Exercises lecture 4: 4.9

Formulation phase 1:

$$
\begin{array}{llcccccccccccccl}
\min w & & & & & & & & & & & x_2^a & & & & \\
\text{s.t} & x_1 & + & x_2 & + & x_3 & + & s_1 & & & & & & & = & 5 \\
 & & & x_2 & + & x_3 & & & - & s_2 & + & x_2^a & & & = & 2 \\
 & 2x_1 & & & + & x_3 & & & & & & & + & s_3 & = & 1 \\
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3,\ x_2^a \geq 0$$

Before starting Phase 1, we express the Phase 1 objective function in non-basic variables (saves a pivot at the beginning of Phase 1).

$$w = x_2^a = 2 - x_2 - x_3 + s_2\,.$$

<u>Phase 1</u>:

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	s_3	x_2^a
s_1	5	1	1	1	1	0	0	0
x_2^a	2	0	1	1	0	-1	0	1
s_3	1	2	0	1	0	0	1	0
$-w$	-2	0	-1	-1	0	1	0	0

Bring x_2 into the basis. min-ratio test: $\min\{5/1, 2/1\} = 2/1$, so x_2^a leaves the basis.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	s_3	x_2^a	
s_1	3	1	0	0	1	1	0	-1	$r_1 - r_2$
x_2	2	0	1	1	0	-1	0	1	r_2
s_3	1	2	0	1	0	0	1	0	r_3
$-w$	0	0	0	0	0	0	0	1	$r_0 + r_2$

$\bar{c}_j \geq 0$ for all j, so we have reached optimum of Phase 1.

x_2^a is not in the basis, so we can remove the column of x_2^a from the tableau, and re-introduce the original objective function z.

$$x = -4x_1 + x_2 - x_3\,.$$

Before setting up the tableau, express z as a function of non-basic variables. From the second row of the simplex tableau at the end of Phase 1, we read $x_2 + x_3 - s_2 = 2$, or $x_2 = 2 - x_3 + s_2$. We now get:

$$z = -4x_1 + x_2 - x_3 = -4x_1 + (2 - x_3 + s_2) - x_3 = -4x_1 - 2x_3 + s_2 + 2\,.$$

The starting tableau of Simplex Phase 2 is:

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	s_3
s_1	3	1	0	0	1	1	0
x_2	2	0	1	1	0	-1	0
s_3	1	2	0	1	0	0	1
$-z$	-2	-4	0	-2	0	1	0

Bring x_1 into the basis. min-ratio test: $\min\{3/1, 1/2\} = 1/2$, so s_3 leaves the basis.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	$2\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	$r_1 - (1/2)r_3$
x_2	2	0	1	1	0	-1	0	r_2
x_1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$r_3/2$
$-z$	0	0	0	0	0	1	2	$r_0 + 2r_3$

$\bar{c}_j \geq 0$ for all j, so we reached an optimal basic feasible solution.

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2 \\ 0 \end{pmatrix}, \quad \text{with } z^* = 0\,.$$

Since there are non-basic variables with $\bar{c}_j = 0$ we see that there are more optimal solutions. We can let x_3 enter the basis ($\bar{c}_{x_3} = 0$). The min-ratio test indicates that x_1 then leaves the basis.

basis	$\bar{b}$	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	3	1	0	0	1	1	0	$r_1 + r_3$
x_2	1	-2	1	0	0	-1	-1	$r_2 - 2r_3$
x_3	1	2	0	1	0	0	1	$2r_3$
$-z$	0	0	0	0	0	1	2	r_0

Another bfs has been found:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{with } z^* = 0\,.$$

These are the only optimal *basic* feasible solutions. All optimal solutions are obtained as convex combinations of these two basic feasible solutions:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \lambda_1 \begin{pmatrix} 1/2 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1 + \lambda_2 = 1,\ \lambda_1,\ \lambda_2 \geq 0, \quad \text{with } z^* = 0\,.$$

Alternatively, we can write:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \lambda \begin{pmatrix} 1/2 \\ 2 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \geq 0, \quad \text{with } z^* = 0\,.$$