IN4344 Advanced Algorithms

Lecture 4 – The Simplex Method

Yuki Murakami

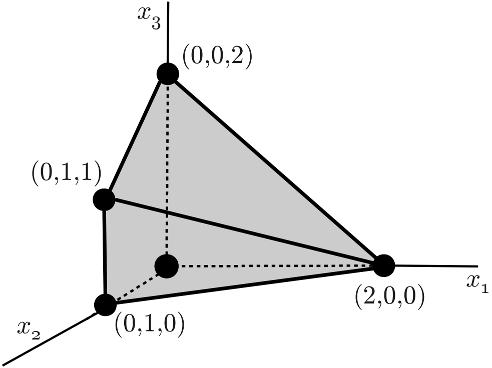
Delft University of Technology

September 18, 2023

LP: Linear Programming

$$\begin{aligned} \min \quad & -x_1 - 2\,x_2 - x_3 \\ \text{s.t.} \quad & x_1 + 2\,x_2 \qquad \leq 2 \\ & x_1 + \quad x_2 + x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The feasible region of an LP is described by a **polyhedron**:



Previous lecture

An LP in **standard form**:

$$\begin{aligned} \min \ & c^T x \\ \text{s.t.} \ & Ax = b \\ & x \geq 0 \end{aligned}$$

with A an $m \times n$ matrix.

- A **basis** is a collection of m linearly independent columns of A, i.e., a submatrix B of A. The variables corresponding to the columns of B are the **basic variables**.
- The corresponding **basic solution** is obtained by making all non-basic variables **0** and solving the system $Bx = b$ to get the values of the basic variables.
- A **basic feasible solution (bfs)** is a basic solution in which all basic variables are nonnegative.

Previous lecture

Theorem

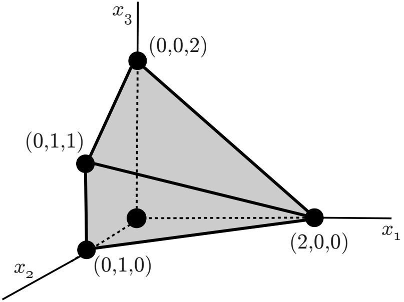
Given a linear programming problem with the constraints in standard form, $F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$, the following statements hold.

(i) If there exists a vector $x \in F$, then there exists a basic feasible solution $\bar{x} \in F$.

(ii) If there exists a finite optimal solution $x \in F$, then there exists an optimal basic feasible solution $x^* \in F$.

Hence, when searching for an optimal solution, we can restrict to bfss.

Previous lecture (2)



- Each LP with an optimal solution has an optimal solution that is an **extreme point of the polyhedron**.
- The extreme points of the polyhedron correspond to the **basic feasible solutions**.
- In theory, you can find an optimal solution by trying each possible basis.
- The **simplex** method walks from one bfs to a neighboring bfs in such a way that the objective function never gets worse.

This lecture

Solving LPs with the Simplex method.

1. How do we go from one bfs to a next bfs? (A **pivot**.)
2. How do we choose the "search direction"? (The **pivot column**.)
3. How do we know if the current bfs is **optimal**?
4. How do we find a **starting bfs**?

Example (1)

$$
\begin{array}{llrlrll}
\max & z = & 2x_1 & + & x_2 & & \\
\text{s.t.} & & 4x_1 & - & 3x_2 & \leq 6 & (1) \\
& & 3x_1 & + & 4x_2 & \leq 12 & (2) \\
& & & x_1, x_2 \geq 0 & & &
\end{array}
$$

In standard form:

$$
\begin{array}{llrlrlll}
\max & z = & 2x_1 & + & x_2 & & & \\
\text{s.t.} & & 4x_1 & - & 3x_2 & + s_1 & & = 6 \\
& & 3x_1 & + & 4x_2 & & + s_2 & = 12 \\
& & & x_1, x_2, s_1, s_2 \geq 0 & & & &
\end{array}
$$

What is an obvious starting bfs?

Answer

Let $x_1 = x_2 = 0$, i.e, $x_1,\ x_2$ are **non-basic** variables.
Solve for $s_1,\ s_2$ (trivial!).
Our starting bfs corresponds the **origin** of the x-coordinate system. $z(0,0) = 0$.

Example (1)

$$\begin{array}{llrlrlrlrl} \max & z = & 2 & x_1 & + & & x_2 & & & \\ \text{s.t.} & & 4 & x_1 & - & 3 & x_2 & + s_1 & & = 6 \\ & & 3 & x_1 & + & 4 & x_2 & & + s_2 & = 12 \\ & & & & & & & & x_1, x_2, s_1, s_2 \geq 0 & \end{array}$$

z will increase if we let x_1 or x_2 increase.
Suppose we increase the value of x_1.
How far can we increase the value of x_1 such that the solution remains feasible?
The other non-basic variable remains 0.

Answer

$$\begin{aligned} s_1 &= 6 - 4x_1 \geq 0 \\ s_2 &= 12 - 3x_1 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4x_1 &\leq 6 \\ 3x_1 &\leq 12 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &\leq 6/4 = 1.5 \\ x_1 &\leq 12/3 = 4 \end{aligned}$$

Example (1)

$$\begin{array}{lrcrcrcrcl}
\max & z = 2 & x_1 & + & & x_2 & & & & \\
\text{s.t.} & 4 & x_1 & - & 3 & x_2 & + & s_1 & & = 6 \\
 & 3 & x_1 & + & 4 & x_2 & & & + \; s_2 & = 12 \\
 & & & & & & & & & x_1, x_2, s_1, s_2 \geq 0
\end{array}$$

Starting bfs was:

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \end{pmatrix}$$

It is profitable to increase value of x_1; x_1 **enters the basis**.
Which variable **leaves the basis**?

The basic variable that first reaches the value 0!

Example (1)

$$\begin{array}{lll} s_1 = 6 - 4x_1 \geq 0 \\ s_2 = 12 - 3x_1 \geq 0 \end{array} \Rightarrow \begin{array}{l} x_1 \leq 6/4 = 1.5 \\ x_1 \leq 12/3 = 4 \end{array}$$

We can increase the value of x_1 to $x_1 = 1.5$.
The variable s_1 will then take value 0 and become non-basic.
The variable s_2 will take value $s_2 = 12 - 3x_1 = 12 - 3 \cdot 1.5 = 7.5$

The new bfs is:

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0 \\ 0 \\ 7.5 \end{pmatrix}$$

We have made a pivot!

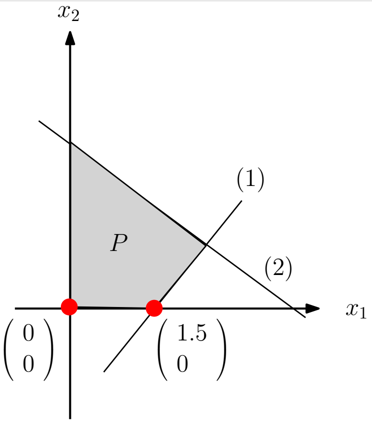
Important about pivot

Important:

- In a pivot **one** variable **enters the basis**, i.e., becomes a basic variable.
- In a pivot **one** variable **leaves the basis**, i.e., becomes a non-basic variable.
- The value of a non-basic variable is **always zero**.
- In this way we go from one extreme point of the feasible region to **a neighboring extreme point** (unless there is degeneracy).

Example (1)

$$\begin{array}{llrcrcl}
\max & z = & 2x_1 & + & x_2 & & \\
\text{s.t.} & & 4x_1 & - & 3x_2 & \leq 6 & (1) \\
& & 3x_1 & + & 4x_2 & \leq 12 & (2) \\
& & & x_1, x_2 \geq 0 & & &
\end{array}$$



Is the current bfs optimal?

To see if the current bfs is optimal, we have to re-write our system of equations so that it reflects the current bfs.

This was the original system:

$$
\begin{array}{llcrcrcrcrl}
\max & z = & 2 & x_1 & + & & x_2 & & & & \\
\text{s.t.} & & 4 & x_1 & - & 3 & x_2 & + & s_1 & & = 6 \\
 & & 3 & x_1 & + & 4 & x_2 & & & + \; s_2 & = 12 \\
 & & & \multicolumn{8}{l}{x_1, x_2, s_1, s_2 \geq 0}
\end{array}
$$

Re-write the objective function slightly, and drop repeating the nonnegativity constraints, since the Simplex-iterations ensure they will be satisfied:

$$
\begin{array}{rrcrcrcrl}
-z & +2x_1 & + & x_2 & & & & & = 0 \\
 & 4x_1 & - & 3x_2 & + & s_1 & & & = 6 \\
 & 3x_1 & + & 4x_2 & & & + & s_2 & = 12
\end{array}
$$

Is the current bfs optimal?

$$\begin{array}{rrrrrrrrrl} -z & +2x_1 & + & x_2 & & & & & = 0 \\ & 4x_1 & - & 3x_2 & + & s_1 & & & = 6 \\ & 3x_1 & + & 4x_2 & & & + & s_2 & = 12 \end{array}$$

When performing Simplex-iterations we will use a form that satisfies the following requirements:

(R1) The system of constraints is in standard form.

(R2) In each constraint row, there is a variable with coefficient $+1$, which in all other rows, including the objective row, has coefficient 0.

Our system above satisfies both these requirements!

Question

Why do we want (R2)?

Answer

Then it is trivial to read out the current bfs!

Is the current bfs optimal?

$$\begin{array}{rrrrrrrrrl} -z & +2x_1 & + & x_2 & & & & & = 0 \\ & 4x_1 & - & 3x_2 & + & s_1 & & & = 6 \\ & 3x_1 & + & 4x_2 & & & + & s_2 & = 12 \end{array}$$

When performing Simplex-iterations we will use a form that satisfies the following requirements:

(R1) The system of constraints is in standard form.

(R2) In each constraint row, there is a variable with coefficient $+1$. In all other rows, including the objective row, this variable has coefficient 0.

In our pivot, variable x_1 was entering, and s_1 was leaving.

To reflect the new bfs, we now need to perform row-operations so that x_1 **gets coefficient** $+1$ **in row 1** (since x_1 replaces s_1 as basic variables in row 1) and zero in all other rows!

Is the current bfs optimal?

$$\begin{array}{rrrrrrrrl} -z & +2x_1 & + & x_2 & & & & & = 0 \\ & \color{red}{4x_1} & - & 3x_2 & + & s_1 & & & = 6 \\ & 3x_1 & + & 4x_2 & & & + & s_2 & = 12 \end{array}$$

To reflect the new bfs, we now need to perform row-operations so that $\color{red}{x_1}$ **gets coefficient $+1$ in row 1** and zero in all other rows!

First, divide constraint row by 4.

$$\begin{array}{rrrrrrrrll} -z & +2x_1 & + & x_2 & & & & & = 0 & \\ & \color{red}{1x_1} & - & (3/4)x_2 & + & (1/4)s_1 & & & = 6/4 & (r_1/4) \\ & 3x_1 & + & 4x_2 & & & + & s_2 & = 12 & \end{array}$$

Then, take row 2 - (3/4) row 1, and objective row (row 0) - (1/2) row 1.

Is the current bfs optimal?

$$
\begin{array}{rrrrrrrrl}
-z & +2x_1 & + & x_2 & & & & & = 0 \\
 & 4x_1 & - & 3x_2 & + & s_1 & & & = 6 \\
 & 3x_1 & + & 4x_2 & & & + & s_2 & = 12
\end{array}
$$

$$
\begin{array}{rrrrrrrrll}
-z & & + & (5/2)x_2 & - & (1/2)s_1 & & & = -3 & (r_0 - (1/2)r_1) \\
 & x_1 & - & (3/4)x_2 & + & (1/4)s_1 & & & = 6/4 & (r_1/4) \\
 & & & (25/4)x_2 & - & (3/4)s_1 & + & s_2 & = 15/2 & (r_2 - (3/4)r_1)
\end{array}
$$

We can now again read out the bfs easily from the system:

$$
\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 6/4 \\ 0 \\ 0 \\ 15/2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0 \\ 0 \\ 7.5 \end{pmatrix}, \text{ with } z = 3\,.
$$

This is how we perform a pivot algebraically!

Is the current bfs optimal?

$$
\begin{array}{rcrcrcrl}
-z & & + & (5/2)x_2 & - & (1/2)s_1 & & = -3 \\
& x_1 & - & (3/4)x_2 & + & (1/4)s_1 & & = 6/4 \\
& & & (25/4)x_2 & - & (3/4)s_1 & + \; s_2 & = 15/2
\end{array}
$$

Is this bfs optimal? **No, if we increase the value of x_2, the objective value will increase!**

x_2 becomes **entering variable**. Which one leaves?

Second pivot: intermezzo

Question

How did we determine the leaving variable?

Answer

It is determined by which of the current basic variables reaches 0 first, when we increase the entering variable.

$$
\begin{array}{rrrrrrrrl}
-z & +2x_1 & + & x_2 & & & & & = 0 \\
 & 4x_1 & - & 3x_2 & + & s_1 & & & = 6 \\
 & 3x_1 & + & 4x_2 & & & + & s_2 & = 12
\end{array}
$$

$$
\begin{aligned}
x_1 &\leq 6/4 = 1.5 \\
x_1 &\leq 12/3 = 4
\end{aligned}
$$

We **check the ratio** between right-hand side values and the coefficients of the entering variable.

Second pivot: intermezzo

$$\begin{array}{rrrrrrrrrl} -z & +2x_1 & + & x_2 & & & & & = & 0 \\ & 4x_1 & - & 3x_2 & + & s_1 & & & = & 6 \\ & 3x_1 & + & 4x_2 & & & + & s_2 & = & 12 \end{array}$$

$$\begin{aligned} x_1 &\leq 6/4 = 1.5 \\ x_1 &\leq 12/3 = 4 \end{aligned}$$

We **check the ratio** between right-hand side values and the coefficients of the entering variable.

The leaving basic variable is the basic variable in the row that restricts the increase of the entering variable the most. In the first pivot this was s_1.

We call this the **"min-ratio test"**.

Second pivot

Back to the current system:

$$\begin{array}{rcrcrcrl} -z & + & (5/2)x_2 & - & (1/2)s_1 & & & = -3 \\ x_1 & - & (3/4)x_2 & + & (1/4)s_1 & & & = 6/4 \\ & & (25/4)x_2 & - & (3/4)s_1 & + & s_2 & = 15/2 \end{array}$$

Variable x_2 is the entering variable.

Perform min-ratio test:

$$\begin{array}{rcll} x_1 & = & (6/4) + (3/4)x_2 & \geq 0 \\ s_2 & = & (15/2) - (25/4)x_2 & \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{not} & \text{meaningful} \\ x_2 & \leq (15/2)/(25/4) = \frac{6}{5} \end{array}$$

$x_2 = 6/5 = 1.2$ and $s_2 = 0$ in the new bfs.

Since the coefficient of x_2 in the first constraint is negative, performing the min-ratio test in the first row does not provide any useful information.

Only do min-ratio test in rows where the entering variable has a positive coefficient!

Second pivot

Now we need to perform the pivot algebraically starting from the system

$$\begin{array}{rcrcrcrl} -z & + & (5/2)x_2 & - & (1/2)s_1 & & & = -3 \\ x_1 & - & (3/4)x_2 & + & (1/4)s_1 & & & = 6/4 \\ & & {\color{red}(25/4)}x_2 & - & (3/4)s_1 & + & s_2 & = 15/2 \end{array}$$

Perform row operations such that the **red coefficient** becomes a $+1$, and such all other coefficients in this column become 0.

Verify as an exercise that this is the outcome.

$$\begin{array}{rrcrcrll} -z & & - & (1/5)s_1 & - & (2/5)s_2 & = -6 & r_0 - (2/5)r_2 \\ x_1 & & + & (4/25)s_1 & + & (3/25)s_2 & = 12/5 & r_1 + (3/25)r_2 \\ & x_2 & - & (3/25)s_1 & + & (4/25)s_2 & = 6/5 & r_2 \cdot (4/25) \end{array}$$

Now, all variables in the objective function have non-positive coefficients. So, increasing any current non-basic variable will decrease the objective function value. This implies: **The current bfs is optimal!**

Second pivot

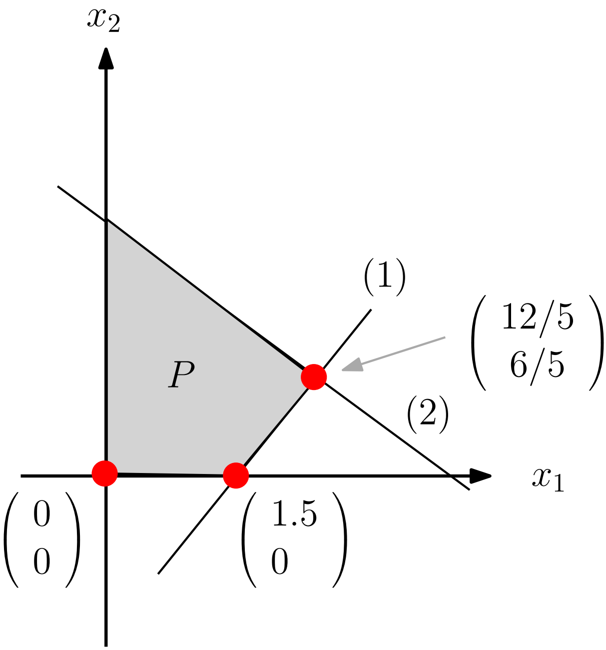
$$\begin{array}{rcrcrcl} -z & & - & (1/5)s_1 & - & (2/5)s_2 & = -6 \\ x_1 & & + & (4/25)s_1 & + & (3/25)s_2 & = 12/5 \\ & x_2 & - & (3/25)s_1 & + & (4/25)s_2 & = 6/5 \end{array}$$

The optimal bfs is

$$\begin{pmatrix} x_1^* \\ x_2^* \\ s_1^* \\ s_2^* \end{pmatrix} = \begin{pmatrix} 12/5 \\ 6/5 \\ 0 \\ 0 \end{pmatrix} \text{ with } z^* = 6$$

It is also fine to just give the answer in the original variables. The optimal solution x^* is

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 12/5 \\ 6/5 \end{pmatrix} \text{ with } z^* = 6$$


x_2
P
(1)
$\begin{pmatrix} 12/5 \\ 6/5 \end{pmatrix}$
(2)
x_1
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 1.5 \\ 0 \end{pmatrix}$

Simplex, more formally

Assume $\bar{A} = [\bar{B} \quad \bar{N}]$ with $\bar{B}$ the columns of the basic variables and $\bar{N}$ the columns of the non-basic variables.
Here $\bar{A},\ \bar{B},\ \bar{N},\ \bar{b}$ denotes the current system of constraints, after row operations, on the original system.
Then

$$\bar{B}x_B + \bar{N}x_N = \bar{b}$$

but the non-basic variables in x_N are all 0, so

$$\bar{B}x_B = \bar{b} \quad \Rightarrow \quad x_B = \bar{B}^{-1}\bar{b}$$

If $\bar{B} = I$ and $\bar{b} \geq 0$ then $\begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix}$ is a **bfs**.

The simplex method makes sure that in each iteration $\bar{B} = I$ and $\bar{b} \geq 0$.

When we bring x_j into the basis, column j is the **pivot column**.

Variable x_j can be increased until a basic variable becomes 0. One such basic variable x_ℓ leaves the basis. Row ℓ is the **pivot row**.

Let $\bar{a}_{ij}$ and $\bar{b}_i$ be the "current" values (after any previous pivots).

pivot column

$$
\begin{array}{lllllllllll}
 & & & & & & & \downarrow & & & \\
-z & & & & & & +\dots+ & \bar{c}_j x_j & + & \dots = -\bar{z} & \leftarrow \text{obj. funct.} \\
 & x_{B_1} & & & & & +\dots+ & \bar{a}_{1j} x_j & + & \dots = \bar{b}_1 & \\
 & & x_{B_2} & & & & +\dots+ & \bar{a}_{2j} x_j & + & \dots = \bar{b}_2 & \\
 & & & \dots & & & \dots & \dots & & \dots & \\
 & & & & x_{B_\ell} & & +\dots+ & \bar{a}_{\ell j} x_j & + & \dots = \bar{b}_\ell & \leftarrow \textbf{pivot row} \\
 & & & & & \dots & \dots & \dots & & \dots & \\
 & & & & & x_{B_m} & +\dots+ & \bar{a}_{mj} x_j & + & \dots = \bar{b}_m &
\end{array}
$$

basic variables **non-basic variables**

$\bar{a}_{\ell j}$ is the **pivot element**

Minimum ratio test: if column j is the pivot column, then we choose as pivot row a row ℓ s.t.

$$\frac{\bar{b}_\ell}{\bar{a}_{\ell j}} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \mid \bar{a}_{ij} > 0 \right\}.$$

Pivot: apply **elementary row operations** such that the columns of the basic variables form the identity matrix again:

1. divide the pivot row by the pivot element

$$(\text{row } \ell) := (\text{row } \ell)/\bar{a}_{\ell j}$$

2. create 0s in all other entries of the pivot column:

$$(\text{row } i) := (\text{row } i) - \frac{\bar{a}_{ij}}{\bar{a}_{lj}}(\text{row } \ell) \quad \forall i \neq \ell$$

2. **How to choose the pivot column?**

Consider the objective function in the current system:

$$-z + \sum_{\{j \mid j \text{ non-basic}\}} \bar{c}_j x_j = -\bar{z}$$

The coefficients $\bar{c}_j$ are called the **reduced costs**
and $\bar{z}$ is the value of the objective function in the current bfs.

2. How to choose the pivot column?

For a **max**imization problem:

- Choose a column j with $\bar{c}_j > 0$.
- Easy selection criterion: choose j with largest $\bar{c}_j > 0$.

For a **min**imization problem:

- Choose a column j with $\bar{c}_j < 0$.
- Easy selection criterion: choose j with smallest $\bar{c}_j < 0$.

3. How do we know whether a bfs is **optimal**?

Theorem

If x is a bfs of a **max** problem with $\bar{c}_j \leq 0$ for all j, then x is **optimal**.

If x is a bfs of a **min**problem with $\bar{c}_j \geq 0$ for all j, then x is **optimal**.

Summary Simplex Algorithm for **max**imization problems:

1. If $\bar{c}_j \leq 0$ for all j then the current bfs is **optimal**. Stop!
2. Choose entering variable $x_{j'}$ with $\bar{c}_{j'} > 0$.
3. If $\bar{a}_{ij'} \leq 0$ for all i then the problem is **unbounded**. Stop!
4. Choose leaving variable $x_{i'}$ for which

$$\frac{\bar{b}_{i'}}{\bar{a}_{i'j'}} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ij'}} \mid \bar{a}_{ij'} > 0 \right\}.$$

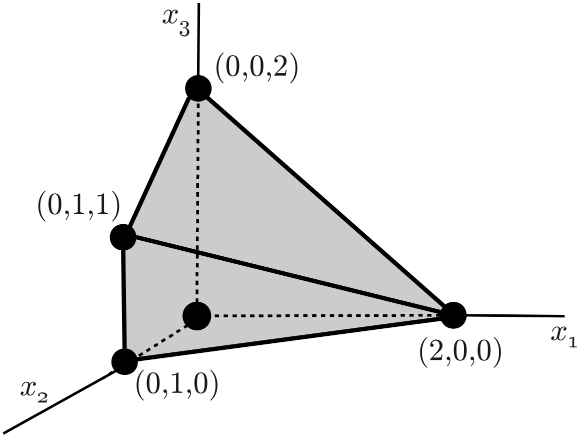
5. Apply elementary row operations such that column j' gets a 1 in row i' and 0s elsewhere. Go to (1).

In a **min**imization problem, only points 1 and 2 change:

1. If $\bar{c}_j \geq 0$ for all j then the current bfs is **optimal**. Stop!
2. Choose entering variable $x_{j'}$ with $\bar{c}_{j'} < 0$.

Example (2)

$$
\begin{array}{lrl}
\min & -x_1 - 2x_2 - x_3 & \\
\text{s.t.} & x_1 + 2x_2 \phantom{{}+x_3} & \leq 2 \\
& x_1 + x_2 + x_3 & \leq 2 \\
& x_1, x_2, x_3 \geq 0 &
\end{array}
$$



Example (2)

$$\begin{array}{lrl} \min & z = & -x_1 - 2x_2 - x_3 \\ \text{s.t.} & & x_1 + 2x_2 \phantom{{}+x_3} + s_1 \phantom{{}+s_2} = 2 \\ & & x_1 + x_2 + x_3 \phantom{{}+s_1} + s_2 = 2 \\ & & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{array}$$

We rewrite the objective function to

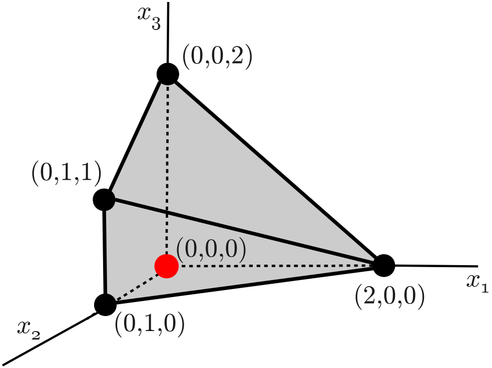
$$-z - x_1 - 2x_2 - x_3 = 0$$

and make a **Simplex tableau** (we just write the coefficients):

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	2	1	2	0	1	0
s_2	2	1	1	1	0	1
$-z$	0	-1	-2	-1	0	0

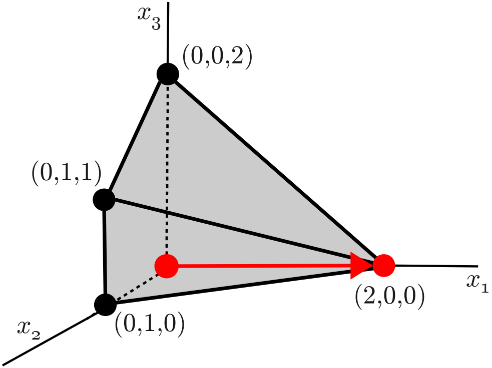
The starting bfs.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	2	1	2	0	1	0
s_2	2	1	1	1	0	1
$-z$	0	-1	-2	-1	0	0



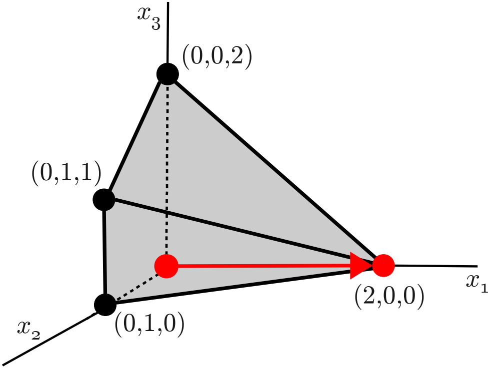
After one pivot.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_1	2	1	2	0	1	0
s_2	0	0	-1	1	-1	1
$-z$	2	0	0	-1	1	0



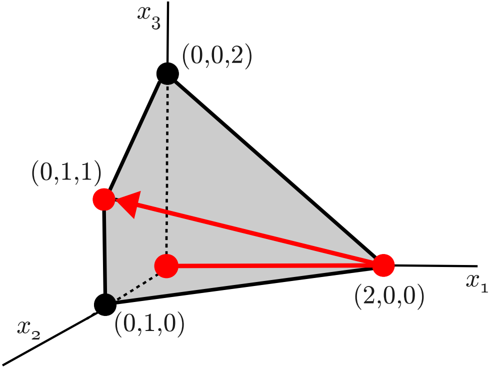
After two pivots.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_1	2	1	2	0	1	0
x_3	0	0	-1	1	-1	1
$-z$	2	0	-1	0	0	1



After three pivots. Optimal solution!

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_2	**1**	1/2	1	0	1/2	0
x_3	**1**	1/2	0	1	-1/2	1
$-z$	**3**	1/2	0	0	1/2	1



4. How to find a starting bfs?

If the original problem has only $\leq$ constraints:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Then each constraint gets a slack variable:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax + Is = b \\ & x, s \geq 0 \end{aligned}$$

and a starting bfs is easy to find:

$$\begin{aligned} x &= 0 \quad \text{non-basic variables} \\ s &= b \quad \text{basic variables} \end{aligned}$$

If the original problem has $\geq$ and/or $=$ constraints:

- For constraints $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$, **first** subtract a surplus variable $s_i \geq 0$ in each constraint.

  $\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i$.

  Why cannot s_i be basic variable for this row?

  Then, we would get $-s_i = b_i \geq 0$, but s_i has to be nonnegative!

  Therefore we need to **add** a so-called "artificial variable" $x_i^a \geq 0$ that serves as basic variable for this row: $\sum_{j=1}^{n} a_{ij}x_j - s_i + x_i^a = b_i$.

- For each equality constraint $\sum_{j=1}^{n} a_{ij}x_j = b_i$,

  add an "artificial variable" $x_i^a \geq 0$: $\sum_{j=1}^{n} a_{ij}x_j + \mathbf{x_i^a} = b_i$

  to have a starting basic variable for that constraint.

Constraint:	New constraint:	Basic variable:
$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$	$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$	s_i
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$\sum_{j=1}^{n} a_{ij}x_j - s_i + x_i^a = b_i$	x_i^a
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j + x_i^a = b_i$	x_i^a

Since the artificial variables are only there to help us get a starting basis, we need to pivot them out of the basis, if possible. They actually represent infeasibilities.

The first phase in which we pivot the artificial variables out of the basis is called Simplex Phase 1.

Simplex, two-phase method:

Phase 1: solve the LP with objective function replaced by

$$\min \quad w = \sum_i x_i^a$$

Let w^* be the optimal value of w.

1. **If $w^* = 0$ and no artificial variable is in the basis** then we have found a starting bfs of the original problem.
   - Remove the columns of artificial variables.
   - Put the original objective function back and express it in non-basic variables.
   - Continue with the Simplex method (this is the **second phase**).

Simplex, two-phase method:

2. **If** $w^* = 0$ **but an artificial variable** x_i^a **is in the basis** (with value 0).
   - Remove x_i^a from the basis by a pivot, with as pivot element an arbitrary nonzero element in the row of x_i^a.
   - Repeat until we get into case (1).
3. **If** $w^* > 0$ then the original problem **is infeasible**. Stop!

Example (3)

$$\begin{aligned}
\min \quad & 3\,x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2\,x_2 \leq 1 \\
& x_1 + x_2 = 1 \\
& x_1, x_2 \geq 0
\end{aligned}$$

First phase problem:

$$\begin{aligned}
\min \quad & x_2^a \\
\text{s.t.} \quad & x_1 + 2\,x_2 + s_1 = 1 \\
& x_1 + x_2 + x_2^a = 1 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Example (3)

First phase problem:

$$
\begin{array}{lrl}
\min & & x_2^a \\
\text{s.t.} & x_1 + 2\,x_2 + s_1 & = 1 \\
 & x_1 + x_2 + x_2^a & = 1 \\
 & x_1, x_2 \geq 0 &
\end{array}
$$

Simplex tableau:

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a
s_1	1	1	2	1	0
x_2^a	1	1	1	0	1
$-w$	0	0	0	0	1

Caution! The objective function has not yet been expressed in non-basic variables.
We have to do this first!

Example (3)

Now the objective function is expressed in non-basic variables:

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a	
s_1	1	1	2	1	0	
x_2^a	1	1	1	0	1	
$-w$	-1	-1	-1	0	0	$r_0 - r_2$

We can bring x_1 or x_2 into the basis. Say, we choose x_1:

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a
x_1	1	1	2	1	0
x_2^a	0	0	-1	-1	1
$-w$	0	0	1	1	0

We have found an optimal solution to the first phase with $w^* = 0$.
Caution! There is still an artificial variable in the basis!

Example (3)

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a
x_1	1	1	2	1	0
x_2^a	0	0	-1	-1	1
$-w$	0	0	1	1	0

Now, pivot x_2 into the basis and x_2^a out.

Notice: Our pivot element is negative! This is normally not allowed.

Why is it OK here? (think about it at home).

We have removed the artificial variable from the basis:

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a	
x_1	1	1	0	-1	2	$r_1 + 2r_2$
x_2	0	0	1	1	-1	$r_2 \cdot (-1)$
$-w$	0	0	0	0	1	

basis	$\bar{b}$	x_1	x_2	s_1	x_2^a
x_1	1	1	0	-1	2
x_2	0	0	1	1	-1
$-w$	0	0	0	0	1

Second phase: remove the column of the artificial variable and put the original objective function back:

Example (3)

basis	$\bar{b}$	x_1	x_2	s_1
x_1	1	1	0	-1
x_2	0	0	1	1
$-z$	0	3	1	0

Caution! Again first express the objective function in non-basic variables! $r_0 - 3r_1 - r_2$

Example (3)

We have expressed the objective function in non-basic variables:

basis	$\bar{b}$	x_1	x_2	s_1
x_1	1	1	0	-1
x_2	0	0	1	1
$-z$	-3	0	0	2

Continue with the **Simplex method**. In this case are all $\bar{c}_j$ nonnegative. Hence, the current solution is optimal.

Optimal solution:

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

with value $z = 3$.

Definition

A basic solution x is **degenerate** if one or more basic variables have value 0.

Example

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ x_1 \quad\quad &\leq 2 \\ 2\,x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

In standard form:

$$\begin{array}{llllll} x_1 + x_2 + s_1 & & & = 4 \\ x_1 & + s_2 & & = 2 \\ 2\,x_1 + x_2 & & + s_3 & = 6 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}$$

Show that (a) the bfs with basic variables $\{x_1, x_2, s_3\}$ and (b) the bfs with basic variables $\{x_1, x_2, s_1\}$ are **degenerate**.

Example (a)

$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 \quad\; &\le 2 \\ 2\,x_1 + x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In standard form:

$$\begin{aligned} x_1 + x_2 + s_1 \qquad\qquad &= 4 \\ x_1 \qquad\quad + s_2 \qquad &= 2 \\ 2\,x_1 + x_2 \qquad\qquad + s_3 &= 6 \\ x_1, x_2, s_1, s_2, s_3 \ge 0 & \end{aligned}$$

Show that the bfs with basic variables $\{x_1, x_2, s_3\}$ is **degenerate**.

Answer

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Basic variable $s_3 = 0$, so the solution is degenerate.

Example (b)

$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 &\le 2 \\ 2\,x_1 + x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In standard form:

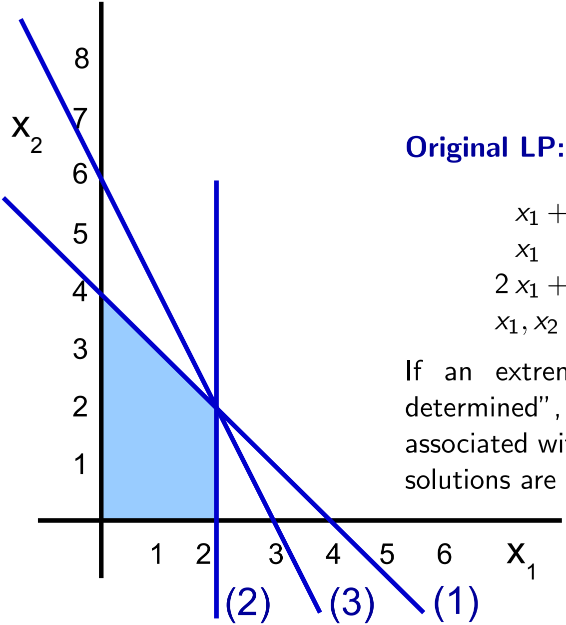
$$\begin{aligned} x_1 + x_2 + s_1 &= 4 \\ x_1 + s_2 &= 2 \\ 2\,x_1 + x_2 + s_3 &= 6 \\ x_1, x_2, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

Show that the bfs with basic variables $\{x_1, x_2, s_1\}$ is **degenerate**.

Answer

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Basic variable $s_1 = 0$, so the solution is degenerate.



Original LP:

$$\begin{aligned} x_1 + x_2 &\leq 4 \quad (1)\\ x_1 &\leq 2 \quad (2)\\ 2\,x_1 + x_2 &\leq 6 \quad (3)\\ x_1, x_2 &\geq 0 \end{aligned}$$

If an extreme point is "over-determined", more than one bfs is associated with that point. These solutions are degenerate.

Theorem

If two different bases have the same basic feasible solution x, then x is degenerate.

Cycling

- The Simplex algorithm as described can take **infinitely** long.
- This happens when at some point we get a bfs that we have had before. This is called **cycling**.

Bland's anti-cycling rules:

- If multiple entering variables are possible (with $\bar{c}_j < 0$), choose the one with smallest index.
- If multiple leaving variables are possible (with the same ratio in de minimum ratio test), choose the one with the smallest index.

Theorem

The Simplex method with Bland's anti-cycling rules terminates after a **finite** number of steps.

To Do

Exercises 4.4, 4.6, 4.7, 4.9