Selected Solutions Module 5

Exercise 5.1.

(a)

$$\begin{array}{ll} \max & 35\,\pi_1 + 2\,\pi_2 \\ s.t. & 5\;\pi_1 - \;\;\pi_2 \;\le 3 \\ & 7\;\pi_1 + 2\,\pi_2 \;\le -6 \\ & \pi_1, \pi_2 \le 0 \end{array}$$

(b)

$$\begin{array}{ll} \max & \pi_1 + 6\,\pi_2 \\ s.t. & \pi_1 + 3\,\pi_2 \le 2 \\ & \pi_1 + 2\,\pi_2 \le 1 \\ & \pi_1 \ge 0, \pi_2 \le 0 \end{array}$$

(c)

$$\begin{array}{ll} \max & 6\,\pi_1 + 4\,\pi_2 - 20 \\ s.t. & -3\,\pi_1 - 8\,\pi_2 \;\le -3 \\ & 3\,\pi_1 + 4\,\pi_2 \;\le 1 \\ & \pi_1, \pi_2 \le 0 \end{array}$$

(d)

$$\begin{array}{ll} \min & 100\,\pi_1 + 100\,\pi_2 + 100\,\pi_3 \\ s.t. & 2\,\pi_1 + \;\;6\,\pi_2 + \;10\,\pi_3 \ge 240 \\ & 2\,\pi_1 + \;\;\;\;\pi_2 \qquad\qquad\;\; \ge 60 \\ & \pi_1, \pi_2, \pi_3 \ge 0 \end{array}$$

(e)

$$\begin{array}{ll} \max & 8\,\pi_1 - 2\,\pi_2 \\ s.t. & 2\,\pi_1 \qquad\quad \le -3 \\ & 3\,\pi_1 + \;\;\pi_2 = -4 \\ & \pi_1 \le 0, \pi_2 \ge 0 \end{array}$$

(f)

$$\begin{array}{ll} \min & 13\,\pi_1 + 20\,\pi_2 \\ s.t. & 2\,\pi_1 - \ \ 5\,\pi_2 \leq -5 \\ & 9\,\pi_1 + \ \ 3\,\pi_2 \geq 7 \\ & \pi_1 \in \mathbb{R}, \pi_2 \geq 0 \end{array}$$

Exercise 5.2.

(a) False. The dual can also be infeasible.

(b) True.

Exercise 5.5.

(b) This is true. A proof is as follows.

The primal (P):

$$\begin{array}{ll} \min & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{R}^n \end{array}$$

The dual (D):

$$\begin{array}{ll} \min & \pi^T b \\ s.t. & \pi^T A \leq c \\ & \pi \in \mathbb{R}^m \end{array}$$

Suppose (D) has a non-degenerate optimum. (D) has n constraints and m variables. To use the definition of "degenerate", we need to add n slack variables to the dual.

$$\begin{array}{ll} \min & \pi^T b \\ s.t. & \pi^T A + Is = c \\ & s \geq 0 \\ & \pi \in \mathbb{R}^m \\ & s \in \mathbb{R}^n \end{array}$$

The dual now has $m + n$ variables. There are still n constraints and hence n basic variables. Hence, in a non-degenerate optimum, exactly m variables are 0. We need at least m hyperplanes to describe a point in $\mathbb{R}^m$. Hence, at least m slack variables are zero in the optimum point $\pi \in \mathbb{R}^m$. Since we have argued that exactly m variables are 0, it follows that exactly m slack variables are 0.

These slack variables we can write as:

$$s_j = c_j - \pi^T A_j = c_j - c_B^T B^{-1} A_j = \bar{c}_j$$

Hence exactly m of the reduced costs $\bar{c}_j$ of the primal problem are zero. These are the $\bar{c}_j$ of the m basic variables. So all non-basic variables have $\bar{c}_j > 0$. This means that the optimal solution is unique.