Artificial Intelligence Techniques
IN4010-12
Planning in partially observable environments

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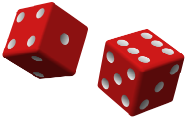
Outline for today

Markov Decision Processes

Sequential decision making under uncertainty

- Uncertainty is abundant in **real-world planning** domains.
- **Bayesian** approach $\Rightarrow$ probabilistic models.



Main assumptions:

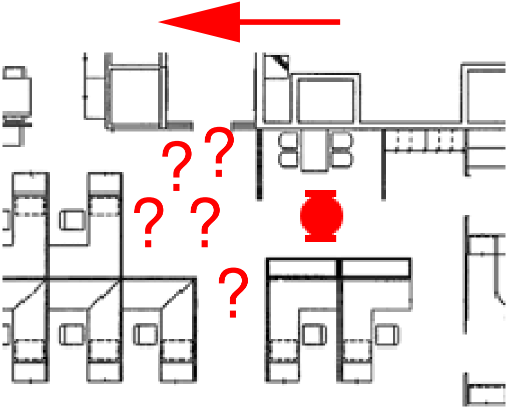
Sequential decisions: problems are formulated as a sequence of “independent” decisions;

Markovian environment: the state at time t depends only on the events at time $t-1$;

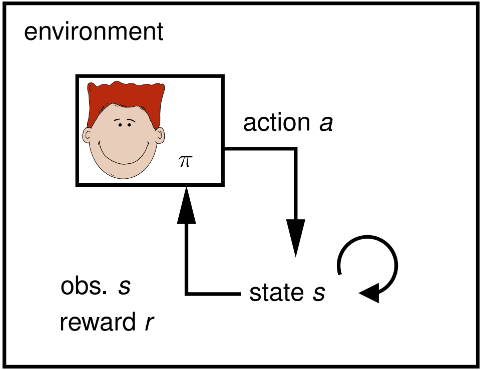
Evaluative feedback: use of a reinforcement signal as performance measure (reinforcement learning);

Transition model

- For instance, robot motion is inaccurate.
- Transitions between states are **stochastic.**
- $p(s'|s, a)$ is the probability to jump from state s to state s' after taking action a.



MDP Agent



Optimality criterion

For instance, agent should maximize the value

$$E\left[\sum_{t=0}^{h} \gamma^t R_t\right], \quad (1)$$

where

- h is the planning horizon, can be finite or ∞
- γ is a discount rate, $0 \leq \gamma < 1$

Reward hypothesis (Sutton and Barto, 1998):
All goals and purposes can be formulated as the maximization of the cumulative sum of a received scalar signal (reward).

Discrete MDP model

Discrete Markov Decision Process model (Puterman, 1994; Bertsekas, 2000):

- Time t is discrete.
- State space S.
- Set of actions A.
- Reward function $R : S \times A \mapsto \mathbb{R}$.
- Transition model $p(s'|s, a)$, $T_a : S \times A \mapsto \Delta(S)$.
- Initial state s_0 is drawn from $\Delta(S)$.

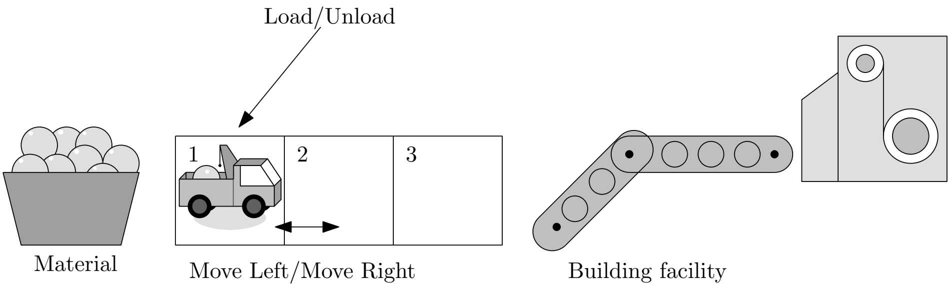
The Markov property entails that the next state s_{t+1} only depends on the previous state s_t and action a_t:

$$p(s_{t+1}|s_t, s_{t-1}, \ldots, s_0, a_t, a_{t-1}, \ldots, a_0) = p(s_{t+1}|s_t, a_t). \tag{2}$$

A simple problem

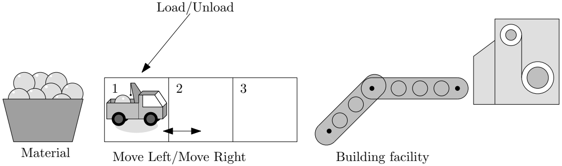
Problem:

An autonomous robot must learn how to transport material from a deposit to a building facility.



(thanks to F. Melo)

Load/Unload as an MDP



- States: $S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$;
  - 1_U Robot in position 1 (unloaded);
  - 2_U Robot in position 2 (unloaded);
  - 3_U Robot in position 3 (unloaded);
  - 1_L Robot in position 1 (loaded);
  - 2_L Robot in position 2 (loaded);
  - 3_L Robot in position 3 (loaded)
- Actions: $A = \{\text{Left, Right, Load, Unload}\}$;

Load/Unload as an MDP (1)

- Transition probabilities: "Left"/"Right" move the robot in the corresponding direction; "Load" loads material (only in position 1); "Unload" unloads material (only in position 3).
  Ex:

$$\begin{aligned} (2_L, \text{Right}) &\rightarrow 3_L; \\ (3_L, \text{Unload}) &\rightarrow 3_U; \\ (1_L, \text{Unload}) &\rightarrow 1_L. \end{aligned}$$

- Reward: We assign a reward of $+10$ for every unloaded package (payment for the service).

Load/Unload as an MDP (2)

- For each action $a \in A$, T_a is a matrix.
  Ex:

$$T_{\text{Right}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Recall: $S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$.

Load/Unload as an MDP (3)

- The reward $R(s, a)$ can also be represented as a matrix
  Ex:

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +10 \end{bmatrix}$$

$$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\},\ A = \{\text{Left, Right, Load, Unload}\}$$

Policies and value

- Policy π: tells the agent how to act.
- A deterministic policy $\pi : S \mapsto A$ is a mapping from states to actions.
- Value: how much reward $E[\sum_{t=0}^{h} \gamma^t R_t]$ does the agent expect to gather.
- Value denoted as $Q^\pi(s, a)$: start in s, do a and follow π afterwards.

Policies and value (1)

- Extracting a policy π from a value function Q is easy:

$$\pi(s) = \arg\max_{a \in A} Q(s, a). \tag{3}$$

- Optimal policy π^*: one that maximizes $E[\sum_{t=0}^{h} \gamma^t R_t]$ (for every state).
- In an infinite-horizon MDP there is always an optimal deterministic stationary (time-independent) policy π^*.
- There can be many optimal policies π^*, but they all share the same optimal value function Q^*.

Dynamic programming

Since S and A are finite, $Q^*(s, a)$ is a matrix.
Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Dynamic programming

Since S and A are finite, $Q^*(s, a)$ is a matrix.
Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9.5 & 0 & 0 \\ 0 & 9.5 & 9.5 & 10 \end{bmatrix}$$

$$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\},\ A = \{\text{Left, Right, Load, Unload}\}$$

Dynamic programming

Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 9.03 & 0 & 0 \\ 0 & 9.5 & 9.03 & 9.03 \\ 9.03 & 9.5 & 9.5 & 10 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Dynamic programming

Iterations of dynamic programming ($\gamma = 0.95$):

$$Q_4 = \begin{bmatrix} 0 & 0 & 8.57 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8.57 & 9.03 & 8.57 & 8.57 \\ 8.57 & 9.5 & 9.03 & 9.03 \\ 9.03 & 9.5 & 9.5 & 10 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{$Left, Right, Load, Unload$\}$

Dynamic programming

Iterations of DP:

$$Q_{10} = \begin{bmatrix} 8.15 & 7.74 & 14.88 & 8.15 \\ 8.15 & 7.35 & 7.74 & 7.74 \\ 7.74 & 7.35 & 7.35 & 7.35 \\ 14.88 & 15.66 & 14.88 & 14.88 \\ 14.88 & 16.48 & 15.66 & 15.66 \\ 15.66 & 16.48 & 16.48 & 17.35 \end{bmatrix}$$

$S = \{1_U, 2_U, 3_U, 1_L, 2_L, 3_L\}$, $A = \{\text{Left, Right, Load, Unload}\}$

Dynamic programming

Final Q^* and policy:

$$Q^* = \begin{bmatrix} 30.75 & 29.21 & 32.36 & 30.75 \\ 30.75 & 27.75 & 29.21 & 29.21 \\ 29.21 & 27.75 & 27.75 & 27.75 \\ 32.36 & 34.07 & 32.36 & 32.37 \\ 32.36 & 35.86 & 34.07 & 34.07 \\ 34.07 & 35.86 & 35.86 & 37.75 \end{bmatrix} \qquad \pi^* = \begin{bmatrix} \text{Load} \\ \text{Left} \\ \text{Left} \\ \text{Right} \\ \text{Right} \\ \text{Unload} \end{bmatrix}$$

Value iteration

- Value iteration: successive approximation technique.
- Start with all values set to 0.
- In order to consider one step deeper into the future, i.e., to compute Q^*_{n+1} from Q^*_n:

$$Q^*_{n+1}(s,a) := R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) \max_{a' \in A} Q^*_n(s',a'), \tag{4}$$

  which is known as the dynamic programming update or Bellman backup.
- Bellman (1957) equation:

$$Q^*(s,a) = R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) \max_{a' \in A} Q^*(s',a'). \tag{5}$$

Value iteration (1)

Initialize Q_0 arbitrarily, e.g., $Q_0(s, a) = 0, \forall s \in S, a \in A$
$n \leftarrow 0$
repeat
  $\delta \leftarrow 0$
  $n \leftarrow n + 1$
  for all $s \in S, a \in A$ **do**
    $Q_n(s, a) \leftarrow R(s, a) + \gamma \sum_{s' \in S} p(s'|s, a) \max_{a' \in A} Q_{n-1}(s', a')$
    $\delta \leftarrow \max(\delta, |Q_{n-1}(s, a) - Q_n(s, a)|)$
  end for
until $\delta < \epsilon$
Return Q_n

Value iteration

Value iteration discussion:

- As $n \to \infty$, value iteration converges.
- Value iteration has converged when the largest update δ in an iteration is below a certain threshold ϵ.
- Exhaustive sweeps are not required for convergence, provided that in the limit all states are visited infinitely often.
- This can be exploited by backing up the most promising states first, known as prioritized sweeping.

Q-learning

- Reinforcement-learning techniques learn from experience, no knowledge of the model is required.
- Q-learning update:

$$Q(s, a) = (1 - \beta)\ Q(s, a) + \beta \left[r + \gamma \max_{a' \in A} Q(s', a') \right], \tag{6}$$

where $0 < \beta \leq 1$ is a learning rate.

Q-learning

Q-learning discussion:

- Q-learning is guaranteed to converge to the optimal Q-values if all $Q(s, a)$ values are updated infinitely often.
- In order to make sure all actions will eventually be tried in all states exploration is necessary.
- A common exploration method is to execute a random action with small probability ϵ, which is known as ϵ-greedy exploration.

Solution methods: MDPs

Model based

- Basic: dynamic programming (Bellman, 1957), value iteration, policy iteration.
- Advanced: prioritized sweeping, function approximators.

Model free, reinforcement learning (Sutton and Barto, 1998)

- Basic: Q-learning, TD(λ), SARSA, actor-critic.
- Advanced: generalization in infinite state spaces, exploration/exploitation issues.

Partially Observable Markov Decision Processes

Beyond MDPs

- Real agents cannot directly observe the state.
- Sensors provide partial and noisy information about the world.

Beyond MDPs

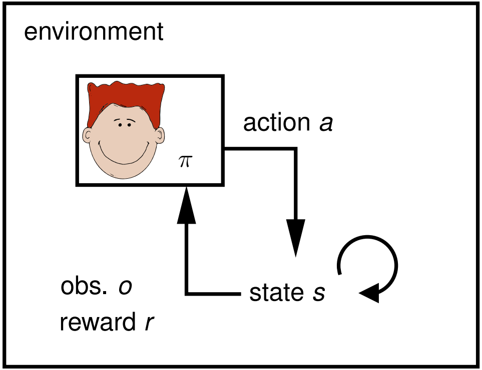
- MDPs have been very successful, but requires to have an observable Markovian state.
- Many domains this is impossible (or expensive) to obtain:

- Diagnosis (medical, maintenance)
- Robot navigation
- Tutoring
- Dialog systems
- Vision-based robotics
- Fault recovery

Observation model

- Imperfect sensors.
- Partially observable environment:
  - Sensors are **noisy**.
  - Sensors have a **limited view**.
- $p(o|s', a)$ is the probability the agent receives observation o in state s' after taking action a.

POMDP Agent



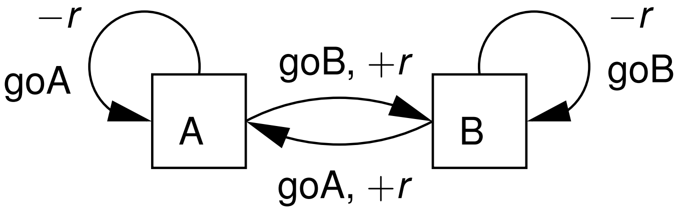
POMDPs

Partially observable Markov decision processes (POMDPs) (Kaelbling et al., 1998):

- Framework for agent planning under uncertainty.
- Typically assumes discrete sets of states S, actions A and observations O.
- Transition model $p(s'|s,a)$: models the effect of **actions**.
- Observation model $p(o|s',a)$: relates **observations** to states.
- Task is defined by a **reward** model $R(s,a)$.
- A planning horizon h (finite or ∞).
- A discount rate $0 \leq \gamma < 1$.
- Goal is to compute plan, or **policy** π, that maximizes long-term reward.

Memory

- In POMDPs memory is required for optimal decision making.
- In this non-observable example (Singh et al., 1994):



Policy	Value
MDP: optimal policy	$V = \sum_{t=0}^{\infty} \gamma^t r = \frac{r}{1-\gamma}$
POMDP: memoryless deterministic	$V_{\max} = r - \frac{\gamma r}{1-\gamma}$
POMDP: memoryless stochastic	$V = 0$
POMDP: memory-based (optimal)	$V_{\min} = \frac{\gamma r}{1-\gamma} - r$

Beliefs

Beliefs:

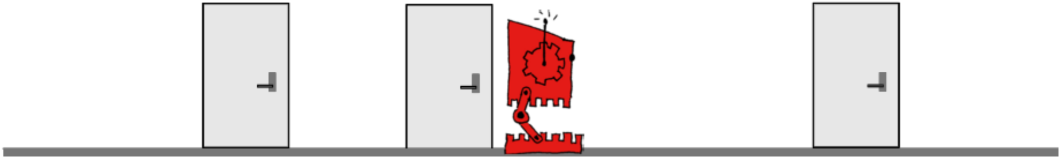
- The agent maintains a **belief** $b(s)$ of being at state s.
- After action $a \in A$ and observation $o \in O$ the belief $b(s)$ can be updated using Bayes' rule:

$$b'(s') = \frac{p(o|s', a) \sum_s p(s'|s, a) b(s)}{p(o|b, a)}$$

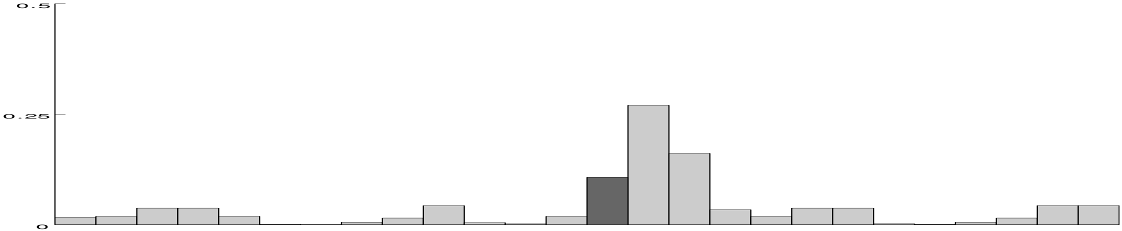
- The belief vector is a **Markov** signal for the planning task.

Belief update example

True situation:



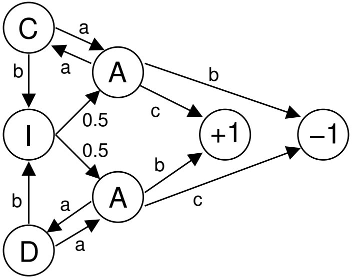
Robot's belief:



- Observations: *door* or **corridor**, 10% noise.
- Action: moves 3 (20%), 4 (60%), or 5 (20%) states.

MDP-based algorithms

- Exploit belief state, and use the MDP solution as a heuristic.
- Most likely state (Cassandra et al., 1996): $\pi_{MLS}(b) = \pi^*(\arg\max_s b(s))$.
- Q_{MDP} (Littman et al., 1995): $\pi_{Q_{\text{MDP}}}(b) = \arg\max_a \sum_s b(s) Q^*(s, a)$.



(Parr and Russell, 1995)

POMDPs as continuous-state MDPs

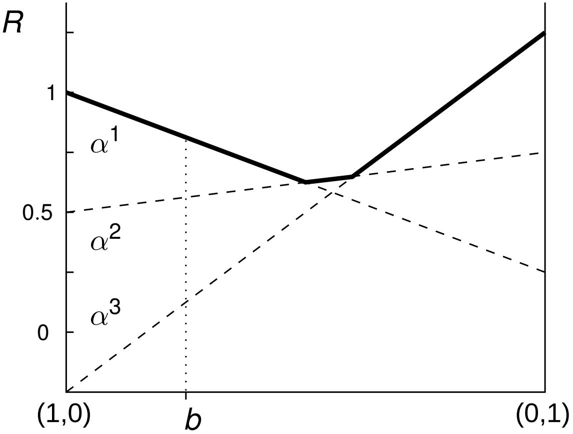
A POMDP can be treated as a continuous-state (belief-state) MDP:

- Continuous state space Δ: a simplex in $[0, 1]^{|S|-1}$.
- Stochastic Markovian transition model $p(b_a^o|b, a) = p(o|b, a)$. This is the normalizer of Bayes' rule.
- Reward function $R(b, a) = \sum_s R(s, a)b(s)$. This is the average reward with respect to $b(s)$.
- The agent fully 'observes' the new belief-state b_a^o after executing a and observing o.

Solving POMDPs

- A solution to a POMDP is a **policy**, i.e., a mapping $\pi : \Delta \mapsto A$ from beliefs to actions.
- The optimal value V^* of a POMDP satisfies the Bellman optimality equation $V^* = HV^*$:
$$V^*(b) = \max_a \left[R(b, a) + \gamma \sum_o p(o|b, a) V^*(b_a^o) \right]$$
- Value iteration repeatedly applies $V_{n+1} = HV_n$ starting from an initial V_0.
- Computing the optimal value function is a hard problem (PSPACE-complete for finite horizon, undecidable for infinite horizon).

Example V_0



$R(s,a)$	a_1	a_2	a_3
s_1	1.00	0.50	-0.25
s_2	0.25	0.75	1.25

PWLC shape of V_n

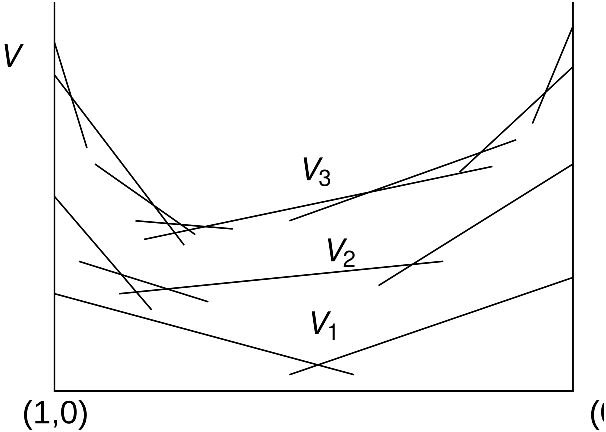
- Like V_0, V_n is as well piecewise linear and convex.
- Rewards $R(b, a) = b \cdot R(s, a)$ are linear functions of b. Note that the value of a point b satisfies:

$$V_{n+1}(b) = \max_a \left[b \cdot R(s, a) + \gamma \sum_o p(o|b, a) V_n(b_a^o) \right]$$

  which involves a maximization over (at least) the vectors $R(s, a)$.
- Intuitively: less uncertainty about the state (low-entropy beliefs) means better decisions (thus higher value).

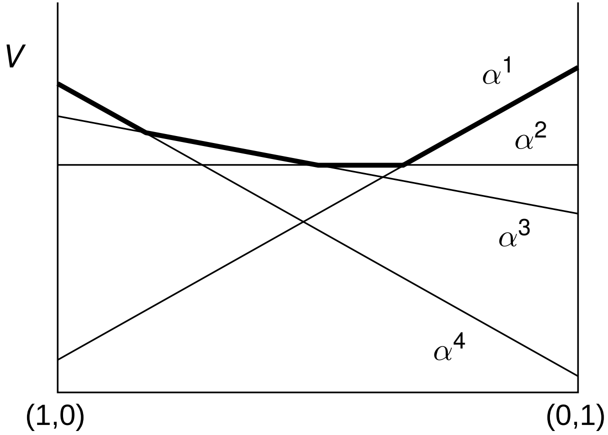
Exact value iteration

Value iteration computes a sequence of value function estimates $V_1, V_2, \ldots, V_n$, using the POMDP backup operator H, $V_{n+1} = HV_n$.

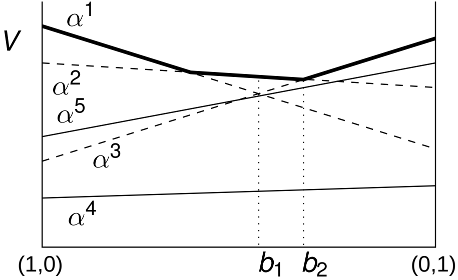


Optimal value functions

The optimal value function of a (finite-horizon) POMDP is piecewise linear and convex: $V(b) = \max_\alpha b \cdot \alpha$.



Vector pruning



Linear program for pruning:

variables: $\forall s \in S, b(s); x$
maximize: x
subject to:

$$b \cdot (\alpha - \alpha') \geq x, \forall \alpha' \in V, \alpha' \neq \alpha$$
$$b \in \Delta(S)$$

Optimal POMDP methods

Enumerate and prune:

- Most straightforward: Monahan (1982)'s enumeration algorithm. Generates a maximum of $|A||V_n|^{|O|}$ vectors at each iteration, hence requires pruning.
- Incremental pruning (Zhang and Liu, 1996; Cassandra et al., 1997; Walraven and Spaan, 2017).

Search for witness points:

- One Pass (Sondik, 1971; Smallwood and Sondik, 1973).
- Relaxed Region, Linear Support (Cheng, 1988).
- Witness (Cassandra et al., 1994).

Sub-optimal techniques

- Grid-based approximations

  (Drake, 1962; Lovejoy, 1991; Brafman, 1997; Zhou and Hansen, 2001; Bonet, 2002).
- Optimizing finite-state controllers

  (Platzman, 1981; Hansen, 1998b; Poupart and Boutilier, 2004).
- Heuristic search in the belief tree

  (Satia and Lave, 1973; Hansen, 1998a).
- Compression or clustering

  (Roy et al., 2005; Poupart and Boutilier, 2003; Virin et al., 2007).
- Point-based techniques

  (Pineau et al., 2003; Smith and Simmons, 2004; Spaan and Vlassis, 2005; Shani et al., 2007; Kurniawati et al., 2008).
- Monte Carlo tree search

  (Silver and Veness, 2010).

Point-based backup

- For finite horizon V^* is piecewise linear and convex, and for infinite horizons V^* can be approximated arbitrary well by a PWLC value function (Smallwood and Sondik, 1973).
- Given value function V_n and a particular belief point b we can easily compute the vector α_{n+1}^b of HV_n such that

$$\alpha_{n+1}^b = \underset{\{\alpha_{n+1}^k\}_k}{\arg\max}\, b \cdot \alpha_{n+1}^k,$$

where $\{\alpha_{n+1}^k\}_{k=1}^{|HV_n|}$ is the (unknown) set of vectors for HV_n. We will denote this operation $\alpha_{n+1}^b = \mathtt{backup}(b)$.

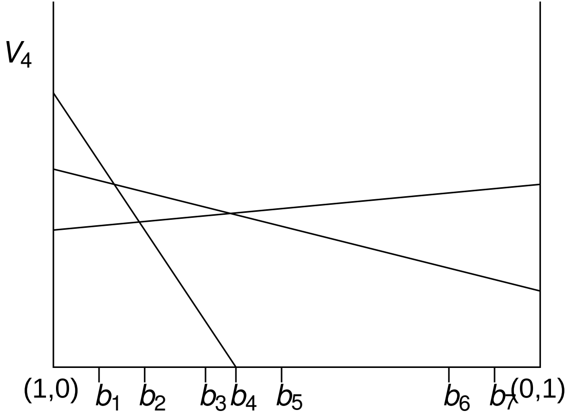
Point-based (approximate) methods

Point-based (approximate) value iteration plans only on a limited set of **reachable** belief points:

1. Let the robot explore the environment.
2. Collect a set B of belief points.
3. Run approximate value iteration on B.

PERSEUS: randomized point-based VI

Idea: at every backup stage improve the value of all $b \in B$.



(Spaan and Vlassis, 2005)

POMDPs in action

- Intention-aware online POMDP planning (Bai et al., 2015)
- ACAS X: Airborne Collision Avoidance System X (Kochenderfer et al., 2012)

Further reading

Books:

- M. Kochenderfer, “Decision Making under Uncertainty”, MIT Press, 2015.
- M. Wiering and M. van Otterlo, editors, “Reinforcement Learning: State of the Art”, Springer, 2012.
- R. S. Sutton and A. G. Barto. “Reinforcement Learning: An Introduction”. MIT Press, 2nd edition, 2018.

Software and other resources:

SolvePOMDP toolbox

MultiAgentDecisionProcess toolbox

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