IN4344 Advanced Algorithms

Lecture 5 – LP Relaxations and Branch & Bound

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Previous lecture: Simplex

- Simplex solves an LP by pivoting from one BFS to another, in a two phase process.
  - Feasibility tested in phase 1.
  - Optimality tested in phase 2.
- In a maximization problem, one reaches an optimum when the coefficients of objective function (reduced cost) is non-positive.
- Problem is unbounded if entering basic variable has all non-positive coefficients in constraint rows.

Recall that our original motivation was to solve ILPs!

LP-relaxation

An **ILP**:

$$\begin{aligned} \min \quad z_{IP} = \; & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

The **LP-relaxation**:

$$\begin{aligned} \min \quad z_{LP} = \; & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Observation

$z^*_{LP} \leq z^*_{IP}$ Similarly for MILP

How good is the lower bound that the LP-relaxation gives us?

Definition

The **integrality gap** of an ILP is the worst-case value of

$$\frac{z^*_{IP}}{z^*_{LP}} \geq 1$$

Solving (M)ILPs in general

In some cases we can prove that the integrality gap is equal to one, and that the LP-solution is integral! We discuss this next week.
In general though, the integrality gap is > 1.

In general (M)ILPs are NP-hard, so we cannot expect to find polynomial time algorithms for them.

The known algorithms are "enumerative", they (implicitly) enumerate feasible solutions until proven optimality.

The most commonly used algorithm for (M)ILPs is **Branch & Bound**.
This algorithm is implemented in all academic and commercial software.

Branch & Bound

Suppose we want to solve the following ILP.

$$\begin{array}{llrcrcl} \max & z = & x_1 & + & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\ & & -x_1 & + & x_2 & \leq & 3 \\ & & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & & \multicolumn{5}{l}{\color{red}{x_1, x_2 \in \mathbb{Z}}} \end{array}$$

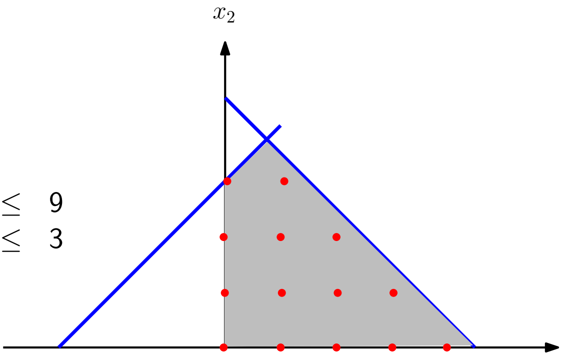
How do we start?

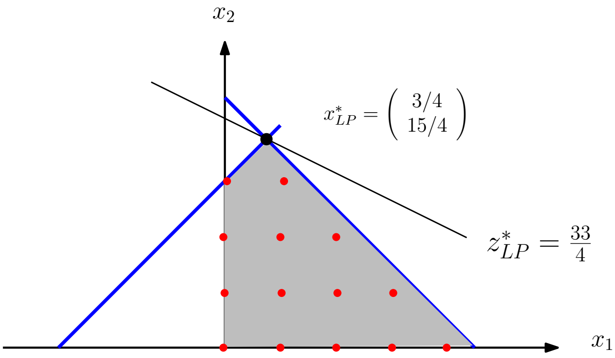
First solve the **LP-relaxation**.

$$\begin{array}{llrcrcr} \max & z = & x_1 & + & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\ & & -x_1 & + & x_2 & \leq & 3 \\ & & \multicolumn{5}{l}{x_1, x_2 \geq 0} \end{array}$$

We can do this with the **Simplex** method.

$$
\begin{array}{llrcrcl}
\max & z = & x_1 & + & 2x_2 & & \\
\text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\
& & -x_1 & + & x_2 & \leq & 3 \\
& & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\
& & \multicolumn{5}{l}{\color{red}{x_1, x_2 \in \mathbb{Z}}}
\end{array}
$$

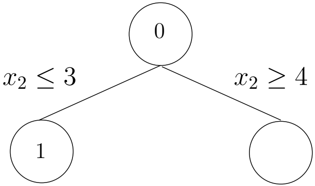




$\mathbf{z^*_{LP} = \frac{33}{4}}$, so $\mathbf{z^*_{IP} \leq \frac{33}{4}}$.

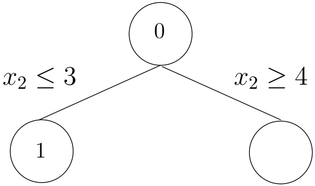
All objective coefficients are integer, so $\mathbf{z^*_{IP} \leq 8}$.

- So we have an **upper bound** on z^*_{IP}, $\bar{\mathbf{z}} = 8$.
- In the optimal solution of the LP-relaxation, $\mathbf{x_2 = \frac{15}{4} = 3\frac{3}{4}}$,
- but in the ILP x_2 should be integral, so $\mathbf{x_2 \leq 3}$ **or** $\mathbf{x_2 \geq 4}$.
- Hence we **branch** into two subproblems:



$\bar{z} = 8$ $\quad\quad$ $\underline{z} = -\infty$

Solve the LP-relaxation for subproblem 1:

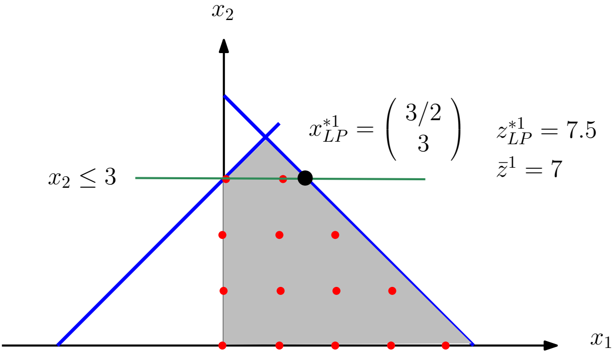


$\bar{z} = 8$ $\qquad$ $\underline{z} = -\infty$

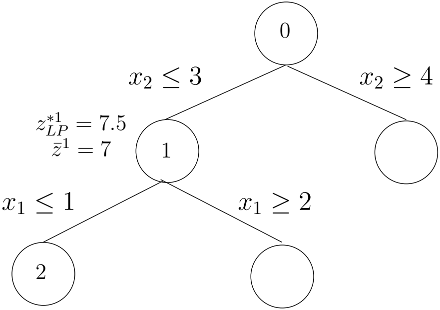
$$\begin{array}{llrcrcl}
\max & z = & x_1 & + & 2x_2 & & \\
\text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\
 & & -x_1 & + & x_2 & \leq & 3 \\
 & & & & \color{red}{x_2} & \color{red}{\leq} & \color{red}{3} \\
 & & x_1, x_2 \geq 0 & & & &
\end{array}$$

Subproblem 1:

$$
\begin{array}{llrcrcr}
\max & z = & x_1 & + & 2x_2 & & \\
\text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\
& & -x_1 & + & x_2 & \leq & 3 \\
& & & & \color{red}{x_2} & \color{red}{\leq} & \color{red}{3} \\
& & \multicolumn{5}{l}{x_1, x_2 \geq 0}
\end{array}
$$



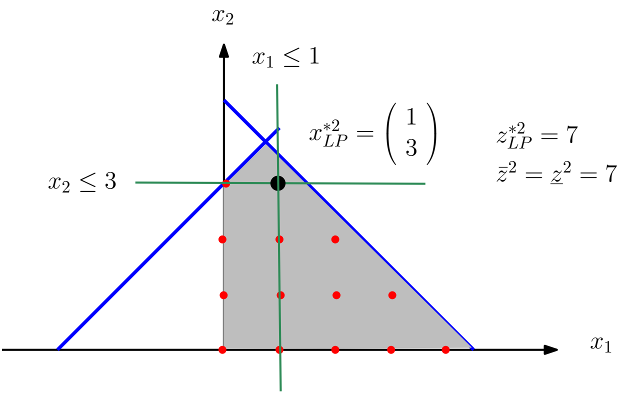
In Subproblem 1, the variable x_1 is not integral.
Create two new subproblems at Subproblem 1 by introducing $x_1 \leq 1$ and $x_1 \geq 2$ respectively.



$\bar{z} = 8$ $\underline{z} = -\infty$

Subproblem 2:

$$\begin{array}{lrcrcl} \max & z = x_1 & + & 2x_2 & & \\ \text{s.t.} & 2x_1 & + & 2x_2 & \leq & 9 \\ & -x_1 & + & x_2 & \leq & 3 \\ & & & \color{red}{x_2} & \color{red}{\leq} & \color{red}{3} \\ & \color{red}{x_1} & & & \color{red}{\leq} & \color{red}{1} \\ & x_1, x_2 \geq 0 & & & & \end{array}$$



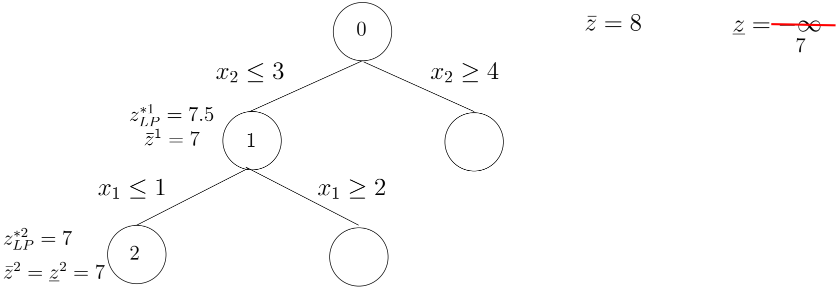
$z_{LP}^{*2} = 7$

$\bar{z}^2 = \underline{z}^2 = 7$

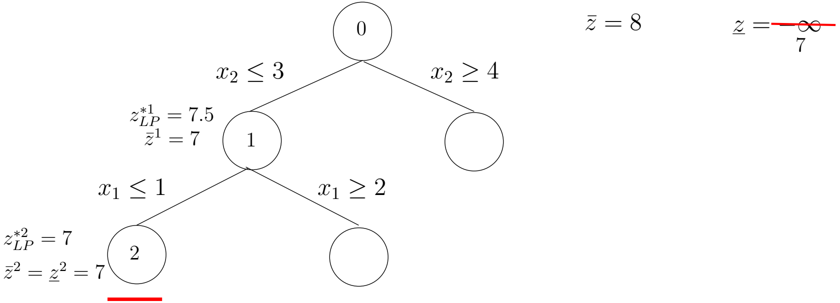
Subproblem 2:

Notice, the solution to Subproblem 2 is integral!!!
So we now also have a feasible solution to our problem.

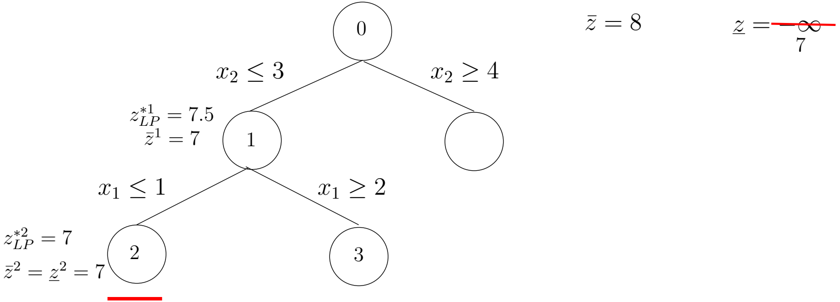
We now need to update the globally valid lower bound $\underline{z}$ to $\underline{z} = 7$.



Since we have solved Subproblem 2 to optimality (the solution is integral!), we do not need to search further below Subproblem 2. We can **prune** the search tree under node 2 **due to optimality of node 2**.

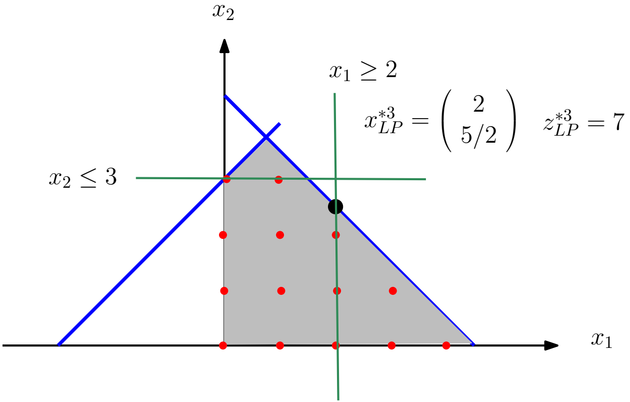


But, we still have unexplored subproblems left. Consider Subproblem 3.

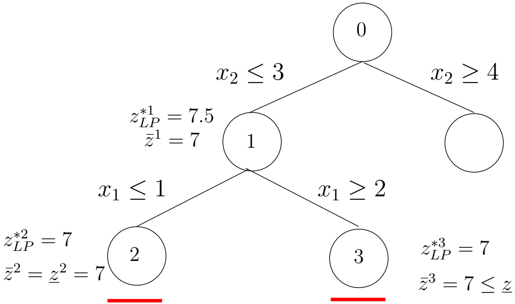


Subproblem 3:

$$\begin{array}{lrcrcr}
\max \quad z = & x_1 & + & 2x_2 & & \\
\text{s.t.} & 2x_1 & + & 2x_2 & \leq & 9 \\
 & -x_1 & + & x_2 & \leq & 3 \\
 & & & \color{red}{x_2} & \color{red}{\leq} & \color{red}{3} \\
 & \color{red}{x_1} & & & \color{red}{\geq} & \color{red}{2} \\
 & x_1, x_2 \geq 0 & & & &
\end{array}$$



Subproblem 3:

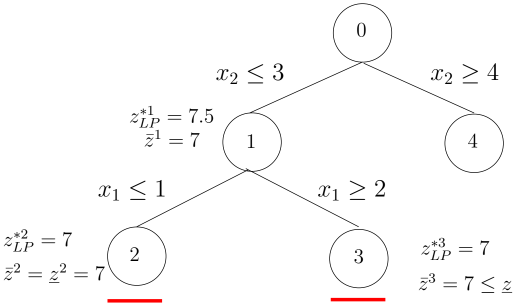


$\bar{z} = 8$ $\quad\quad$ $\underline{z} = \cancel{-\infty}$ 7

Since $\bar{z}^3 \le \underline{z}$ we will never find a feasible solution below Subproblem 3 with better value than 7, so we can **prune** the tree under node 3 **due to bound**.

We still have one subproblem left to investigate.

Subproblem 4:



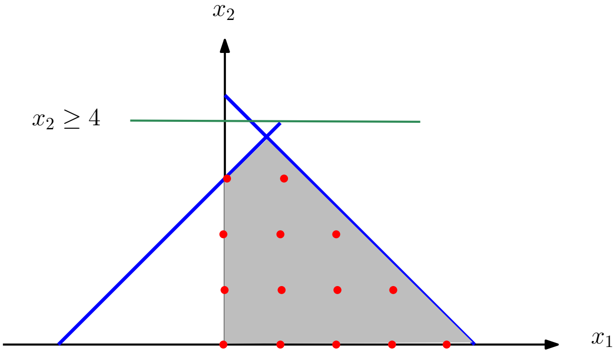
$\bar{z} = 8$

$\underline{z} = \cancel{-\infty}$ 7

$$
\begin{array}{llrcrcr}
\max & z = & x_1 & + & 2x_2 & & \\
\text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\
& & -x_1 & + & x_2 & \leq & 3 \\
& & & & {\color{red}x_2} & {\color{red}\geq} & {\color{red}4} \\
& & \multicolumn{5}{l}{x_1, x_2 \geq 0}
\end{array}
$$

Subproblem 4:

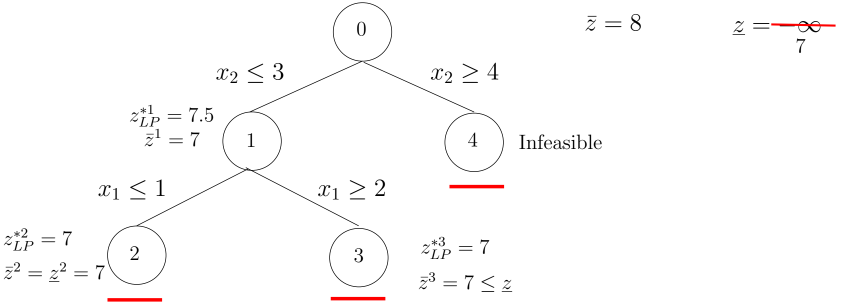
$$\begin{array}{llrcrcl} \max & z = & x_1 & + & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & 2x_2 & \leq & 9 \\ & & -x_1 & + & x_2 & \leq & 3 \\ & & & & \color{red}{x_2} & \color{red}{\geq} & \color{red}{4} \\ & & & & x_1, x_2 \geq 0 & & \end{array}$$



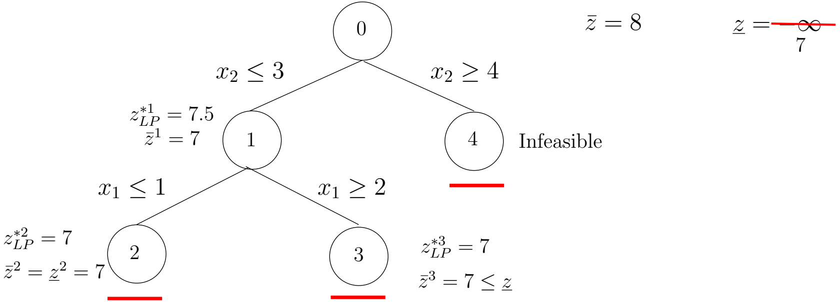
Subproblem 4:

Since Subproblem 4 has an infeasible LP-relaxation, there is no hope to find a feasible integer solution in this branch of the tree, so we can prune the tree under node 4 due to **infeasibility**.

This is the search tree.



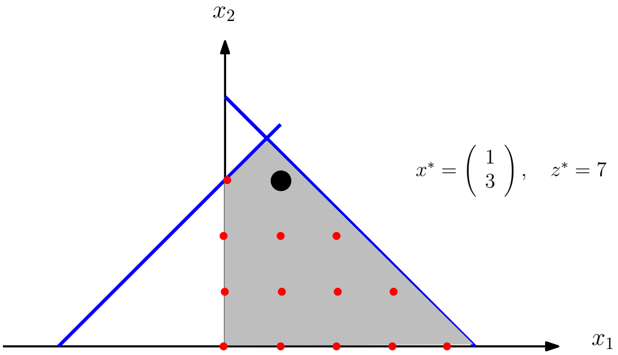
Optimal solution:



Since all leaves of the tree have been pruned, the problem has been solved. we have maintained the value $\underline{z}$ of the best solution found, which is $\underline{z} = 7$, corresponding to the integer solution:

$$x^* = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{with value } z^* = 7\,.$$

Optimal solution:



$$x^* = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad z^* = 7$$

Summary Branch & Bound method (for a max-problem):

- Solve LP-relaxations of subproblems (starting with the original problem).
    - The optimal value of the LP-relaxation is an **upper bound** on the optimal value of the subproblem.
    - If the optimal solution is integral, then the optimal value is a **lower bound** $\underline{z}$ on the optimal value of the original ILP. If an integer solution is found that is better than the current best integer solution **update** $\underline{z}$!
- If we can't prune a subproblem, we **branch** on a variable that is not yet integral.

Min-problems are handled similarly but **lower** bounds become **upper** bounds and vice versa.

When can we prune a node (in a max-problem)?

1. The LP-relaxation has **no feasible solution**.
   Prune by infeasibility. (See subproblem 4)

2. The optimal solution to the LP-relaxation is **integral**.
   Prune by optimality. (See subproblem 2).
   - If the value of the integral solution found is larger than $\underline{z}$, also **update the lower bound** $\underline{z}$.

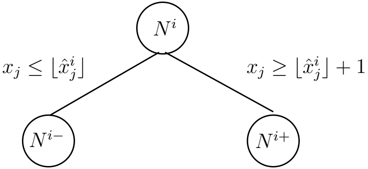
3. The **upper bound** $\bar{z}^k$ in node k is smaller than or equal to the current **lower bound** $\underline{z}$.
   Prune by bound. (See Subproblem 3).

Implementation aspects

- **Which B&B node do we consider next?**
  (a) **Depth-first search**: go down the tree as fast as possible, hoping to find an integral solution and hence an upper/lower bound (min/max). Notice that re-optimizing is very simple, since one adds one more constraint to the previous subproblem. **Dual simplex!**
  (b) **Best-node-first**: choose a node with best value of the LP-relaxation.
  (c) Use (a) until an integral solution has been found and then (b).
- **On which variable should we branch?**
  - Simple rule: choose a variable with fractional part closest to $\frac{1}{2}$.
  - More advanced rule: "Pseudocost branching" (see next slide).

There are many advanced variable- and node selection rules!

Pseudocost branching (min-problem)



Objective gain per unit change in variable j at node N^i in:
"downward" branch:

$$P_j^{i-} = \frac{z_{LP}^{i-} - z_{LP}^i}{f_j^{i-}} \quad \text{where } f_j^{i-} = \hat{x}_j^i - \lfloor \hat{x}_j^i \rfloor$$

"upward" branch:

$$P_j^{i+} = \frac{z_{LP}^{i+} - z_{LP}^i}{f_j^{i+}} \quad \text{where } f_j^{i+} = \lceil \hat{x}_j^i \rceil - \hat{x}_j^i$$

If an LP-relaxation is infeasible we set $z_{LP} = +\infty$.

Pseudocost branching (min-problem)

- At a branch point N^i, we decide to branch on variable x_j by looking at the historical performance of branching on x_j in other nodes explored before.

We now want to calculate the average performance in the upward, resp. downward branches, and then weigh these together to determine a so-called variable score. It involves a bit of notation that is easier to read than to see on a slide.

See pdf-file for more details!

CUTTING PLANES

Cutting Planes

- **Cutting planes** are constraints that can be added to an LP-relaxation of a (M)ILP **without cutting away any integral points**.
- Cutting planes are often used in combination with **Branch & Bound** to strengthen the LP-relaxations so as to increase the chances of pruning.

Consider an ILP:

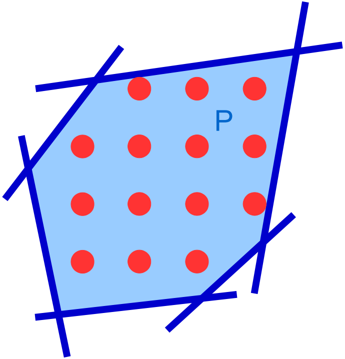
$$\begin{aligned} z_{IP} = \ & \min \ c^{\mathsf{T}} x \\ & \text{s.t.} \ Ax \leq b \\ & \qquad x \geq 0 \\ & \qquad \mathbf{x} \in \mathbb{Z}^{\mathbf{n}} \end{aligned}$$

The optimal value of the **LP-relaxation** gives a **lower bound** on the optimal value of the ILP.

We can try to **improve** this lower bound by adding **cutting planes** to the LP-relaxation.

$\mathbf{P} = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ is the corresponding **polyhedron** and

$\mathbf{S} = P \cap \mathbb{Z}^n$ the set of **feasible solutions** of the ILP.



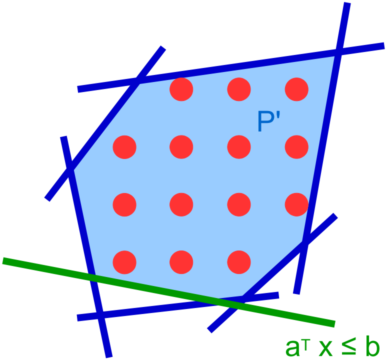
Definition

A polyhedron Q is a **formulation** for S if $Q \cap \mathbb{Z}^n = S$.

So P is **one** formulation for S,
but there exist **infinitely many** formulations for S!

Definition

An inequality $a^\mathsf{T} x \leq b$ is **valid** for S if $a^\mathsf{T} x \leq b$ for all $x \in S$.



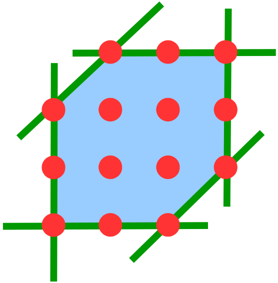
Adding a valid inequality $a^\mathsf{T} x \leq b$ to P gives a formulation P' for S which is
at least as strong, i.e., $P' \subseteq P$.

Question

What is the strongest possible formulation of S?

Question

What is the strongest possible formulation of S?



The **convex hull** of S: the smallest convex set containing all points in S.

In general, we cannot find the convex hull of S in polynomial time,
so we try to approximate it.

Gomory's Cutting Planes

Consider the following ILP.

$$
\begin{aligned}
z_{IP} = \min\ & -4\,x_1 + \ x_2 & \\
\text{s.t.}\quad & 7\,x_1 - 2\,x_2 \le 14 & (1)\\
& x_2 \le 3 & (2)\\
& 2\,x_1 - 2\,x_2 \le 3 & (3)\\
& x_1, x_2 \ge 0 & \\
& x_1, x_2 \in \mathbb{Z} &
\end{aligned}
$$

The **optimal tableau** of the LP-relaxation:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_1	20/7	1	0	1/7	2/7	0
x_2	3	0	1	0	1	0
s_3	23/7	0	0	-2/7	10/7	1
$-z$	59/7	0	0	4/7	1/7	0

Definition

Let $a \in \mathbb{R}$.

- $\lfloor a \rfloor$ is the largest integer $n \in \mathbb{Z}$ with $n \leq a$ (round down).
- We call $\lfloor a \rfloor$ the **integral part** of a
- and $a - \lfloor a \rfloor$ the **fractional part** of a.

Observe that for the fractional part of a holds that

$$0 \leq a - \lfloor a \rfloor < 1.$$

Be careful when a is negative!

For example if $a = -4\frac{1}{4}$ then the **integral part** is $\lfloor a \rfloor = -5$ and the **fractional part** is $a - \lfloor a \rfloor = \frac{3}{4}$.

For example, the first row of the Simplex tableau is:

$$x_1 + \frac{1}{7}s_1 + \frac{2}{7}s_2 = 2\frac{6}{7}$$

Split each coefficient a into the **integral** part $\lfloor a \rfloor$
and the **fractional** part $a - \lfloor a \rfloor$.

$$1x_1 + (0 + \frac{1}{7})s_1 + (0 + \frac{2}{7})s_2 = 2 + \frac{6}{7}$$

Bring all **integral** parts to the **left**
and the **fractional** parts to the **right**.

$$x_1 - 2 = \frac{6}{7} - \frac{1}{7}s_1 - \frac{2}{7}s_2$$

Since the left-hand side is integral in all feasible solutions,
the **right-hand side** should also be **integral**.

The right-hand side $\frac{6}{7} - \frac{1}{7}s_1 - \frac{2}{7}s_2$ should be integral.

Since $\frac{6}{7} < 1$ and $s_1, s_2 \geq 0$, the only possible values for the right-hand side are $0, -1, -2, \ldots$, so:

$$\frac{6}{7} - \frac{1}{7}s_1 - \frac{2}{7}s_2 \leq 0$$

Rewrite:

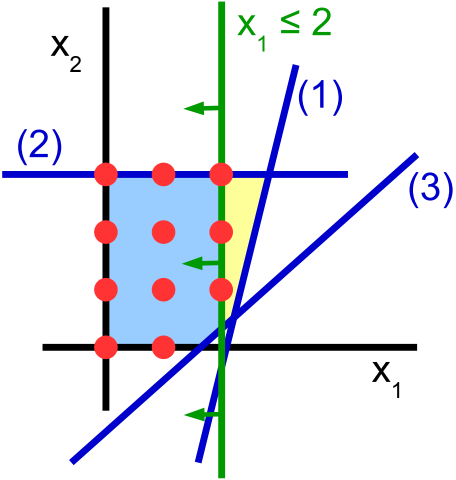
$$-\frac{1}{7}s_1 - \frac{2}{7}s_2 \leq -\frac{6}{7}$$

(or

$$-s_1 - 2s_2 \leq -6)$$

This is a **Gomory cut** or **Gomory cutting plane**.

The Gomory cut expressed in the original variables, is: $x_1 \leq 2$:



The yellow part of the polyhedron has been cut off.

To solve the new LP-relaxation algorithmically, add a slack variable to the Gomory cut as it was derived:

$$-\frac{1}{7}s_1 - \frac{2}{7}s_2 + s_4 = -\frac{6}{7}$$

and add the row to the **current Simplex tableau** of the LP relaxation: (Notice: The Gomory cut is “automatically” expressed in non-basic variables!)

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	$\mathbf{s_4}$
x_1	20/7	1	0	1/7	2/7	0	**0**
x_2	3	0	1	0	1	0	**0**
s_3	23/7	0	0	-2/7	10/7	1	**0**
$\mathbf{s_4}$	$-6/7$	**0**	**0**	$-1/7$	$-2/7$	**0**	**1**
$-z$	59/7	0	0	4/7	1/7	0	**0**

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	s_4
x_1	$20/7$	1	0	$1/7$	$2/7$	0	0
x_2	3	0	1	0	1	0	0
s_3	$23/7$	0	0	$-2/7$	$10/7$	1	0
s_4	$-6/7$	0	0	$-1/7$	$-2/7$	0	1
$-z$	$59/7$	0	0	**$4/7$**	**$1/7$**	0	0

The solution remains **dual feasible**
but is **no longer primal feasible**.
Apply the **dual Simplex method**. s_4 is the leaving basis variable.

Determine the entering basis variable:

$$\min\left\{\left|\frac{\bar{c}_{s_1}}{\bar{a}_{4,s_1}}\right|,\ \left|\frac{\bar{c}_{s_2}}{\bar{a}_{4,s_2}}\right|\right\} = \min\left\{\left|\frac{4/7}{-1/7}\right|, \left|\frac{1/7}{-2/7}\right|\right\} = \left|\frac{1/7}{-2/7}\right|$$

So s_2 enters the basis.

Apply row operations:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	s_4
x_1	$\mathbf{2}$	1	0	$\mathbf{0}$	$\mathbf{0}$	0	$\mathbf{1}$
x_2	$\mathbf{0}$	0	1	$-1/2$	$\mathbf{0}$	0	$\mathbf{7/2}$
s_3	-1	0	0	-1	$\mathbf{0}$	1	$\mathbf{5}$
$\mathbf{s_2}$	$\mathbf{3}$	0	0	$\mathbf{1/2}$	$\mathbf{1}$	0	$-7/2$
$-z$	$\mathbf{8}$	0	0	$\mathbf{1/2}$	$\mathbf{0}$	0	$\mathbf{1/2}$

New dual pivot:

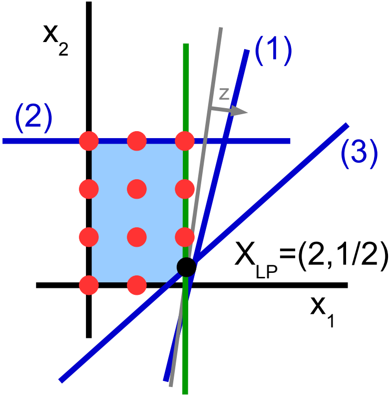
basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	s_4
x_1	2	1	0	0	0	0	1
x_2	0	0	1	$-1/2$	0	0	$7/2$
s_3	-1	0	0	-1	0	1	5
s_2	3	0	0	$1/2$	1	0	$-7/2$
$-z$	8	0	0	$1/2$	0	0	$1/2$

Now s_1 enters the basis for s_3.

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3	s_4
x_1	2	1	0	0	0	0	1
x_2	$\mathbf{1/2}$	0	1	$\mathbf{0}$	0	$-1/2$	$\mathbf{1}$
$\mathbf{s_1}$	$\mathbf{1}$	0	0	$\mathbf{1}$	0	-1	-5
s_2	$\mathbf{5/2}$	0	0	$\mathbf{0}$	1	$\mathbf{1/2}$	-1
$-z$	$\mathbf{15/2}$	0	0	$\mathbf{0}$	0	$\mathbf{1/2}$	$\mathbf{3}$

The current solution, $x_{LP} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$, is dual and primal feasible, and hence optimal.

The optimal solution of the LP-relaxation after adding the Gomory cut ($x_{\mathrm{LP}} = (3, 1/2)$).



The objective value increased from $z_{LP} = -59/7 = -8.42857...$ to $z_{LP} = -15/2 = -7.5$.

Since the LP-relaxation improved, we are more likely to being able to prune by bound (or optimality) in B&B.

Final remarks

- We can formulate a Gomory cut for each row of the Simplex tableau, including the objective function row.
- We can add all these to the Simplex tableau simultaneously.
- In practice, Branch & Bound is **combined** with Cutting Planes.

To Do

Solve exercises: 13.1 (you may solve all subproblems graphically), 14.1