CS4070: ASSIGNMENT 2: POISSON REGRESSION

Hand in before 22 December, 23:59. Include code as an appendix.

Consider the Poisson regression model, which is a basic model for count-data. So we assume data $\{(x_i, y_i)\}_{i=1}^n$, where $y_i \in \{0, 1, 2, \ldots\}$ and $x_i \in \mathbb{R}^p$. The model is given by

$$y_i \sim Pois\,(\mu_i)\,, \qquad \text{where} \quad \log \mu_i = x_i^T \theta$$

for an unknown parameter vector $\theta \in \mathbb{R}^p$. Hence, for $k \in \{0, 1, 2, \ldots\}$ we have

$$\mathbb{P}(Y_i = k) = e^{-\mu_i} \mu_i^k / (k!).$$

(1) Give the loglikelihood, assuming all y_i are independent.
(2) Derive an expression for the gradient and Hessian of the loglikelihood.
(3) In the following we consider the dataset `dataexercise2.csv`. We take a Bayesian point of view, where we assume

$$\begin{aligned} y_i \mid \theta &\overset{\text{ind}}{\sim} Pois\,(\mu_i)\,, \qquad \text{where} \quad \log \mu_i = x_i^T \theta \\ \theta &\sim N\left(0, \tilde{\sigma}^2 I_p\right). \end{aligned}$$

Assume the prior standard deviation is given by $\tilde{\sigma} = 4$. Implement a Newton algorithm for computing the Laplace approximation to the posterior distribution. Report mean and covariance matrix of the approximation.

(4) Implement a random walk Metropolis–Hastings algorithm to sample from the posterior. Take proposals of the form $\theta^\circ := \theta + \sigma_{\text{proposal}} N\,(0, I_p)$. Tune σ_{proposal} to achieve an acceptance rate of about $25\% - 50\%$. Make a plot of the iterates where you plot θ_2 versus θ_1, with colour indicating the iteration number. Report the Monte-Carlo estimate of the posterior mean (where you "throw away" burn-in samples, i.e. initial samples where the chain has not reached its stationary region).

(5) The results may be sensitive to the choice of $\tilde{\sigma}$. For that reason we add an extra layer to the hierarchical model in the following way:

$$\begin{aligned} y_i \mid \theta &\overset{\text{ind}}{\sim} Pois\,(\mu_i)\,, \qquad \text{where} \quad \log \mu_i = x_i^T \theta, \\ \theta \mid \tilde{\sigma} &\sim N\left(0, \tilde{\sigma}^2 I_p\right), \\ \tilde{\sigma}^2 &\sim IG\,(\alpha, \beta)\,. \end{aligned}$$

Here $IG\,(\alpha, \beta)$ denotes the inverse Gamma distribution with parameters α and β (its density function is given in Exercise 3.12 in RG). Take $\alpha = \beta = 0.2$. Implement a Gibbs sampler that iteratively samples from the full conditionals of θ and $\tilde{\sigma}$.

Include a derivation for the update step for $\tilde{\sigma}^2$ in your report. Also include a traceplot of the posterior samples of $\tilde{\sigma}^2$ (a traceplot is a plot of iterate value versus iterate number).

Date: December 7, 2023.