Selected solutions Module 9

Exercise 12.3.

(a) Matrix X is TUM by Theorem 12.3 (or note that X is the incidence matrix of a bipartite graph). Matrix Y is not TUM because Y has a submatrix

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

with determinant -2.

(b) A^T is TUM by the observation that $det(A^T) = det(A)$ and the same holds for each square submatrix. It follows directly that $-A^T$ is also TUM.

Now take a square submatrix C of $\begin{bmatrix} I \\ A^T \\ -A^T \end{bmatrix}$.

If C contains only elements of A^T, or only of $-A^T$, or only of I then it is clear that $det(C) \in \{0, 1, -1\}$.

If C contains elements of both A^T and $-A^T$ then we can replace the elements of $-A^T$-rows by the corresponding elements of A^T. This can only change the sign of $det(C)$.

The case that we still need to consider is that C contains elements of A^T and I. Assume that C is non-singular. Then C looks like:

$$C = \begin{bmatrix} A^T_{\text{sub}} \\ I_{\text{sub}} \end{bmatrix}$$

By column permutations we can get the following:

$$\begin{bmatrix} B & D \\ 0 & I_k \end{bmatrix}$$

with B a square submatrix of A^T, so $det(B) \in \{-1, 1\}$. So $det(C) \in \{1, -1\}$ because the column permutations can only change the sign of the determinant.

(c) From the known result follows directly that the primal problem has an integral optimal solutions (assuming it has an optimal solution). The dual problem can be rewritten to:

$$\begin{array}{ll} \min & b^T\pi \\ \text{s.t.} & -I\pi \leq 0 \\ & A^T\pi \leq c \\ & -A^T\pi \leq -c \end{array}$$

From (b) follows that the matrix

$$\begin{bmatrix} -I \\ A^T \\ -A^T \end{bmatrix}$$

is TUM. The vector c is integral and so it now follows from the given result that also the dual problem has an integral optimal solutions (assuming it has an optimal solution).

Exercise 2.7. Decision variables:
x_{in} is the number of instruments of type i that is assigned to tray n
$y_{ns} = 1$ if tray n if used for surgery s and $y_{ns} = 0$ otherwise.
$z_n = 1$ if tray n is used at all and $z_n = 0$ otherwise.
q_{ins} is the number of instruments of type i on tray n if tray n is used for surgery s, and $q_{ins} = 0$ if tray n is not used for surgery s (so $q_{ins} = x_{in}y_{ns}$ but we need to enforce this by linear constraints).

The objective function:

$$\min \quad \sum_{n=1}^{N} Fz_n + S\sum_{s=1}^{T} n_s \sum_{n=1}^{N} y_{ns}$$

The first constraint states that for each surgery all necessary instruments should be available.

$$\sum_{n=1}^{N} q_{ins} \geq r_{is} \quad \text{for } s = 1, \ldots, T \text{ and } i = 1, \ldots, I$$

The second constraint states that the instruments assigned to a tray should fit on that tray.

$$\sum_{i=1}^{I} f_i x_{in} \leq 1 \quad \text{for } n = 1, \ldots, N$$

The third constraint states that each tray to which instruments are assigned should be used (and so fixed costs need to be paid for it).

$$x_{in} \leq M z_n \quad \text{for } i = 1, \ldots, I \text{ and } n = 1, \ldots, N$$

with M a large enough number.

Now we still need to enforce that $q_{ins} = x_{in} y_{ns}$. We can do that as follows with linear constraints:

$$\begin{aligned} q_{ins} &\leq M y_{ns} \\ q_{ins} &\leq x_{in} \\ q_{ins} &\geq x_{in} + M(y_{ns} - 1) \end{aligned}$$

Finally, we need the integrality constraints and binarity constraints.

$$z_n, y_{ns} \in \{0, 1\} \quad \text{for } n = 1, \ldots, N \text{ and } s = 1, \ldots, T$$

$$x_{in}, q_{ins} \geq 0 \text{ for } n = 1, \ldots, N, s = 1, \ldots, T \text{ and } i = 1, \ldots, I$$

$$x_{in}, q_{ins} \in \mathbb{Z} \text{ for } n = 1, \ldots, N, s = 1, \ldots, T \text{ and } i = 1, \ldots, I$$