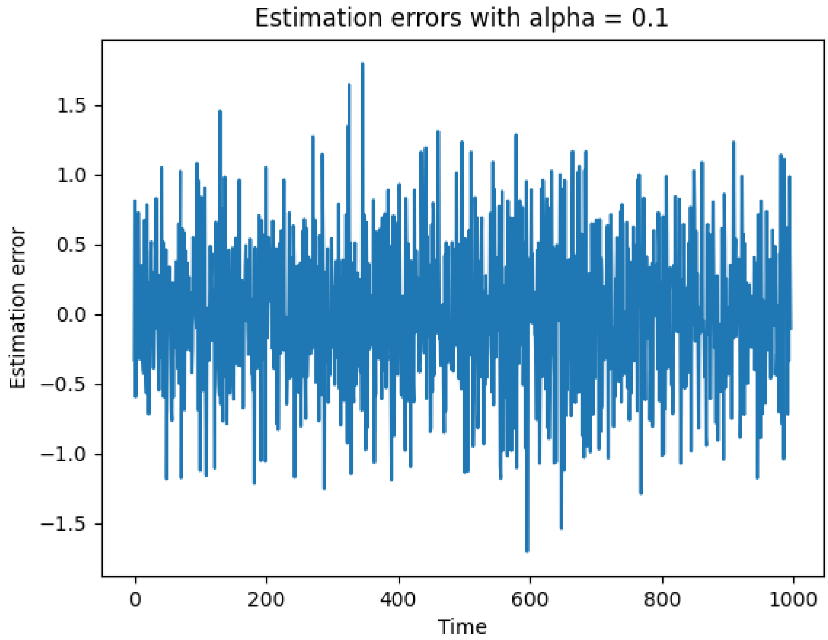
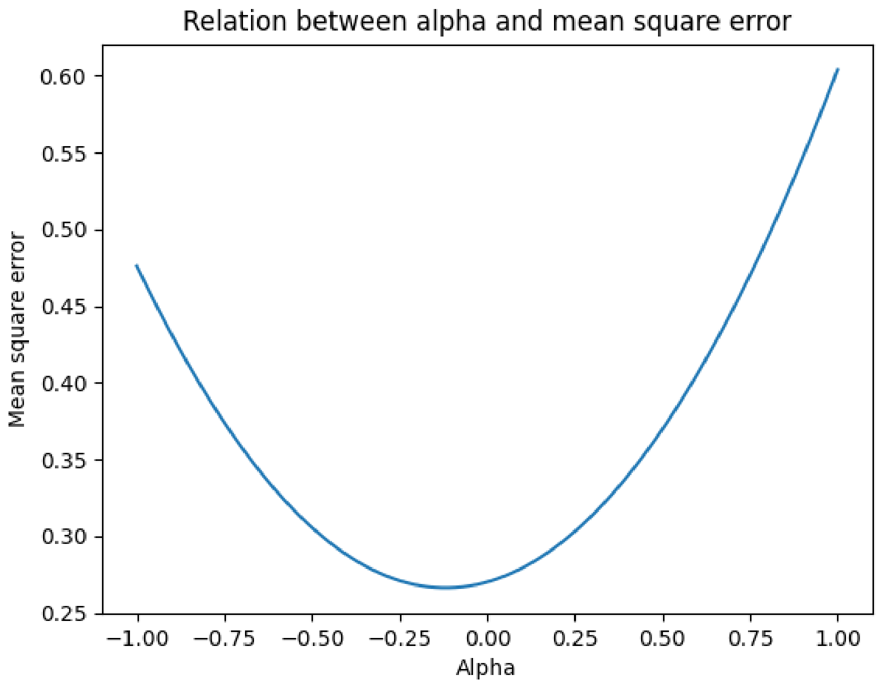
Final exercise

A: *Calculate and plot the position estimation error $\varepsilon(n+1)$ for $\alpha = 0.1$ as a function of n ($n = 1, ..., N-2$). In other words, use the pair of measurements $(X(n), X(n-1))$ (for all N) to estimate the position at time instance $n+1$ using eq. (1) with $\alpha = 0.1$, and subtract the result from the actual (and given) position measurement $X(n+1)$.*



B: *Estimate the variance of position estimation error $\varepsilon(n+1)$ from the given measured car positions for the values of the tuning parameter $\alpha = -1.00, -0.99, -0.98, -0.97, \ldots 0.97, 0.98, 0.99, 1.00$. Plot the resulting relation between α and the estimated $\sigma^2{}_\varepsilon$. Determine a suitable value of α from this figure.*



- From this figure we can get that the best possible α value is -0.12 sin is the value with min mean square error

C: *In (b.) you have calculated and plotted the variance of the estimation error directly from the data. The variance of the estimation error can also be expressed in terms of the autocorrelation function of the process X(n). Show that the variance can be expressed as:*
$\sigma^2 = E[\varepsilon(n + 1)^2] = (2 + 2\alpha + 2\alpha^2)R_X(0) + (-2 - 4\alpha - 2\alpha^2)R_X(1) + (2\alpha)R_X(2)$

- $\sigma^2 = E[\varepsilon(n + 1)^2] = E[(X_{n+1} - \hat{X}_{n+1})^2]$
- $= E[(X_{n+1} - (X_n + \alpha (X_n - X_{n-1})))^2]$
- $= E[(X_{n+1} - X_n - \alpha X_n + \alpha X_{n-1})^2]$
- $= E[(X_{n+1} - X_n - \alpha X_n + \alpha X_{n-1}) * (X_{n+1} - X_n - \alpha X_n + \alpha X_{n-1})]$
- $= E[(X_{n+1})^2 - 2(X_{n+1}*X_n) - 2\alpha(X_{n+1}*X_n) + 2\alpha(X_{n+1}*X_{n-1}) + (X_n)^2 + 2\alpha(X_n)^2 - 2\alpha(X_n*X_{n-1}) + \alpha^2(X_n)^2 - 2\alpha^2(X_n*X_{n-1}) + \alpha^2(X_{n-1})^2]$
- $= 2R_X(0) + 2\alpha R_X(0) + 2\alpha^2 R_X(0) - 2R_X(1) - 4\alpha R_X(1) - 2\alpha^2 R_X(1) + 2\alpha R_X(2)$
- $= (2 + 2\alpha + 2\alpha^2)R_X(0) + (-2 -4\alpha -2\alpha^2)R_X(1) + (2\alpha)R_X(2)$

D: *Find an analytical expression for the optimal value of α by minimizing Equation (4) (taking derivative with respect to α).*

First find the derivative:

- $f(\alpha) = (2 + 2\alpha + 2\alpha^2)R_X(0) + (-2 -4\alpha -2\alpha^2)R_X(1) + (2\alpha)R_X(2)$
- $f'(\alpha) = (2 + 4\alpha)R_X(0) + (-4 -4\alpha)R_X(1) + (2)R_X(2)$
- $= 4\alpha R_X(0) -4\alpha R_X(1) + 2R_X(0) - 4R_X(1) + 2R_X(2)$
- $= \alpha[4R_X(0) -4R_X(1)] + 2R_X(0) - 4R_X(1) + 2R_X(2)$

$f'(\alpha) = 0$ to find a possible minimum:

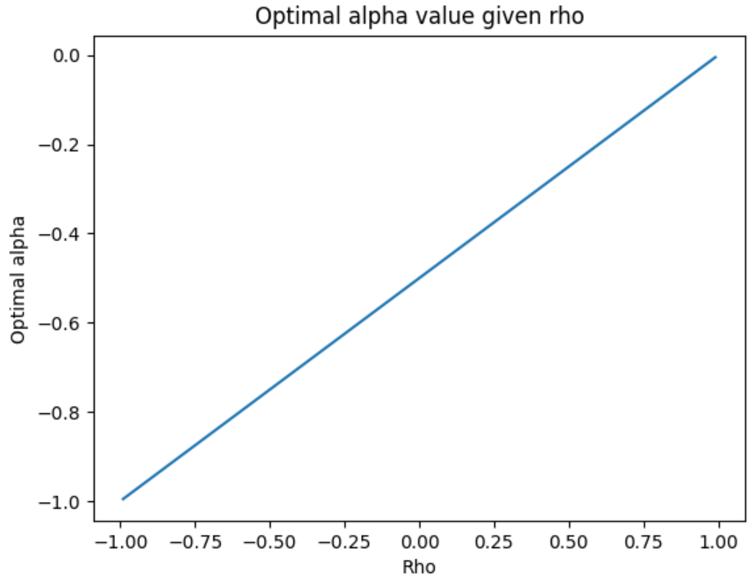
- $0 = \alpha[4R_X(0) -4R_X(1)] + 2R_X(0) - 4R_X(1) + 2R_X(2)$
- $\alpha[4R_X(0) -4R_X(1)] = - 2R_X(0) + 4R_X(1) - 2R_X(2)$
- $\alpha = [- 2R_X(0) + 4R_X(1) - 2R_X(2)] / [4R_X(0) -4R_X(1)]$

E: *Assume that the autocorrelation function of the random process X(n) can be modeled as follows: $R_X(k) = \sigma_X^2 * \rho$^|k| , $|\rho| < 1$. Use this expression to calculate and plot the optimal value of α as a function of ρ.*

We have the expression of optimal α from the last exercise so we can substitute the correlations with the provided formula:

- $\alpha = [- 2R_X(0) + 4R_X(1) - 2R_X(2)] / [4R_X(0) -4R_X(1)]$
- $\alpha = [- 2\sigma_X^2 \rho^0 + 4\sigma_X^2\rho^1 - 2\sigma_X^2\rho^2] / [4\sigma_X^2\rho^0 -4\sigma_X^2\rho^1]$

- α = [− 2σ_X^2 + 4$\sigma_X^2\rho$ − 2$\sigma_X^2\rho^2$] / [4σ_X^2 −4$\sigma_X^2\rho$]
- α = [2σ_X^2 * (−1 + 2ρ − ρ^2)] / [4σ_X^2 *(1−ρ)]
- α = [−1 + 2ρ − ρ^2] / [2−2ρ]



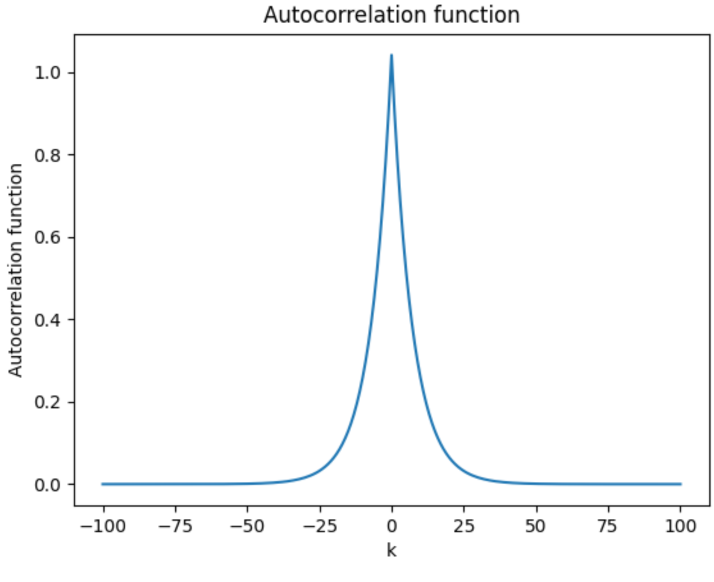
F: *Estimate the value of ρ from the data X(n), and find the optimal value of α*

To estimate the value of ρ from the data we can first calculate $R_X(1)$ which is equal to: $\sigma_X^2\rho$. Once we have this value we can conclude that $\rho = R_X(1) / \sigma_X^2$.

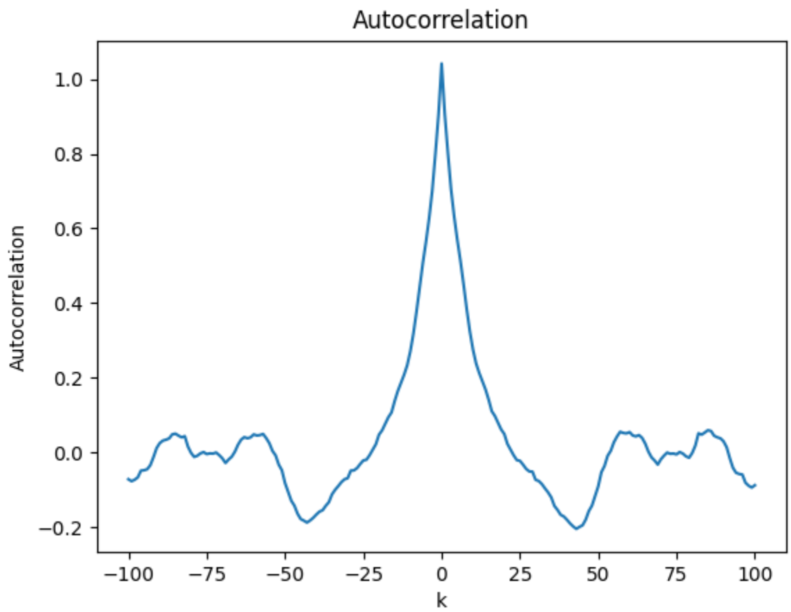
- $\rho = R_X(1) / \sigma_X^2$
- = 0.9064462161041652 / 1.0418186912944625
- = −0.06496930623412836

G: *Estimate the autocorrelation function $R_X(k)$ of the position measurements X(n) in the file positions1.mat. Plot the resulting autocorrelation function.*

- Autocorrelation function using the formula $R_X(k) = \sigma_X^2 * \rho$^|k|



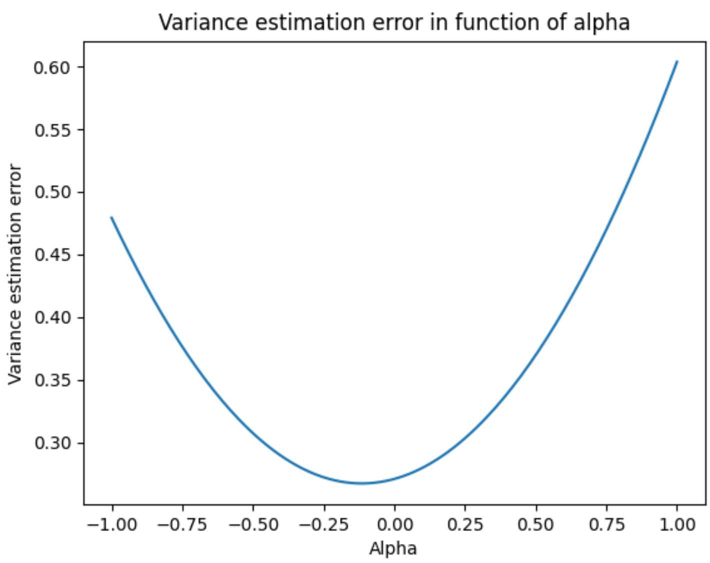
- Autocorrelation function using the the provided data with the approximation function: $R_X(m) = \frac{1}{n}\sum_{i=1}^{n} X_i(k)X_i(k+m)$



H: *Use the result under part (g.) an Equation (4) to plot the variance of position estimation error $\varepsilon(n + 1)$ as a function of α. Determine a suitable value of α from this figure.*

From equation 4 we have that:

$\sigma^2_{\varepsilon(n+1)} = E[\varepsilon(n+1)^2] = (2 + 2\alpha + 2\alpha^2)R_X(0) + (-2 - 4\alpha - 2\alpha^2)R_X(1) + (2\alpha)R_X(2)$



- From this figure we can get that the best possible α value is -0.12 sin is the value with min mean square error

I: *Use the result under parts (d.) and (g.) to find the optimal value of α.*

Given the formula in part (d) using the correlation approximation function and the results obtained in part (g) we can evaluate the function in part(d) with a result of:

- -0.11496957527195101

Given the formula in part (d) using the correlation function provided with the previous calculated and the results obtained in part (g) we can evaluate the function in part(d) with a result of:

- -0.06496930623412836

J: Explain the differences (if they exist) between the calculated optimal/suitable values of α in parts (b.), (f.), and (i.).

There is a noticeable difference between b) and f) on the optimal alpha value. That is given because the autocorrelation function provided has some differences with the approximation function applied to calculate the correlation: $R_X(m) = \frac{1}{n}\sum_{i=1}^{n} X_i(k)X_i(k+m)$ on the provided data.
The calculation in i) has 2 different results depending on the autocorrelation function used. If we use the one provided in the exercise we get the same result as in g) while if we use the approximation on the data its more similar to the result on b). Is not exactly the same value due to the fact that in be we are treating alpha as a discrete value and not as a continuous one.

Tomàs Osarte
Multivariate Analysis
29/10/2023