Tree decomposition of triangulated cycle graphs (10.4 from Kleinberg)

Consider the problem of finding a tree decomposition of a triangulated cycle graph $G = (V, E)$. (See the exercise in the book for a definition.) Show that the tree width of such graphs is 2, and give an efficient algorithm to construct such a tree decomposition.

Solution

We will show that such a graph G has a tree decomposion $(T, \{V_t\})$ in which each piece V_t corresponds uniquely to an internal triangular face of G. We prove this by induction on the number of nodes in G.

The base case is a triangle $u - v - w$. In this case we can construct a tree T with just one node t, and $V_t = \{u, v, w\}$.

Suppose now that for any triangulated cycle graph with less than n nodes we can find a tree decomposition in which each piece corresponds uniquely to an internal triangular face (this is the induction hypothesis).

Given a graph G of size n, choose any internal edge $e = (u, v)$ of G. Deleting u and v produces two components A and B. Let G_1 be the subgraph induced by $A \cup \{u, v\}$ and G_2 be the subgraph induced by $B \cup \{u, v\}$. It is straightforward to see that G_1 and G_2 are also triangulated cycle graphs. By the induction hypothesis there are thus tree decompositions $(T_1, \{X_t\})$ and $(T_2, \{Y_t\})$ of G_1 and G_2, respectively, in which the pieces correspond uniquely to internal faces. Thus there are nodes $t_1 \in T_1$and $t_2 \in T_2$ that correspond to the faces containing the edge (u, v). If we let T denote the tree obtained by adding an edge (t_1, t_2) to $T_1 \cup T_2$, then $(T, X_t \cup Y_t)$ is a tree decomposition having the desired properties.

In this case each piece has three nodes, so the width of the decomposition is 2.