Selected solutions Module 3

Exercise 3.2.

Let $x, y \in A \cap B$ and let z be a convex combination of x and y. We need to show that $z \in A \cap B$. Since $x, y \in A \cap B$, we know that $x, y \in A$. Since A is convex, it follows that $z \in A$. Similarly, we find that $x, y \in B$ and, since B is convex, it follows that $z \in B$. Hence, we conclude that $z \in A \cap B$.

Exercise 3.3.

A polyhedron P is the intersection of a finite number of half-spaces. Each half-space is convex because, if $a^T x \leq b$ and $a^T y \leq b$, then

$$a^T(\lambda x + (1-\lambda)y) = \lambda a^T x + (1-\lambda)a^T y \leq b.$$

Hence, the intersection P is also convex by Exercise 3.2.

Exercise 3.4.

The feasible region of an LP is a polyhedron P. The set of optimal solutions of the LP is the intersection of P with a supporting hyperplane $H = \{x \in \mathbb{R}^n \mid c^T x = d\}$, with $c^T x$ the objective function of the LP.

Hyperplane H is convex because, if $c^T x = d$ and $c^T y = d$ then

$$c^T(\lambda x + (1-\lambda)y) = \lambda c^T x + (1-\lambda)c^T y = \lambda d + (1-\lambda)d = d.$$

The polyhedron P is convex by Exercise 3.3 and hence the intersection of P and H is also convex by Exercise 3.2.

Exercise 3.5.

$$O_n = \{x \in \mathbb{R}^n \mid c^T x \leq 1 \text{ for all } c \in \{-1, 1\}^n\}$$