IN4344 Advanced Algorithms

Lecture 6: Multiple optimal solutions, Duality, Complementary Slackness

Yuki Murakami

Delft University of Technology

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Simplex so far

We have learned how to recognize:

- A bounded optimal solution:
  - The reduced costs $\bar{c}_j \leq 0$ for all j (maximization)
  - The reduced costs $\bar{c}_j \geq 0$ for all j (minimization)
- An unbounded solution:
  If there is a candidate entering basic variable $x_{j'}$ that has $\bar{a}_{ij'} \leq 0$ for all i.
- The problem is infeasible: The Simplex Phase 1 objective function is greater than zero in the Phase 1 optimal solution ($w^* > 0$).

Multiple optimal solutions

Suppose we maximize and that we have reached an optimal bfs, i.e.,

$$\bar{c}_j \leq 0 \text{ for all } j\,.$$

Notice: $\bar{c}_j = 0$ for all j such that x_j is basic.

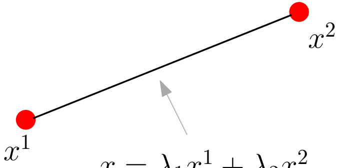
If one or more **non-basic variables** have $\bar{c}_j = 0$, then we can let one of them enter the basis to reach another optimal bfs.

Suppose $x^1,\ x^2, \ldots, x^k$ are optimal basic feasible solutions. Then the following is an expression of **all optimal solutions**:

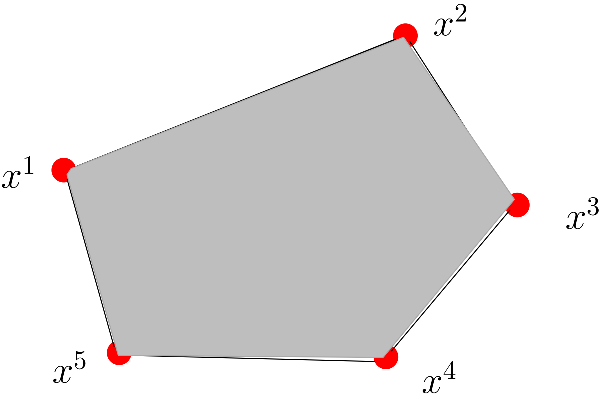
$$x^* = \lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_k x^k, \quad \sum_{j=1}^{k} \lambda_j = 1,\ \lambda_1, \ldots, \lambda_k \geq 0\,.$$

So, x^* is expressed as a **convex combination** of all the optimal basic feasible solutions.

Multiple optimal solutions



$$x = \lambda_1 x^1 + \lambda_2 x^2,$$
$$\lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0$$



$$x = \sum_{j=1}^{5} \lambda_j x^j,$$

$$\sum_{j=1}^{5} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1, \ldots, 5$$

Duality

Each LP has a corresponding **dual** LP.

Primal (original) LP	**Dual** LP
min	max
max	min
n variables	n constraints
m constraints	m variables

Duality is used to compute **bounds** on the optimal objective function value and to verify **optimality**.

These bounds are also useful if we **approximate**.

Diet problem

- Different **food types** contain different amounts of certain **nutrients**.
- You want to take in enough of each nutrient.
- For as little money as possible.
- How much should you eat from each food type?

Example

	chips	muesli	sausage	required (RDI)
costs	3	3	4	
carbohydrates	2	1	0	3
protein	0	1	4	2
fat	4	0	8	9

Diet problem

Example (1)

	chips	muesli	sausage	required (RDI)
costs	3	3	4	
carbohydrates	2	1	0	3
protein	0	1	4	2
fat	4	0	8	9

LP formulation:

$$
\begin{array}{llrl}
\min & 3\,x_1 + 3\,x_2 + 4\,x_3 & & \\
\text{s.t.} & 2\,x_1 + x_2 & \geq 3 & (1) \\
 & x_2 + 4\,x_3 & \geq 2 & (2) \\
 & 4\,x_1 + 8\,x_3 & \geq 9 & (3) \\
 & x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

Can you determine a lower bound on how much you will need to spend on your diet?

Example (1)

$$
\begin{array}{lllll}
\min & 3x_1 + 3x_2 + 4x_3 & & & \\
\text{s.t.} & 2x_1 + x_2 & \geq 3 & & (1) \\
 & \phantom{2x_1 +{}} x_2 + 4x_3 & \geq 2 & & (2) \\
 & 4x_1 \phantom{{}+ x_2} + 8x_3 & \geq 9 & & (3) \\
 & x_1, x_2, x_3 \geq 0 & & &
\end{array}
$$

$$\Rightarrow \quad 2x_1 + 2x_2 + 4x_3 \geq 5 \qquad (1) + (2)$$

$$\Rightarrow \quad 3x_1 + 3x_2 + 4x_3 \geq 5 \qquad \text{(lhs=objective function!)}$$

So an optimal diet costs at least 5.

Can you find a **better lower bound**?

Example (1)

$$\begin{array}{llr}
\min & 3x_1 + 3x_2 + 4x_3 & \\
\text{s.t.} & 2x_1 + x_2 \geq 3 & (1) \\
 & x_2 + 4x_3 \geq 2 & (2) \\
 & 4x_1 + 8x_3 \geq 9 & (3) \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}$$

$$\Rightarrow \quad 3x_1 + 2\tfrac{1}{2}x_2 + 4x_3 \geq 6\tfrac{1}{2} \qquad 1\tfrac{1}{2} \times (1) + (2)$$

$$\Rightarrow \quad 3x_1 + 3x_2 + 4x_3 \geq 6\tfrac{1}{2}$$

So an optimal diet costs at least $6\frac{1}{2}$.

Can you find **an even better** lower bound?

Example (1)

$$
\begin{array}{llcl}
\min & 3x_1 + 3x_2 + 4x_3 & & \\
\text{s.t.} & 2x_1 + x_2 & \geq 3 & (1) \\
& x_2 + 4x_3 & \geq 2 & (2) \\
& 4x_1 + 8x_3 & \geq 9 & (3) \\
& x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

For all $\pi_1, \pi_2, \pi_3 \geq 0$ we have:

$$
\begin{array}{ll}
\pi_1 (2x_1 + x_2) & \pi_1 \times (1) \\
+ \pi_2 (x_2 + 4x_3) & \pi_2 \times (2) \\
+ \pi_3 (4x_1 + 8x_3) & \pi_3 \times (3) \\
\geq 3\pi_1 + 2\pi_2 + 9\pi_3 &
\end{array}
$$

So $3\pi_1 + 2\pi_2 + 9\pi_3$ is a lower bound on the optimal value when

$$
\begin{array}{ll}
\pi_1 (2x_1 + x_2) & \pi_1 \times (1) \\
+ \pi_2 (x_2 + 4x_3) & \pi_2 \times (2) \\
+ \pi_3 (4x_1 + 8x_3) & \pi_3 \times (3) \\
\leq 3x_1 + 3x_2 + 4x_3 & \leftarrow \text{Primal objective function}
\end{array}
$$

Example (1)

So $3\pi_1 + 2\pi_2 + 9\pi_3$ **is a lower bound on the optimal value for all** $\pi_1, \pi_2, \pi_3 \geq 0$ **with**

$$\begin{array}{rlllll} & \pi_1\,(2\,x_1 + & x_2 & &) & \pi_1 \times (1) \\ + & \pi_2\,(& x_2 & + 4\,x_3 &) & \pi_2 \times (2) \\ + & \pi_3\,(4\,x_1 & & + 8\,x_3 &) & \pi_3 \times (3) \\ \leq & 3\,x_1 & + 3\,x_2 & + 4\,x_3 & & \end{array}$$

Rewrite $\pi_1 \times (1) + \pi_2 \times (2) + \pi_3 \times (3)$ in x-variables:

$$\begin{array}{rlllll} & x_1\,(2\,\pi_1 & & + 4\,\pi_3 &) \\ + & x_2\,(\ \pi_1 + & \pi_2 & &) \\ + & x_3\,(& 4\pi_2 & + 8\,\pi_3 &) \\ \leq & 3\,x_1 + & 3x_2 & + 4\,x_3 & \end{array}$$

We can find the best lower bound by

maximizing $3\pi_1 + 2\pi_2 + 9\pi_3$.

Example

The **dual** LP (finding the best lower bound):

$$\max 3\pi_1 + 2\pi_2 + 9\pi_3$$

for all π that satisfies:

$$\begin{array}{rl} & x_1\,(2\,\pi_1 \qquad\qquad + 4\,\pi_3\,) \\ + & x_2\,(\ \ \pi_1 + \ \ \pi_2 \qquad\qquad) \\ + & x_3\,(\qquad\qquad 4\pi_2 + 8\,\pi_3\,) \\ \leq & \quad 3\,x_1 + 3x_2 + 4\,x_3 \end{array}$$

$$\begin{array}{lrrrl} \max & 3\,\pi_1 + & 2\,\pi_2 + & 9\,\pi_3 & \\ \text{s.t.} & 2\,\pi_1 & & +4\,\pi_3 \leq 3 & (D1) \\ & \pi_1 + & \pi_2 & \leq 3 & (D2) \\ & & 4\,\pi_2 + & 8\,\pi_3 \leq 4 & (D3) \\ & \pi_1, \pi_2, \pi_3 \geq 0 & & & \end{array}$$

Primal:

$$
\begin{array}{lrcl}
\min & 3\,x_1 + 3\,x_2 + 4\,x_3 & & \\
\text{s.t.} & 2\,x_1 + \quad x_2 \qquad\quad & \geq 3 & (1) \\
 & x_2 + 4\,x_3 & \geq 2 & (2) \\
 & 4\,x_1 \qquad\quad + 8\,x_3 & \geq 9 & (3) \\
 & x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

Dual:

$$
\begin{array}{lrcl}
\max & 3\,\pi_1 + 2\,\pi_2 + 9\,\pi_3 & & \\
\text{s.t.} & 2\,\pi_1 \qquad\quad + 4\,\pi_3 & \leq 3 & (D1) \\
 & \pi_1 + \quad \pi_2 \qquad\quad & \leq 3 & (D2) \\
 & 4\,\pi_2 + 8\,\pi_3 & \leq 4 & (D3) \\
 & \pi_1, \pi_2, \pi_3 \geq 0 & &
\end{array}
$$

Notice: The objective function coefficients of the dual are the right-hand side coefficients of the primal (and vice versa).

The constraint matrix in the dual is the transpose of the constraint matrix in the primal (and vice versa).

Example (1)

The **primal** (We call the original problem “the primal problem”):

$$
\begin{array}{llr}
\min & 3\,x_1 + 3\,x_2 + 4\,x_3 & \\
s.t. & 2\,x_1 + x_2 \geq 3 & (1) \\
 & x_2 + 4\,x_3 \geq 2 & (2) \\
 & 4\,x_1 + 8\,x_3 \geq 9 & (3) \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

The **dual** (finding the best lower bound):

$$
\begin{array}{llr}
\max & 3\,\pi_1 + 2\,\pi_2 + 9\,\pi_3 & \\
\text{s.t.} & 2\,\pi_1 + 4\,\pi_3 \leq 3 & (D1) \\
 & \pi_1 + \pi_2 \leq 3 & (D2) \\
 & 4\,\pi_2 + 8\,\pi_3 \leq 4 & (D3) \\
 & \pi_1, \pi_2, \pi_3 \geq 0 &
\end{array}
$$

How can we **interpret** the dual of the diet problem?

Dual of the diet problem (interpretation)

- A pill maker produces pills for each nutrient.
- For each food type, it should be cheaper to buy the corresponding pills than the food.
- The pill maker wants to maximize the price of the pills necessary to get enough intake from each nutrient.
- What are the optimal prices (π_1, π_2, π_3) of the pills?

	chips	muesli	sausage	required (RDI)
costs	3	3	4	
carbohydrates	2	1	0	3
protein	0	1	4	2
fat	4	0	8	9

- For each food type, it should be cheaper to buy the corresponding pills than the food.
- The pill maker wants to maximize the price of the pills necessary to get enough intake from each nutrient.

Example

$$\begin{array}{llr}
\max & 3\,\pi_1 + 2\,\pi_2 + 9\,\pi_3 & \\
\text{s.t.} & 2\,\pi_1 \qquad\quad + 4\,\pi_3 \le 3 & (D1) \\
& \pi_1 + \quad \pi_2 \qquad\qquad \le 3 & (D2) \\
& 4\,\pi_2 + 8\,\pi_3 \le 4 & (D3) \\
& \pi_1, \pi_2, \pi_3 \ge 0 &
\end{array}$$

Example (1)

The **primal** LP:

$$\begin{array}{lrl} \min & 3\,x_1 + 3\,x_2 + 4\,x_3 & \\ \text{s.t.} & 2\,x_1 + x_2 \phantom{{}+4\,x_3} & \geq 3 \\ & x_2 + 4\,x_3 & \geq 2 \\ & 4\,x_1 \phantom{{}+x_2} + 8\,x_3 & \geq 9 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

The **dual** LP:

$$\begin{array}{lrl} \max & 3\,\pi_1 + 2\,\pi_2 + 9\,\pi_3 & \\ \text{s.t.} & 2\,\pi_1 \phantom{{}+\pi_2} + 4\,\pi_3 & \leq 3 \\ & \pi_1 + \pi_2 \phantom{{}+8\,\pi_3} & \leq 3 \\ & 4\,\pi_2 + 8\,\pi_3 & \leq 4 \\ & \pi_1, \pi_2, \pi_3 & \geq 0 \end{array}$$

We have seen that solution $(\pi_1, \pi_2, \pi_3) = (1\frac{1}{2}, 1, 0)$ for the **dual** gives a lower bound of **6.5** on the optimal value of the primal.
Is there an even better lower bound?

No! Since $(x_1, x_2, x_3) = (1\frac{1}{2}, 0, \frac{1}{2})$ is a solution of the **primal** with value **6.5**.

So **both solutions are optimal!**

Example

Example (2)

Formulate the dual of the following problem:

$$
\begin{array}{llrcrcrcr}
(\mathrm{P}) & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
\end{array}
$$

$$x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}$$

We have 3 variables and 4 constraints,
so the **dual has 4 variables and 3 constraints**.

Example

$$
\begin{array}{llrcrcrcr}
(\mathrm{P}) & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
& & \multicolumn{7}{l}{x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}}
\end{array}
$$

1. Primal is a max-problem, so dual becomes a min-problem.
2. The right-hand side coefficients in the primal are the objective coefficients in the dual.

$$
(\mathrm{D}) \quad \min \quad 3\pi_1 + 2\pi_2 - 3\pi_3 + 4\pi_4
$$

Example

$$
\begin{array}{llrcrcrcr}
\text{(P)} & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
& & \multicolumn{7}{l}{x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}}
\end{array}
$$

1. Primal is a max-problem, so dual becomes a min-problem.
2. The right-hand side coefficients in the primal are the objective coefficients in the dual.
3. The objective coefficients in the primal are the right-hand side coefficients in the dual

$$
\begin{array}{llrcrcrcrr}
\text{(D)} & \min & 3\pi_1 & + & 2\pi_2 & - & 3\pi_3 & + & 4\pi_4 & \\
& \text{s.t.} & & & & & & & & 3 \\
& & & & & & & & & -5 \\
& & & & & & & & & 2
\end{array}
$$

Example

$$
\begin{array}{llrcrcrcr}
(P) & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
& & \multicolumn{7}{l}{x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}}
\end{array}
$$

4. The constraint matrix becomes transposed.

$$
\begin{array}{llrcrcrcrr}
(D) & \min & 3\pi_1 & + & 2\pi_2 & - & 3\pi_3 & + & 4\pi_4 & \\
& \text{s.t.} & \pi_1 & - & 2\pi_2 & + & 3\pi_3 & + & 4\pi_4 & 3 \\
& & & + & \pi_2 & + & 3\pi_3 & + & 5\pi_4 & -5 \\
& & 2\pi_1 & - & 3\pi_2 & - & 7\pi_3 & - & 4\pi_4 & 2
\end{array}
$$

Example

$$
\begin{array}{llrcrcrcr}
\text{(P)} & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
& & \multicolumn{5}{c}{x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}} & &
\end{array}
$$

4. The constraint matrix becomes transposed.
5. The signs of the variables in the dual depend on the constraint signs in the primal.

$$
\begin{array}{llrcrcrcrcr}
\text{(D)} & \min & 3\pi_1 & + & 2\pi_2 & - & 3\pi_3 & + & 4\pi_4 & & \\
& \text{s.t.} & \pi_1 & - & 2\pi_2 & + & 3\pi_3 & + & 4\pi_4 & & 3 \\
& & & + & \pi_2 & + & 3\pi_3 & + & 5\pi_4 & & -5 \\
& & 2\pi_1 & - & 3\pi_2 & - & 7\pi_3 & - & 4\pi_4 & & 2 \\
& & \multicolumn{7}{c}{\pi_1 \geq 0,\ \pi_2, \pi_3 \in \mathbb{R},\ \pi_4 \leq 0} & &
\end{array}
$$

Example

$$
\begin{array}{llrcrcrcr}
\text{(P)} & \max & 3x_1 & - & 5x_2 & + & 2x_3 & & \\
& \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 3 \\
& & -2x_1 & + & x_2 & - & 3x_3 & = & 2 \\
& & 3x_1 & + & 3x_2 & - & 7x_3 & = & -3 \\
& & 4x_1 & + & 5x_2 & - & 4x_3 & \geq & 4 \\
& & \multicolumn{7}{l}{x_1 \leq 0,\ x_2 \geq 0,\ x_3 \in \mathbb{R}}
\end{array}
$$

4. The constraint matrix becomes transposed.
5. The signs of the variables in the dual depend on the constraint signs in the primal.
6. The constraint signs in the dual depend on the variable signs in the primal.

$$
\begin{array}{llrcrcrcrcr}
\text{(D)} & \min & 3\pi_1 & + & 2\pi_2 & - & 3\pi_3 & + & 4\pi_4 & & \\
& \text{s.t.} & \pi_1 & - & 2\pi_2 & + & 3\pi_3 & + & 4\pi_4 & \leq & 3 \\
& & & + & \pi_2 & + & 3\pi_3 & + & 5\pi_4 & \geq & -5 \\
& & 2\pi_1 & - & 3\pi_2 & - & 7\pi_3 & - & 4\pi_4 & = & 2 \\
& & \multicolumn{9}{l}{\pi_1 \geq 0, \pi_2, \pi_3 \in \mathbb{R},\ \pi_4 \leq 0}
\end{array}
$$

Duality for general LPs

Assuming the primal is a **min** problem.

The **primal** (P):

$$
\begin{array}{lll}
\textbf{min} & c^\mathsf{T} x & \\
\text{s.t.} & a_i x = b_i & i \in M \\
 & a_i x \geq b_i & i \in \bar{M} \\
 & a_i x \leq b_i & i \in \tilde{M} \\
 & x_j \geq 0 & j \in N \\
 & x_j \leq 0 & j \in \tilde{N} \\
 & x_j \in \mathbb{R} & j \in \bar{N}
\end{array}
$$

The **dual** (D):

$$
\begin{array}{lll}
\textbf{max} & b^\mathsf{T} \pi & \\
\text{s.t.} & \pi_i \in \mathbb{R} & i \in M \\
 & \pi_i \geq 0 & i \in \bar{M} \\
 & \pi_i \leq 0 & i \in \tilde{M} \\
 & (A_j)^\mathsf{T} \pi \leq c_j & j \in N \\
 & (A_j)^\mathsf{T} \pi \geq c_j & j \in \tilde{N} \\
 & (A_j)^\mathsf{T} \pi = c_j & j \in \bar{N}
\end{array}
$$

Theorem

The dual of the dual is the primal.

So to find the dual of a **max** problem, you go from right to left.

Theorem (weak duality theorem)

Suppose we are given a primal and corresponding dual in the following form:

The **primal** (P):

$$\begin{array}{lrl} \min & z = c^{\mathsf{T}} x & \\ \text{s.t.} & Ax \geq b & (1) \\ & x \geq 0 & \end{array}$$

The **dual** (D):

$$\begin{array}{lrl} \max & w = b^{\mathsf{T}} \pi & \\ \text{s.t.} & A^{\mathsf{T}} \pi \leq c & (2) \\ & \pi \geq 0 & \end{array}$$

If $\hat{x}$ is a feasible solution of (P) and $\hat{\pi}$ of (D), then $z(\hat{x}) \geq w(\hat{\pi})$.

Proof

Multiply (1) by $\hat{\pi}^{\mathsf{T}}$: $\hat{\pi}^{\mathsf{T}} A \hat{x} \geq \hat{\pi}^{\mathsf{T}} b$. This holds since $\hat{\pi} \geq 0$.

Multiply (2) by $\hat{x}^{\mathsf{T}}$: $\hat{x}^{\mathsf{T}} A^{\mathsf{T}} \hat{\pi} \leq \hat{x}^{\mathsf{T}} c$. This holds since $\hat{x} \geq 0$.

We now have $z(\hat{x}) = c^{\mathsf{T}} \hat{x} = \hat{x}^{\mathsf{T}} c \geq \hat{x}^{\mathsf{T}} A^{\mathsf{T}} \hat{\pi} = (\hat{\pi}^{\mathsf{T}} A \hat{x})^{\mathsf{T}} \geq (\hat{\pi}^{\mathsf{T}} b)^{\mathsf{T}} = b^{\mathsf{T}} \hat{\pi} = w(\hat{\pi})$.

Before strong duality (page 67 of lecture notes)

Suppose now we have a primal problem of the form:

$$\begin{aligned} \min \quad & z = c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

To solve this using simplex, we introduce slack variables:

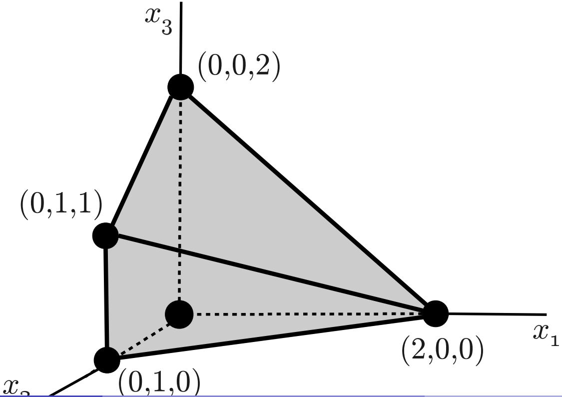
$$\begin{aligned} \min \quad & z = c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax + Is = b \\ & x, s \geq 0 \end{aligned}$$

Suppose we know that some optimal solution x^* has a basis B. For this basis, let x_B be the basic variables and let x_N denote the other variables. Assume that $A = [B \quad N]$ and that $c^T = [c_B^T \quad c_N^T]$.

An example from Lecture 4

Example (2)

$$\begin{array}{lrl} \min & -x_1 - 2x_2 - x_3 & \\ \text{s.t.} & x_1 + 2x_2 & \leq 2 \\ & x_1 + x_2 + x_3 & \leq 2 \\ & x_1, x_2, x_3 \geq 0 & \end{array}$$



Example (2)

$$\begin{array}{lrl} \min & z = & -x_1 - 2\,x_2 - x_3 \\ \text{s.t.} & & x_1 + 2\,x_2 \qquad + s_1 \qquad = 2 \\ & & x_1 + \,x_2 + x_3 \qquad + s_2 = 2 \\ & & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{array}$$

We rewrite the objective function to

$$-z - x_1 - 2x_2 - x_3 = 0$$

and make a **Simplex tableau** (we just write the coefficients):

basis	$\bar{b}$	x_2	x_3	x_1	s_1	s_2
s_1	2	2	0	1	1	0
s_2	2	1	1	1	0	1
$-z$	0	-2	-1	-1	0	0

Before strong duality ctd.

General:

basis	$\bar{b}$	x_B	x_N	s
s	b	B	N	I
$-z$	0	c_B^T	c_N^T	0^T

Example:

basis	$\bar{b}$	x_2	x_3	x_1	s_1	s_2
s_1	2	2	0	1	1	0
s_2	2	1	1	1	0	1
$-z$	0	-2	-1	-1	0	0

Before strong duality ctd.

General:

basis	$\bar{b}$	x_B	x_N	s
x_B	$B^{-1}b$	I	$B^{-1}N$	B^{-1}
$-z$	$-c_B^T B^{-1} b$	0	$c_N^T - c_B^T B^{-1} N$	$-c_B^T B^{-1}$

Example:

basis	$\bar{b}$	x_2	x_3	x_1	s_1	s_2
x_2	1	1	0	1/2	1/2	0
x_3	1	0	1	1/2	-1/2	1
$-z$	3	0	0	1/2	1/2	1

Pick $\pi^T = c_B^T B^{-1}$. Then this gives an optimal solution to the dual problem, since

$$\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c_B^T x_B + c_N^T x_N = c^T x^*.$$

Strong duality

The **primal** (P):

$$\begin{array}{ll} \min & z = c^{\mathsf{T}} x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

The **dual** (D):

$$\begin{array}{ll} \max & w = b^{\mathsf{T}} \pi \\ \text{s.t.} & A^{\mathsf{T}} \pi \leq c \\ & \pi \geq 0 \end{array}$$

Theorem (strong duality theorem)

If (P) has an optimal solution x^*,
then (D) has an optimal solution π^* and

$$z(x^*) = w(\pi^*).$$

Possible primal-dual combinations

		Dual		
		Bounded optimum	Unbounded	Infeasible
Primal	Bounded optimum	✓	x	x
	Unbounded	x	x	✓
	Infeasible	x	✓	✓

Combinations with an x are **not possible**!

The Dual Simplex Method

Very useful when **re-optimizing** an LP after adding a constraint

This will be done when we solve (M)ILPs

Example

Suppose we have solved the following problem to optimality:

$$
\begin{array}{llrcrcl}
\min & z = & 2x_1 & + & x_2 & & \\
\text{s.t.} & & x_1 & + & x_2 & \geq & 1 \\
& & 3x_1 & + & 2x_2 & \leq & 6 \\
& & & & \multicolumn{3}{r}{x_1 \geq 0,\ x_2 \geq 0\,.}
\end{array}
$$

The optimal Simplex-tableau is:

basis	$\bar{b}$	x_1	x_2	s_1	s_2
x_2	1	1	1	-1	
s_2	4	1		2	1
$-z$	-1	1		1	

Suppose we now add an extra constraint $x_2 \leq 1/2$.

Example

basis	$\bar{b}$	x_1	x_2	s_1	s_2
x_2	1	1	1	-1	
s_2	4	1		2	1
$-z$	-1	1		1	

Suppose we now add an extra constraint $x_2 \leq 1/2$.
Add a slack variable: $x_2 + s_3 = 1/2$, and express x_2 in non-basic variables:

$$x_1 + x_2 - s_1 = 1 \Rightarrow \quad x_2 = 1 - x_1 + s_1 .$$

The new constraint is:

$$-x_1 + s_1 + s_3 = -1/2 .$$

Add it to the Simplex tableau:

Example

basis	$\bar{b}$	x_1	x_2	s_1	s_2
x_2	1	1	1	-1	
s_2	4	1		2	1
$-z$	-1	1		1	

The new constraint is:

$$-x_1 + s_1 + s_3 = -1/2\,.$$

Add it to the Simplex tableau:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_2	1	1	1	-1		
s_2	4	1		2	1	
s_3	$-1/2$	-1		1		1
$-z$	-1	1		1		

Example

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_2	1	1	1	-1		
s_2	4	1		2	1	
s_3	$-1/2$	-1		1		1
$-z$	-1	1		1		

This solution is infeasible in the primal problem!

But, all $\bar{c}_j \geq 0$, which we can interpret as dual feasibility.

We will pivot to **maintain dual feasibility**, and **obtain primal feasibility**.

We choose a basic variable i' with $\bar{b}_i < 0$ as a leaving basic variable. Here, s_3.

If $\bar{a}_{i'j} \geq 0$ for all j, then no feasible solution exists since $\sum_j \bar{a}_{i'j} x_j \geq 0$ in all feasible solutions, and $\bar{b}_{i'} < 0$.

Example

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_2	1	1	1	-1		
s_2	4	1		2	1	
s_3	$-1/2$	-1		1		1
$-z$	-1	1		1		

We choose a basic variable i' with $\bar{b}_i < 0$ as a leaving basic variable. Here, s_3.

If $\bar{a}_{i'j} \geq 0$ for all j, then no feasible solution exists since $\sum_j \bar{a}_{i'j} x_j \geq 0$ in all feasible solutions, and $\bar{b}_{i'} < 0$.

So, we **pivot only on elements $\bar{a}_{i'j} < 0$!** Here we only have one choice: $\bar{a}_{31} = -1$. x_1 **becomes entering basis variable**.

Example

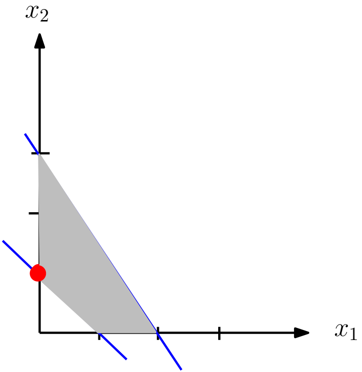
basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_2	1	1	1	-1		
s_2	4	1		2	1	
s_3	$-1/2$	-1		1		1
$-z$	-1	1		1		

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	
x_2	$1/2$		1			1	$r_1 + r_3$
s_2	$7/2$			3	1	1	$r_2 + r_3$
x_1	$1/2$	1		-1		-1	$-1 \cdot r_3$
$-z$	$-3/2$			2		1	$r_0 + r_3$

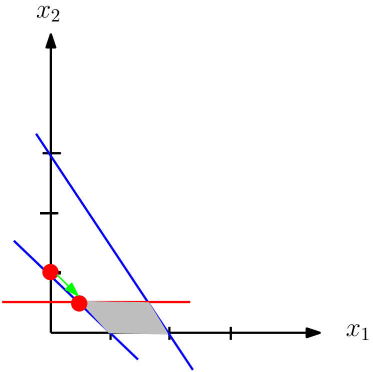
Now, $\bar{b}_i \geq 0$ for all i, and $\bar{c}_j \geq 0$ for all j, so we have a new optimal solution!

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \text{with } z^* = 3/2\,.$$

Graphically



Original problem



Problem after adding $x_2 \leq 1/2$

- **(Primal) Simplex method**
  - Go from **primal feasible** basic solution (bfs) to a next (not worse) bfs.
  - Stop when $\bar{c}_j \geq 0$ for all j (for a min problem), so when a solution is found that is **dual feasible**.
- **Dual Simplex method**
  - Go from **dual feasible** basic solution (a solution with correct values of the $\bar{c}_j$s) to a next (not worse) dual feasible basic solution.
  - Stop when $\bar{b}_i \geq 0$ for all i, so when a solution is found that is **primal feasible**.
  - You use the **primal** tableau.

Dual Simplex Algorithm for **min**imization problems:

Given is a basic solution, not necessary feasible, with $\bar{c}_j \geq 0$ for all j.

1. If $\bar{b}_i \geq 0$ for all i then the current solution is **feasible** and **optimal**. Stop!
2. Choose **leaving** basic variable corresponding to a **row** i' **with** $\bar{b}_{i'} < 0$.
3. If $\bar{a}_{i'j} \geq 0$ for all j then there is **no feasible solution**. Stop!
4. Choose **entering** variable $x_{j'}$ such that

$$\frac{\bar{c}_{j'}}{\bar{a}_{i'j'}} = \min_j \left\{ \quad \left| \frac{\bar{c}_j}{\bar{a}_{i'j}} \right| \quad \middle| \quad \bar{a}_{i'j} < 0 \right\}.$$

   Why do we divide the objective function instead of the b column?

5. Apply elementary row operations such that column j' gets a 1 in row i' and 0s elsewhere. Go to (1).

Dual Simplex Algorithm for **max**imization problems:

Given is a basic solution, not necessary feasible, with $\bar{c}_j \leq 0$ for all j.

1. If $\bar{b}_i \geq 0$ for all i then the current solution is **feasible** and **optimal**. Stop!
2. Choose **leaving** basic variable corresponding to a **row** i' **with** $\bar{b}_{i'} < 0$.
3. If $\bar{a}_{i'j} \geq 0$ for all j then there is **no feasible solution**. Stop!
4. Choose **entering** variable $x_{j'}$ such that

$$\frac{\bar{c}_{j'}}{\bar{a}_{i'j'}} = \min_j \left\{ \quad \left| \frac{\bar{c}_j}{\bar{a}_{i'j}} \right| \quad \middle| \quad \bar{a}_{i'j} < 0 \right\}.$$

5. Apply elementary row operations such that column j' gets a 1 in row i' and 0s elsewhere. Go to (1).

Complementary Slackness

The **primal** LP (P):

$$
\begin{array}{lll}
\textbf{min} & c^{\mathsf{T}}x & \\
\text{s.t.} & a_i x = b_i & i \in M \\
 & a_i x \geq b_i & i \in \bar{M} \\
 & x_j \geq 0 & j \in N \\
 & x_j \in \mathbb{R} & j \in \bar{N}
\end{array}
$$

The **dual** LP (D):

$$
\begin{array}{lll}
\textbf{max} & b^{\mathsf{T}}\pi & \\
\text{s.t.} & \pi_i \in \mathbb{R} & i \in M \\
 & \pi_i \geq 0 & i \in \bar{M} \\
 & \pi^{\mathsf{T}} A_j \leq c_j & j \in N \\
 & \pi^{\mathsf{T}} A_j = c_j & j \in \bar{N}
\end{array}
$$

Theorem (Complementary Slackness)

Let $\hat{x}$ be a feasible solution of (P) and $\hat{\pi}$ a feasible solution of (D). Solutions $\hat{x}$ and $\hat{\pi}$ are both **optimal if and only if**:

$$\hat{\pi}_i(a_i\hat{x} - b_i) = 0 \qquad i = 1, \ldots, m \qquad (1)$$

$$\hat{x}_j(c_j - \hat{\pi}^{\mathsf{T}} A_j) = 0 \qquad j = 1, \ldots, n \qquad (2)$$

Example (1. Diet Problem)

$$
\begin{array}{llrrrl}
\text{(P): min} & 3\,x_1 & +3\,x_2 & +4\,x_3 & & \\
\text{s.t.} & 2\,x_1 + & x_2 & & \geq 3 \\
 & & x_2 & +4\,x_3 & \geq 2 \\
 & 4\,x_1 & & +8\,x_3 & \geq 9 \\
 & & x_1, x_2, x_3 \geq 0 & & &
\end{array}
\qquad
\begin{array}{llrrrl}
\text{(D): max} & 3\,\pi_1 + & 2\,\pi_2 & +9\,\pi_3 & & \\
\text{s.t.} & 2\,\pi_1 & & +4\,\pi_3 & \leq 3 \\
 & \pi_1 + & \pi_2 & & \leq 3 \\
 & & 4\,\pi_2 & +8\,\pi_3 & \leq 4 \\
 & & \pi_1, \pi_2, \pi_3 \geq 0 & & &
\end{array}
$$

Suppose we know an optimal solution $\pi^* = (1\frac{1}{2}, 1, 0)$ of the dual (D).
Use CS to find an optimal solution of the primal (P).

Formulate the CS conditions:

$$
\begin{array}{lllllllll}
\pi_1^*(2x_1^* & + & x_2^* & & & -3) & = & 0 & (1) \\
\pi_2^*(& & x_2^* & + & 4x_3^* & -2) & = & 0 & (2) \\
\pi_3^*(4x_1^* & & & + & 8x_3^* & -9) & = & 0 & (3)
\end{array}
$$

$$
\begin{array}{llllllllll}
x_1^*(3 & - & 2\pi_1^* & & & - & 4\pi_3^*) & = & 0 & (4) \\
x_2^*(3 & - & \pi_1^* & - & \pi_2^* & &) & = & 0 & (5) \\
x_3^*(4 & & & - & 4\pi_2^* & - & 8\pi_3^*) & = & 0 & (6)
\end{array}
$$

Suppose we know an optimal solution $\pi^* = (1\frac{1}{2}, 1, 0)$ of the dual (D).
Use CS to find an optimal solution of the primal (P).
Formulate the CS conditions:

$$\begin{array}{llllllll}
\pi_1^*(2x_1^* & + & x_2^* & & & -3) & = & 0 \quad (1) \\
\pi_2^*(& & x_2^* & + & 4x_3^* & -2) & = & 0 \quad (2) \\
\pi_3^*(4x_1^* & & & + & 8x_3^* & -9) & = & 0 \quad (3)
\end{array}$$

$$\begin{array}{llllllll}
x_1^*(3 & - & 2\pi_1^* & & & - & 4\pi_3^*) & = \ 0 \quad (4) \\
x_2^*(3 & - & \pi_1^* & - & \pi_2^* & &) & = \ 0 \quad (5) \\
x_3^*(4 & & & - & 4\pi_2^* & - & 8\pi_3^*) & = \ 0 \quad (6)
\end{array}$$

Since we know that $\pi_1^*, \pi_2^* \neq 0$, we know from (1) & (2) that

$$\begin{array}{llllllll}
2x_1^* & + & x_2^* & & & = & 3 & (7) \\
& & x_2^* & + & 4x_3^* & = & 2 & (8)
\end{array}$$

Insert the dual solution in the dual constraints. In (5), we see that
$3 - 1.5 - 1 = 0.5 \neq 0$, so $x_2^* = 0$.
Set $x_2^* = 0$ in (7) & (8), and we obtain $x_1^* = 3/2, \ x_3^* = 1/2$.

Check the objective values

$$
\begin{aligned}
\text{(P): min} \quad & z = 3x_1 + 3x_2 + 4x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 3 \\
& x_2 + 4x_3 \geq 2 \\
& 4x_1 + 8x_3 \geq 9 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
\qquad
\begin{aligned}
\text{(D): max} \quad & w = 3\pi_1 + 2\pi_2 + 9\pi_3 \\
\text{s.t.} \quad & s.t.\, 2\pi_1 + 4\pi_3 \leq 3 \\
& \pi_1 + \pi_2 \leq 3 \\
& 4\pi_2 + 8\pi_3 \leq 4 \\
& \pi_1, \pi_2, \pi_3 \geq 0
\end{aligned}
$$

$$
\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ 1/2 \end{pmatrix}, \quad z^* = 6.5\,, \qquad \begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_3^* \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix}, \quad w^* = 6.5\,.
$$

Example (2. Now the other way round)

$$\begin{array}{llrrrl} \text{(P): min} & 3x_1 & +3x_2 & +4x_3 & & \\ \text{s.t.} & 2x_1 + & x_2 & & \geq & 3 \\ & & x_2 & +4x_3 & \geq & 2 \\ & 4x_1 & & +8x_3 & \geq & 9 \\ & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{array}{llrrrl} \text{(D): max} & 3\pi_1 + & 2\pi_2 & +9\pi_3 & & \\ \text{s.t.} & 2\pi_1 & & +4\pi_3 & \leq & 3 \\ & \pi_1 + & \pi_2 & & \leq & 3 \\ & & 4\pi_2 & +8\pi_3 & \leq & 4 \\ & & & \pi_1, \pi_2, \pi_3 & \geq & 0 \end{array}$$

Suppose we know an optimal solution $x^* = (1\frac{1}{2}, 0, \frac{1}{2})$ of the **primal (P)**.
Use CS to find an optimal solution of the **dual (D)**.

Practice at home!

To Do

Solve Exercises 5.1, 5.2, 5.5 (a), 6.1, 8.1