IN4344 Advanced Algorithms

Lecture 3 – The geometry and linear algebra of LP

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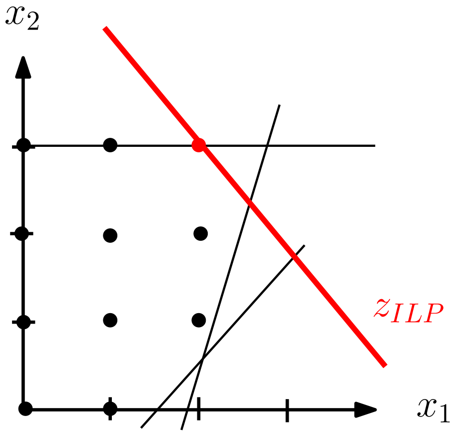
Why are Linear Programming Problems important?

- They occur a lot in practice.
- They give rise to many nice mathematical and algorithmic questions.
- **They are VERY important in analysing and solving (M)ILPs!**

Relation (M)ILP – LP

Consider the following ILP ($x_1,\ x_2 \in \mathbb{Z}_{\geq 0}$)



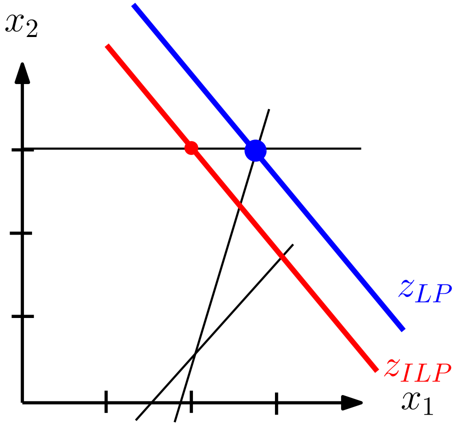
The **red point** is optimal.

How can we find this optimum? (It is not an extreme point of the polyhedron.)

This will be the topic for later, but the question serves as motivation!

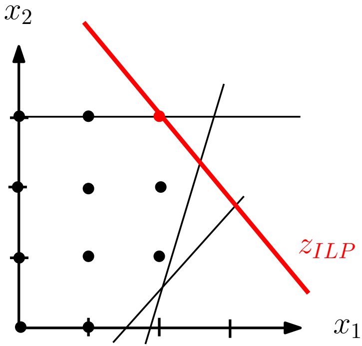
Relation (M)ILP – LP

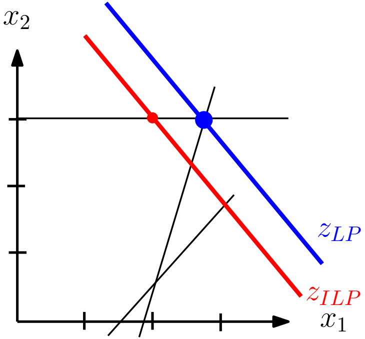
Suppose that $x_1,\ x_2 \in \mathbb{R}_{\geq 0}$ **instead of** $x_1,\ x_2 \in \mathbb{Z}_{\geq 0}$.
Then we get the following problem:



The blue point is optimal.

Relation (M)ILP – LP





What is the relation between z_{ILP} and z_{LP}?

Notice, the set of feasible solutions to the ILP is the same as the LP **except** that we have an additional constraint $x \in \mathbb{Z}^2$.

The set of feasible solutions of the LP contains **all ILP-solutions, plus more**!

Relation (M)ILP – LP

We call a problem RP a **relaxation** of problem P if the objective functions are the same and if the set of feasible solutions to RP contains the set of feasible solutions to P (plus possibly more).

Nice feature: each (M)ILP has a natural so-called *LP-relaxation*!
Just remove the integrality restrictions!

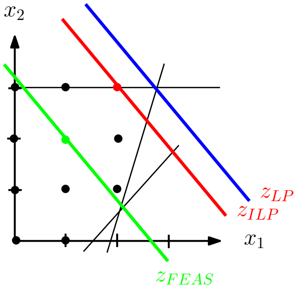
If the (M)ILP is a maximization problem, then the following holds:

$$z_{(M)ILP} \leq z_{LP}$$

This relation is heavily used in algorithms for **optimizing and approximating (M)ILPs**!

In addition, we have the following nice relation:

Relation (M)ILP – LP



The **green** point is a feasible ILP-solution with objective value z_{FEAS}.
If ILP is a **maximization problem**, the following holds:

$$z_{FEAS} \leq z_{(M)ILP}$$

We now have: $$z_{FEAS} \leq z_{(M)ILP} \leq z_{LP}$$

VERY IMPORTANT ALGORITHMICALLY!

Linear Programming (LP)

In LP, the following situations can occur:

1. Problem is **feasible**, and has a **bounded** optimal objective value.
2. Problem is **feasible**, and has an **unbounded** optimal objective value.
3. Problem is **infeasible**, i.e., the set of feasible solutions is the empty set.

In Case 1, there is an optimal solution in an **extreme point** of the feasible region.
How does a feasible region look like?
A constraint in an LP has the form:

$$3x_1 - 4x_2 + \cdots + 8x_{12} \leq 47,$$

or more formally: $\sum_{j=1}^{n} a_j x_j \leq b$ or $ax \leq b$,
where a, b is input.

Linear Programming (LP)

Definition

A set $H = \{x \in \mathbb{R}^n \mid ax = b\}$ is called a *hyperplane.*

Definition

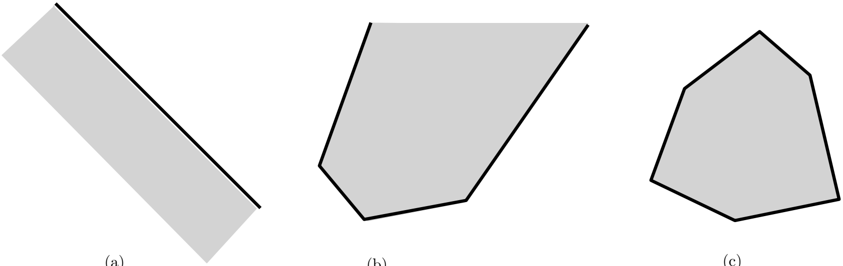
A set $HS = \{x \in \mathbb{R}^n \mid ax \leq b\}$ is called a *half-space.*

Definition

A set $P = \{x \in \mathbb{R}^n \mid a_i x \leq b_i,\ \ i = 1, \ldots, m\}$ is called a *polyhedron*.
If the polyhedron is bounded it is called a *polytope*.

So, a **polyhedron** is a set that is defined by
finitely many hyperplanes/half-spaces.
The feasible region of an LP is a polyhedron.

Linear Programming (LP)



(a) (b) (c)

- (a) Half-space
- (b) Polyhedron
- (c) Polytope

Linear Programming (LP)

If we have a **bounded optimum**, there is an optimal solution in an **extreme point** of the polyhedron defined by the constraints.

There may be **exponentially many** extreme points!

The Simplex method: Goes from one extreme point to a neighboring extreme point, **along an edge of the polyhedron**, such that the objective function value never deteriorates.

Going from one extreme point to a neighboring one is called a **pivot**.

LP-problems are **polynomially solvable** but no polynomial implementation of Simplex is known.

Yet, Simplex works **very well** in practice!

Linear Programming (LP)

Before giving the **Simplex** method, we address the following questions:

1. How do we represent an extreme point **algebraically**?
2. How do we know whether an extreme point is **optimal**?
3. How do we make a **pivot**?

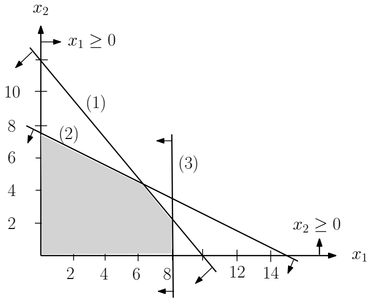
Representing extreme points algebraically

In dimension n, an extreme point is defined by an intersection of n hyperplanes.

Example

$$
\begin{array}{llrcrcrl}
\max & z = & 50x_1 & + & 45x_2 & & & \\
\text{s.t.} & & 6x_1 & + & 5x_2 & \leq & 60 & (1) \\
 & & x_1 & + & 2x_2 & \leq & 15 & (2) \\
 & & x_1 & & & \leq & 8 & (3) \\
 & & \multicolumn{5}{l}{x_1,\ x_2 \geq 0} &
\end{array}
$$

Representing extreme points algebraically



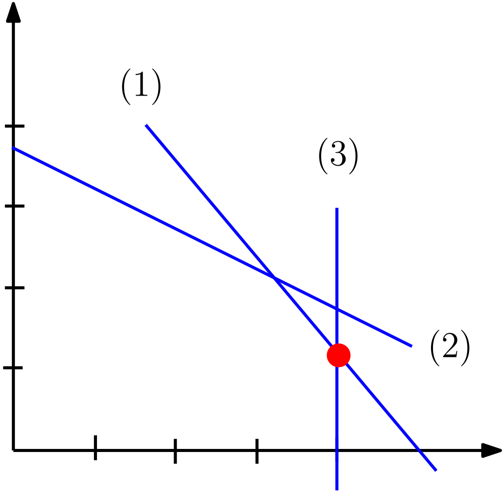
$$
\begin{array}{llrcrcrr}
\max & z = & 50x_1 & + & 45x_2 & & & \\
\text{s.t.} & & 6x_1 & + & 5x_2 & \leq & 60 & (1) \\
& & x_1 & + & 2x_2 & \leq & 15 & (2) \\
& & x_1 & & & \leq & 8 & (3) \\
& & \multicolumn{5}{l}{x_1,\ x_2 \geq 0} &
\end{array}
$$

The extreme points are:

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 2\frac{2}{5} \end{pmatrix}, \begin{pmatrix} 6\frac{3}{7} \\ 4\frac{2}{7} \end{pmatrix}, \begin{pmatrix} 0 \\ 7\frac{1}{2} \end{pmatrix}.
$$

Each of the extreme points is defined by an **intersection of two lines**.

Representing extreme points algebraically

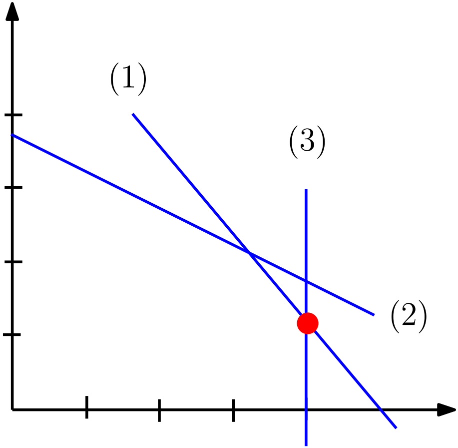


Take extreme point

$$\begin{pmatrix} 8 \\ 2\frac{2}{5} \end{pmatrix}.$$

This point is obtained by taking the intersection of the lines defining Constraints (1) & (3).

Representing extreme points algebraically



Solve:

$$\begin{array}{rcrcrr} 6x_1 & + & 5x_2 & = & 60 & (1) \\ x_1 & & & = & 8 & (3) \end{array}$$

This is a system of **two** equations in **two** variables. The solution is:

$$\begin{pmatrix} 8 \\ 2\frac{2}{5} \end{pmatrix}$$

Linear Programming (LP)

It is clear that we need to work with **equations** rather than inequalities. But, we cannot just change all constraints to equations! That would really change the problem.

For algorithmic simplicity, we will also assume, without loss of generality, that the feasible region lies in the **nonnegative orthant**.

Most optimization problems are, however, formulated with a mix of equations and inequalities, and not all variables are "naturally" nonnegative. How to proceed?

Example (LP)

$$
\begin{array}{llcllcll}
\min & x_1 & + & 2x_2 & + & x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & & & \leq & 2 \\
& x_1 & + & x_2 & + & x_3 & \geq & 1 \\
& x_1 & + & x_2 & + & x_3 & = & -4 \\
& \multicolumn{7}{l}{x_1, x_2 \geq 0, x_3 \in \mathbb{R}}
\end{array}
$$

LP: standard form

Before applying Simplex, we need to put the problem in **standard form**.

An LP is in **standard form** if:

- all variables are restricted to be **nonnegative**;
- all other constraints are **equalities**;
- each right-hand side constants b_i **is nonnegative**.

Example (LP in standard form)

$$
\begin{array}{lrcrcrcl}
\min & x_1 & & & + & 4x_3 & & \\
\text{s.t.} & 5x_1 & + & 2x_2 & - & x_3 & = & 5 \\
 & & & x_2 & + & x_3 & = & 6 \\
 & x_1 & + & x_2 & + & x_3 & = & 4 \\
 & & & & & & & x_1, x_2, x_3 \geq 0
\end{array}
$$

LP: standard form

An LP is in **standard form** if:

- all variables are restricted to be **nonnegative**;
- all other constraints are **equalities**;
- each right-hand side constants b_i **is nonnegative**.

An LP in standard form is written as:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

with A an $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m_{\geq 0}$.

We will see that
every linear optimization problem can be written in standard form!

Writing an LP in standard form

Question (1)

Suppose a constraint has a negative right-hand side constant $b_i < 0$, e.g.

$$x_1 + x_2 = -5\,.$$

How can we rewrite this into standard form?

Answer

Multiply the constraint by -1. This gives:

$$-x_1 - x_2 = 5\,.$$

Question (2)

How can we rewrite a $\leq$-constraint into an equality constraint? E.g.

$$x_1 + x_2 \leq 5\,.$$

Answer

Add a so-called **"slack variable"** s for the constraint. This variable represents the difference between the right-hand side and the left-hand side of the inequality.

In every feasible solution $s \geq 0$!

$$x_1 + x_2 + s = 5, \quad s \geq 0\,.$$

Question (3)

How can we rewrite a $\geq$-constraint into an equality constraint? E.g.

$$x_1 + x_2 \geq 5\,.$$

Answer

Subtract a so-called **"surplus variable"** s for the constraint. This variable represents the difference between the left-hand side and the right-hand side of the inequality.

In every feasible solution $s \geq 0$!

$$x_1 + x_2 - s = 5, \quad s \geq 0\,.$$

Question (4)

How can we substitute a **nonpositive** variable by a nonnegative variable?

$$\begin{aligned} \max z = 4x_1 - 2x_2 & \\ \text{s.t. } 2x_1 + 3x_2 &= 5, \\ x_1 \geq 0,\ x_2 &\leq 0\,. \end{aligned}$$

Answer

Substitute x_2 by its nonnegative counterpart in the whole problem formulation, i.e. substitute x_2 by $x_2' = -x_2 \geq 0$.

$$\begin{aligned} \max 4x_1 + 2x_2' & \\ \text{s.t. } 2x_1 - 3x_2' &= 5, \\ x_1,\ x_2' &\geq 0\,. \end{aligned}$$

Question (5)

How can we replace a free (=unrestricted) variable by a nonnegative variable? E.g.

$$2x_1 + 3x_2 = 5$$
$$x_1 \geq 0, \mathbf{x_2} \in \mathbb{R}$$

Answer

Substitute x_2 by a composition of the nonnegative part and the negative part in the whole problem formulation, i.e., $x_2 = x_2^+ - x_2^-$, with $x_2^+,\ x_2^- \geq 0$.

$$2x_1 + 3x_2^+ - 3x_2^- = 5$$
$$x_1,\ x_2^+,\ x_2^- \geq 0$$

Theorem

Each LP can be written in standard form.

Proof

1. For each constraint with negative right-hand side constant $b_i < 0$, multiply the constraint by -1.
2. Replace each constraint $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ by $\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$ with $s_i \geq 0$ a new **slack variable**.
3. Replace each constraint $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ by $\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i$ with $s_i \geq 0$ a new **surplus variable**.
4. For each variable $x_j \leq 0$, substitute $x_j = -x_j'$ with $x_j' \geq 0$.
5. For each **free variable** $x_j \in \mathbb{R}$, substitute $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$.

Exercise (1)

Write in standard form.

$$\begin{array}{llcl}
\min & x_1 + 2x_2 + x_3 & & \\
\text{s.t.} & x_1 + 2x_2 & \leq & 2 \\
& x_1 + x_2 + x_3 & \geq & 1 \\
& x_1 + x_2 + x_3 & = & -4 \\
& x_1,\ x_2 \geq 0, x_3 \in \mathbb{R} & &
\end{array}$$

Answer

First, fix 3rd constraint:

$$\begin{array}{llcl}
\min & x_1 + 2x_2 + x_3 & & \\
\text{s.t.} & x_1 + 2x_2 & \leq & 2 \\
& x_1 + x_2 + x_3 & \geq & 1 \\
& -x_1 - x_2 - x_3 & = & 4 \\
& x_1,\ x_2 \geq 0, x_3 \in \mathbb{R} & &
\end{array}$$

Exercise (1)

$$
\begin{array}{lrcrcrcl}
\min & x_1 & + & 2x_2 & + & x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & & & \leq & 2 \\
 & x_1 & + & x_2 & + & x_3 & \geq & 1 \\
 & -x_1 & - & x_2 & - & x_3 & = & 4 \\
 & \multicolumn{7}{c}{x_1,\ x_2 \geq 0, x_3 \in \mathbb{R}}
\end{array}
$$

Answer

Next, make equality constraints of 1st and 2nd constraint:

$$
\begin{array}{lrcrcrcrcrcl}
\min & x_1 & + & 2x_2 & + & x_3 & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & & & + & s_1 & & & = & 2 \\
 & x_1 & + & x_2 & + & x_3 & & & - & s_2 & = & 1 \\
 & -x_1 & - & x_2 & - & x_3 & & & & & = & 4 \\
 & \multicolumn{11}{c}{x_1,\ x_2,\ s_1,\ s_2 \geq 0, x_3 \in \mathbb{R}}
\end{array}
$$

Exercise (1)

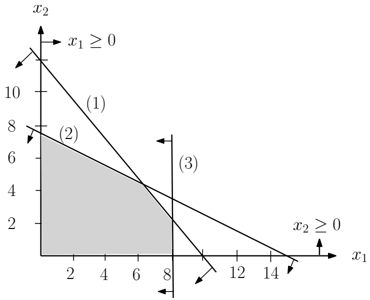
$$
\begin{array}{llllllll}
\min & x_1 & + 2x_2 & + x_3 & & & & \\
\text{s.t.} & x_1 & + 2x_2 & & + s_1 & & = & 2 \\
& x_1 & + x_2 & + x_3 & & - s_2 & = & 1 \\
& -x_1 & - x_2 & - x_3 & & & = & 4 \\
& & & x_1, \ x_2, \ s_1, \ s_2 \geq 0, x_3 \in \mathbb{R} & & & &
\end{array}
$$

Answer

Finally, substitute x_3:

$$
\begin{array}{lllllllll}
\min & x_1 & + 2x_2 & + x_3^+ & - x_3^- & & & & \\
\text{s.t.} & x_1 & + 2x_2 & & & + s_1 & & = & 2 \\
& x_1 & + x_2 & + x_3^+ & - x_3^- & & - s_2 & = & 1 \\
& -x_1 & - x_2 & - x_3^+ & + x_3^- & & & = & 4 \\
& & & x_1, \ x_2, \ x_3^+, \ x_3^-, \ s_1, \ s_2 \geq 0 & & & & &
\end{array}
$$

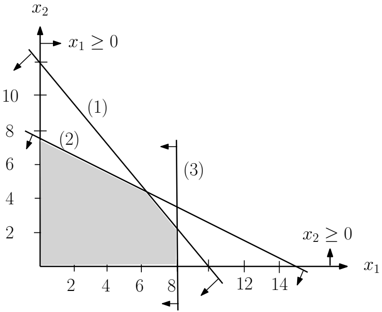
Back to earlier example



$$
\begin{array}{llrcrcrl}
\max & z = & 50x_1 & + & 45x_2 & & & \\
\text{s.t.} & & 6x_1 & + & 5x_2 & \leq & 60 & (1) \\
 & & x_1 & + & 2x_2 & \leq & 15 & (2) \\
 & & x_1 & & & \leq & 8 & (3) \\
 & & \multicolumn{5}{l}{x_1,\ x_2 \geq 0} &
\end{array}
$$

Back to earlier example

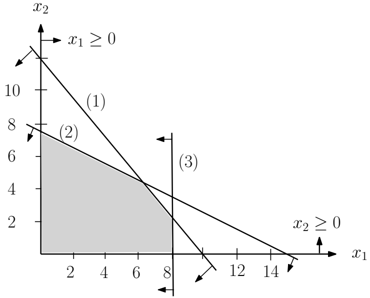
Write problem in standard form.



$$
\begin{array}{llrcrcrcrcrcl}
\max & z = & 50x_1 & + & 45x_2 & & & & & & & & \\
\text{s.t.} & & 6x_1 & + & 5x_2 & + & s_1 & & & & & = & 60 \quad (1) \\
 & & x_1 & + & 2x_2 & & & + & s_2 & & & = & 15 \quad (2) \\
 & & x_1 & & & & & & & + & s_3 & = & 8 \quad (3)
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0$$

Back to earlier example



We have 3 equations and 5 variables. If we set 2 variables equal to zero, we get a system of 3 equations in 3 variables.

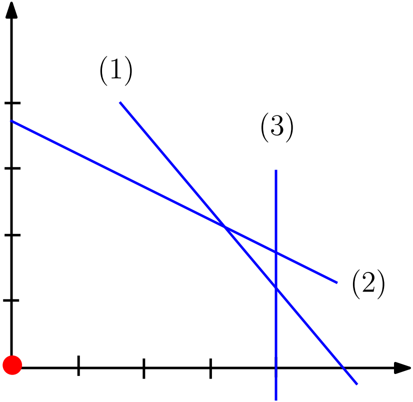
$$
\begin{array}{llrcrcrcrcrcl}
\max & z = & 50x_1 & + & 45x_2 & & & & & & & & \\
\text{s.t.} & & 6x_1 & + & 5x_2 & + & s_1 & & & & & = & 60 \quad (1) \\
& & x_1 & + & 2x_2 & & & + & s_2 & & & = & 15 \quad (2) \\
& & x_1 & & & & & & & + & s_3 & = & 8 \quad (3)
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0$$

Set $x_1 = x_2 = 0$ and solve the constraint system for other variables.

$$
\begin{array}{llllllllll}
\text{s.t.} & 6x_1 & + & 5x_2 & + & s_1 & & & = & 60 \quad (1) \\
& x_1 & + & 2x_2 & & & + & s_2 & = & 15 \quad (2) \\
& x_1 & & & & & + & s_3 & = & 8 \quad (3) \\
& & & & & x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0 & & & &
\end{array}
$$

This easily gives the feasible solution



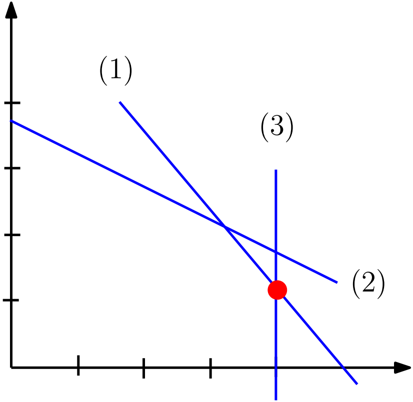
$$
x = \begin{pmatrix} 0 \\ 0 \\ 60 \\ 15 \\ 8 \end{pmatrix}
$$

Next, set $s_1 = s_3 = 0$ and solve the constraint system for other variables.

$$\begin{array}{lrcrcrcrcrcl} \text{s.t.} & 6x_1 & + & 5x_2 & + & s_1 & & & & & = & 60 \quad (1) \\ & x_1 & + & 2x_2 & & & + & s_2 & & & = & 15 \quad (2) \\ & x_1 & & & & & & & + & s_3 & = & 8 \quad (3) \\ & & & & & & & & & & & \end{array}$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0$$

This gives the feasible solution



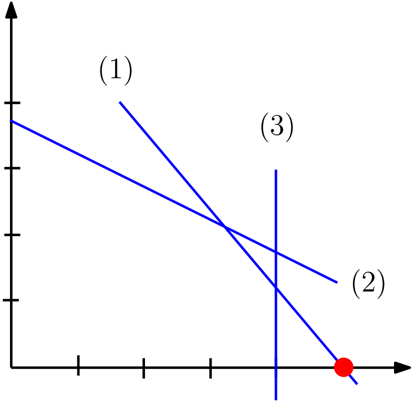
$$x = \begin{pmatrix} 8 \\ 2\frac{2}{5} \\ 0 \\ 2\frac{1}{5} \\ 0 \end{pmatrix}$$

Finally, set $x_2 = s_1 = 0$ and solve the constraint system for other variables.

$$
\begin{array}{lrcrcrcrcrcrl}
\text{s.t.} & 6x_1 & + & 5x_2 & + & s_1 & & & & & = & 60 & (1) \\
 & x_1 & + & 2x_2 & & & + & s_2 & & & = & 15 & (2) \\
 & x_1 & & & & & & & + & s_3 & = & 8 & (3) \\
 & & & & & & & & & & & & \\
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0$$

This gives the **infeasible** solution



$$x = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 5 \\ -2 \end{pmatrix}$$

Basic solutions

We have seen examples of so-called **basic** solutions. Two of them were feasible, one was infeasible.

They all represent intersections of constraints in the original space of variables.

The **basic feasible solutions** correspond to **extreme points** of the feasible region!

Basic solutions more formally

Consider an LP in standard form:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

with A an $m \times n$ matrix. Let A_j denote column j of matrix A.

Assumption

There are m linearly independent columns A_j of A.

Definition

A **basis** B of the $m \times n$ matrix A is an $m \times m$ non-singular submatrix of A, i.e., B contains m linearly independent columns of A, $B = [A_{j_1}, \ldots, A_{j_m}]$.

Basic solutions more formally

Definition

The **basic solution** corresponding to a basis B, is the vector $x \in \mathbb{R}^n$ with:

- $x_j = 0$ for all A_j that are not basis-columns (the **non-basic variables**) and
- x_j for A_j being a basis-column, is uniquely determined by the system $Bx = b$.
  (the **basic variables**).
  The solution vector x is computed as $x = B^{-1}b$.

Back to example

In the example, the basis consisted of the following set of columns:

1. $\{3,4,5\}$
2. $\{1,2,4\}$
3. $\{1,4,5\}$

Definition

A **basic feasible solution (bfs)** is a basic solution in which all variables are nonnegative.

Definition

A basic solution x is **degenerate** if one or more basic variables have value 0.

Example

$$
\begin{aligned}
x_1 + x_2 &\le 4 \\
x_1 \qquad &\le 2 \\
2\,x_1 + x_2 &\le 6 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

In standard form:

$$
\begin{array}{lcl}
x_1 + x_2 + s_1 & & = 4 \\
x_1 \qquad\qquad + s_2 & & = 2 \\
2\,x_1 + x_2 \qquad\qquad & + s_3 & = 6 \\
x_1, x_2, s_1, s_2, s_3 \ge 0 & &
\end{array}
$$

Show that (a) the bfs with basic variables $\{x_1, x_2, s_3\}$ and (b) the bfs with basic variables $\{x_1, x_2, s_1\}$ are **degenerate**.

Example (a)

$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 \quad &\le 2 \\ 2\,x_1 + x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In standard form:

$$\begin{aligned} x_1 + x_2 + s_1 \qquad\qquad &= 4 \\ x_1 \qquad + s_2 \qquad &= 2 \\ 2\,x_1 + x_2 \qquad\qquad + s_3 &= 6 \\ x_1, x_2, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

Show that the bfs with basic variables $\{x_1, x_2, s_3\}$ is **degenerate**.

Answer

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Basic variable $s_3 = 0$, so the solution is degenerate.

Example (b)

$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 \quad\quad &\le 2 \\ 2\,x_1 + x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In standard form:

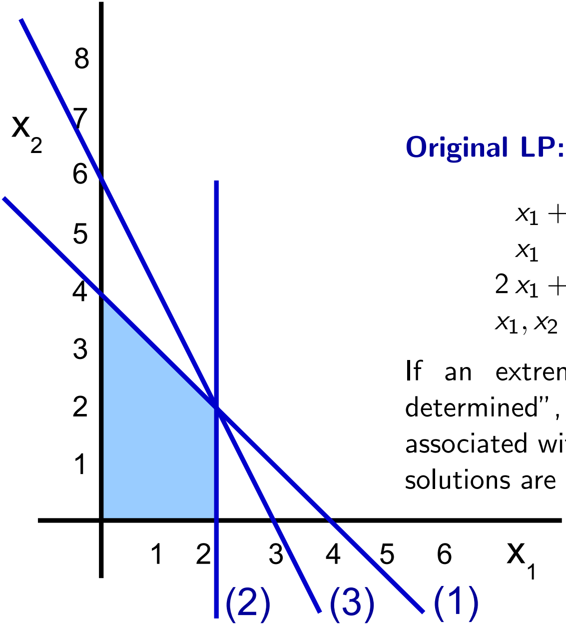
$$\begin{array}{lllll} x_1 + x_2 + s_1 & & & = 4 \\ x_1 & + s_2 & & = 2 \\ 2\,x_1 + x_2 & & + s_3 & = 6 \\ x_1, x_2, s_1, s_2, s_3 \ge 0 & & & \end{array}$$

Show that the bfs with basic variables $\{x_1, x_2, s_1\}$ is **degenerate**.

Answer

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Basic variable $s_1 = 0$, so the solution is degenerate.



Original LP:

$$\begin{aligned} x_1 + x_2 &\le 4 \quad &(1)\\ x_1 &\le 2 \quad &(2)\\ 2\,x_1 + x_2 &\le 6 \quad &(3)\\ x_1, x_2 &\ge 0 \end{aligned}$$

If an extreme point is "over-determined", more than one bfs is associated with that point. These solutions are degenerate.

Theorem

If two different bases have the same basic feasible solution x, then x is degenerate.

Theorem

Given a linear programming problem with the constraints in standard form, $F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$, the following statements hold.

(i) If there exists a vector $x \in F$, then there exists a basic feasible solution $\bar{x} \in F$.

(ii) If there exists a finite optimal solution $x \in F$, then there exists an optimal basic feasible solution $x^* \in F$.

Hence, when searching for an optimal solution, we can restrict to bfss.

Polyhedra and LP

- Each LP with an optimal solution has an **optimal solution** that is an **extreme point** of the corresponding polyhedron P.
- Given a polyhedron P and the corresponding feasible set $F = \{x \in R^n \mid Ax = b,\ x \geq 0\}$ written in standard form. The following holds.
  - Each bfs of F corresponds to an extreme point of P.
  - Each extreme point of P corresponds to one (or possibly more, if the bfs is degenerate) bfs of F.
- In theory, you can find an optimal solution by trying each possible basis.
- The **Simplex algorithm** (introduced next lecture) pivots from bfs to bfs such that the objective function value never deteriorates.

To Do

Solve exercises 4.1, 4.2.