Partial Exam 1, IN4344 Advanced Algorithms

October 4, 2022, 18:30-21:30

This partial exam consists of 6 questions on 8 pages. The total number of points is 50 . If your score is n points, your grade will be $1 + \frac{9 \cdot n}{50}$. You are allowed to bring your own pens and a standard, non-graphical, calculator. No other devices such as smart phones, tablets, smartwatches, are allowed, nor are notes and books.

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1. You want to solve a linear *maximization* problem with the Simplex algorithm. After a couple of iterations, the Simplex tableau looks as follows:

basis	$\bar{b}$	x_1	x_2	s_1	s_2
s_1	4			1	1
x_1	2	1	-1		1
$-z$	-4		3		-2

   (a) (3 points) Is the current basic solution optimal? Motivate your answer.

   Solution: No, since $\bar{c}_2 > 0$.

   (b) (3 points) Does the problem have a well-defined optimal solution? Motivate your answer.

   Solution: No, since $\bar{a}_{i2} \leq 0$ for $i = 1, 2$, and x_2 is a possible entering basis variable, we know that the solution is unbounded.

   (c) (3 points) Consider the following linear *minimization* problem.

$$
\begin{aligned}
\min z = -3x_1 - x_2 & \\
\text{s.t. } x_1 - x_2 &\leq -1 \\
-x_1 - x_2 &= -3 \\
2x_1 + x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

   Formulate the problem in standard form.

Solution:

$$\begin{array}{llllllllll}
\min z = & -3x_1 & - & x_2 & & & & & & \\
\text{s.t.} & -x_1 & + & x_2 & - & s_1 & & & = & 1 \\
& x_1 & + & x_2 & & & & & = & 3 \\
& 2x_1 & + & x_2 & & & + & s_3 & = & 4 \\
& x_1, & & x_2, & & s_1 & & s_3 & \geq & 0
\end{array}$$

(d) (3 points) Formulate the Simplex Phase 1 problem and express the Phase 1 objective function in non-basic variables.

Solution:

$$\begin{array}{llllllllllllll}
\min w = & a_1 & + & a_2 & & & & & & & & & & \\
\text{s.t.} & -x_1 & + & x_2 & - & s_1 & & & + & a_1 & & & = & 1 \\
& x_1 & + & x_2 & & & & & & & + & a_2 & = & 3 \\
& 2x_1 & + & x_2 & & & + & s_3 & & & & & = & 4 \\
& x_1, & & x_2, & & s_1, & & s_3, & & a_1, & & a_2 & \geq & 0
\end{array}$$

From the first and second equation we can derive $a_1 = 1 + x_1 - x_2 + s_1$ and $a_2 = 3 - x_1 - x_2$, which gives the Phase 1 objective function expressed in non-basic variables:

$$w = 4 - 2x_2 + s_1 \,.$$

2. (a) (3 points) Formulate the following problem as a linear optimization problem:

$$\begin{array}{lll}
\max & \min\{x, y\} & \\
\text{s.t.} & x + 3y & \geq 7 \\
& 3x - 2y & \leq 9 \\
& x, y & \geq 0
\end{array}$$

Solution:

$$\begin{array}{ll}
\max & z \\
\text{s.t.} & z \leq x \quad (\text{ or } x - z \geq 0) \\
& z \leq y \quad (\text{ or } y - z \geq 0) \\
& x + 3y \geq 7 \\
& 3x - 2y \leq 9 \\
& x, y, z \geq 0
\end{array}$$

(b) (3 points) Formulate the following problem as an integer linear optimization problem:

$$\begin{array}{ll}
\min & 4x + 5y \\
\text{s.t.} & 2x + 3y \leq 5 \quad \text{or} \quad 3x - 4y \leq 7 \\
& x, y \in \mathbb{Z}_{\geq 0}
\end{array}$$

Solution:

$$\begin{array}{ll} \min & 4x + 5y \\ \text{s.t.} & 2x + 3y \leq 5 + Mz \\ & 3x - 4y \leq 7 + M(1 - z) \\ & x, y \in \mathbb{Z}_{\geq 0} \\ & z \in \{0, 1\}\,, \end{array}$$

for sufficiently large values of $M > 0$.

3. Given is the following optimization problem:

$$\begin{array}{lrcrcr} \text{max } z = & 8x_1 & + & 4x_2 & & \\ \text{s.t.} & 3x_1 & - & x_2 & \leq & 7 \\ & 9x_1 & + & 5x_2 & \leq & -2 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \leq & 0 \end{array}$$

(a) (5 points) Formulate the corresponding dual problem.

Solution:

$$\begin{array}{lrcrcr} \min w = & 7\pi_1 & - & 2\pi_2 & & \\ \text{s.t.} & 3\pi_1 & + & 9\pi_2 & \geq & 8 \\ & -\pi_1 & + & 5\pi_2 & \leq & 4 \\ & \pi_1,\ \pi_2 \geq 0 & & & & \end{array}$$

(b) (2 points) Formulate all complementary slackness conditions for the specific primal-dual pair.

Solution:

$$\begin{aligned} \pi_1(7 - 3x_1 + x_2) &= 0 \\ \pi_2(-2 - 9x_1 - 5x_2) &= 0 \\ x_1(3\pi_1 + 9\pi_2 - 8) &= 0 \\ x_2(4 + \pi_1 - 5\pi_2) &= 0 \end{aligned}$$

(c) (2 points) The dual optimal solution is $(\pi_1^*,\ \pi_2^*)^\mathsf{T} = (1/6,\ 5/6)$, with dual optimal value $w^* = -0.5$. Use this solution and the complementary slackness conditions in (b) to determine the primal optimal solution. Give both x^* and z^*.

Solution: Since both π_1^* and π_2^* are non-zero, we know that both primal constraints have to be satisfied with equality. Solving the system

$$\begin{aligned} 3x_1 - x_2 &= 7 \\ 9x_1 + 5x_2 &= -2 \end{aligned}$$

yields $(x_1^*,\ x_2^*)^\mathsf{T} = (11/8,\ -23/8)$ with objective value $z^* = -0.5$

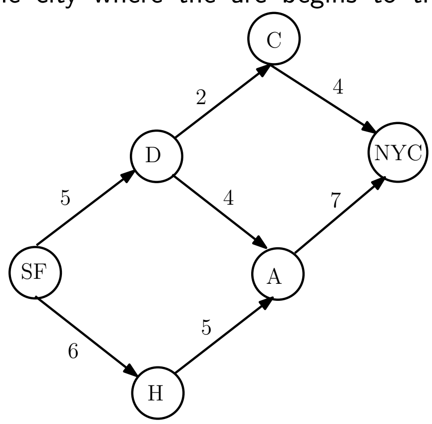
From	To	# of seats
SF	Denver	5
SF	Houston	6
Denver	Atlanta	4
Denver	Chicago	2
Houston	Atlanta	5
Atlanta	NYC	7
Chicago	NYC	4

Table 1: Seat availability.

4. Suppose a large group of people needs to go from San Francisco (SF) to New York City (NYC) on a short notice. All direct flights are booked, but it is till possible to go from SF to NYC by changing planes one or more times. The seat availability on the various flight legs are as shown in Table 1.

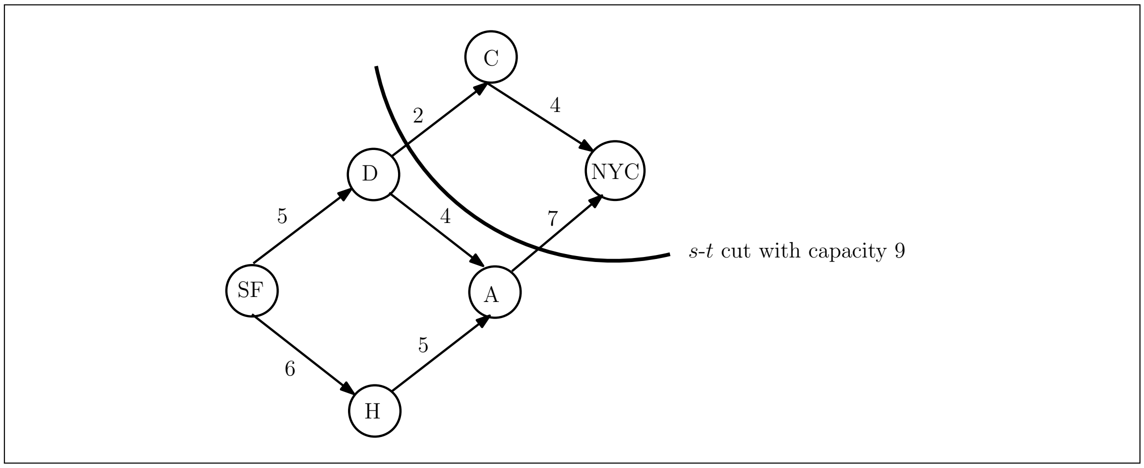
   (a) (3 points) Assume all connections as suggested in the table are possible to reach. Suppose we want to send as many people as possible from our group from SF to NYC. Give a directed graph that models the problem for this specific input. Explain the notation you use and state in addition which optimization problem you need to solve on this graph.

   Solution: This can be modeled as a max flow problem with starting vertex s being SF and terminal vertex t being NYC. The intermediate vertices are labeled with the first letter of the respective city name, and the number of the arcs is the number of seats on the flight leg from the city where the arc begins to the city where the arc ends.



   (b) (3 points) Use weak duality to argue that no more than nine people can reach NYC from SF using the mentioned possible connections. Give a short explanation.

   Solution: Any s-t cut in the directed graph gives an upper bound on the max flow through the network from s to t (in this case from SF to NYC). Take the partition of the vertices: $(W, \overline{W})$ with $W =$(SF, D, H, A). The capacity of the cut is $2 + 7 = 9$.



5. Consider the ILP below.

$$\begin{array}{llrcrcr} z_{IP} = \max & z = & 5x_1 & + & 2x_2 & & \\ \text{s.t.} & & 3x_1 & + & x_2 & \leq & 12 \\ & & x_1 & + & x_2 & \leq & 5 \\ & & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & & \multicolumn{5}{l}{x_1, x_2 \in \mathbb{Z}} \end{array}$$

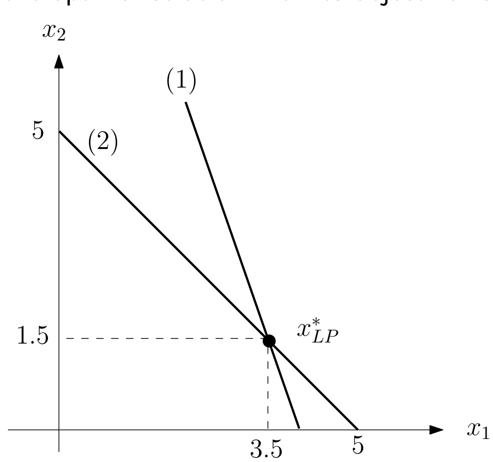
The optimal solution to the LP relaxation is:

$$x^*_{LP} = \begin{pmatrix} 3.5 \\ 1.5 \end{pmatrix} \qquad \text{with } z^*_{LP} = 20.5\,.$$

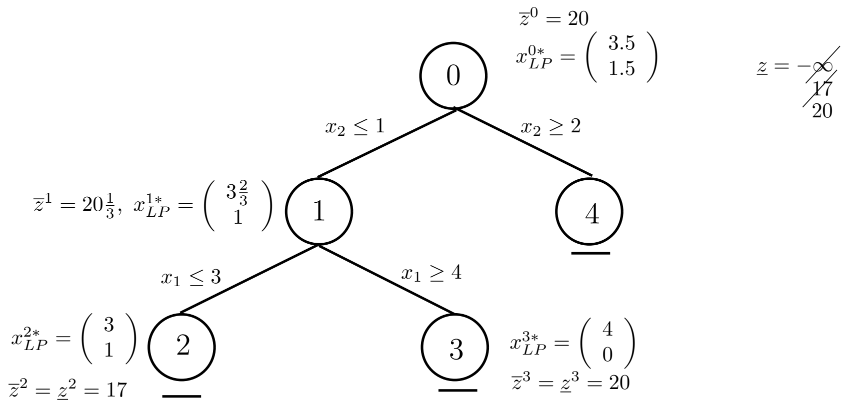
(a) (6 points) Determine one optimal solution to the ILP with the **branch & bound** method. Use the following search strategy:

- Whenever given a choice, branch first on variable x_2.
- Choose the $\leq$-branch first.
- Go depth first.

LP relaxations may be solved graphically. For simplicity, the feasible region is illustrated below. Do not forget to give the optimal solution with its objective value!



Solution: From the solution of the LP-relaxation we get a first upper bound $\overline{z} = \lfloor 20.5 \rfloor = 20$. The lower bound at the root node is $\underline{z} = -\infty$. The B&B tree looks as follows:



The optimal solution is $(x^*)^\mathsf{T} = (4, 0)$ with $z^* = 20$. Node 2 and 3 are pruned by optimality, node 4 by bound, since the upper bound of node 0 is 20, and the best lower bound is 20 as well. So there is no hope to get a better solution under node 4.

(b) (3 points) The Simplex tableau of the optimal solution to the LP-relaxation of the problem in (a) is given below.

basis	$\overline{b}$	x_1	x_2	s_1	s_2
x_1	3.5	1		1/2	−1/2
x_2	1.5		1	−1/2	3/2
$-z_{LP}$	−20.5			−3/2	−1/2

Derive a Gomory cut from the first row of the optimal Simplex tableau (the row in which variable x_1 is basic).

Solution: The first row reads:

$$x_1 + \frac{1}{2}s_1 - \frac{1}{2}s_2 = 3\frac{1}{2}\,.$$

Splitting the coefficients in integer and fractional parts yields:

$$x_1 + \frac{1}{2}s_1 + (-1 + \frac{1}{2})s_2 = (3 + \frac{1}{2})\,.$$

Integer parts left, and fractional parts right yields:

$$x_1 - s_2 - 3 = \frac{1}{2} - \frac{1}{2}s_1 - \frac{1}{2}s_2\,.$$

The Gomory cut is:

$$-\frac{1}{2}s_1 - \frac{1}{2}s_2 \leq -\frac{1}{2}\,.$$

(c) (4 points) Add the Gomory cut to the optimal Simplex tableau and re-optimize using dual simplex. What is the new optimal solution? Give both x^* and z^*_{LP}.

Solution: After adding a slack variable s_3 we obtain $-(1/2)s_1 - (1/2)s_2 + s_3 = -(1/2)$. Add this row to the Simplex tableau:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_1	3.5	1		$1/2$	$-1/2$	
x_2	1.5		1	$-1/2$	$3/2$	
s_3	-0.5			$-1/2$	$-1/2$	1
$-z_{LP}$	-20.5			$-3/2$	$-1/2$	

The leaving basic variable is s_3. The entering variable is s_2 since

$$\min\left\{\left|\frac{-3/2}{-1/2}\right|, \left|\frac{-1/2}{-1/2}\right|\right\} = \left|\frac{-1/2}{-1/2}\right| .$$

After the pivot we obtain the new tableau:

basis	$\bar{b}$	x_1	x_2	s_1	s_2	s_3
x_1	4	1		1		-1
x_2	0		1	-2		3
s_2	1			1	1	-2
$-z_{LP}$	-20			-1		-1

We have now reached an optimal solution, since it is primal feasible. The new optimal solution is $(x^*)^\mathsf{T} = (4, 0)$ with $z^*_{LP} = 20$. Notice that this is also the optimal solution to the ILP.

6. (4 points) Answer the following questions with True or False accompanied by a brief answer (1-2 sentences).

   (a) Suppose you have an integer minimization problem IP and that you have designed a polynomial time algorithm A, such that for every instance it holds that $z(A) \leq \rho \cdot \bar{z}_D$. Then $z(A) \leq \rho \cdot z_{IP}$. Here $z(A)$ is the value of the integer feasible solution produced by A, z_{IP} is the optimal integer solution to IP, ρ is a constant ≥ 1, and $\bar{z}_D$ is the value of a feasible dual solution, where the dual is taken with respect to the LP-relaxation of IP.

   (b) Consider a primal-dual pair in which the primal is a maximization problem. When the dual is infeasible, the only possibility is that the primal is infeasible as well.

   (c) Suppose you have solved the shortest path problem to optimality. Then all arcs (a, b) used in a shortest path satisfy $\pi_a - \pi_b = \ell_{ab}$, where π_v is the optimal dual variable corresponding to vertex $v \in V$, and ℓ_{ab} is the length of arc (a, b).

   (d) Given is a totally unimodular matrix A. Then the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ has integral extreme points for all vectors $b \in \mathbb{R}^n$.

Solution:

(a) True, since $\bar{z}_D \leq z_{LP} \leq z_{IP}$, where z_{LP} is the optimal solution to the LP relaxation of IP.

(b) False, the primal can also be unbounded.

(c) True, this follows from the complementary slackness constraints $f_{ab} \cdot (\ell_{ab} - \pi_a + \pi_b) = 0$.

(d) False, the statement is only true if $b \in \mathbb{Z}^n$.

End of exam