Selected solutions Module 7

Exercise 9.4.

Replace ach vertex v with a capactiy c_v by two vertices v' and v'' with an arc from v' to v''. All incoming arcs of v become incoming arcs of v' and all outgoing arcs of v become outgoing arcs of v''. We given the arc (v', v'') the capacity c_v. We can then use Ford-Fulkerson to solve the problem.

Exercise 9.6.

Make a directed graph with a vertex u_i for each male student and a vertex v_j for each female student and two additional vertices s and t. Add arcs from s to each u_i and from each v_j to t. In addition, add an arc from u_i to v_j if the i-th male and the j-th female student want to collaborate. Give each arc capacity 1.

Exercise 10.2.

- (i) $\Rightarrow$ (ii): If there exists a shortest s-t walk, then the primal LP has a bounded optimum, so the dual LP also has a bounded optimum (strong duality theorem), and hence a feasible solution.
- (ii) $\Rightarrow$ (iii): Suppose there exists a cycle $(v_1, v_2, \ldots, v_k = v_1)$ with negative total length, then from the dual constraints it follows that $\pi_i \leq \pi_j + c_{ij}$ for each arc (i, j) on the cycle. So $\pi_1 \leq \pi_k + C = \pi_1 + C$ with C the sum of the lengths of the arcs on the circuit. This is not possible when $C < 0$ and so the dual has no feasible solution.
- (iii) $\Rightarrow$ (i): The exists a directed path from each vertex to t and so there exists an s-t walk. If there is no *shortest* s-t walk, then there must be unbounded short s-t walks. Hence, there must be a directed cycle of negative length.