Selected solutions Module 6

Exercise 6.3.

a)

$$\begin{array}{ll} \max & 3\,\pi_1 - 2\,\pi_2 \\ s.t. & 2\,\pi_1 \leq 2 \\ & \pi_1 - \pi_2 \leq 3 \\ & -4\,\pi_2 \leq 4 \\ & \pi_2 \geq 0 \end{array}$$

or

$$\begin{array}{ll} \max & 3\,\pi_1 + 2\,\pi_2 \\ s.t. & 2\,\pi_1 \leq 2 \\ & \pi_1 + \pi_2 \leq 3 \\ & 4\,\pi_2 \leq 4 \\ & \pi_2 \leq 0 \end{array}$$

b) Starting tableau:

basis	$\bar{b}$	π_1^+	π_1^-	π_2	s_1	s_2	s_3
s_1	2	$\underline{2}$	-2	0	1	0	0
s_2	3	1	-1	-1	0	1	0
s_3	4	0	0	-4	0	0	1
$-z$	0	3	-3	-2	0	0	0

Pivot:

basis	$\bar{b}$	π_1^+	π_1^-	π_2	s_1	s_2	s_3
π_1^+	1	1	-1	0	$\frac{1}{2}$	0	0
s_2	2	0	0	-1	$-\frac{1}{2}$	1	0
s_3	4	0	0	-4	0	0	1
$-z$	-3	0	0	-2	$-1\frac{1}{2}$	0	0

Optimal solution: $\pi_1 = 1$, $\pi_2 = 0$.

(c) Complementary slackness conditions:

$$\begin{aligned}
(3 - 2x_1 - x_2)\pi_1 &= 0 \\
(2 - x_2 - 4x_3)\pi_2 &= 0 \\
x_1(2 - 2\pi_1) &= 0 \\
x_2(3 - \pi_1 + \pi_2) &= 0 \\
x_3(4 - \pi_2) &= 0
\end{aligned}$$

Since $3 - \pi_1 + \pi_2 = s_2 > 0$, we have $x_2 = 0$.

Since $4 + \pi_2 = s_3 > 0$, we have $x_3 = 0$.

From the first primal constraint $2x_1 + x_2 = 3$ now follows that $x_1 = 1\frac{1}{2}$.

Exercise 7.1.

Consider the following primal dual pair.

The primal LP (P):

$$\begin{array}{ll}
\max & 0^T x \\
s.t. & Ax \leq b \\
 & x \in \mathbb{R}^n
\end{array}$$

The dual LP (D):

$$\begin{array}{ll}
\min & b^T y \\
s.t. & y^T A = 0 \\
 & y \geq 0 \\
 & y \in \mathbb{R}^m
\end{array}$$

First assume that (i) holds. Then (P) is infeasible. Hence (D) is infeasible or unbounded. The first option is not possible because $y = 0$ is a feasible solution of (D). So (D) is unbounded. So it is definitely possible to find feasible solutions with negative objective function value. This means that (ii) holds.

Now assume that (ii) holds. Then there is a $y_0 \in \mathbb{R}^m$ with $y_0^T A = 0$, $y_0 \geq 0$ and $b^T y_0 < 0$. In that case, λy_0 is a feasible solution of (D) for all $\lambda \geq 0$ and $\lim_{\lambda \to \infty} b^T(\lambda y_0) = -\infty$. So (D) is unbounded. So (P) is infeasible and hence (i) holds.

Exercise 7.2.

Apply Farkas' lemma to $A' = [A \quad -A]$ and $x' = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}$. Farkas' lemma then states that precisely one of the following two statements is true:

(a) $\exists x^-, x^+ \in \mathbb{R}^n$ such that $A(x^+ - x^-) = b$ and $x^+, x^- \geq 0$;

(b) $\exists \pi \in \mathbb{R}^m$ such that $\pi^T A \geq 0$ and $-\pi^T A \geq 0$ and $\pi^T b < 0$.

- (1) $\Rightarrow$ (a): take $x^+ = x$, $x^- = 0$ if $x \geq 0$, and take $x^- = x$, $x^+ = 0$ if $x < 0$.
- (a) $\Rightarrow$ (1): take $x = x^+ - x^-$.
- (2) $\Rightarrow$ (b): take $\pi = -y$.
- (b) $\Rightarrow$ (2): take $y = \frac{1}{\pi^T b}\pi$.