Exam Part 1 of Advanced Algorithms (IN4344)

5 October 2023, 18:30 – 21:30

This exam consists of 7 questions on 12 pages. You can earn 70 points in total. The obtained grade is equal to $1 + 9 \times (\text{points}/70)$.

- You may use a non-graphical calculator. You are **not** allowed to use the lecture notes, your own notes or any electronic devices except for a non-graphical calculator.
- Write each solution in the solution box corresponding to that question. If you need more space, you may continue your solution in one of the extra boxes at the end. Indicate this clearly.
- For all questions, you need to use the right method and describe all steps and arguments clearly.

Responsible examiner: Dr. Yukihiro Murakami **Good luck!**

1. (8 points) Answer the following questions with True or False accompanied by a brief justification (1-2 sentences).

   (i) If an LP-relaxation is unbounded, then the original ILP is also unbounded.

   (ii) Let z_A^* be a solution to a minimization ILP, obtained via an approximation algorithm with approximation ratio 3. If z_{ILP}^* is an optimal solution to the ILP and z_{LP}^* is an optimal solution to the LP-relaxation, then the following is always true.

$$z_{\text{LP}}^* \leq z_A^* \leq 4z_{\text{ILP}}^*.$$

   (iii) The matrix below is totally unimodular.

$$\begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

   (iv) Suppose that an LP is feasible and unbounded. Then its dual must also be unbounded.

Solution:

(i) False. Consider an ILP where the feasible region contains exactly one point.

$$\min \quad x_1 \quad \text{s.t.} \quad x_1 + \sqrt{2}x_2 = 0.$$

(ii) True. We have

$$z_{\text{LP}}^* \leq z_{\text{ILP}}^* \leq z_A^* \leq 3z_{\text{ILP}}^* \leq 4z_{\text{ILP}}^*$$

(iii) True. Consider the partition $\{\rho_1, \rho_3\} \cup \{\rho_2, \rho_4\}$ with Theorem 12.3. If the partition is used, the condition of the Theorem should be stated.

(iv) False. By weak duality, the dual must be infeasible.

2. Consider the following final Simplex tableau of the LP-relaxation of a minimization ILP.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
x_2	5/3	1/2	1	7/3	-4/3	0
s_2	7	-3	0	-12	-2/3	1
$-z$	-5/3	0	0	10	4/3	0

(a) (4 points) Does the LP-relaxation have an optimal solution? If yes, state the optimal solution, and also state if this optimal solution is unique. Give a short justification at each step (1-2 sentences).

Solution: From the tableau, we see that $(x_1, x_2, x_3, s_1, s_2) = (0, 5/3, 0, 0, 7)$ is an optimal BFS, since the coefficients of the objective row are all non-negative (for a minimization problem). The optimal solution is non-unique, as there exists a non-basic variable x_1 whose reduced cost is 0.

(b) (6 points) Find the **Gomory cutting plane** corresponding to the first row, expressed in non-basic variables.

Solution: The first row is $(1/2)x_1 + x_2 + (7/3)x_3 + (-4/3)s_1 = 5/3$.
Split into integral and fractional parts: $(1/2)x_1 + 1x_2 + (2+1/3)x_3 + (-2+2/3)s_1 = 1 + 2/3$.
Separate integral and fractional parts: $x_2 + 2x_3 - 2s_1 - 1 = 2/3 - (1/2)x_1 - (1/3)x_3 - (2/3)s_1$.
The left hand side is integral, so the right hand side must be integral. Since the right hand side is also less than 1, we get the following cutting plane.

$$2/3 - (1/2)x_1 - (1/3)x_3 - (2/3)s_1 \leq 0$$

You may rewrite this to:

$$3x_1 + 2x_3 + 4s_1 \geq 4$$

3. (8 points) Consider the following Simplex tableau of a minimization problem. Apply **one** iteration of the **dual Simplex method**. Indicate whether the obtained solution is feasible and why (or why not).

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	12	-5	0	2	1	6
x_2	-7	1	1	-2	0	-3
$-z$	7	1	0	3	0	3

Solution: x_2 leaves the basis since $-7 < 0$. s_2 enters the basis since $|3/-3| < |3/-2|$. We get the following Simplex tableau.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2
s_1	-2	-3	2	-2	1	0
s_2	7/3	-1/3	-1/3	2/3	0	1
$-z$	0	2	1	1	0	0

The solution found is not yet feasible since $s_1 = -2 < 0$.

4. Each morning, two trains travel along a line from Delft (station s_1) to Amsterdam (station s_{n+1}), passing n stations $s_1, s_2, \ldots, s_n$ in that order. The parameter q_j denotes the number of passengers who wish to travel from station s_j to Amsterdam. We need to decide the route of the two trains. Each station s_j has to be visited by exactly one train, and all q_j passengers will get on that train. The travel time from station s_j to Amsterdam is p_j minutes without stops. Each stop takes an additional 2 minutes. The aim is to minimize the total travel time summed over all passengers. To give an **ILP formulation** of this problem, the following decision variables are introduced. A decision variable t_j indicates the travel time for a passenger starting at station s_j. We also introduce a binary decision variable x_{ij}, where $x_{ij} = 1$ if train i stops at station s_j and $x_{ij} = 0$ otherwise. Do not introduce any new variables in your solutions. **During the exam, a clarification was given. Delft was named station s_0. Both trains would start at s_0, with no passengers. Both trains would arrive in station s_{n+1}.**

   (a) (3 points) Formulate the objective function.

   Solution:

   $$\min \sum_{j=1}^{n} q_j t_j$$

   (b) (3 points) Formulate constraints enforcing that each station is visited by exactly one train.

   Solution:

   $$x_{1j} + x_{2j} = 1 \quad \text{for all } j \in \{1, \ldots, n\}$$

   (c) (6 points) Formulate constraints enforcing that the variables t_j get the right values, i.e. that t_j is at least equal to the time it takes to travel from station s_j to Amsterdam, including stops, using the train that stops at station s_j. When passengers board the train at station s_j, they must wait 2 minutes before the train starts moving (boarding is instantaneous as soon as the train arrives).

   Solution:

   $$t_j \geq p_j + \sum_{h=j}^{n} 2x_{ih} - (1 - x_{ij})M \quad \text{for all } i \in \{1, 2\}, j \in \{1, \ldots, n\}$$

   With M a sufficiently large constant ($2n$ is enough). If train i stops at station s_j, then this constraint enforces that t_j is at least the travel time from station s_j to Amsterdam, including stops, using this train. If train i does not stop at station s_j, then $x_{ij} = 0$ and the constraint does not restrict the possible values for t_j (except perhaps $t_j \geq p_j$ which should be satisfied anyway).

5. Given is the following optimization problem.

$$\begin{array}{llrcrcrcl} \min & z = & 5x_1 & + & 8x_2 & + & 3x_3 & & \\ \text{s.t.} & & x_1 & - & x_2 & & & \leq & 1 \\ & & & & x_2 & + & x_3 & \geq & 1 \\ & & \multicolumn{7}{l}{x_1 \geq 0,\ x_2, x_3 \in \mathbb{R}} \end{array}$$

(a) (4 points) Formulate the corresponding dual problem. A justification for each variable / constraint is not required.

Solution:

$$\begin{array}{llrcrcl} \max & w = & \pi_1 & + & \pi_2 & & \\ \text{s.t.} & & \pi_1 & & & \leq & 5 \\ & & -\pi_1 & + & \pi_2 & = & 8 \\ & & & & \pi_2 & = & 3 \\ & & \multicolumn{5}{l}{\pi_1 \leq 0,\ \pi_2 \geq 0} \end{array}$$

(b) (6 points) Using part (a), show that $(0, -1, 2)$ is an optimal solution of the original LP.

Solution: The dual problem has a unique feasible solution $(-5, 3)$. Since the objective function value of this solution is $w^* = -2$ and the objective function of the primal for the feasible solution $(0, -1, 2)$ is also -2, it follows by the weak duality theorem that $(0, -1, 2)$ is an optimal solution to the primal (this can alternatively be argued using complementary slackness).

Students can also argue by complementary slackness to find the dual solution, as long as they check that the output solution is feasible.

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6. (12 points) Consider the ILP below.

$$\begin{array}{llrcrcr} z_{IP} = & \max z = & x_1 & + & 5x_2 & & \\ & \text{s.t.} & -4x_1 & + & 3x_2 & \leq & 3 \\ & & 7x_1 & + & 6x_2 & \leq & 42 \\ & & x_1, x_2 \geq 0 & & & & \\ & & x_1, x_2 \in \mathbb{Z} & & & & \end{array}$$

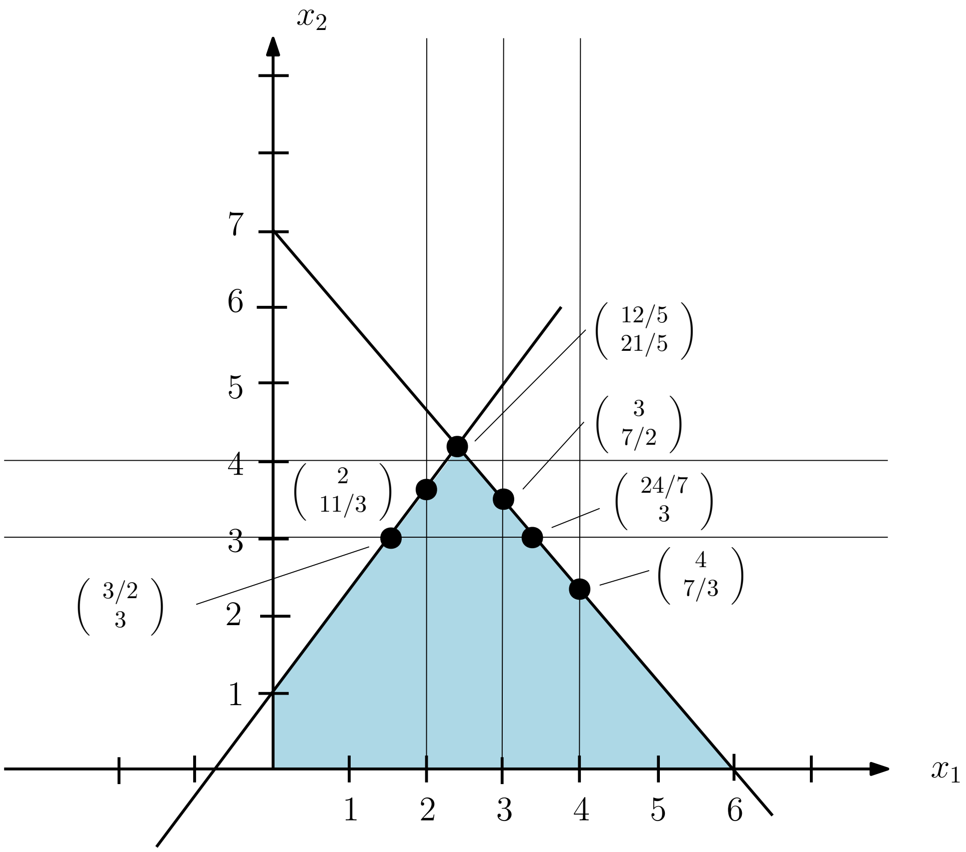
The optimal solution to the LP relaxation is:

$$x^*_{LP} = \begin{pmatrix} 12/5 \\ 21/5 \end{pmatrix} \qquad \text{with } z^*_{LP} = 117/5\,.$$

Determine one optimal solution to the ILP with the **branch & bound** method. Use the following search strategy:

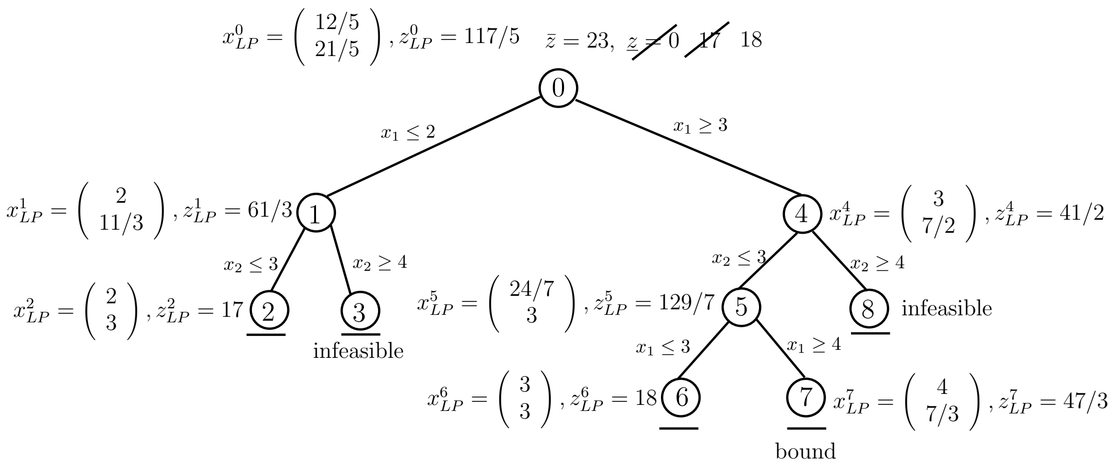
- Whenever given a choice, branch first on variable x_1.
- Choose the $\leq$-branch first.
- Go depth first.

LP relaxations may be solved graphically using the feasible region illustrated below along with *some* useful points. Indicate clearly if / where pruning has taken place, together with justification. Write your solution on the next page.



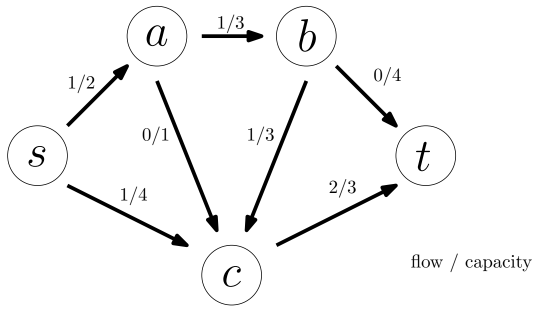
Solution to Question 6.

Solution: From the solution of the LP-relaxation we get a first upper bound $\overline{z} = \lfloor 117/5 \rfloor = 23$. The lower bound at the root node is $\underline{z} = 0$ (since $x_1, x_2 \geq 0$ and the objective coefficients are positive). The B&B tree looks as follows:



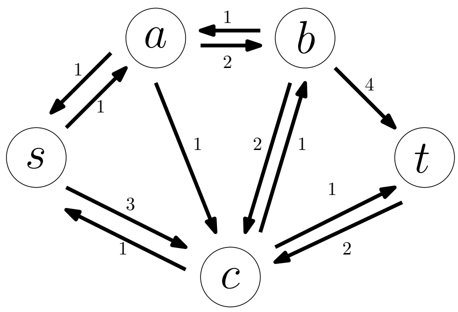
The optimal solution is $(x^*) = (3, 3)$ with $z^* = 18$. Nodes 3 and 8 are pruned by infeasibility, nodes 2 and 6 by optimality (optimal solutions LP-relaxation are integer), and node 7 by bound.

7. The question is about st maximum flow and the Ford-Fulkerson algorithm. In the directed graph G below, each arc is labelled by current flow / capacity. For example, the arc ab has flow 1 and capacity 3.



(a) (3 points) Draw the auxiliary graph of G. Find a shortest augmenting path and a longest augmenting path, where the length is the number of arcs in the path. For each augmenting path, determine the increase in flow as a result of adding it.

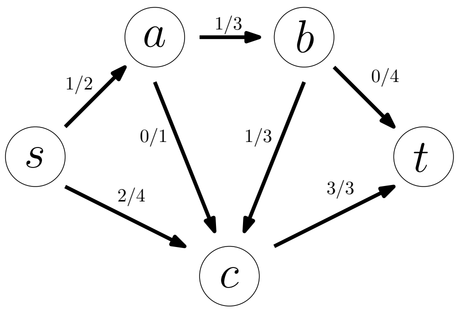
Solution:



There are six augmenting paths in total: $sct, sabt, scbt, sact, sacbt, sabct$, all with increase in flow 1. The shortest: sct; Two longest: $sacbt, sabct$.

(b) (3 points) Of the augmenting paths found in part (a), add a shortest one. Draw the graph G with updated flow values.

Solution:



Add the augmenting path sct.

(c) (4 points) Use an st-cut to argue that the flow cannot exceed 5. Give a short explanation.

Solution: Take the st-cut $W = \{s, c\}$ and $\bar{W} = \{a, b, t\}$. Argue by weak duality, or another method, that the flow cannot exceed 5.