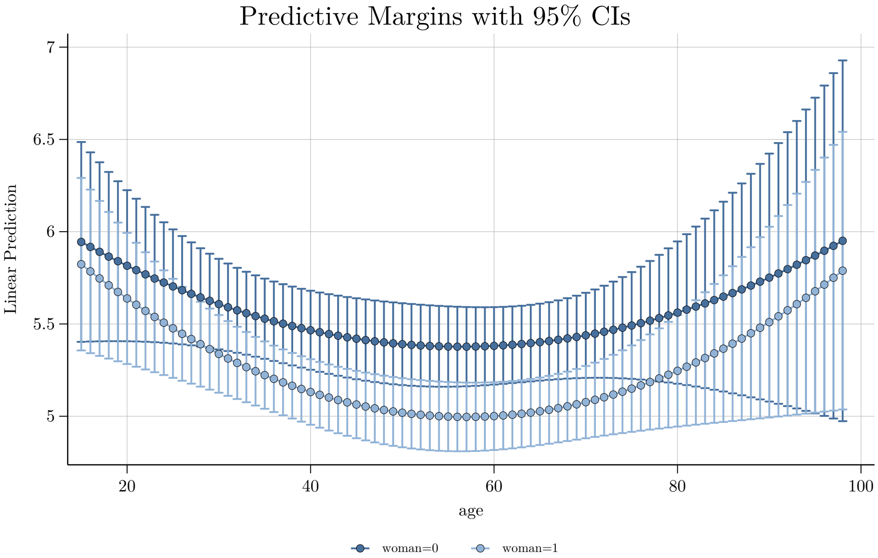
```
*Regression including squared term and interaction effect
regress trstlgl c.age##c.age##i.woman political_interest religious

margins, at (age=(15(1)98) woman=(0 1))
marginsplot
```

```
. *Regression including squared term and interaction effect
. regress trstlgl c.age##c.age##i.woman political_interest religious

      Source |       SS           df       MS      Number of obs   =     1,902
-------------+----------------------------------   F(7, 1894)      =     15.64
       Model |  576.785799         7  82.3979713   Prob > F        =    0.0000
    Residual |  9979.19948     1,894  5.26884872   R-squared       =    0.0546
-------------+----------------------------------   Adj R-squared   =    0.0511
       Total |  10555.9853     1,901  5.55285917   Root MSE        =    2.2954

------------------------------------------------------------------------------------
           trstlgl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------------+----------------------------------------------------------------
               age |  -.0373657   .0236385    -1.58   0.114    -.0837259    .0089946
                   |
       c.age#c.age |   .0003313   .0002306     1.44   0.151    -.0001209    .0007836
                   |
           1.woman |   .0910325   .7326664     0.12   0.901    -1.345885     1.52795
                   |
       woman#c.age |
                1  |  -.0161969   .0305791    -0.53   0.596    -.0761692    .0437754
                   |
 woman#c.age#c.age |
                1  |   .0001389   .0002951     0.47   0.638    -.0004399    .0007178
                   |
political_interest |   .4723383   .0584197     8.09   0.000     .3577645    .5869122
         religious |   .0870871   .0217246     4.01   0.000     .0444804    .1296938
             _cons |   4.831897    .577209     8.37   0.000     3.699865    5.963929
------------------------------------------------------------------------------------
```


Predictive Margins with 95% CIs
Linear Prediction
7
6.5
6
5.5
5
20
40
60
80
100
age
woman=0
woman=1

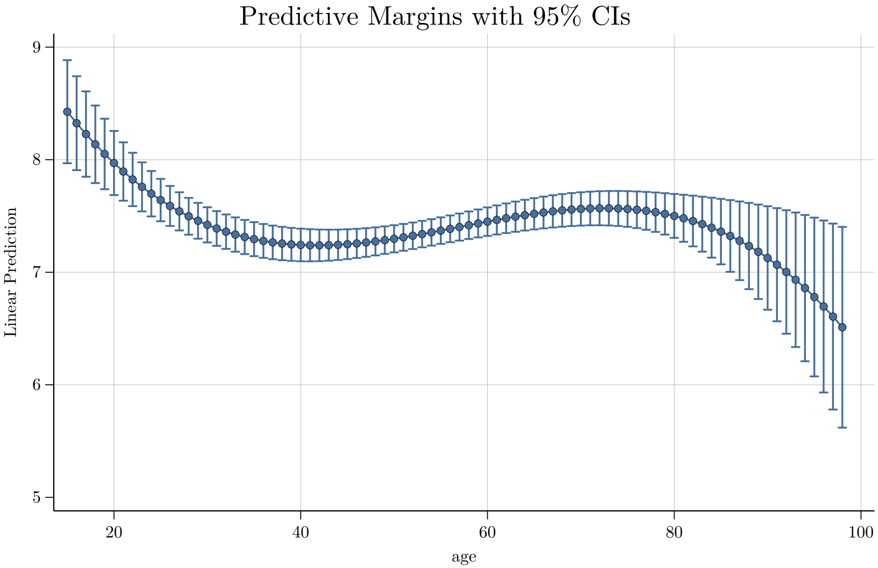
```
*Curvilinear effect with two bends
regress happy c.age##c.age##c.age woman political_interest religious
    leftright

margins, at(age=(15(1)98))
marginsplot, noci
```

```
. *Curvilinear effect with two bends
. regress happy c.age##c.age##c.age woman political_interest religious leftright

      Source |       SS           df       MS      Number of obs   =     1,649
-------------+----------------------------------   F(7, 1641)      =      9.18
       Model |  205.511812         7  29.3588303   Prob > F        =    0.0000
    Residual |   5250.7344     1,641  3.19971627   R-squared       =    0.0377
-------------+----------------------------------   Adj R-squared   =    0.0336
       Total |  5456.24621     1,648  3.31082901   Root MSE        =    1.7888

------------------------------------------------------------------------------------
             happy |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------------+----------------------------------------------------------------
               age |   -.206582   .0491139    -4.21   0.000    -.3029145   -.1102495
                   |
       c.age#c.age |   .0038972   .0009895     3.94   0.000     .0019563    .0058381
                   |
 c.age#c.age#c.age |  -.0000227   6.21e-06    -3.66   0.000    -.0000349   -.0000106
                   |
             woman |  -.0239869   .0906505    -0.26   0.791    -.2017898     .153816
political_interest |   .1368392   .0509382     2.69   0.007     .0369284    .2367499
         religious |   .0942766   .0183395     5.14   0.000     .0583054    .1302479
         leftright |   .0520857   .0257479     2.02   0.043     .0015835    .1025879
             _cons |   9.651854   .7685288    12.56   0.000     8.144453    11.15925
------------------------------------------------------------------------------------
```



Predictive Margins with 95% CIs
Linear Prediction
age
9
8
7
6
5
20
40
60
80
100

Additivity in Regression

Definition:

- The assumption of additivity means that the effect of an independent variable (X_i) on Y is constant, regardless of the values of other independent variables.
- This implies that each X-variable has a separate, independent effect on Y.

When Additivity is Violated:

- If the effect of one X-variable depends on the level of another X-variable, the assumption is breached.
- This situation is known as an interaction effect.

Solution: Introducing Interaction Terms (See Previous Lecture)

- Interaction effects can be modeled by including interaction terms in the regression:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \times X_2) + \varepsilon$$

- The coefficient β_3 captures the interaction effect between X_1 and X_2.

Absence of Multicollinearity

Definition:

- Multicollinearity occurs when two or more X-variables in a regression model are highly correlated.
- Perfect multicollinearity means that one X-variable can be perfectly explained by a linear combination of other X-variables.

Problems Caused by Multicollinearity:

- High correlation (above 0.8) makes it difficult to assess the individual impact of explanatory variables.
- Inflated standard errors, leading to unreliable significance tests.
- Explanatory power is shared, making it difficult to interpret coefficients.

How to Detect Multicollinearity in Stata:

- Correlation Matrix:

```
correlate X1 X2 X3
```

- Variance Inflation Factor (VIF):

```
regress  Y X1 X2 X3
vif
```

- High VIF values (> 10) indicate severe multicollinearity.

Addressing Multicollinearity

How to Fix Multicollinearity:

- **Remove one of the correlated variables** if they measure the same phenomenon.
- **Combine variables into an index or scale** using factor analysis or reliability tests (advanced not covered in this course).
- **Center variables** (for interaction terms) to reduce multicollinearity.

```
*Multicollienarity
quietly regress trstlgl age woman political_interest religious
estat vif
estat vce
```

Residuals Assumptions

Zero Conditional Mean Assumption

$$E[\varepsilon | X_1, X_2, ..., X_N] = 0 \qquad (1)$$

Why This Assumption Holds in Sample:

- Due to the least-squares method, the residuals in an OLS model always balance out in the sample.
- If the errors have a non-zero mean, this will be absorbed by the constant term.

When OLS May Not Be the Best Estimator:

- The estimated coefficients may not represent the true population relationships.
- This can happen when explanatory variables are related to the error term.

Addressing the Problem: ▸ This is subject to the Applied Economics!

Constant Variance Assumption Holds - (aka Homoscedasticity)

Homoskedasticity means that the variance of the error term remains constant across all values of the independent variables.

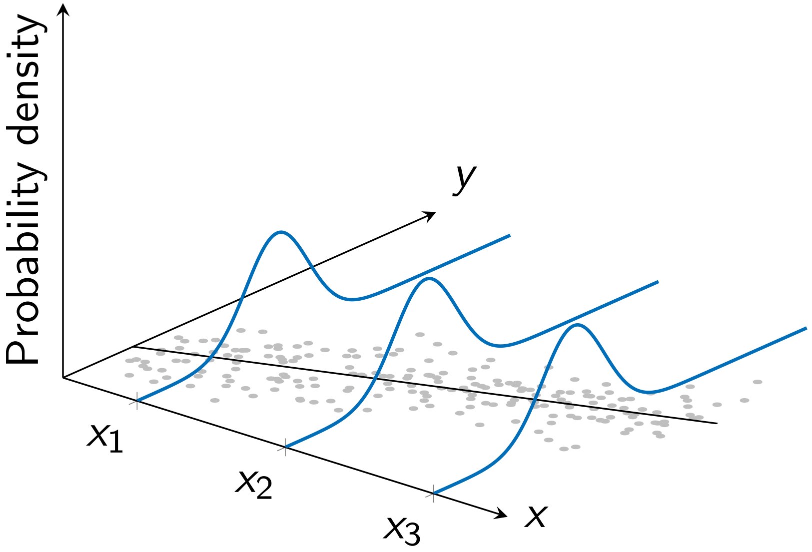
$$\text{Var}(\varepsilon_i | X) = \sigma^2$$

where σ^2 is finite and positive.

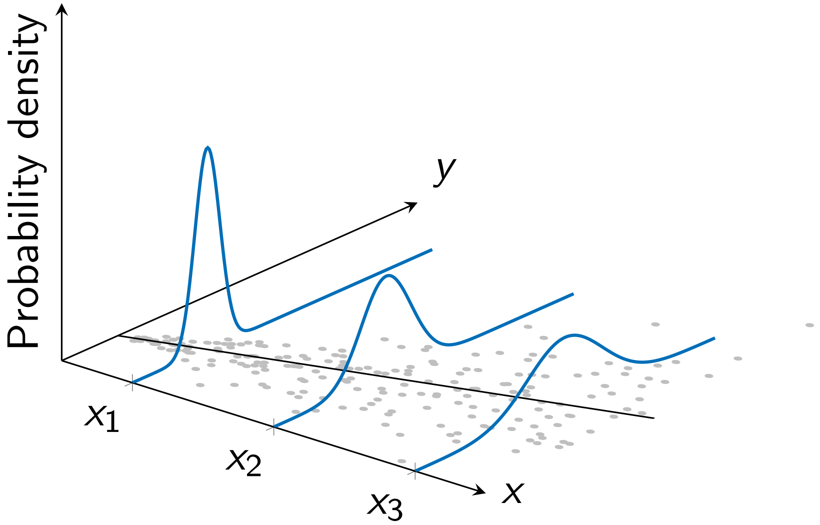
Why This Assumption Matters:

- Ensures valid statistical inference and generalization from sample to population.
- If the variance of residuals changes across levels of X, the model has heteroskedasticity, which leads to:
  - Biased standard errors.
  - Inefficient OLS estimates.
  - Incorrect hypothesis test results.

Constant Variance Assumption Holds - (aka Homoscedasticity)



Constant Variance Assumption Does Not Hold - (aka Heteroscedasticity)



▸ What's that?

Detecting and Testing for Heteroskedasticity in Stata

Step 1: Run the Regression Model

```
quietly regress trstlgl age woman political_interest religious
```

Step 2: Residual vs. Fitted Plot

```
rvfplot
```

- A funnel shape in the plot suggests heteroskedasticity.

Step 3: Breusch-Pagan / Cook-Weisberg Test

```
estat hettest
```

- A significant test result indicates heteroskedasticity.

Solutions for Heteroskedasticity:

- Use robust standard errors:

  ```
  regress trstlgl age woman political_interest religious, vce(robust)
  ```

- Consider log transformation if appropriate.
- Use weighted least squares (WLS) if needed.

Uncorrelated Errors

The errors in a regression model should be uncorrelated across observations.

$$E(\varepsilon_i \varepsilon_j | X_1, \ldots, X_n) = 0, \quad i \neq j$$

If errors are correlated, we call this autocorrelation.

When This Assumption is Violated:

- Often occurs in time series or geographically nested data.
- Example: Values from the previous year influence the current year.
- In cross-sectional data, autocorrelation is usually not a concern.

Implications of Autocorrelation:

- Standard errors are underestimated, leading to inflated significance.
- Model predictions may be biased if time dependence is ignored.

Testing for Autocorrelation in Stata

Dataset: `datasets/Durbin_Watson.dta`

Step 1: Set the Data for Time Series Analysis

```
tsset year
```

Step 2: Run the Regression Model

```
regress FDI GDPperCapita GDPGrowth incidence
```

Step 3: Perform Durbin-Watson Test

```
estat dwatson
```

Interpreting the Durbin-Watson Statistic:

- A value near 2 suggests no autocorrelation.
- A value near 0 suggests positive autocorrelation.
- A value near 4 suggests negative autocorrelation.

Addressing Autocorrelation

Solutions for Autocorrelation:

- Use robust standard errors for time-series data:

```
regress FDI GDPperCapita GDPGrowth incidence, vce(robust)
```

- Apply a first-difference transformation:

```
gen d_FDI = D.FDI
regress d_FDI GDPperCapita GDPGrowth incidence
```

- Use Generalized Least Squares (GLS) or Newey-West standard errors for time dependence.

Normally Distributed Errors

The residuals in an OLS regression should follow a normal distribution:

$$\varepsilon_i \sim N(0, \sigma^2), \quad \text{for all } i$$

This assumption ensures valid statistical inference, particularly in small samples.

Why This Assumption Matters:

- Normal errors are not required for OLS to be unbiased, but they affect:
  - The accuracy of t-tests and F-tests.
  - The efficiency of OLS estimates.
- Highly skewed distributions of the dependent variable or residuals can be problematic.

Testing for Normality in Stata

Step 1: Run the Regression Model

```
use ESSGBdiagnostics.dta, clear
quietly regress trstlgl age woman political_interest religious
```

Step 2: Generate Residuals

```
    predict res, residual
```

Step 3: Visual Inspection with Histogram

```
histogram res, normal
```

- This plots a histogram of residuals overlaid with a normal curve.

Step 4: Statistical Tests for Normality

```
summarize res, detail
sktest res
```

- The skewness/kurtosis test (sktest) checks if residuals follow a normal distribution.

Addressing Non-Normal Residuals

Solutions for Non-Normal Residuals:

- Use robust standard errors:

```
regress trstlgl age woman political_interest religious, vce(robust)
```

- Apply transformations to the dependent variable (e.g., log transformation for right-skewed distributions).
- Check for outliers and high-leverage points (we will talk about them on the next lecture) that distort normality.

When Normality is Less Important:

- In large samples, the Central Limit Theorem ensures that standard errors remain valid even when residuals are not perfectly normal.
- Focus should be on heteroskedasticity and omitted variable bias rather than strict normality.

References I

Laffers, L. (2021). *Draft poznámok k predmetu Moderná Aplikovaná regresia 1*. UMB Banská Bystrica.

Mehmetoglu, M. and Jakobsen, T. G. (2022). *Applied Statistics using Stata: a Guide for the Social Sciences*. Sage.