Selected solutions Module 11

Exercise 16.6. 0,1-Knapsack $\in$ NP because, given as a certificate a vector $x \in \{0,1\}^m$, it can be verified in polynomial time whether $\sum_{j=1}^{m} c_j x_j = k$.

We prove that 0,1-Knapsack is NP-hard by showing a reduction from Partition.

Let $(a_1, \ldots a_n)$ be an arbitrary instance of Partition. We construct an instance of 0,1-Knapsack as follows. Let $m := n$ and $c_j := a_j$ for all $j \in \{1, \ldots m\}$. Let $k := \frac{1}{2} \sum_{j=1}^{n} a_j$.

It remains to show that $(a_1, \ldots a_n)$ is a yes-instance of Partition if and only if $(c_1, \ldots, c_m, k)$ is a yes-instance of 0,1-Knapsack.

First assume that $(a_1, \ldots a_n)$ is a yes-instance of Partition. Then there exists a subset $S \subseteq \{1, \ldots n\}$ such that $\sum_{j \in S} a_j = \sum_{j \notin S} a_j$, which implies that $\sum_{j \in S} a_j = \frac{1}{2} \sum_{j=1}^{n} a_j$. We can define a vector $x \in \{0,1\}^m$ by setting $x_j = 1$ if and only if $j \in S$ for all $j \in \{1, \ldots m\}$. Now

$$\sum_{j=1}^{m} c_j x_j = \sum_{j \in S} c_j = \sum_{j \in S} a_j = \frac{1}{2} \sum_{j=1}^{n} a_j = k$$

and so $(c_1, \ldots, c_m, k)$ is a yes-instance of 0,1-Knapsack.

For the other direction, assume that $(c_1, \ldots, c_m, k)$ is a yes-instance of 0,1-Knapsack. Then there exists a vector $x \in \{0,1\}^m$ such that $\sum_{j=1}^{m} c_j x_j = k$. We can define a set $S \subseteq \{1, \ldots, n\}$ by setting $S = \{j \mid x_j = 1\}$. Now

$$\sum_{j \in S} a_j = \sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{m} c_j x_j = k = \frac{1}{2} \sum_{j=1}^{n} a_j$$

which implies that $\sum_{j \in S} a_j = \sum_{j \notin S} a_j$. Thus $(a_1, \ldots a_n)$ is a yes-instance of Partition.

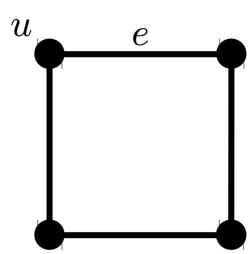
Exercise 16.7. Genes $\in$ NP because, given as certificate a set $J \subseteq \{1, \ldots, m\}$, it can be verified in polynomial time whether $|J| \leq k$ and whether there exists a $j \in J$ with $M_{ij} = 1$ for each $i \in \{1, \ldots, n\}$.

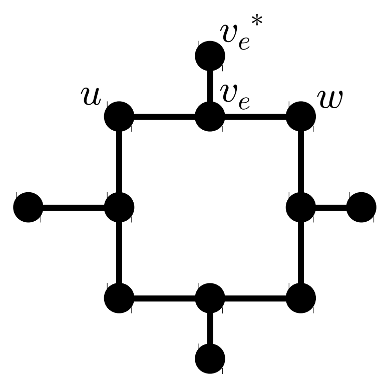
We prove that GENES is NP-hard by showing a reduction from VERTEX COVER DECISION.

Let $G = (V, E), k'$ be an instance of VERTEX COVER DECISION. Let M be the transposed incidence matrix of G and $k := k'$. Then G has a vertex cover of cardinality k' if and only if there exists a set $J \subseteq \{1, \ldots, m\}$ of cardinality k such that for all $i \in \{1, \ldots, n\}$ there exists at least one $j \in J$ with $M_{ij} = 1$.

Exercise 16.8. NEIGHBOUR INCIDENT $\in$ NP because, given as certificate a set $V' \subseteq V$, it can be verified in polynomial time whether each edge in E is incident with a vertex in V' or incident with a neighbour of a vertex in V'.

We prove NP-hardness by a reduction from VERTEX COVER DECISION. Given an instance $(G = (V, E), k)$ of VERTEX COVER DECISION, we create an instance $(G^* = (V^*, E^*), k)$ of NEIGHBOUR INCIDENT by adding, for each edge $e = \{u, w\}$, two vertices v_e, v_e^* and replacing the edge e by three edges $\{u, v_e\}$, $\{v_e, v_e^*\}$, $\{v_e, w\}$. See the example below.





First assume that G has a vertex cover V' with cardinality at most k. Then $V' \subseteq V^*$ and $|V'| \leq k$. For each edge $e = \{u, w\} \in E$, at least one of u and w is in V' since V' is a vertex cover. Hence, in G^*, v_e is a neighbour of a vertex in V'. So all three the edges $\{u, v_e\}$, $\{v_e, v_e^*\}$, $\{v_e, w\}$ are incident with a neighbour of a vertex in V'. So each edge in E^* is incident with a vertex in V' or with a neighbour of a vertex in V' (in G^*).

Now assume there exists a set $V' \subseteq V^*$ with $|V'| \leq k$ such that each edge in E^* is incident with a vertex in V' or with a neighbour of a vertex in V'

(in G^*). For each edge $e = \{u, w\}$ of the original graph G, at least one of the vertices u, w, v_e, v_e^* is in V' because otherwise the edge $\{v_e, v_e^*\}$ would not be incident with a vertex in V' or with a neighbour of a vertex in V'. If u or w is in V' then we put the same vertex in K. If v_e or v_e^* is in V' then we put one of u and w in K, arbitrarily. We do this for each edge $e = \{u, w\}$ of G. Then we have $|K| \leq k$ and moreover that for each edge $e = \{u, w\}$ of G at least one of u and w is in K. Hence, K is a vertex cover of G with cardinality at most k.