DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

Exam IN4344 Advanced Algorithms – Part II

November 7, 2022 | 9:00–12:00

- This is a closed-book individual examination with 6 questions worth 50 points in total.
- If your score is n points, then your grade for this exam part will be $1 + \frac{9}{50}n$.
- Use of (graphical) calculators is not permitted.
- Write clearly, use correct English, and avoid verbose explanations. Giving irrelevant information may lead to a reduction in your score. *Almost all question parts can be answered in a few lines!*
- This exam covers Chapters 10 of Kleinberg, J. and Tardos, E. (2005), *Algorithm Design*, all information on the slides of the Part II of the course, everything discussed in lectures of Part II, and the supplemental study material provided via BrightSpace.
- The total number of pages of this exam is 5 (excluding this front page).
- Exam prepared by M.M. de Weerdt. ©2022 TU Delft. (With thanks to Leo van Iersel.)

1. (8 points) The *vaccination problem* has as input parameters d and k both in $\mathbb{N}$, and an undirected graph $G = (V, E)$. The output of the decision problem is YES iff a set $C \subseteq V$ exists with $|C| \leq k$ such that for each $v \in V \setminus C$ holds that at most d neighbors of v are not in C.

   The interpretation of the problem is that the vertices of the graph G represent persons and its edges represent contacts (potential virus transmissions). The goal is to vaccinate at most k people in such a way that each person that is not vaccinated has at most d contacts that are also not vaccinated (to limit the speed the virus spreads).

   Describe a bounded search tree algorithm for this problem (include bases cases and explain why the answers to the subproblems provide an answer to the parent problem). Provide and explain an analysis of the runtime and explain why this problem is FPT.

2. Consider the following problem called Colourful Path, where we look for paths that contain all colours exactly once.

   Let $C = \{1, 2, \ldots, k\}$ be a set of k colours, $G = (V, E)$ an undirected graph and $f : V \to C$ a function that assigns a colour $f(v) \in C$ to each vertex in V. Let all of these be given to us. For $S \subseteq C$, define an S-path to be a path P such that it contains for each colour $i \in S$ exactly one vertex $v \in P$ with $f(v) = i$. The decision problem we consider is: does G contain a C-path?

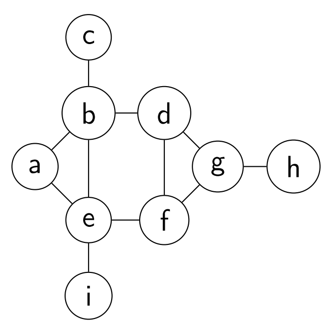
   (a) (1 point) If a C-path exists, what is its length?

   (b) (3 points) Let $S \subseteq C$ and $v \in V$ be given. Prove that there exists an S-path starting at v if and only if there exists a $S \setminus \{f(v)\}$-path starting at a neighbour of v.

   (c) (4 points) Give a recursive dynamic programming function for Colourful Path with a runtime not exponential in the size of the graph. (Hint: do not forget the base case and how you would use the function to find the solution.)

   (d) (1 point) Determine the runtime of your algorithm.

3. (8 points) Give a tree decomposition of the following graph that has the lowest width you can find, *and* explain why this is a correct tree decomposition (hint: you don't need to give the definition itself, but you may use it for your explanation). (There will be $|w' - w|$ point deduction if the width of your tree decomposition is w' while the treewidth is w.)



4. Let $G = (V, E)$ be a clique.

   (a) (2 points) What is the treewidth $tw(G)$ of G?

   (b) (6 points) Give a proof of your answer.

5. (7 points) Consider the well-known problem of finding a maximum-weight independent set in an undirected graph $G = (V, E)$ with for every vertex $i \in V$ a weight w_i.

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Consider the exact decision diagram constructed using some ordering of vertices $i \in V$ and the following recursive function:

$$\mathrm{OPT}_i(S) = \begin{cases} \max\{\mathrm{OPT}_{i+1}(S \setminus \{i\}), \mathrm{OPT}_{i+1}(S \setminus N^*(i)) + w_i\} & \text{if } i \in S \\ \mathrm{OPT}_{i+1}(S) & \text{otherwise} \end{cases}$$

Where $N^*(i)$ denotes the set of i and its neighbors. Additionally, the base case $\mathrm{OPT}_i(\emptyset) = 0$ for every $i \in V$.

Give a possible merge operator $\oplus$ to construct a relaxed diagram with smaller width, explain how this is used and why/when this is a relaxation.

6. Consider the following parameterized problem of Set Splitting. We are given a finite set U (the "universe"), a set $\mathcal{F}$ of subsets of U and an integer k. Can we colour the elements of U with two colours such that at least k sets of $\mathcal{F}$ are bi-chromatic (i.e. contain vertices of both colours)?
   (a) (2 points) Consider the following reduction rule. If $S \in \mathcal{F}$ is size 1, then delete S from $\mathcal{F}$. The parameter k remains the same. Show that the reduced instance has a solution *if and only if* the original instance has a solution.
   (b) (4 points) Prove that, if the condition of the previous reduction rule does not apply, and there exists an $x \in U$ that is in at least k sets in $\mathcal{F}$, then the answer is YES.
   (c) (4 points) Consider the following reduction rule. If $S \in \mathcal{F}$ has $|S| \geq 2k$ then delete S from $\mathcal{F}$ and reduce k by 1. Show that the reduced instance has a solution *if and only if* the original instance has a solution.

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