Ex. 18.1

(a) positive definite

(b) negative definite

(c) indefinite

(d) indefinite

Ex. 18.3

(a) $\nabla f(x) = \begin{bmatrix} 3x_1^2 - 3 \\ 3x_2^2 - 12 \end{bmatrix}$

$Hf(x) = \begin{bmatrix} 6x_1 & 0 \\ 0 & 6x_2 \end{bmatrix}$

critical points:

$(1, 2)$ strict local minimizer

$(1, -2)$ saddle point

$(-1, 2)$ saddle point

$(-1, -2)$ strict local maximizer

Ex 18.4

(a) $\nabla f(x) = \begin{bmatrix} 2x_1 - 4 \\ 4x_2 \end{bmatrix}$ so $(2, 0)$ is a critical point.

$Hf(x) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite on $\mathbb{R}^2$

so $(2, 0)$ is a global minimizer

Ex. 18.5

(a) no

(b) yes

(c) no

(d) yes

Ex. 19.1 $f(x) = 4x_1^2 - 4x_1x_2 + 2x_2^2$

$x^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\nabla f(x^{(1)}) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

$$\begin{aligned} \varphi_1(t) &= f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} - t\begin{bmatrix} -4 \\ 4 \end{bmatrix}\right) \\ &= f\left(\begin{bmatrix} 4t \\ 1-4t \end{bmatrix}\right) \\ &= 160t^2 - 32t + 2 \end{aligned}$$

$\varphi_1'(t) = 320t - 32$

$t_1 = \frac{1}{10}$ is a global minimizer of φ_1 since $\varphi''(t) = 320 > 0$

$x^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

continuing like this

gives $x^{(3)} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$

Ex. 19.2 $f(x) = 2x_1^4 + x_2^2 - 4x_1x_2 + 5x_2$

$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\nabla f(x) = \begin{bmatrix} 8x_1^3 - 4x_2 \\ 2x_2 - 4x_1 + 5 \end{bmatrix}$ $\qquad \nabla f(x^{(0)}) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$$\varphi_0(t) = f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - t\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right)$$
$$= 25t^2 - 25t$$

$$\varphi_0'(t) = 50t - 25$$

$t_0 = \frac{1}{2}$ is a global minimizer

since $\varphi_0''(t) = 50 > 0$

$$x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5/2 \end{bmatrix}$$

Ex. 19.3 Yes, since f is strictly convex, coercive and has continuous first partial derivatives (check all three conditions)

Ex. 19.4 $\nabla f(x) = b + Ax$

so x^* with $Ax^* = -b$ is a critical point

$Hf(x) = A$ is positive definite
so x^* is a strict global minimizer

$$x^{(1)} = x^{(0)} - A^{-1}(b + Ax^{(0)}) = -A^{-1}b = x^*$$

Ex. 19.5 $f(x) = \frac{2}{3}|x|^{3/2}$

$$\nabla f(x) = \begin{cases} x^{\frac{1}{2}} & \text{if } x > 0 \\ -(-x)^{\frac{1}{2}} & \text{if } x < 0 \end{cases}$$

$$Hf(x) = \begin{cases} \frac{1}{2}x^{-\frac{1}{2}} & \text{if } x > 0 \\ \frac{1}{2}(-x)^{-\frac{1}{2}} & \text{if } x < 0 \end{cases}$$

$x^{(0)} = 1$

$$x^{(1)} = x^{(0)} - [Hf(x^{(0)})]^{-1} \nabla f(x^{(0)}) = 1 - 2 \cdot 1 = -1$$

$$x^{(2)} = -1 - 2(-1) = 1$$

$$x^{(3)} = -1$$

$\vdots$

etc.