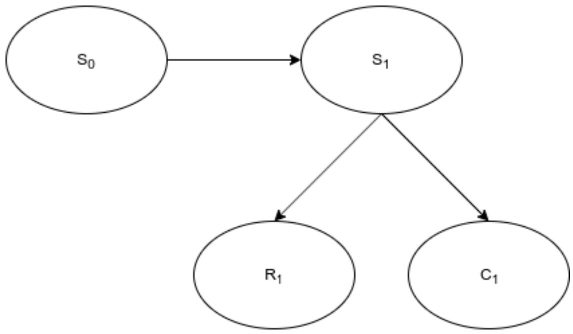

Figure 1: DBN for exercise 1.

Exercise Sheet - Reasoning over Time

Exercises are obtained from Russell & Norvig.

Exercise 1. A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following domain theory:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 if not.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 if not.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 if not.

Formulate this information as a dynamic Bayesian network that the professor could use to filter or predict from a sequence of observations. Then reformulate it as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.

Solution: The DBN has three variables: S_t, whether the student gets enough sleep; R_t, whether they have red eyes in class; C_t, whether the student sleeps in class. S_t is a parent of S_{t+1}, R_t, and C_t. The DBN is shown in Figure 1. The CPTs are given by

- $P(s_0) = 0.7$
- $P(s_{t+1}|s_t) = 0.8$
- $P(s_{t+1}|\neg s_t) = 0.3$
- $P(r_t|s_t) = 0.2$
- $P(r_t|\neg s_t) = 0.7$
- $P(c_t|s_t) = 0.1$
- $P(c_t|\neg s_t) = 0.3$

To reformulate as an HMM with a single observation node, simply combine the 2-valued variables "having red eyes" and "sleeping in class" into a single 4-valued variable, multiplying together the emission probabilities. The HMM is shown in Figure 1, O is the single 4-valued variable. For S we again have

1. $P(s_0) = 0.7$
2. $P(s_{t+1}|s_t) = 0.8$
3. $P(s_{t+1}|\neg s_t) = 0.3$

Writing the 4 possible instances of the single 4-valued variable as rc, $r\neg c$, $\neg rc$, $\neg r\neg c$, we get

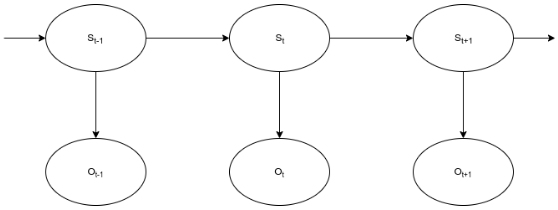

Figure 2: HMM for exercise 1.

S	$P(rc)$	$P(r\neg c)$	$P(\neg rc)$	$P(\neg rnegc)$
t	0.02	0.18	0.08	0.72
f	0.21	0.49	0.09	0.21

Exercise 2. For the DBN specified in Exercise 1 and for the evidence values:

- e_1 = not red eyes, not sleeping in class
- e_2 = red eyes, not sleeping in class
- e_3 = red eyes, sleeping in class

perform the following computations:

a) State estimation: Compute $P(EnoughSleep_t|e_{1:t})$ for each of t = 1, 2, 3.

b) Smoothing: Compute $P(EnoughSleep_t|e_{1:3})$ for each of t = 1, 2, 3.

c) Compare the filtered and smoothed probabilities for t = 1 and t = 2.

Solution: a. We apply the foreard algorithm to compute these probabilities.

$$\begin{aligned}
P(S_0) &= \langle 0.7, 0.3\rangle \\
P(S_1) &= \sum_{s_0} P(S_1 \mid s_0) P(s_0) \\
&= (\langle 0.8, 0.2\rangle 0.7 + \langle 0.3, 0.7\rangle 0.3) \\
&= \langle 0.65, 0.35\rangle \\
P(S_1 \mid e_1) &= \alpha P(e_1 \mid S_1) P(S_1) \\
&= \alpha\langle 0.8 \times 0.9, 0.3 \times 0.7\rangle\langle 0.65, 0.35\rangle \\
&= \alpha\langle 0.72, 0.21\rangle\langle 0.65, 0.35\rangle \\
&= \langle 0.8643, 0.1357\rangle \\
P(S_2 \mid e_1) &= \sum_{s_1} P(S_2 \mid s_1) P(s_1 \mid e_1) \\
&= \langle 0.7321, 0.2679\rangle \\
P(S_2 \mid e_{1:2}) &= \alpha P(e_2 \mid S_2) P(S_2 \mid e_1) \\
&= \langle 0.5010, 0.4990\rangle \\
P(S_3 \mid e_{1:2}) &= \sum_{s_2} P(S_3 \mid s_2) P(s_2 \mid e_{1:2}) \\
&= \langle 0.5505, 0.4495\rangle \\
P(S_3 \mid e_{1:3}) &= \alpha P(e_3 \mid S_3) P(S_3 \mid e_{1:2}) \\
&= \langle 0.1045, 0.8955\rangle
\end{aligned}$$

Similar to many students during the course of the school term, the student observed here seems to have a higher likelihood of being sleep deprived as time goes on!

b. First we compute the backwards messages:

$$\begin{aligned}
P(e_3 \mid S_3) &= \langle 0.2 \times 0.1, 0.7 \times 0.3\rangle \\
&= \langle 0.02, 0.21\rangle \\
P(e_3 \mid S_2) &= \sum_{s_3} P(e_3 \mid s_3) P(\mid s_3) P(s_3 \mid S_2) \\
&= \langle 0.02 \times 0.8 + 0.21 \times 0.2, 0.02 \times 0.3 + 0.21 \times 0.7\rangle \\
&= \langle 0.0588, 0.153\rangle \\
P(e_{2:3} \mid S_1) &= \sum_{s_2} P(e_2 \mid s_2) P(e_3 \mid s_2) P(s_2 \mid S_1) \\
&= \langle 0.0233, 0.0556\rangle
\end{aligned}$$

$P(e_3|S_1)$ should be $P(e_3|S_2)$ and it's $< 0.058, 0.153 >$.
Then we combine these with the forwards messages computed previouly and normalize:

$$\begin{aligned}
P(S_1 \mid e_{1:3}) &= \alpha P(S_1 \mid e_1) P(e_{2:3} \mid S_1) \\
&= \langle 0.7277, 0.2723\rangle \\
P(S_2 \mid e_{1:3}) &= \alpha P(S_2 \mid e_{1:2}) P(e_3 \mid S_1) \\
&= \langle 0.2757, 0.7243\rangle \\
P(S_3 \mid e_{1:3}) &= \langle 0.1045, 0.8955\rangle
\end{aligned}$$

c. The smoothed analysis places the time the student started sleeping poorly one step earlier than filtered analysis, integrating future observations indicating lack of sleep at the last step.

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