131

Exam Part 1 of ADVANCED ALGORITHMS (IN4344)

5 October 2023, 18:30 – 21:30

This exam consists of 7 questions on 13 pages. You can earn 70 points in total. The obtained grade is equal to $1 + 9 \times (\text{points}/70)$.

- You may use a non-graphical calculator. You are **not** allowed to use the lecture notes, your own notes or any electronic devices except for a non-graphical calculator.
- Write each solution in the solution box corresponding to that question. If you need more space, you may continue your solution in one of the extra boxes at the end. Indicate this clearly.
- For all questions, you need to use the right method and describe all steps and arguments clearly.

Responsible examiner: Dr. Yukihiro Murakami **Good luck!**

Student number :

0	☐	☒	☐	☐	☐	☐	☐
1	☐	☐	☐	☐	☐	☐	☐
2	☐	☐	☐	☐	☐	☐	☐
3	☐	☐	☐	☒	☐	☐	☐
4	☐	☐	☐	☐	☐	☐	☐
5	☐	☐	☐	☐	☐	☐	☐
6	☒	☐	☐	☐	☐	☐	☒
7	☐	☐	☒	☐	☒	☐	☐
8	☐	☐	☐	☐	☐	☒	☐
9	☐	☐	☐	☐	☐	☐	☐

First name : Tomás

Last name : Osarte

Legend:

→ ∀ : for all

EXCUSE ME FOR THE PRESENTATION!

I'm sorry :(

1. (8 points) Answer the following questions with True or False accompanied by a brief justification (1-2 sentences).

   (i) If an LP-relaxation is unbounded, then the original ILP is also unbounded.

   (ii) Let z_A^* be a solution to a minimization ILP, obtained via an approximation algorithm with approximation ratio 3. If z_{ILP}^* is an optimal solution to the ILP and z_{LP}^* is an optimal solution to the LP-relaxation, then the following is always true.

$$z_{\text{LP}}^* \leq z_A^* \leq 4z_{\text{ILP}}^*.$$

   (iii) The matrix below is totally unimodular.

$$\begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

   (iv) Suppose that an LP is feasible and unbounded. Then its dual must also be unbounded.

(iv) False, the dual can also be Infeasible

(iii) True since every column has at most 2 non zero elements and the matrix can be divided by rows in 2 sets such ~~tet~~ that:

→ if a ~~(row has ca)~~ column has 2 different sign numbers are in the same set

→ if a column has 2 same sign numbers are in different sets.

$S_1 = \{R_1, R_3\}$ $S_2 = \{R_2, R_4\}$

(ii) TRUE since a ρ-approximation ~~means~~ is always

(for min) $\underset{\text{Alg}(i)}{\cancel{Z_A}} \leq \rho \cdot \text{opt}(i) \rightarrow Z_A^* \leq 3 Z_{ILP}^* \rightarrow Z_A^* \leq 4 Z_{ILP}^*$

and also the relaxation solution is always smaller ~~(than~~) or equal than an ILP solution.

→ Continue in ~~box)~~ EXTRA BOX 2!

(i) ~~TRUE~~ (~~False, I can give you a counter example in~~)

2. Consider the following final Simplex tableau of the LP-relaxation of a minimization ILP.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_2	5/3	1/2	1	7/3	-4/3	0
s_2	7	-3	0	-12	-2/3	1
$-z$	-5/3	0	0	10	4/3	0

(a) (4 points) Does the LP-relaxation have an optimal solution? If yes, state the optimal solution, and also state if this optimal solution is unique. Give a short justification at each step (1-2 sentences).

Yes, the optimal solution is $\cancel{\begin{pmatrix} x_2^* \\ s_2 \end{pmatrix}}$ $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 5/3 \end{pmatrix}$. This is optimal ~~because~~ since is a minimization problem and $\bar{c}_j \geqslant 0 \ \forall j$.

This optimal solution is unique since the basis is non-degenerate.

(b) (6 points) Find the **Gomory cutting plane** corresponding to the first row, expressed in non-basic variables.

1rst row: $1/2\, x_1 + x_2 + 7/3\, x_3 - 4/3\, s_1 = 5/3$

- Seperate integral from frachonal parts:

$$(0 + 1/2)x_1 + (1+0)x_2 + (2 + 1/3)x_3 + (-2 + 2/3)s_1 = 1 + 2/3$$

- Left side integral palt, night side frachonal part:

$$x_2 + 2x_3 - 2s_1 - 1 = 2/3 - x_1 \cdot \frac{1}{2} - 1/3\, x_3 - 2/3\, s_1$$

- Lef side is integral so night side should also be integral

$$0 \geqslant 2/3 - 1/2\, x_1 - 1/3\, x_3 - 2/3\, s_1$$

- Gomory cutt: $-2/3 \geqslant -1/2\, x_1 - 1/3\, x_3 - 2/3\, s_1$

3. (8 points) Consider the following Simplex tableau of a minimization problem. Apply **one** iteration of the **dual Simplex method**. Indicate whether the obtained solution is feasible and why (or why not).

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	12	-5	0	2	1	6
x_2	-7	1	1	-2	0	-3
$-z$	7	1	0	3	0	3

• x_2 is leaving the base since it is negative
• Candidates to enter the base: x_3 ; s_2
→ s_2 entering since: $\min\{|3/-2|, |3/-3|\} = |3/-3| = 1$

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	
s_1	-2	-3	2	-2	1	0	→ $r_1 + 2r_2$
s_2	7/3	-1/3	-1/3	2/3	0	1	→ $r_2 \cdot (-1/3)$
$-z$	0	2	1	1	0	0	→ $r_0 + r_2$

The obtained solution is not primal feasible because we still have a negative basic variable, but it is still ~~dual~~ dual feasible since $\forall j\ \bar{c}_j \geq 0$ and we also have some candidates to enter the base since the leaving variable ~~x~~ would be s_1 and x_1 and x_3 are negative, so for further conclusion ~~into~~ if the problem is feasible we should continue iterating.

$j = 1, \dots n$

4. Each morning, two trains travel along a line from Delft (station s_1 $\nearrow s_0$) to Amsterdam (station s_{n+1}), passing n stations $s_1, s_2, \dots, s_n$ in that order. The parameter q_j denotes the number of passengers who wish to travel from station s_j to Amsterdam. We need to decide the route of the two trains. Each station s_j has to be visited by exactly one train, and all q_j passengers will get on that train. The travel time from station s_j to Amsterdam is p_j minutes without stops. Each stop takes an additional 2 minutes. The aim is to minimize the total travel time summed over all passengers. To give an **ILP formulation** of this problem, the following decision variables are introduced. A decision variable t_j indicates the travel time for a passenger starting at station s_j. We also introduce a binary decision variable x_{ij}, where $x_{ij} = 1$ if train i stops at station s_j and $x_{ij} = 0$ otherwise. Do not introduce any new variables in your solutions.

   (a) (3 points) Formulate the objective function.

   $$\min z = \sum_{j=1}^{n} t_j q_j$$

   (b) (3 points) Formulate constraints enforcing that each station is visited by exactly one train.

   $$(1) \quad \sum_{i=1}^{2} x_{ij} = 1 \qquad j = 1, \dots, n$$

   (c) (6 points) Formulate constraints enforcing that the variables t_j get the right values, i.e. that t_j is at least equal to the time it takes to travel from station s_j to Amsterdam, including stops, using the train that stops at station s_j. When passengers board the train at station s_j, they must wait 2 minutes before the train starts moving (boarding is instantaneous as soon as the train arrives).

   $$(2) \quad t_j \geq s_j + \sum_{i=1}^{2} \sum_{j=1}^{n} 2 \cdot x_{ij} \qquad j = 1, \dots, n$$

5. Given is the following optimization problem.

$$\begin{array}{llcccccl}
\min & z = & 5x_1 & + & 8x_2 & + & 3x_3 & \\
\text{s.t.} & & x_1 & - & x_2 & & & \leq 1 \\
 & & & & x_2 & + & x_3 & \geq 1 \\
 & & \multicolumn{5}{c}{x_1 \geq 0,\ x_2, x_3 \in \mathbb{R}} &
\end{array}$$

(a) (4 points) Formulate the corresponding dual problem. A justification for each variable / constraint is not required.

$$\begin{array}{llcl}
\max & W = \Pi_1 + \Pi_2 & & \\
 & \Pi_1 & \leq & 5 \\
 & -\Pi_1 + \Pi_2 & = & 8 \\
 & \Pi_2 & = & 3 \\
 & \Pi_1 \leq 0,\ \Pi_2 \geq 0 & &
\end{array}$$

(b) (6 points) Using part (a), show that $(0, -1, 2)$ is an optimal solution of the original LP.

→ Due to the theorem of Complementery & Slackness, ~~if an optimal solution in the primal if~~ a solution in the primal is optimal if and only if ~~th~~ all complementary slackness restrictions are met and the ~~val~~ objective value in the primal is equal to the objective value in the dual.

$\boxed{\text{\& strong duality}}$

* Continue in EXTRA BOX 1 !!!

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6. (12 points) Consider the ILP below.

$$\begin{array}{llrcrcr} z_{IP} = & \max z = & x_1 & + & 5x_2 & & \\ & \text{s.t.} & -4x_1 & + & 3x_2 & \leq & 3 \\ & & 7x_1 & + & 6x_2 & \leq & 42 \\ & & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & & \multicolumn{5}{l}{x_1, x_2 \in \mathbb{Z}} \end{array}$$

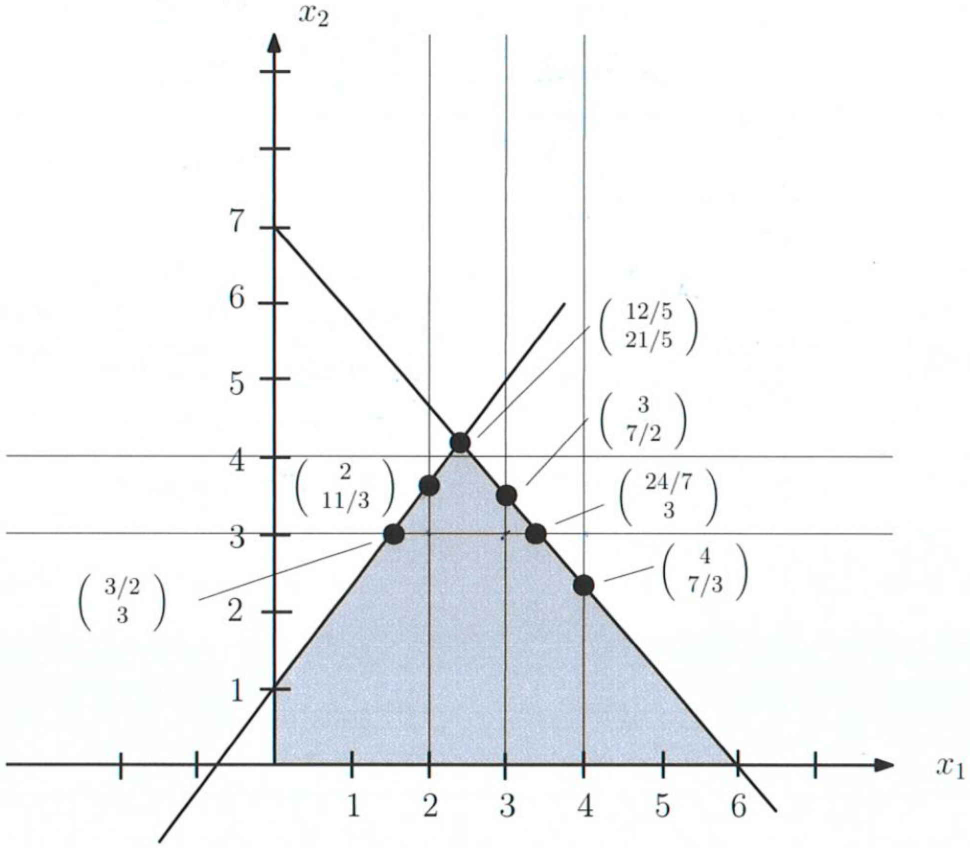
The optimal solution to the LP relaxation is:

$$x^*_{LP} = \begin{pmatrix} 12/5 \\ 21/5 \end{pmatrix} \qquad \text{with } z^*_{LP} = 117/5\,.$$

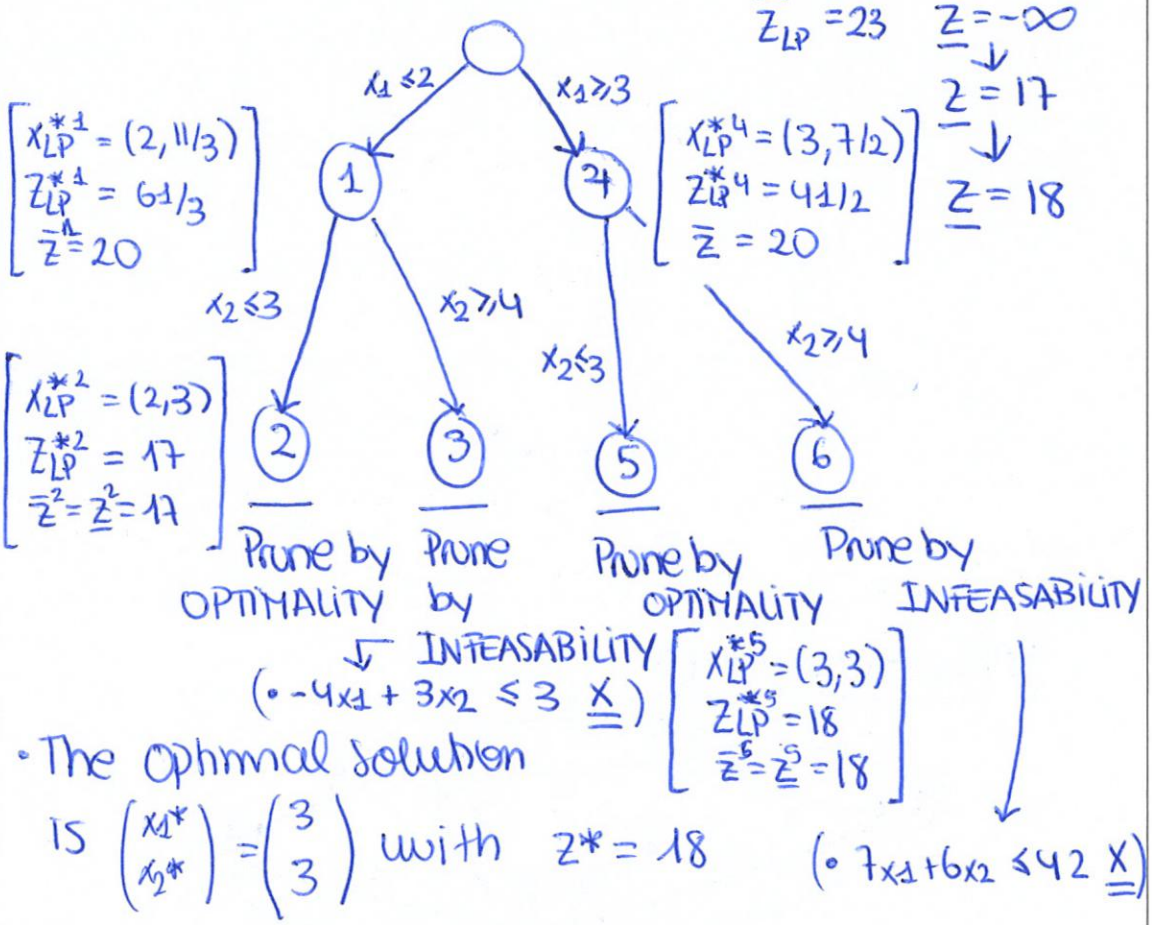
Determine one optimal solution to the ILP with the **branch & bound** method. Use the following search strategy:

- Whenever given a choice, branch first on variable x_1.
- Choose the $\leq$-branch first.
- Go depth first.

LP relaxations may be solved graphically using the feasible region illustrated below along with *some* useful points. Indicate clearly if / where pruning has taken place, together with justification. Write your solution on the next page.



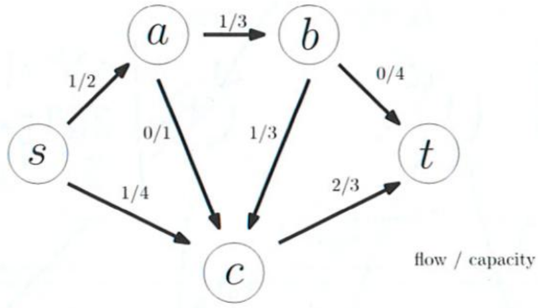
Solution to Question 6.



- The optimal solution is $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ with $z^* = 18$

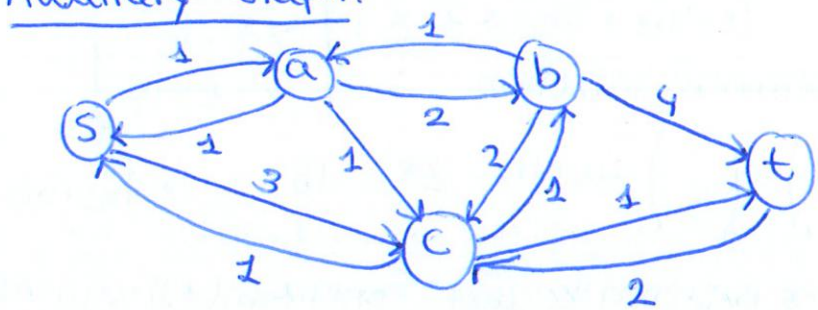
→ Node 3 is pruned because constraint (1) was violated
→ Node 6 is pruned because constraint (2) was violated
→ Node 2 is pruned by optimality at the time
→ Node 5 is pruned by optimality at the time and also resulted the final solution.

7. The question is about st maximum flow and the Ford-Fulkerson algorithm. In the directed graph G below, each arc is labelled by current flow / capacity. For example, the arc ab has flow 1 and capacity 3.



(a) (3 points) Draw the auxiliary graph of G. Find a shortest augmenting path and a longest augmenting path, where the length is the number of arcs in the path. For each augmenting path, determine the increase in flow as a result of adding it.

Auxilrary Graph:

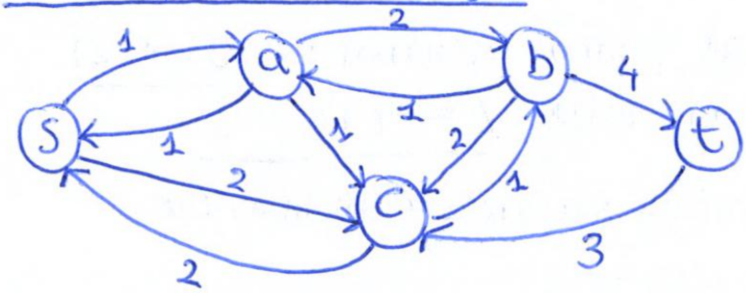


→ The shortest augmenting path is: $s \to c \to t$ ~~(3)~~ with 2 arcs. The increase of flow would be 1. since is the lowest 'ponderation' in an arc in all the path

→ The longest augmenting path is: ~~s→c~~ $s \to a \to c \to b \to t$ with 4 arcs. The increase of flow would also be 1 for the same reason stated before.

(b) (3 points) Of the augmenting paths found in part (a), add a shortest one. Draw the graph G with updated flow values.

Auxiliary Graph actualized:



(c) (4 points) Use an st-cut to argue that the flow cannot exceed 5. Give a short explanation.

Due to weak duality (since the min-cut problem is the dual problem of max flow) any ~~(min)~~ s-t cut is an upper bound to the max flow of a graph.

Given this cut:

s = < s, c >
t = < t, a, b >

the capacity of the cut is 2+3 = 5 ~~(capacit)~~ capacity

~~cap~~ so 5 is an upper bound to the max flow

Extra box 1. Use the box below to continue your solution to one of the questions.

5 b) Continuation !!!

We know that primal solution is $(0,-1,2)$ and its objective value $Z = -8+6 = -2$

All complementary slackness restrictions are

(1) $\pi_1 (x_1 - x_2 - 1) = 0$

(2) $\pi_2 (x_2 + x_3 - 1) = 0$ ~~since /~~

(3) $x_1 (\pi_1 - 5) = 0$ ~~$\pi_1 - 5 = 0$~~

(4) $x_2 (-\pi_1 + \pi_2 - 8) = 0$

(5) $x_3 (\pi_2 - 3) = 0 \rightarrow$ since $x_3 \neq 0 : \pi_2 - 3 = 0$ $\boxed{\pi_2 = 3}$

(4) since $x_2 \neq 0 \rightarrow -\pi_1 + \pi_2 - 8 = 0$ $\boxed{\pi_1 = -5}$

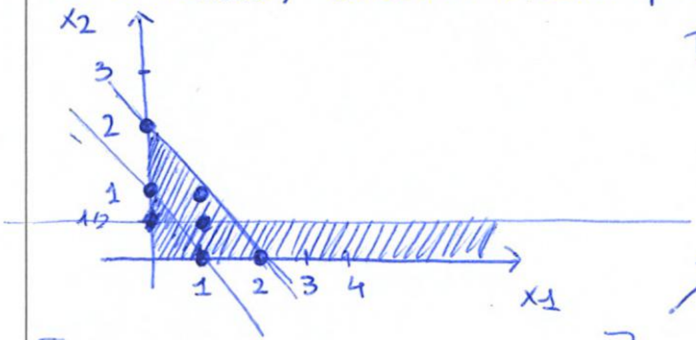
The found solution is $(-5, 3)$ for the dual, now we only need to prove that the objective value of this solution has the same value as the primal solution

$W = \pi_1 + \pi_2 = -5 + 3 = 2$

Since they have the same objective value we can conclude that $(0,-1,2)$ is an optimal solution to the primal/original LP.

Extra box 2. Use the box below to continue your solution to one of the questions.

1 i) False, Counter example



The feasible region for this problem for the LP-relaxation is unbounded but for the ILP is only $\{(0,0),(0,1),(0,2)$ $(1,1)(1,0),(2,0)\}$ So it's bounded.

$$\begin{aligned} &\max\ x_1 \\ &\text{s.t.}\quad x_2 \leq 1/2 + Mz \\ &\qquad x_1 + x_2 \leq 2 + Mz \\ &\qquad x_1, x_2 \geq 0,\ z \in \{0,1\} \\ &\qquad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

For M sufficently larger