Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Third Edition

MARKOV CHAINS SUPPLEMENT (MCS)

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1 Discrete-Time Markov Chains

Discrete-time Markov chains are discrete-value random sequences such that the current value of the sequence is a system state that summarizes the past history of the sequence with respect to predicting the future values.

In Chapter 13, we introduced discrete-time random processes and we emphasized iid random sequences. Now we will consider a discrete-value random sequence $\{X_n | n = 0, 1, 2, \ldots\}$ that is *not* an iid random sequence. In particular, we will examine systems, called *Markov chains*, in which X_{n+1} depends on X_n but not on the earlier values $X_0, \ldots, X_{n-1}$ of the random sequence. To keep things reasonably simple, we restrict our attention to the case where each X_n is a discrete random variable with range $S_X = \{0, 1, 2, \ldots\}$. In this case, we make the following definition.

Definition 1 — Discrete-Time Markov Chain

*A **discrete-time Markov chain** $\{X_n | n = 0, 1, \ldots\}$ is a discrete-value random sequence such that given $X_0, \ldots, X_n$, the next random variable X_{n+1} depends only on X_n through the transition probability*

$$\mathrm{P}\left[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right] = \mathrm{P}\left[X_{n+1} = j | X_n = i\right] = P_{ij}.$$

The value of X_n summarizes all of the past history of the system needed to predict the next variable X_{n+1} in the random sequence. We call X_n the *state* of the system at time n, and the sample space of X_n is called the *set of states* or *state space*. In short, there is a fixed *transition probability* P_{ij} that the next state will be j given that the current state is i. These facts are reflected in the next theorem.

Theorem 1

The transition probabilities P_{ij} of a Markov chain satisfy

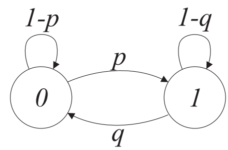
$$P_{ij} \geq 0, \qquad \sum_{j=0}^{\infty} P_{ij} = 1.$$

We can represent a Markov chain by a graph with nodes representing the sample space of X_n and directed arcs (i, j) for all pairs of states (i, j) such that $P_{ij} > 0$.

Example 1

The two-state Markov chain can be used to model a wide variety of systems that alternate between ON and OFF states. After each unit of time in the OFF state, the system turns ON with probability p. After each unit of time in the ON state, the system turns OFF with probability q. Using 0 and 1 to denote the OFF and ON states, what is the Markov chain for the system?

The Markov chain for this system is shown on the right. The transition probabilities are

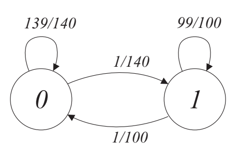


$$P_{00} = 1 - p, \quad P_{01} = p, \quad P_{10} = q, \quad P_{11} = 1 - q.$$

Example 2

A packet voice communications system transmits digitized speech only during "talkspurts" when the speaker is talking. In every 10-ms interval (referred to as a timeslot) the system decides whether the speaker is talking or silent. When the speaker is talking, a speech packet is generated; otherwise no packet is generated. If the speaker is silent in a slot, then the speaker is talking in the next slot with probability $p = 1/140$. If the speaker is talking in a slot, the speaker is silent in the next slot with probability $q = 1/100$. If states 0 and 1 represent silent and talking, sketch the Markov chain for this packet voice system.

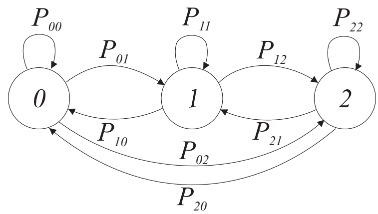
For this system, the two-state Markov chain is shown on the right. The transition probabilities are



$$P_{00} = \frac{139}{140}, \quad P_{01} = \frac{1}{140}, \quad P_{10} = \frac{1}{100}, \quad P_{11} = \frac{99}{100}.$$

Example 3

A computer disk drive can be in one of three possible states: 0 (IDLE), 1 (READ), or 2 (WRITE). When a unit of time is required to read or write a sector on the disk, the Markov chain is



The values of the transition probabilities will depend on factors such as the number of sectors in a read or a write operation and the length of the idle periods.

Example 4

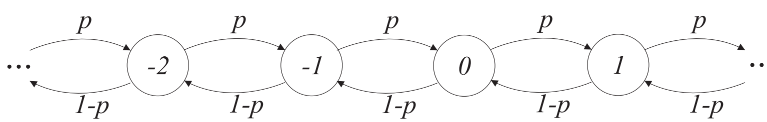
In a discrete random walk, a person's position is marked by an integer on the real line. Each unit of time, the person randomly moves one step, either to the right (with probability p) or to the left. Sketch the Markov chain.

. . . .

The Markov chain has state space $\{\ldots, -1, 0, 1, \ldots\}$ and transition probabilities

$$P_{i,i+1} = p, \qquad P_{i,i-1} = 1 - p. \tag{1}$$

The Markov chain is



Another name for this Markov chain is the "Drunken Sailor."

The graphical representation of Markov chains encourages the use of special terminology. We will often call the transition probability P_{ij} a *branch probability* because it equals the probability of following the branch from state i to state j. When we examine Theorem 1, we see that it says the sum of the branch probabilities leaving any state i must sum to 1. A state transition is also called a *hop* because a transition from i to j can be viewed as hopping from i to j on the Markov chain. In addition, the state sequence resulting from a sequence of hops in the Markov chain will frequently be called a *path*. For example, a state sequence i, j, k corresponding to a sequence of states $X_n = i$, $X_{n+1} = j$, and $X_{n+2} = k$ is a two-hop path from i to k. Note that the state sequence i, j, k is a path in the Markov chain only if each corresponding state transition has nonzero probability.

The random walk of Example 4 shows that a Markov chain can have a countably infinite set of states. We will see in Section 7 that countably infinite Markov chains introduce complexities that initially get in the way of understanding and using Markov chains. Hence, until Section 7, we focus on Markov chains with a finite set of states $\{0, 1, \ldots, K\}$. In this case, we represent the one-step transition probabilities by the matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0K} \\ P_{10} & P_{11} & & \vdots \\ \vdots & & \ddots & \\ P_{K0} & \cdots & & P_{KK} \end{bmatrix}. \tag{2}$$

By Theorem 1, $\mathbf{P}$ has nonnegative elements and each row sums to 1. A nonnegative matrix $\mathbf{P}$ with rows that sum to 1 is called a *state transition matrix* or a *stochastic matrix*.

Example 5

The two-state ON/OFF Markov chain of Example 1 has state transition matrix

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}. \tag{3}$$

Quiz 1

A wireless packet communications channel suffers from clustered errors. That is, whenever a packet has an error, the next packet will have an error with probability 0.9. Whenever a packet is error-free, the next packet is error-free with probability 0.99. Let $X_n = 1$ if the nth packet has an error; otherwise, $X_n = 0$. Model the random process $\{X_n | n \geq 0\}$ using a Markov chain. Sketch the chain and find the transition probability matrix $\mathbf{P}$.

2 Discrete-Time Markov Chain Dynamics

> The n-step transition probabilities are conditional probabilities for the sequence value n steps in the future given the current sequence value.

In an electric circuit or any physical system described by differential equations, the system dynamics describe the short-term response to a set of initial conditions. For a Markov chain, we use the word *dynamics* to describe the variation of the state over a short time interval starting from a given initial state. The initial state of the chain represents the initial condition of the system. Unlike a circuit, the evolution of a Markov chain is a random process and so we cannot say exactly what sequence of states will follow the initial state. However, there are many applications in which it is desirable or necessary to predict the future states given the current state X_m. A prediction of the future state X_{n+m} given the current state X_m usually requires knowledge of the conditional PMF of X_{n+m} given X_m. This information is contained in the n-step transition probabilities.

Definition 2 **n-step transition probabilities**

For a finite Markov chain, the n-step transition probabilities are given by the matrix $\mathbf{P}(n)$ which has i,jth element

$$P_{ij}(n) = \mathrm{P}\left[X_{n+m} = j | X_m = i\right].$$

The i,jth element of $\mathbf{P}(n)$ tells us the probability of going from state i to state j in exactly n steps. For $n = 1$, $\mathbf{P}(1) = \mathbf{P}$, the state transition matrix. Keep in mind that $P_{ij}(n)$ must account for the probability of every n-step path from state i to state j. As a result, it is easier to define than to calculate the n-step transition probabilities. The Chapman-Kolmogorov equations give a recursive procedure for calculating the n-step transition probabilities. The equations are based on the observation that going from i to j in $n+m$ steps requires being in some state k after n steps. We state this result, and others, in two equivalent forms: as a sum of probabilities, and in matrix/vector notation.

Theorem 2 Chapman-Kolmogorov equations

For a finite Markov chain, the n-step transition probabilities satisfy

$$P_{ij}(n+m) = \sum_{k=0}^{K} P_{ik}(n) P_{kj}(m), \qquad \mathbf{P}(n+m) = \mathbf{P}(n)\mathbf{P}(m).$$

Proof By the definition of the n-step transition probability,

$$\begin{aligned} P_{ij}(n+m) &= \sum_{k=0}^{K} \mathrm{P}\left[X_{n+m} = j, X_n = k | X_0 = i\right] \\ &= \sum_{k=0}^{K} \mathrm{P}\left[X_n = k | X_0 = i\right] \mathrm{P}\left[X_{n+m} = j | X_n = k, X_0 = i\right] \end{aligned} \tag{4}$$

By the Markov property, $\mathrm{P}[X_{n+m} = j | X_n = k, X_0 = i] = \mathrm{P}[X_{n+m} = j | X_n = k] = P_{kj}(m)$. With the additional observation that $\sum_{k=0}^{K} \mathrm{P}[X_n = k | X_0 = i] = P_{ik}(n)$, we see that Equation (4) is the same as the statement of the theorem.

For a finite Markov chain with K states, the Chapman-Kolmogorov equations can be expressed in terms of matrix multiplication of the transition matrix $\mathbf{P}$. For $m = 1$, the matrix form of the Chapman-Kolmogorov equations yield $\mathbf{P}(n+1) = \mathbf{P}(n)\mathbf{P}$. This implies our next result.

Theorem 3

For a finite Markov chain with transition matrix $\mathbf{P}$, the n-step transition matrix is

$$\mathbf{P}(n) = \mathbf{P}^n.$$

Example 6

For the two-state Markov chain described in Example 1, find the n-step transition matrix $\mathbf{P}^n$. Given the system is OFF at time 0, what is the probability the system is OFF at time $n = 33$?

. . . .

The state transition matrix is

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}. \tag{5}$$

The eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1-(p+q)$. Since p and q are probabilities, $|\lambda_2| \le 1$. We can express $\mathbf{P}$ in the diagonalized form

$$\mathbf{P} = \mathbf{S}^{-1}\mathbf{D}\mathbf{S} = \begin{bmatrix} 1 & \frac{-p}{p+q} \\ 1 & \frac{q}{p+q} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ -1 & 1 \end{bmatrix}. \tag{6}$$

Note that $\mathbf{s}_i$, the ith row $\mathbf{S}$, is the left eigenvector of $\mathbf{P}$ corresponding to λ_i. That is, $\mathbf{s}_i\mathbf{P} = \lambda_i\mathbf{s}_i$. Some straightforward algebra will verify that the n-step transition matrix is

$$\mathbf{P}^n = \begin{bmatrix} P_{00}(n) & P_{01}(n) \\ P_{10}(n) & P_{11}(n) \end{bmatrix} = \mathbf{S}^{-1}\mathbf{D}^n\mathbf{S} = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{\lambda_2^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}. \tag{7}$$

Given the system is OFF at time 0, the conditional probability the system is OFF at time $n = 33$ is simply

$$P_{00}(33) = \frac{q}{p+q} + \frac{\lambda_2^{33}p}{p+q} = \frac{q + [1-(p+q)]^{33}p}{p+q}. \tag{8}$$

The n-step transition matrix is a complete description of the evolution of probabilities in a Markov chain. Given that the system is in state i, we learn the probability the system is in state j n steps later just by looking at $P_{ij}(n)$. On the other hand, calculating the n-step transition matrix is nontrivial. Even for the two-state chain, it was desirable to identify the eigenvalues and diagonalize $\mathbf{P}$ before computing $\mathbf{P}(n)$.

Often, the n-step transition matrix provides more information than we need. When working with a Markov chain $\{X_n|n \ge 0\}$, we may need to know only the state probabilities $\mathrm{P}[X_n = i]$. Since each X_n is a random variable, we could express this set of state probabilities in terms of the PMF $P_{X_n}(i)$. This representation can be a little misleading since the states $0, 1, 2, \ldots$ may not correspond to sample values of a random variable but rather labels for the states of a system. In the two-state chain of Example 1, the states 0 and 1 corresponded to OFF and ON states of a system and have no numerical significance. Consequently, we represent the state probabilities at time n by the set $\{p_j(n)|j = 0, 1, \ldots, K\}$ where $p_j(n) = \mathrm{P}[X_n = j]$. An equivalent representation of the state probabilities at time n is the vector $\mathbf{p}(n) = \begin{bmatrix} p_0(n) & \cdots & p_K(n) \end{bmatrix}'$.

Definition 3 — State Probability Vector

*A vector $\mathbf{p} = \begin{bmatrix} p_0 & \cdots & p_K \end{bmatrix}'$ is a **state probability vector** if each element p_j is nonnegative and $\sum_{j=0}^{K} p_j = 1$.*

Starting at time $n = 0$ with the a priori state probabilities $\{p_j(0)\}$, or, equivalently, the vector $\mathbf{p}(0)$, of the Markov chain, the following theorem shows how to calculate the state probability vector $\mathbf{p}(n)$ for any time n in the future. We state this theorem, as well as others, in terms of summations over states in parallel with the equivalent matrix/vector form. In this text, we assume the state vector $\mathbf{p}(n)$ is a column vectors. In the analysis of Markov chains, it is a common convention to use P_{ij} rather than P_{ji} for the probability of a transition *from i to j*. The combined effect is that our matrix calculations will involve left multiplication by the row vector $\mathbf{p}'(n-1)$.

Theorem 4

The state probabilities $p_j(n)$ at time n can be found by either one iteration with the n-step transition probabilities:

$$p_j(n) = \sum_{i=0}^{K} p_i(0) P_{ij}(n), \qquad \mathbf{p}'(n) = \mathbf{p}'(0)\mathbf{P}^n,$$

or n iterations with the one-step transition probabilities:

$$p_j(n) = \sum_{i=0}^{K} p_i(n-1)P_{ij}, \qquad \mathbf{p}'(n) = \mathbf{p}'(n-1)\mathbf{P}.$$

Proof From Definition 2,

$$\begin{aligned} p_j(n) &= \mathrm{P}[X_n = j] \\ &= \sum_{i=0}^{K} \mathrm{P}[X_n = j|X_0 = i]\,\mathrm{P}[X_0 = i] = \sum_{i=0}^{K} P_{ij}(n)p_i(0). \end{aligned} \tag{9}$$

From the definition of the transition probabilities,

$$\begin{aligned} p_j(n) &= \mathrm{P}[X_n = j] \\ &= \sum_{i=0}^{K} \mathrm{P}[X_n = j|X_{n-1} = i]\,\mathrm{P}[X_{n-1} = i] = \sum_{i=0}^{K} P_{ij}p_i(n-1). \end{aligned} \tag{10}$$

Example 7

For the two-state Markov chain described in Example 1 with initial state probabilities $\mathbf{p}(0) = \begin{bmatrix} p_0 & p_1 \end{bmatrix}$, find the state probability vector $\mathbf{p}(n)$.

. . . .

By Theorem 4, $\mathbf{p}'(n) = \mathbf{p}'(0)\mathbf{P}^n$. From $\mathbf{P}(n)$ found in Equation (7) of Example 6, we can write the state probabilities at time n as

$$\begin{aligned} \mathbf{p}'(n) &= \begin{bmatrix} p_0(n) & p_1(n) \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{q}{p+q} & \dfrac{p}{p+q} \end{bmatrix} + \lambda_2^n \begin{bmatrix} \dfrac{p_0 p - p_1 q}{p+q} & \dfrac{-p_0 p + p_1 q}{p+q} \end{bmatrix}, \end{aligned} \tag{11}$$

where $\lambda_2 = 1 - (p+q)$.

Quiz 2

Stock traders pay close attention to the “ticks” of a stock. A stock can trade on an uptick, even tick, or downtick, if the trade price is higher, the same, or lower than the previous trade price. For a particular stock, traders have observed that the ticks are accurately modeled by a Markov chain. Following an even tick, the next trade is an even tick with probability 0.6, an uptick with probability 0.2, or a downtick with probability 0.2. After a downtick, another downtick has probability 0.4, while an even tick has probability 0.6. After an uptick, another uptick occurs with probability 0.4, while an even tick occurs with probability 0.6. Using states $0, 1, 2$ to denote the previous trade being a downtick, an even tick, or an uptick, sketch the Markov chain, and find the state transition matrix $\mathbf{P}$ and the n-step transition matrix $\mathbf{P}^n$.

3 Limiting State Probabilities for a Finite Markov Chain

Limiting state probabilities comprise a probability model of state occupancy in the distant future. The limiting state probabilities may depend the initial system state.

An important task in analyzing Markov chains is to examine the state probability vector $\mathbf{p}(n)$ as n becomes very large.

Definition 4 Limiting State Probabilities

For a finite Markov chain with initial state probability vector $\mathbf{p}(0)$, *the* ***limiting state probabilities****, when they exist, are defined to be the vector*

$$\boldsymbol{\pi} = \lim_{n\to\infty} \mathbf{p}(n).$$

The jth element, π_j, of $\boldsymbol{\pi}$ is the probability the system will be in state j in the distant future.

Example 8

For the two-state packet voice system of Example 2, what is the limiting state probability vector $\begin{bmatrix}\pi_0 & \pi_1\end{bmatrix}' = \lim_{n\to\infty} \mathbf{p}(n)$?

For initial state probabilities $\mathbf{p}'(0) = \begin{bmatrix} p_0 & p_1 \end{bmatrix}'$, the state probabilities at time n are given in Equation (11) with $p = 1/140$ and $q = 1/100$. Note that the second eigenvalue is $\lambda_2 = 1 - (p+q) = 344/350$. Thus

$$\mathbf{p}'(n) = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} + \lambda_2^n \begin{bmatrix} \frac{5}{12}p_0 - \frac{7}{12}p_1 & \frac{-5}{12}p_0 + \frac{7}{12}p_1 \end{bmatrix}. \tag{12}$$

Since $|\lambda_2| < 1$, the limiting state probabilities are

$$\lim_{n\to\infty} \mathbf{p}'(n) = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix}. \tag{13}$$

For this system, the limiting state probabilities are the same regardless of how we choose the initial state probabilities.

The two-state packet voice system is an example of a well-behaved system in which the limiting state probabilities exist and are independent of the initial state of the system $\mathbf{p}(0)$. In general, π_j may or may not exist and if it exists, it may or may not depend on the initial state probability vector $\mathbf{p}(0)$. As we see in the next theorem, limiting state probability vectors must satisfy certain constraints.

Theorem 5

If a finite Markov chain with transition matrix $\mathbf{P}$ *and initial state probability* $\mathbf{p}(0)$ *has limiting state probability vector* $\boldsymbol{\pi} = \lim_{n\to\infty} \mathbf{p}(n)$, *then*

$$\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}.$$

Proof By Theorem 4, $\mathbf{p}'(n+1) = \mathbf{p}'(n)\mathbf{P}$. In the limit of large n,

$$\lim_{n\to\infty} \mathbf{p}'(n+1) = \left(\lim_{n\to\infty} \mathbf{p}'(n)\right)\mathbf{P}. \tag{14}$$

Given that the limiting state probabilities exist, $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$.

Closely related to limiting state probabilities are *stationary probabilities*.

Definition 5 Stationary Probability Vector

For a finite Markov chain with transition matrix $\mathbf{P}$, *a state probability vector* $\boldsymbol{\pi}$ *is* ***stationary*** *if* $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$.

If the system is initialized with a stationary probability vector, then the state probabilities are stationary; i.e., they never change: $\mathbf{p}(n) = \boldsymbol{\pi}$ for all n. We can also prove a much stronger result that the Markov chain X_n is a stationary process.

Theorem 6

If a finite Markov chain X_n with transition matrix $\mathbf{P}$ is initialized with stationary probability vector $\mathbf{p}(0) = \boldsymbol{\pi}$, then $\mathbf{p}(n) = \boldsymbol{\pi}$ for all n and the stochastic process X_n is stationary.

Proof First, we show by induction that $\mathbf{p}(n) = \boldsymbol{\pi}$ for all n. Since $\mathbf{p}(0) = \boldsymbol{\pi}$, assume $\mathbf{p}(n-1) = \boldsymbol{\pi}$. By Theorem 4, $\mathbf{p}'(n) = \mathbf{p}'(n-1)\mathbf{P} = \boldsymbol{\pi}'\mathbf{P} = \boldsymbol{\pi}'$. Now we can show stationarity of the process X_n. By Definition 13.14, we must show that for any set of time instances $n_1, \ldots, n_m$ and time offset k that

$$P_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = P_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m). \tag{15}$$

Because the system is a Markov chain and $P_{X_{n_1}}(x_1) = \pi_{x_1}$, we observe that

$$\begin{aligned} P_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) &= P_{X_{n_1}}(x_1)\, P_{X_{n_2}|X_{n_1}}(x_2|x_1) \cdots P_{X_{n_m}|X_{n_{m-1}}}(x_2|x_1) \\ &= \pi_{x_1} P_{x_1 x_2}(n_2 - n_1) \cdots P_{x_{m-1} x_m}(n_m - n_{m-1}). \end{aligned} \tag{16}$$

By the first part of this theorem, $P_{X_{n_1+k}}(x_1) = \pi_{x_1}$. Once again, because the system is a Markov chain,

$$\begin{aligned} P_{X_{n_j+k}|X_{n_{j-1}+k}}(x_j|x_{j-1}) &= P_{x_{j-1} x_j}(n_j + k - (n_{j-1} + k)) \\ &= P_{x_{j-1} x_j}(n_j - n_{j-1}). \end{aligned} \tag{17}$$

It follows that

$$\begin{aligned} P_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m) &= \pi_{x_1} P_{x_1 x_2}(n_2 - n_1) \cdots P_{x_{m-1} x_m}(n_m - n_{m-1}) \\ &= P_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m). \end{aligned} \tag{18}$$

A Markov chain in which the state probabilities are stationary is often said to be in *steady-state*. When the limiting state probabilities exist and are unique, we can assume the system has reached steady-state by simply letting the system run for a long time. The following example presents common terminology associated with Markov chains in steady-state.

Example 9

A queueing system is described by a Markov chain in which the state X_n is the number of customers in the queue at time n. The Markov chain has a unique stationary distribution $\boldsymbol{\pi}$. The following questions are all equivalent.

- What is the steady-state probability of at least 10 customers in the system?
- If we inspect the queue in the distant future, what is the probability of at least 10 customers in the system?
- What is the stationary probability of at least 10 customers in the system?
- What is the limiting probability of at least 10 customers in the queue?

For each statement of the question, the answer is just $\sum_{j\geq 10} \pi_j$.

Although we have seen that we can calculate limiting state probabilities, the significance of these probabilities may not be so apparent. For a system described by a "well-behaved" Markov chain, the key idea is that π_j, the probability the system is in state j after a very long time, should depend only on the fraction of time the system spends in state j. In particular, after a very long time, the effect of an initial condition should have worn off and π_j should *not* depend on the system having started in state i at time $n = 0$.

This intuition may seem reasonable but, in fact, it depends critically on a precise definition of a "well-behaved" chain. In particular, for a finite Markov chain, there are three distinct possibilities:

- $\lim_{n\to\infty} \mathbf{p}(n)$ exists, independent of the initial state probability vector $\mathbf{p}(0)$,
- $\lim_{n\to\infty} \mathbf{p}(n)$ exists, but depends on $\mathbf{p}(0)$,
- $\lim_{n\to\infty} \mathbf{p}(n)$ does not exist.

We will see that the "well-behaved" first case corresponds to a Markov chain $\mathbf{P}$ with a unique stationary probability vector $\boldsymbol{\pi}$. The latter two cases are considered "ill-behaved." The second case occurs when the Markov chain has multiple stationary probability vectors; the third case occurs when there is no stationary probability vector. In the following example, we use the two-state Markov chain to demonstrate these possibilities.

Example 10

Consider the two-state Markov chain of Example 1 and Example 6. For what values of p and q does $\lim_{n\to\infty} \mathbf{p}(n)$

(a) exist, independent of the initial state probability vector $\mathbf{p}(0)$;
(b) exist, but depend on $\mathbf{p}(0)$;
(c) or not exist?

. . . .

In Equation (7) of Example 6, we found that the n-step transition matrix $\mathbf{P}^n$ could be expressed in terms of the eigenvalue $\lambda_2 = 1 - (p+q)$ as

$$\mathbf{P}^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{\lambda_2^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}. \tag{19}$$

We see that whether a limiting state distribution exists depends on the value of the eigenvalue λ_2. There are three basic cases described here and shown in Figure 1.

(a) $0 < p + q < 2$

This case is described in Example 6. In this case, $|\lambda_2| < 1$ and

$$\lim_{n\to\infty} \mathbf{P}^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix}. \tag{20}$$

No matter how the initial state probability vector $\mathbf{p}'(0) = \begin{bmatrix} p_0 & p_1 \end{bmatrix}$ is chosen, after a long time we are in state 0 with probability $q/(p+q)$, or we are in state 1 with probability $p/(p+q)$ independent of the initial state distribution $\mathbf{p}'(0)$.

(b) $p = q = 0$

In this case, $\lambda_2 = 1$ and

$$\mathbf{P}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

When we start in state i, we stay in state i forever since no state changing transitions are possible. Consequently, $\mathbf{p}(n) = \mathbf{p}(0)$ for all n and the initial conditions completely dictate the limiting state probabilities.

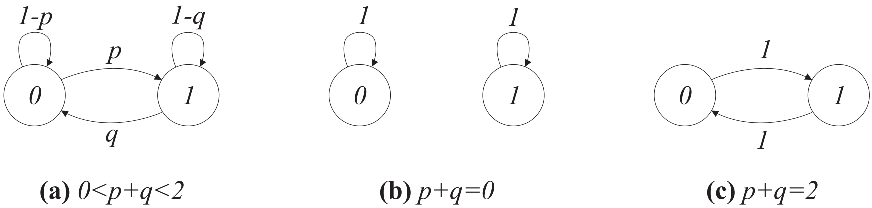


Figure 1 Three possibilities for the two-state Markov chain

(c) $p+q=2$

In this instance, $\lambda_2=-1$ so that

$$\mathbf{P}^n = \frac{1}{2}\begin{bmatrix} 1+(-1)^n & 1-(-1)^n \\ 1-(-1)^n & 1+(-1)^n \end{bmatrix}. \tag{22}$$

We observe that

$$\mathbf{P}^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{P}^{2n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{23}$$

In this case, $\lim_{n\to\infty}\mathbf{P}^n$ doesn't exist. Physically, if we start in state i at time 0, then we are in state i at every time $2n$. In short, the sequence of states has a periodic behavior with a period of two steps. Mathematically, we have the state probabilities $\mathbf{p}(2n)=[p_0\ p_1]$ and $\mathbf{p}(2n+1)=[p_1\ p_0]$. This periodic behavior does not permit the existence of limiting state probabilities.

The characteristics of these three cases should be apparent from Figure 1.

In the next section, we will see that the ways in which the two-state chain can fail to have unique limiting state probabilities are typical of Markov chains with many more states.

Quiz 3

A microchip fabrication plant works properly most of the time. After a day in which the plant is working, the plant is working the next day with probability

0.9. Otherwise, a day of repair followed by a day of testing is required to restore the plant to working status. Sketch a Markov chain for the plant states (0) working, (1) repairing, and (2) testing. Given an initial state distribution $\mathbf{p}(0)$, find the limiting state probabilities $\boldsymbol{\pi} = \lim_{n\to\infty} \mathbf{p}(n)$.

4 State Classification

> A *communicating class* is a set of states such that every state can be reached from any other state. A Markov chain with just one communicating class is called *irreducible*. A communicating class has*period d* if the time until the system can return to a state is always a multiple of d with $d > 1$. A chain is *aperiodic* if there is no such d.

In Example 10, we saw that the chain does not have unique limiting state probabilities when either the chain disconnects into two separate chains or when the chain causes periodic behavior in the state transitions. We will see that these two ways in which the two-state chain fails to have unique limiting state probabilities are typical of Markov chains with many more states. In particular, we will see that for Markov chains with certain structural properties, the state probabilities will converge to a unique stationary probability vector independent of the initial distribution $\mathbf{p}(0)$. In this section, we describe the structure of a Markov chain by classifying the states.

Definition 6 — Accessibility

State j is ***accessible*** *from state i, written $i \to j$, if $P_{ij}(n) > 0$ for some $n > 0$.*

When j is not accessible from i, we write $i \not\to j$. In the Markov chain graph, $i \to j$ if there is a path from i to j.

Definition 7 — Communicating States

States i and j ***communicate****, written $i \leftrightarrow j$, if $i \to j$ and $j \to i$.*

We adopt the convention that state i always communicates with itself since the system can reach i from i in zero steps. Hence for any state i, there is a set of states that communicate with i. Moreover, if both j and k communicate with i, then j and k must communicate. To verify this, we observe that we can go from j to i to k or we can go from k to i to j. Thus associated with any state i there is a *set* or a *class* of states that all communicate with each other.

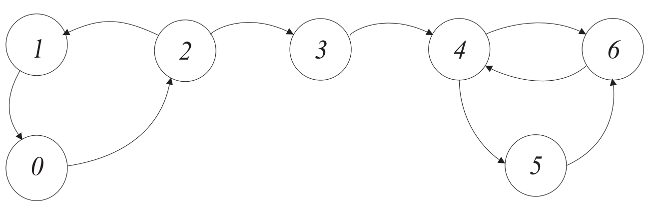
Definition 8 Communicating Class

*A **communicating class** is a nonempty subset of states C such that if $i \in C$, then $j \in C$ if and only if $i \leftrightarrow j$.*

A communicating class includes all possible states that communicate with a member of the communicating class. That is, a set of states C is not a communicating class if there is a state $j \notin C$ that communicates with a state $i \in C$.

Example 11

In the following Markov chain, we draw the branches corresponding to transition probabilities $P_{ij} > 0$ without labeling the actual transition probabilities. For this chain, identify the communicating classes.



. . . .

This chain has three communicating classes. First, we note that states 0, 1, and 2 communicate and form a class $C_1 = \{0, 1, 2\}$. Second, we observe that $C_2 = \{4, 5, 6\}$ is a communicating class since every pair of states in C_2 communicates. The state 3 communicates only with itself and $C_3 = \{3\}$.

In Example 10, we observed that we could not calculate the limiting state probabilities when the sequence of states had a periodic behavior. The periodicity is defined by $P_{ii}(n)$, the probability that the system is in state i at time n given that the system is in state i at time 0.

Definition 9 — Periodic and Aperiodic States

*State i has **period** d if d is the largest integer such that $P_{ii}(n) = 0$ whenever n is not divisible by d. If $d = 1$, then state i is called **aperiodic**.*

The following theorem says that all states in a communicating class have the same period.

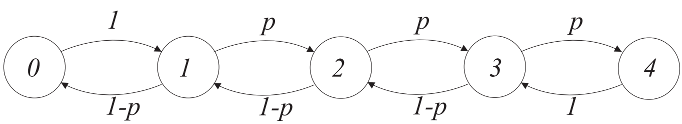
Theorem 7

Communicating states have the same period.

Proof Let $d(i)$ and $d(j)$ denote the periods of states i and j. For some m and n, there is an n-hop path from j to i and an m-hop path from i to j. Hence $P_{jj}(n+m) > 0$ and the system can go from j to j in $n+m$ hops. This implies $d(j)$ divides $n+m$. For any k such that $P_{ii}(k) > 0$, the system can go from j to i in n hops, from i back to i in k hops, and from i to j in m hops. This implies $P_{jj}(n+k+m) > 0$ and so $d(j)$ must divide $n+k+m$. Since $d(j)$ divides $n+m$ and also $n+k+m$, $d(j)$ must divide k. Since this must hold for any k divisible by $d(i)$, we must have $d(j) \le d(i)$. Reversing the labels of i and j in this argument will show that $d(i) \le d(j)$. Hence they must be equal.

Example 12

Consider the five-position discrete random walk with Markov chain



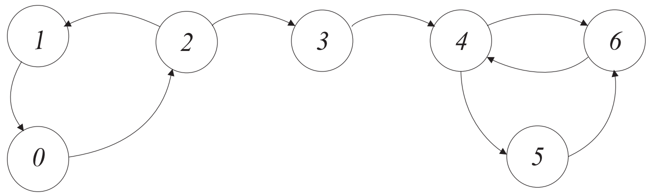
What is the period of each state i?

. . . .

In this discrete random walk, we can return to state i only after an even number of transitions because each transition away from state i requires a transition back toward state i. Consequently, $P_{ii}(n) = 0$ whenever n is not divisible by 2 and every state i has period 2.

Example 13

Example 11 presented the Markov chain on the right. Find the periodicity of each communicating class.



By inspection of the Markov chain, states 0, 1, and 2 in communicating class C_1 have period $d = 3$. States 4, 5, and 6 in communicating class C_2 are aperiodic.

To analyze the long-term behavior of a Markov chain, it is important to distinguish between those *recurrent* states that may be visited repeatedly and *transient* states that are visited perhaps only a small number of times.

Definition 10 Transient and Recurrent States

In a finite Markov chain, a state i is ***transient*** *if there exists a state j such that $i \to j$ but $j \not\to i$; otherwise, if no such state j exists, then state i is* ***recurrent****.*

As in the case of periodicity, such properties will be coupled to the communicating classes. The next theorem verifies that if two states communicate, then both must be either recurrent or transient.

Theorem 8

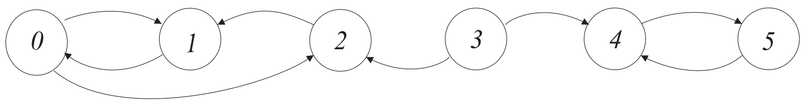
If i is recurrent and $i \leftrightarrow j$, then j is recurrent.

Proof For any state k, we must show $j \to k$ implies $k \to j$. Since $i \leftrightarrow j$, and $j \to k$, there is a path from i to k via j. Thus $i \to k$. Since i is recurrent, we must have $k \to i$. Since $i \leftrightarrow j$, there is a path from j to i and then to k.

The implication of Theorem 8 is that if any state in a communicating class is recurrent, then all states in the communicating class must be recurrent. Similarly, if any state in a communicating class is transient, then all states in the class must be transient. That is, recurrence and transience are properties of the communicating class. When the states of a communicating class are recurrent, we say we have a *recurrent class*.

Example 14

In the following Markov chain, the branches indicate transition probabilities $P_{ij} > 0$.



Identify each communicating class and indicate whether it is transient or recurrent.

By inspection of the Markov chain, the communicating classes are $C_1 = \{0, 1, 2\}$, $C_2 = \{3\}$, and $C_3 = \{4, 5\}$. Classes C_1 and C_3 are recurrent while C_2 is transient.

In the next theorem, we verify that the expected number of visits to a transient state must be finite since the system will eventually enter a state from which the transient state is no longer reachable.

Theorem 9

If state i is transient, then N_i, the number of visits to state i over all time, has expected value $\mathrm{E}[N_i] < \infty$.

Proof Given an initial state distribution, let V_i denote the event that the system eventually goes to state i. Obviously $\mathrm{P}[V_i] \leq 1$. If V_i does not occur, then $N_i = 0$, implying $\mathrm{E}[N_i|V_i^c] = 0$. Otherwise, there exists a time n when the system first enters state i. In this case, given that the state is i, let V_{ii} denote the event that the system eventually returns to state i. Thus V_{ii}^c is the event that the system never returns to state i. Since i is transient, there exists state j such that for some n, $P_{ij}(n) > 0$ but i is not accessible from j. Thus if we enter state j at time n, the event V_{ii}^c will occur. Since this is one possible way that V_{ii}^c can occur, $\mathrm{P}[V_{ii}^c] \geq P_{ij}(n) > 0$. After each return to i, there is a probability $\mathrm{P}[V_{ii}^c] > 0$ that state i will never be reentered. Hence, given V_i, the expected number of visits to i is geometric with conditional expected value $\mathrm{E}[N_i|V_i] = 1/\,\mathrm{P}[V_{ii}^c]$. Finally,

$$\mathrm{E}\left[N_i\right] = \mathrm{E}\left[N_i|V_i^c\right]\mathrm{P}\left[V_i^c\right] + \mathrm{E}\left[N_i|V_i\right]\mathrm{P}\left[V_i\right] = \mathrm{E}\left[N_i|V_i\right]\mathrm{P}\left[V_i\right] < \infty. \tag{24}$$

Consequently, in a *finite state* Markov chain, not all states can be transient; otherwise, we would run out of states to visit. Thus a finite Markov chain must always have a set of recurrent states.

Theorem 10

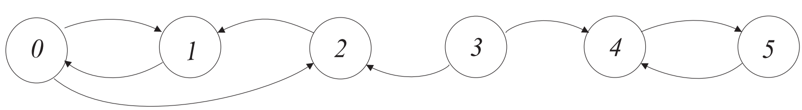
A finite-state Markov chain always has a recurrent communicating class.

This implies we can partition the set of states into a set of transient states T and a set of r recurrent communicating classes $\{C_1, C_2, \ldots, C_r\}$. In particular, in the proof of Theorem 9, we observed that each transient state is either never visited, or if it is visited once, then it is visited a geometric number of times. Eventually, one of the recurrent communicating classes is entered and the system remains forever in that communicating class. In terms of the evolution of the system state, we have the following possibilities.

- If the system starts in a recurrent class C_l, the system stays forever in C_l.
- If the system starts in a transient state, the system passes through transient states for a finite period of time until the system randomly lands in a recurrent class C_l. The system then stays in the class C_l forever.

Example 15

Consider again the Markov chain of Example 14:



We can make the following observations.

- If the system starts in state $j \in C_1 = \{0, 1, 2\}$, the system never leaves C_1.
- Similarly, if the system starts in communicating class $C_3 = \{4, 5\}$, the system never leaves C_3.
- If the system starts in the transient state 3, then in the first step there is a random transition to either state 2 or to state 4 and the system then remains forever in the corresponding communicating class.

One can view the different recurrent classes as individual component systems. The only connection between these component systems is the initial transient phase that results in a random selection of one of these systems. For a Markov chain with multiple recurrent classes, it behooves us to treat the recurrent classes as individual systems and to examine them separately. When we focus on the behavior of an individual communicating class, this communicating class might just as well be the whole Markov chain. Thus we define a chain with just one communicating class.

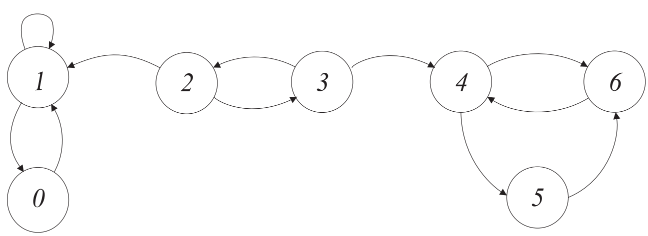
Definition 11 Irreducible Markov Chain

A Markov chain is ***irreducible*** *if there is only one communicating class.*

In the following section, we focus on the properties of a single recurrent class.

Quiz 4

In this Markov chain, all transitions with nonzero probability are shown.



(a) What are the communicating classes?

(b) For each communicating class, identify whether the states are periodic or aperiodic.

(c) For each communicating class, identify whether the states are transient or recurrent.

5 Limit Theorems For Irreducible Finite Markov Chains

> Stationary probabilities comprise a probability model that does not change with time. Limiting state probabilities are stationary probabilities. An aperiodic, irreducible, finite discrete-time Markov chain has unique limiting state probabilities, independent of the initial system state.

Section 3 introduced limiting state probabilities for discrete-time Markov chains. For Markov chains with multiple recurrent classes, we have observed that the limiting state probabilities depend on the initial state distribution. For a complete understanding of a system with multiple communicating classes, we need to examine each recurrent class separately as an irreducible system consisting of just that class. In this section, we focus on irreducible, aperiodic chains and their limiting state probabilities.

Theorem 11

For an irreducible, aperiodic, finite Markov chain with states $\{0, 1, \dots, K\}$, the limiting n-step transition matrix is

$$\lim_{n\to\infty} \mathbf{P}^n = \mathbf{1}\boldsymbol{\pi}' = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_K \\ \pi_0 & \pi_1 & \cdots & \pi_K \\ \vdots & & \ddots & \vdots \\ \pi_0 & \pi_1 & & \pi_K \end{bmatrix}$$

where $\mathbf{1}$ *is the column vector* $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}'$ *and* $\boldsymbol{\pi} = \begin{bmatrix}\pi_0 & \cdots & \pi_K\end{bmatrix}'$ *is the unique vector satisfying*

$$\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}, \qquad \boldsymbol{\pi}'\mathbf{1} = 1.$$

Proof The steps of a proof are outlined in Problem 5.19.

When the set of possible states of the Markov chain is $\{0, 1, \ldots, K\}$, the system of equations $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ has $K+1$ equations and $K+1$ unknowns. Normally, $K+1$ equations are sufficient to determine the $K+1$ unknowns uniquely. However, the particular set of equations, $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$, does not not have a unique solution. If $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$, then for any constant c, $\mathbf{x} = c\boldsymbol{\pi}$ is also a solution since $\mathbf{x}'\mathbf{P} = c\boldsymbol{\pi}'\mathbf{P} = c\boldsymbol{\pi}' = \mathbf{x}'$. In this case, there is a redundant equation. To obtain a unique solution we use the fact that $\boldsymbol{\pi}$ is a state probability vector by explicitly including the equation $\boldsymbol{\pi}'\mathbf{1} = \sum_j \pi_j = 1$. Specifically, to find the stationary probability vector $\boldsymbol{\pi}$, we must replace one of the equations in the system of equations $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ with the equation $\boldsymbol{\pi}'\mathbf{1} = 1$.

Note that the result of Theorem 11 is precisely what we observed for the two-state Markov chain in Example 6 when $|\lambda_2| < 1$ and the two-state chain is irreducible and aperiodic. Moreover, Theorem 11 implies convergence of the limiting state probabilities to the probability vector $\boldsymbol{\pi}$, independent of the initial state probability vector $\mathbf{p}(0)$.

Theorem 12

For an irreducible, aperiodic, finite Markov chain with transition matrix $\mathbf{P}$ *and initial state probability vector* $\mathbf{p}(0)$, $\lim_{n\to\infty} \mathbf{p}(n) = \boldsymbol{\pi}$.

Proof Recall that $\mathbf{p}'(n) = \mathbf{p}'(0)\mathbf{P}^n$. Since $\mathbf{p}'(0)\mathbf{1} = 1$, Theorem 11 implies

$$\lim_{n\to\infty} \mathbf{p}'(n) = \mathbf{p}'(0)\left(\lim_{n\to\infty} \mathbf{P}(n)\right) = \mathbf{p}'(0)\mathbf{1}\boldsymbol{\pi}' = \boldsymbol{\pi}'. \tag{25}$$

Example 16

For the packet voice communications system of Example 8, use Theorem 12 to calculate the stationary probabilities $\begin{bmatrix}\pi_0 & \pi_1\end{bmatrix}$.

. . . .

From the Markov chain depicted in Example 8, Theorem 12 yields the following three equations:

$$\pi_0 = \pi_0 \frac{139}{140} + \pi_1 \frac{1}{100}, \tag{26}$$

$$\pi_1 = \pi_0 \frac{1}{140} + \pi_1 \frac{99}{100}, \tag{27}$$

$$1 = \pi_0 + \pi_1. \tag{28}$$

We observe that Equations (26) and (27) are dependent in that each equation can be simplified to $\pi_1 = (100/140)\pi_0$. Applying $\pi_0 + \pi_1 = 1$ yields

$$\pi_0 + \pi_1 = \pi_0 + \frac{100}{140}\pi_0 = 1. \tag{29}$$

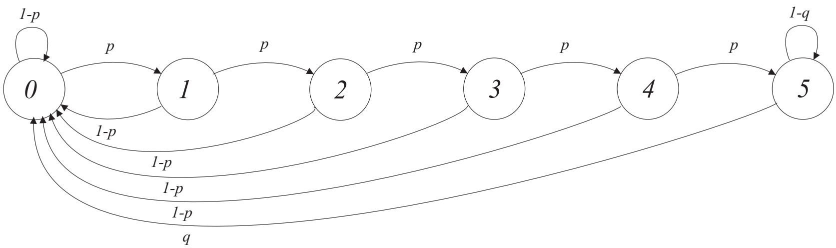
Thus $\pi_0 = 140/240 = 7/12$ and $\pi_1 = 5/12$.

Example 17

A digital mobile phone transmits one packet in every 20-ms time slot over a wireless connection. With probability $p = 0.1$, a packet is received in error, independent of any other packet. To avoid wasting transmitter power when the link quality is poor, the transmitter enters a timeout state whenever five consecutive packets are received in error. During a timeout, the mobile terminal performs an independent Bernoulli trial with success probability $q = 0.01$ in every slot. When a success occurs, the mobile terminal starts transmitting in the next slot as though no packets had been in error. Construct a Markov chain for this system. What are the limiting state probabilities?

. . . .

For the Markov chain, we use the 20-ms slot as the unit of time. For the state of the system, we can use the number of consecutive packets in error. The state corresponding to five consecutive corrupted packets is also the timeout state. The Markov chain is



The limiting state probabilities satisfy

$$\pi_n = p\pi_{n-1}, \qquad n = 1, 2, 3, 4, \tag{30}$$

$$\pi_5 = p\pi_4 + (1-q)\pi_5. \tag{31}$$

These equations imply that for $n = 1, 2, 3, 4$,

$$\pi_n = \begin{cases} p^n \pi_0 & n = 1, 2, 3, 4, \\ p^5\pi_0/(1-q) & n = 5. \end{cases} \tag{32}$$

Since $\sum_{n=1}^{5} \pi_n = 1$, we have

$$\pi_0 + \cdots + \pi_5 = \pi_0 \left[1 + p + p^2 + p^3 + p^4 + p^5/(1-q)\right] = 1. \tag{33}$$

Since $1 + p + p^2 + p^3 + p^4 = (1-p^5)/(1-p)$,

$$\pi_0 = \frac{1}{(1-p^5)/(1-p) + p^5/(1-q)} = \frac{(1-q)(1-p)}{1-q+qp^5-p^6}, \tag{34}$$

and the limiting state probabilities are

$$\pi_n = \begin{cases} \frac{(1-q)(1-p)p^n}{1-q+qp^5-p^6} & n = 0, 1, 2, 3, 4, \\ \frac{(1-p)p^5}{1-q+qp^5-p^6} & n = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$\approx \begin{cases} 9 \times 10^{-(n+1)} & n = 0, 1, 2, 3, 4, \\ 9.09 \times 10^{-6} & n = 5, \\ 0 & \text{otherwise}. \end{cases} \tag{35}$$

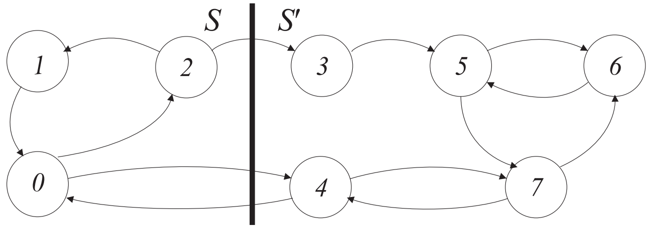


Figure 2 The vertical bar indicates a partition of the Markov chain state space into mutually exclusive subsets $S = \{0, 1, 2\}$ and $S' = \{3, 4, 5, \ldots\}$.

In Example 16, we were fortunate in that the stationary probabilities $\boldsymbol{\pi}$ could be found directly from Theorem 11 by solving $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$. It is more often the case that Theorem 11 leads to a lot of messy equations that cannot be solved by hand. On the other hand, there are some Markov chains where a little creativity can yield simple closed-form solutions for $\boldsymbol{\pi}$ that may not be obvious from the direct method of Theorem 11.

We now develop a simple but useful technique for calculating the stationary probabilities. The idea is that we partition the state space of an irreducible, aperiodic, finite Markov chain into mutually exclusive subsets S and S', as depicted in Figure 2. We will count crossings back and forth across the $S - S'$ boundary. The key observation is that the cumulative number of $S \to S'$ crossings and $S' \to S$ crossings cannot differ by more than 1 because we cannot make two $S \to S'$ crossings without an $S' \to S$ crossing in between. It follows that the expected crossing rates must be the same. This observation is summarized in the following theorem, with the details of this argument appearing in the proof.

Theorem 13

Consider an irreducible, aperiodic, finite Markov chain with transition probabilities $\{P_{ij}\}$ and stationary probabilities $\{\pi_i\}$. For any partition of the state space into mutually exclusive subsets S and S',

$$\sum_{i \in S} \sum_{j \in S'} \pi_i P_{ij} = \sum_{j \in S'} \sum_{i \in S} \pi_j P_{ji}.$$

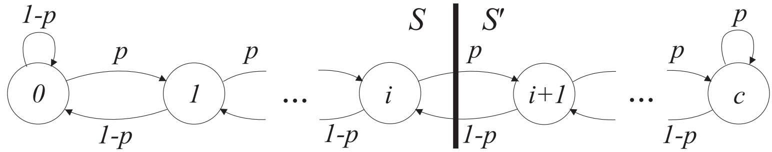


Figure 3 The Markov chain for the packet buffer of Example 18. Also shown is the $S - S'$ partition we use to calculate the stationary probabilities.

Proof To track the occurrence of an $S \to S'$ crossing at time m, we define the indicator $I_{SS'}(m) = 1$ if $X_m \in S$ and $X_{m+1} \in S'$; otherwise $I_{SS'}(m) = 0$. The expected value of the indicator is

$$\mathrm{E}\left[I_{SS'}(m)\right] = \mathrm{P}\left[I_{SS'}(n) = 1\right] = \sum_{i \in S} \sum_{j \in S'} p_i(m) P_{ij}. \tag{36}$$

The cumulative number of $S \to S'$ crossings by time n is

$$N_{SS'}(n) = \sum_{m=0}^{n-1} I_{SS'}(m), \tag{37}$$

which has expected value

$$\mathrm{E}\left[N_{SS'}(n)\right] = \sum_{m=0}^{n-1} \mathrm{E}\left[I_{SS'}(m)\right] = \sum_{m=0}^{n-1} \sum_{i \in S} \sum_{j \in S'} p_i(m) P_{ij}. \tag{38}$$

Dividing by n and changing the order of summation, we obtain

$$\frac{1}{n} \mathrm{E}\left[N_{SS'}(n)\right] = \sum_{i \in S} \sum_{j \in S'} P_{ij} \frac{1}{n} \sum_{m=0}^{n-1} p_i(m). \tag{39}$$

Note that $p_i(m) \to \pi_i$ implies that $\frac{1}{n} \sum_{m=0}^{n-1} p_i(m) \to \pi_i$. This implies

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{E}\left[N_{SS'}(n)\right] = \sum_{i \in S} \sum_{j \in S'} \pi_i P_{ij}. \tag{40}$$

By the same logic, the number of $S' \to S$ crossings, $N_{S'S}(n)$, satisfies

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{E}\left[N_{S'S}(n)\right] = \sum_{j\in S'} \sum_{i\in S} \pi_j P_{ji}. \tag{41}$$

Since we cannot make two $S \to S'$ crossings without an $S' \to S$ crossing in between,

$$N_{S'S}(n) - 1 \le N_{SS'}(n) \le N_{S'S}(n) + 1. \tag{42}$$

Taking expected values, and dividing by n, we have

$$\frac{\mathrm{E}\left[N_{S'S}(n)\right] - 1}{n} \le \frac{\mathrm{E}\left[N_{SS'}(n)\right]}{n} \le \frac{\mathrm{E}\left[N_{S'S}(n)\right] + 1}{n}. \tag{43}$$

As $n \to \infty$, we have $\lim_{n\to\infty} \mathrm{E}[N_{SS'}(n)]/n = \lim_{n\to\infty} \mathrm{E}[N_{S'S}(n)]/n$. The theorem then follows from Equations (40) and (41).

Example 18

In each time slot, a router can either store an arriving data packet in its buffer or forward a stored packet (and remove that packet from its buffer). In each time slot, a new packet arrives with probability p, independent of arrivals in all other slots. This packet is stored as long as the router is storing fewer than c packets. If c packets are already buffered, then the new packet is discarded by the router. If no new packet arrives and $n > 0$ packets are buffered by the router, then the router will forward one buffered packet. That packet is then removed from the buffer. Let X_n denote the number of buffered packets at time n. Sketch the Markov chain for X_n and find the stationary probabilities.

. . . .

From the description of the system, the buffer occupancy is given by the Markov chain in Figure 3. The figure shows the $S - S'$ boundary where we apply Theorem 13, yielding

$$\pi_{i+1} = \frac{p}{1-p}\pi_i. \tag{44}$$

Since Equation (44) holds for $i = 0, 1, \ldots, c-1$, we have that $\pi_i = \pi_0 \alpha^i$ where $\alpha = p/1 - p$.

Requiring the state probabilities to sum to 1, we have

$$\sum_{i=0}^{c} \pi_i = \pi_0 \sum_{i=0}^{c} \alpha^i = \pi_o \frac{1-\alpha^{c+1}}{1-\alpha} = 1. \tag{45}$$

The complete state probabilities are

$$\pi_i = \frac{1-\alpha}{1-\alpha^{c-1}} \alpha^i, \qquad i = 0, 1, 2, \ldots, c. \tag{46}$$

Quiz 5

Let N be a integer-valued positive random variable with range

$$S_N = \{1, \ldots, K+1\}. \tag{47}$$

We use N to generate a Markov chain in the following way. When the system is in state 0, we generate a sample value of random variable N. If $N = n$, the system transitions from state 0 to state $n-1$. In any state $i \in \{1, \ldots, K\}$, the next system state is $n-1$. Sketch the Markov chain and find the stationary probability vector.

6 Periodic States and Multiple Communicating Classes

A periodic Markov chain does not have limiting state probabilities. A Markov chain with multiple communicating classes has limiting state probabilities that depend on the initial state.

In Section 5, we analyzed the limiting state probabilities of irreducible, aperiodic, finite Markov chains. In this section, we consider problematic finite chains with periodic states and multiple communicating classes.

We start with irreducible Markov chains with periodic states. The following theorem for periodic chains is equivalent to Theorem 11 for aperiodic chains.

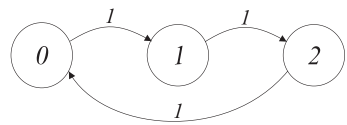
Theorem 14

For an irreducible, recurrent, periodic, finite Markov chain with transition probability matrix $\mathbf{P}$, *the stationary probability vector* $\boldsymbol{\pi}$ *is the unique non-negative solution of*

$$\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}, \qquad \boldsymbol{\pi}'\mathbf{1} = 1.$$

Example 19

Find the stationary probabilities for the Markov chain shown to the right.



We observe that each state has period 3. Applying Theorem 14, the stationary probabilities satisfy

$$\pi_1 = \pi_0, \qquad \pi_2 = \pi_1, \qquad \pi_0 + \pi_1 + \pi_2 = 1. \tag{48}$$

The stationary probabilities are $\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}1/3 & 1/3 & 1/3\end{bmatrix}$. Although the system does not have limiting state probabilities, the stationary probabilities reflect the fact that the fraction of time spent in each state is $1/3$.

Multiple communicating classes are more complicated. For a Markov chain with multiple recurrent classes, we can still use Theorem 4 to calculate the state probabilities $\mathbf{p}(n)$. Further, we will observe that $\mathbf{p}(n)$ will converge to a stationary distribution $\boldsymbol{\pi}$. However, we do need to be careful in our interpretation of these stationary probabilities because they will depend on the initial state probabilities $\mathbf{p}(0)$.

Suppose a finite Markov chain has a set of transient states and a set of recurrent communicating classes $C_1, \ldots, C_m$. In this case, each communicating class C_k acts like a mode for the system. That is, if the system starts at time 0 in a recurrent class C_k, then the system stays in class C_k and an observer of the process sees only states in C_k. Effectively, the observer sees only the mode of operation for the system associated with class C_k. If the system starts in a transient state, then the initial random transitions eventually lead to a state belonging to a recurrent communicating class. The subsequent state transitions reflect the mode of operation associated with that recurrent class.

When the system starts in a recurrent communicating class C_k, there is a set of limiting state probabilities $\boldsymbol{\pi}^{(k)}$ such that $\pi_j^{(k)} = 0$ for $j \notin C_k$. Starting in a transient state i, the limiting probabilities reflect the likelihood of ending up in each possible communicating class.

Theorem 15

For a Markov chain with recurrent communicating classes $C_1, \ldots, C_m$, let $\boldsymbol{\pi}^{(k)}$ denote the limiting state probabilities associated with class C_k. Given that the system starts in a transient state i, the limiting probability of state j is

$$\lim_{n\to\infty} P_{ij}(n) = \pi_j^{(1)}\,\mathrm{P}\left[B_{i1}\right] + \cdots + \pi_j^{(m)}\,\mathrm{P}\left[B_{im}\right]$$

where $\mathrm{P}[B_{ik}]$ is the conditional probability that the system enters class C_k.

Proof The events $B_{i1}, B_{i2}, \ldots, B_{im}$ form a partitions. For each positive recurrent state j, the law of total probability says that

$$\mathrm{P}\left[X_n = j|X_0 = i\right] = \mathrm{P}\left[X_n = j|B_{i1}\right]\mathrm{P}\left[B_{i1}\right] + \cdots + \mathrm{P}\left[X_n = j|B_{im}\right]\mathrm{P}\left[B_{im}\right]. \quad (49)$$

Given B_{ik}, the system ends up in communicating class k and

$$\lim_{n\to\infty} \mathrm{P}\left[X_n = j|B_{ik}\right] = \pi_j^{(k)}. \quad (50)$$

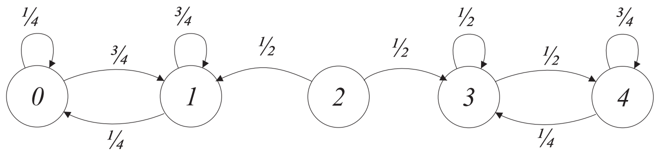
This implies

$$\lim_{n\to\infty} \mathrm{P}\left[X_n = j|X_0 = i\right] = \pi_j^{(1)}\,\mathrm{P}\left[B_{i1}\right] + \cdots + \pi_j^{(m)}\,\mathrm{P}\left[B_{im}\right]. \quad (51)$$

Theorem 15 says that if the system starts in transient state i, the limiting state probabilities will be a weighted combination of limiting state probabilities associated with each communicating class where the weights represent the likelihood of ending up in the corresponding class. These conclusions are best demonstrated by a simple example.

Example 20

For each possible starting state $i \in \{0, 1, \ldots, 4\}$, find the limiting state probabilities for the following Markov chain.



We could solve this problem by forming the 5×5 state transition matrix $\mathbf{P}$ and evaluating $\mathbf{P}^n$ as we did in Example 6, but a simpler approach is to recognize the communicating classes $C_1 = \{0, 1\}$ and $C_2 = \{3, 4\}$. Starting in state $i \in C_1$, the system operates like a two-state chain with transition probabilities $p = 3/4$ and $q = 1/4$. Let $\pi_j^{(1)} = \lim_{n\to\infty} \mathrm{P}[X_n = j | X_0 \in C_1]$ denote the limiting state probabilities. From Example 6, the limiting state probabilities for this embedded two-state chain are

$$\begin{bmatrix}\pi_0^{(1)} & \pi_1^{(1)}\end{bmatrix} = \begin{bmatrix}q/(q+p) & p/(q+p)\end{bmatrix} = \begin{bmatrix}1/4 & 3/4\end{bmatrix}. \qquad (52)$$

Since this two-state chain is within the five-state chain, the limiting state probability vector is

$$\boldsymbol{\pi}^{(1)} = \begin{bmatrix}\pi_0^{(1)} & \pi_1^{(1)} & \pi_2^{(1)} & \pi_3^{(1)} & \pi_4^{(1)}\end{bmatrix}' = \begin{bmatrix}1/4 & 3/4 & 0 & 0 & 0\end{bmatrix}'. \qquad (53)$$

When the system starts in state 3 or 4, let $\pi_j^{(2)} = \lim_{n\to\infty} \mathrm{P}[X_n = j | X_0 \in C_2]$ denote the limiting state probabilities. In this case, the system cannot leave class C_2. The limiting state probabilities are the same as if states 3 and 4 were a two-state chain with $p = 1/2$ and $q = 1/4$. Starting in class C_2, the limiting state probabilities are $\begin{bmatrix}\pi_3^{(2)} & \pi_4^{(2)}\end{bmatrix} = \begin{bmatrix}1/3 & 2/3\end{bmatrix}$. The limiting state probability vector is

$$\boldsymbol{\pi}^{(2)} = \begin{bmatrix}\pi_0^{(2)} & \pi_1^{(2)} & \pi_2^{(2)} & \pi_3^{(2)} & \pi_4^{(2)}\end{bmatrix}' = \begin{bmatrix}0 & 0 & 0 & 1/3 & 2/3\end{bmatrix}'. \qquad (54)$$

Starting in state 2, we see that the limiting state behavior depends on the first transition we make out of state 2. Let event B_{2k} denote the event that our first transition is to a state in class C_k. Given B_{21}, the system enters state 1 and the limiting probabilities are given by $\boldsymbol{\pi}^{(1)}$. Given B_{22}, the system enters state 3 and the limiting probabilities are given by $\boldsymbol{\pi}^{(2)}$. Since $\mathrm{P}[B_{21}] = \mathrm{P}[B_{22}] = 1/2$, Theorem 15 says that the limiting probabilities are

$$\lim_{n\to\infty} \mathrm{P}\left[X_n = j | X_0 = 2\right] = \frac{1}{2}\left(\pi_j^{(1)} + \pi_j^{(2)}\right). \tag{55}$$

In terms of vectors, the limiting state probabilities are

$$\boldsymbol{\pi} = \frac{1}{2}\boldsymbol{\pi}^{(1)} + \frac{1}{2}\boldsymbol{\pi}^{(2)} = \begin{bmatrix}1/8 & 3/8 & 0 & 1/6 & 1/3\end{bmatrix}'. \tag{56}$$

In Section 13.9, we introduced the concept of an ergodic wide sense stationary process in which the time average (as time t goes to infinity) of the process always equals $\mathrm{E}[X(t)]$, the process ensemble average. A Markov chain with multiple recurrent communicating classes is an example of *nonergodic* process. Each time we observe such a system, the system eventually lands in a recurrent communicating class and any long-term time averages that we calculate would reflect that particular mode of operation. On the other hand, an ensemble average, much like the limiting state probabilities we calculated in Example 20, is a weighted average over all the modes (recurrent classes) of the system.

For an irreducible finite Markov chain, the stationary probability π_n of state n does in fact tell us the fraction of time the system will spend in state n. For a chain with multiple recurrent communicating classes, π_n does tell us the probability that the system will be in state n in the distant future, but π_n is not the long-term fraction of time the system will be in state n. In that sense, when a Markov chain has multiple recurrent classes, the stationary probabilities lose much of their significance.

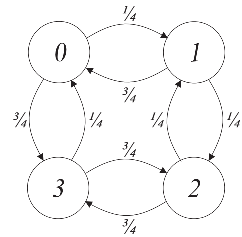
Quiz 6

Consider the Markov chain shown on the right.

(a) What is the period d of state 0?

(b) What are the stationary probabilities π_0, π_1, π_2, and π_3?

(c) Given the system is in state 0 at time 0, what is the probability the system is in state 0 at time nd in the limit as $n \to \infty$?



7 Countably Infinite Chains

> Continuous-time Markov chains are continuous-time, discrete-value processes in which the time spent in each state is an exponential random variable.

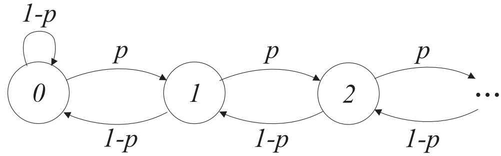
Until now, we have focused on finite-state Markov chains. In this section, we begin to examine Markov chains with a countably infinite set of states $\{0, 1, 2, \ldots\}$. We will see that a single communicating class is sufficient to describe the complications offered by an infinite number of states and we ignore the possibility of multiple communicating classes. As in the case of finite chains, multiple communicating classes represent distinct system modes that are coupled only through an initial transient phase that results in the system landing in one of the communicating classes.

An example of countably infinite Markov chain is the discrete random walk of Example 4. Many other simple yet practical examples are forms of queueing systems in which customers wait in line (queue) for service. These queueing systems have a countably infinite state space because the number of waiting customers can be arbitrarily large.

Example 21

Suppose that the router in Example 18 has unlimited buffer space. In each time slot, a router can either store an arriving data packet in its buffer or forward a stored packet (and remove that packet from its buffer). In each time slot, a new packet is stored with probability p, independent of arrivals in all other slots. If no new packet arrives, then one packet will be removed from the buffer and forwarded. Sketch the Markov chain for X_n, the number of buffered packets at time n.

From the description of the system, the buffer occupancy is given by the Markov chain:



For the general countably infinite Markov chain, we will assume the state space is the set $\{0, 1, 2, \ldots\}$. Unchanged from Definition 2, the n-step transition probabilities are given by $P_{ij}(n)$. The state probabilities at time n are specified by the set $\{p_j(n)|j = 0, 1, \ldots\}$. The Chapman-Kolmogorov equations and the iterative methods of calculating the state probabilities $p_j(n)$ in Theorem 4 also extend directly to countably infinite chains. We summarize these results here.

Theorem 16 Chapman-Kolmogorov equations

The n-step transition probabilities satisfy

$$P_{ij}(n+m) = \sum_{k=0}^{\infty} P_{ik}(n)P_{kj}(m).$$

Theorem 17

The state probabilities $p_j(n)$ at time n can be found by either one iteration with the n-step transition probabilities

$$p_j(n) = \sum_{i=0}^{\infty} p_i(0)P_{ij}(n)$$

or n iterations with the one-step transition probabilities

$$p_j(n) = \sum_{i=0}^{\infty} p_i(n-1)P_{ij}.$$

Just as for finite chains, a primary issue is the existence of limiting state probabilities $\pi_j = \lim_{n\to\infty} p_j(n)$. In Example 21, we will see that the existence of a limiting state distribution depends on the parameter p. When p is near zero, we would expect the system to have very few customers and the distribution of the number of customers to be well defined. On the other hand, if p is close to 1, we would expect the number of customers to grow steadily because most slots would have an arrival and very few slots would have departures. In the most extreme case of $p = 1$, there will be an arrival each unit of time and never any departures. When the system is such that the number of customers grows steadily, stationary probabilities do not exist.

We will see that the existence of a stationary distribution depends on the recurrence properties of the chain; however, the recurrence or transience of the system states is somewhat more complicated. For the finite chain, it was sufficient to look simply at the nonzero transition probabilities and verify that a state i communicated with every state j that was accessible from i. For the infinite Markov chain, this is not enough. For example, in the infinite buffer of Example 21, the chain has a single communicating class and state 0 communicates with every state; however, whether state 0 is recurrent will depend on the parameter p.

In this section, we develop a new definition for transient states and we define two types of recurrent states. For purposes of discussion, we make the following definition.

Definition 12 — First Return Time

Given that the system is in state i at an arbitrary time, T_{ii} is the time (number of transitions) until the system first returns to i.

Using the definition of T_{ii}, we can define transient and recurrent states for countably infinite chains.

Definition 13 — Transient and Recurrent States

For a countably infinite Markov chain, state i is ***recurrent*** *if*

$$P[T_{ii} < \infty] = \lim_{t\to\infty} F_{T_{ii}}(t) = 1;$$

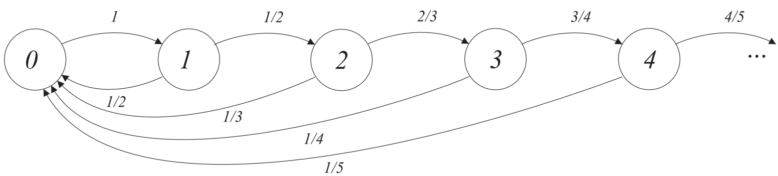
*otherwise state i is **transient**.*

When state i is transient, there is a nonzero probability that an experiment that starts the system in state i and counts how many transitions are needed to return to state i will never terminate. This is akin to the *improper experiment* in Example 3.4 in which we launched a rocket from Earth, with initial speed greater than the Earth's escape velocity, and timed how long until rocket returned to Earth.

Definition 13 can be applied to both finite and countably infinite Markov chains. In both cases, the idea is that a state is recurrent if the system is certain to return to the state. The difference is that for a finite chain it is easy to test recurrence for state i by checking that state i communicates with every state j that is accessible from i. For the countably infinite chain, the verification that $\mathrm{P}[T_{ii} < \infty] = 1$ is a far more complicated test.

Example 22

A system with states $\{0, 1, 2, \ldots\}$ has Markov chain



Note that for any state $i > 0$, $P_{i,0} = 1/(i+1)$ and $P_{i,i+1} = i/(i+1)$. Is state 0 transient, or recurrent?

Assume the system starts in state 0 at time 0. Note that $T_{00} > n$ if the system reaches state n before returning to state 0, which occurs with probability

$$\mathrm{P}\left[T_{00} > n\right] = 1 \times \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-1}{n} = \frac{1}{n}. \tag{57}$$

Thus the CDF of T_{00} satisfies $F_{T_{00}}(n) = 1 - \mathrm{P}[T_{00} > n] = 1 - 1/n$. To determine whether state 0 is recurrent, we calculate

$$\mathrm{P}\left[T_{ii} < \infty\right] = \lim_{n\to\infty} F_{T_{00}}(n) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1. \tag{58}$$

Thus state 0 is recurrent.

When state i is transient, then there is a probability

$$p = 1 - \mathrm{P}\left[T_{ii} < \infty\right] > 0 \tag{59}$$

that the system starting in state i never returns to state i. If we refer to this event (that we never return) as a "success," then we can define N_{ii} as a random variable equal to the the number of failures before the first success. Put another way, N_{ii} is the number of return visits to state i over all time. With each failure, i.e. each return to state i, the system starts over and there is again a probability $1-p$, independent of past history, that we return to state i. Thus N_{ii} is shifted geometric random variable with PMF

$$P_{N_{ii}}(n) \begin{cases} (1-p)^n p & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{60}$$

It is easy to show that $\mathrm{E}[N_{ii}] = (1-p)/p$. Thus when state i is transient and $p > 0$, the expected number of return visits $\mathrm{E}[N_{ii}]$ is finite. On the other hand, when state i is recurrent and the system is certain to return to i, then $p = 0$ and over an infinite time, the expected number of returns $\mathrm{E}[N_{ii}]$ must be infinite. These observations yield the following theorem.

Theorem 18

State i is recurrent if and only if $\mathrm{E}[N_{ii}] = \infty$.

Theorem 18 can be useful when we can calculate $\mathrm{E}[N_{ii}]$ from the n-step transition probabilities. Specifically, Problem 7.3 asks the reader to verify that the expected number of return visits to state i over all time is

$$\mathrm{E}\left[N_{ii}\right] = \sum_{n=1}^{\infty} P_{ii}(n). \tag{61}$$

Determining whether the infinite sum in Equation (61) converges or diverges is another way to determine whether state i is recurrent. For example, Problem 7.4 asks the reader to show for the random walk introduced in Example 4 that state 0 is recurrent if $p = 1/2$ and otherwise is transient.

Curiously, a countably infinite chain permits two kinds of recurrent states.

Definition 14 Positive Recurrence and Null Recurrence

A recurrent state i is ***positive recurrent*** *if $\mathrm{E}[T_{ii}] < \infty$; otherwise, state i is* ***null recurrent****.*

Both positive recurrent and null recurrent states are called *recurrent*. The distinguishing property of a recurrent state i is that when the system leaves state i, it is certain to return eventually to i; however, if i is null recurrent, then the expected time to re-visit i is infinite. This difference is demonstrated in the following example.

Example 23

In Example 22, we found that state 0 is recurrent. Is state 0 positive recurrent or null recurrent?

. . . .

In Example 22, we found for $n = 1, 2, \ldots$ that $\mathrm{P}[T_{00} > n] = 1/n$. For $n > 1$, the PMF of T_{00} satisfies

$$P_{T_{00}}(n) = \mathrm{P}\left[T_{00} > n-1\right] - \mathrm{P}\left[T_{00} > n\right] = \frac{1}{(n-1)n}, \quad n = 2, 3, \ldots \tag{62}$$

The expected time to return to state 0 is

$$\mathrm{E}\left[T_{00}\right] = \sum_{n=2}^{\infty} n P_{T_{00}}(n) = \sum_{n=2}^{\infty} \frac{1}{n-1} = \infty. \tag{63}$$

From Definition 14, we can conclude that state 0 is null recurrent.

It is possible to show that positive recurrence, null recurrence, and transience are class properties.

Theorem 19

For a communicating class of a Markov chain, either (a) all states are transient, (b) all states are null recurrent, or (c) all states are positive recurrent.

Example 24

In the Markov chain of Examples 22 and 23, is state 33 positive recurrent, null recurrent, or transient?

Since Example 23 showed that 0 is null recurrent, Theorem 19 implies that state 33, as well as every other state, is null recurrent.

From our examples, we can conclude that classifying the states of a countably infinite Markov chain is decidedly nontrivial. In this section, the examples were carefully chosen in order to simplify the calculations required for state classification. However, some intuition was necessary to determine which calculations to perform. For countably infinite Markov chains, the tests for recurrence, $\mathrm{P}[T_{ii} < \infty] = 1$, and positive recurrence, $\mathrm{E}[T_{ii}] < \infty$ of state i are difficult simply because it is difficult to find the PMF or CDF of T_{ii}. In practice, however, these tests are rarely used. We will see shortly that identifying the stationary probabilities is sufficient to verify positive recurrence. Moreover, in the event that the states of a communicating class are either null recurrent or transient, it is typically not crucial to distinguish between the two possibilities.

We now examine the stationary probabilities of a countably infinite discrete-time Markov chain. First, we observe that the Chapman-Kolmogorov equations (Theorem 2) as well as Theorem 4 apply to both finite and countably infinite Markov chains. In particular, given the state probabilities $\{p_j(n)\}$ at time n, the state probabilities at time $n+1$ are given by

$$p_j(n+1) = \sum_{i=0}^{\infty} p_i(n) P_{ij}. \tag{64}$$

From Equation (64), we see that if the limiting state probabilities $limtynp_i(n) = \pi_n$ exist, then they must satisfy $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$.

Theorem 20

An irreducible Markov chain with states $\{0, 1, \ldots\}$ and transition probabilities P_{ij} is positive recurrent if and only if there exist stationary probabilities π_i

satisfying

$$\sum_{j=0}^{\infty} \pi_j = 1, \qquad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots.$$

Furthermore, if such a solution π_i exists, then it is unique.

Theorem 20 is powerful because it says that if we can find the stationary probabilities π_i, then we know the Markov chain is positive recurrent and we need not bother with the difficult tests associated with Definition 13 and Definition 14. Conversely, if we can show there is no solution for the π_i in Theorem 20, then we know that the chain is not positive recurrent. In this case, the chain will be either null recurrent or transient; but in practical settings, it rarely matters which it is, as both represent a system in which the steady-state behavior is defective.

We further note that it can be shown that if the stationary probabilities π_j exist, then $\pi_j = 1/\operatorname{E}[T_{jj}]$. Since $\operatorname{E}[T_{jj}]$ is the expected time between visits to state j, we can interpret π_j as the fraction of time the system spends in state j. Lastly, we observe that Theorem 20 holds even if the chain is periodic. In this case, π_j is still the fraction of time the system spends in state j; however, π_j cannot be equated to a limiting state probabilities $\lim_{n\to\infty} P_{ij}(n)$, since the n step probabilities will always vary with the periodicity of the chain.

Sometimes the transition probabilities have a structure that leads to a simple direct calculation of π_j. For other transition probabilities it is helpful to partition the state space into mutually exclusive subsets S and S' and use Theorem 13 to simplify the calculation.

Example 25 — Router Buffer Revisited

Find the stationary probabilities of the router buffer described in Example 21. Make sure to identify for what values of p that the stationary probabilities exist.

. . . .

We apply Theorem 13 by partitioning the state space between $S = \{0, 1, \ldots, i\}$ and $S' = \{i+1, i+2, \ldots\}$ as shown in Figure 4. For any state $i \geq 0$,

$$\pi_i p = \pi_{i+1}(1-p). \tag{65}$$

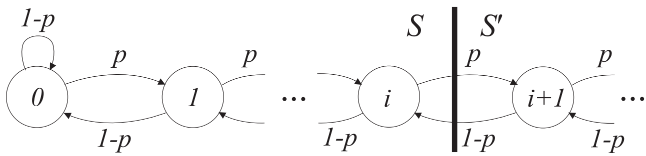


Figure 4 A partition for the discrete-time queue of Example 25.

Defining $\alpha = p/(1-p)$, it follows that

$$\pi_{i+1} = \alpha \pi_i. \tag{66}$$

Since Equation (66) holds for $i = 0, 1, \ldots$, we have that $\pi_i = \pi_0 \alpha^i$. Requiring the state probabilities to sum to 1, we have that for $\alpha < 1$,

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \alpha^i = \frac{\pi_0}{1-\alpha} = 1. \tag{67}$$

Thus for $\alpha < 1$, the complete state probabilities are

$$\pi_i = (1-\alpha)\alpha^i, \qquad i = 0, 1, 2, \ldots \tag{68}$$

For $\alpha \geq 1$ or, equivalently, $p \geq 1/2$, the limiting state probabilities do not exist.

Quiz 7

In each one-second interval at a convenience store, a new customer arrives with probability p, independent of the number of customers in the store and also other arrivals at other times. The clerk gives each arriving customer a friendly "Hello." In each unit of time in which there is no arrival, the clerk can provide a unit of service to a waiting customer. Given that a customer has received a unit of service, the customer departs with probability q. When the store is empty, the clerk sits idle. Sketch a Markov chain for the number of customers in the store. Under what conditions on p and q do limiting state probabilities exist? Under those conditions, find the limiting state probabilities.

8 Continuous-Time Markov Chains

Continuous-time Markov chains are continuous-time, discrete-value processes in which the time spent in each state is an exponential random variable. An irreducible, positive-recurrent, continuous-time Markov chain has unique limiting state probabilities.

For many systems characterized by state transitions, the transitions naturally occur at discrete time instants. These processes are naturally modeled by discrete-time Markov chains. In this section, we relax our earlier requirement that transitions can occur exactly once each unit of time. In particular, we examine a class of *continuous-time processes* in which state transitions can occur at any time.

These systems are described by a stochastic process $\{X(t)|t \geq 0\}$, where $X(t)$ is the state of the system at time t. Although state transitions can occur at any time, the model for state transitions is not completely arbitrary.

Definition 15 Continuous-Time Markov Chain

A continuous-time Markov chain $\{X(t)|t \geq 0\}$ *is a continuous-time, discrete-value random process such that for an infinitesimal time step of size* Δ*,*

$$\mathrm{P}\left[X(t+\Delta) = j | X(t) = i\right] = q_{ij}\Delta$$
$$\mathrm{P}\left[X(t+\Delta) = i | X(t) = i\right] = 1 - \sum_{j \neq i} q_{ij}\Delta$$

Note that this model assumes that only a single transition can occur in the small time Δ. In addition, we observe that Definition 15 implies that

$$\mathrm{P}\left[X(t+\Delta) \neq i | X(t) = i\right] = \sum_{j \neq i} q_{ij}\Delta \tag{69}$$

In short, in every infinitesimal interval of length Δ, a Bernoulli trial determines whether the system exits state i.

The continuous-time Markov chain is closely related to the Poisson process. In Section 13.4, we derived a Poisson process of rate λ as the limiting case of a process that for any small time interval of length Δ, a Bernoulli trial with success probability $\lambda\Delta$ indicated whether an arrival occurred. We also found in Theorem 13.3 that the time until the next arrival is an exponential (λ) random variable.

In the limit as Δ approaches zero, we can conclude that for a Markov chain in state i, the time until the next transition will be an exponential random variable with parameter

$$\nu_i = \sum_{j \neq i} q_{ij}. \tag{70}$$

We call ν_i the *departure rate* of state i. Because the exponential random variable is memoryless, we know that no matter how long the system has been in state i, the time until the system departs state i is always an exponential (ν_i) random variable. In particular, this says that the time the system has spent in state i has no influence on the future sample path of the system. Recall that in Definition 1, the key idea of a discrete-time Markov chain was that at time n, the X_n summarized the past history of the system. In the same way, $X(t)$ for a continuous-time Markov chain summarizes the state history prior to time t.

We can further interpret the state transitions for a continuous-time Markov chain in terms of the sum of independent Poisson processes. We recall from Theorem 13.7 of Section 13.5 that the sum of independent Poisson processes $N_1(t)+N_2(t)$ could be viewed as a single Poisson process with rate $\lambda = \lambda_1+\lambda_2$. For this combined process starting at time 0, the system waits a random time with an exponential (λ) PDF for an arrival. When there is an arrival, an independent trial determines whether the arrival was from process $N_1(t)$ or $N_2(t)$.

For a continuous-time Markov chain, when the system enters state i at time 0, we start a Poisson process $N_{ik}(t)$ of rate q_{ik} for every other state k. If the process $N_{ij}(t)$ is the first to have an arrival, then the system transitions to state j. The process then resets and starts a Poisson process $N_{jk}(t)$ for each state $k \neq j$. Effectively, when the system is in state i, the time until a transition is an exponential (ν_i) random variable. Given the event D_i that the system departs state i in the time interval $(t, t+\Delta]$, the conditional probability of the event D_{ij} that the system went to state j is

$$\mathrm{P}\left[D_{ij}|D_i\right] = \frac{\mathrm{P}\left[D_{ij}\right]}{\mathrm{P}\left[D_i\right]} = \frac{q_{ij}\Delta}{\nu_i\Delta} = \frac{q_{ij}}{\nu_i}. \tag{71}$$

Thus for a continuous-time Markov chain, the system spends an exponential (ν_i) time in state i, followed by an independent trial that specifies that the next state is j with probability $P_{ij} = q_{ij}/\nu_i$. When we ignore the time spent in each state, the transition probabilities P_{ij} can be viewed as the transition probabilities of a discrete-time Markov chain.

Definition 16 Embedded Discrete-Time Markov Chain

For a continuous-time Markov chain with transition rates q_{ij} and state i departure rates ν_i, the ***embedded discrete-time Markov chain*** *has transition probabilities $P_{ij} = q_{ij}/\nu_i$ for states i with $\nu_i > 0$ and $P_{ii} = 1$ for states i with $\nu_i = 0$.*

For discrete-time chains, we found that the limiting state probabilities depend on the number of communicating classes. For continuous-time Markov chains, the issue of communicating classes remains.

Definition 17 Continuous-Time Communicating Classes

The communicating classes of a continuous-time Markov chain are given by the communicating classes of its embedded discrete-time Markov chain.

Definition 18 Continuous-Time Irreducible Markov Chain

A continuous-time Markov chain is ***irreducible*** *if the embedded discrete-time Markov chain is irreducible.*

At this point, we focus on irreducible continuous-time Markov chains; we will not consider multiple communicating classes. In a continuous-time chain, multiple communicating classes still result in multiple modes of operation for the system. These modes can and should be evaluated as separate irreducible systems.

Definition 19 Continuous-Time Positive Recurrence

An irreducible continuous-time Markov chain is ***positive recurrent*** *if for all states i, the time T_{ii} to return to state i satisfies $\mathrm{E}[T_{ii}] < \infty$.*

For continuous-time chains, issues of irreducibility and positive recurrence are essentially the same as for discrete-time chains. Unlike discrete-time chains, however, in a continuous-time chain we need not worry about periodicity because the time spent in each state is a continuous random variable.

Example 26

In a continuous-time ON-OFF process, alternating OFF and ON (states 0 and 1) periods have independent exponential durations. The average ON period lasts $1/\mu$ seconds, while the average OFF period lasts $1/\lambda$ seconds. Sketch the continuous-time Markov chain.

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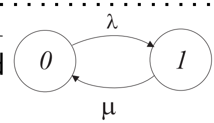
In the continuous-time chain, we have states 0 (OFF) and 1 (ON). The chain, as shown, has transition rates $q_{01} = \lambda$ and $q_{10} = \mu$.

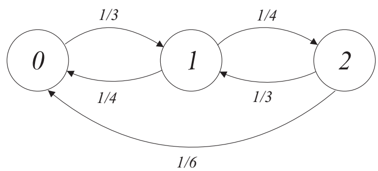


Example 27

In the summer, an air conditioner is in one of three states: (0) off, (1) low, or (2) high. While off, transitions to low occur after an exponential time with expected time 3 minutes. While in the low state, transitions to off or high are equally likely and transitions out of the low state occur at rate 0.5 per minute. When the system is in the high state, it makes a transition to the low state with probability $2/3$ or to the off state with probability $1/3$. The time spent in the high state is an exponential $(1/2)$ random variable. Model this air conditioning system using a continuous-time Markov chain.

. . . .

Each fact provides information about the state transition rates. First, we learn that $q_{01} = 1/3$ and $q_{02} = 0$. Second, we are told that $\nu_1 = 0.5$ and that $q_{10}/\nu_1 = q_{12}/\nu_1 = 1/2$. Thus $q_{10} = q_{12} = 1/4$. Next, we see that $q_{21}/\nu_2 = 2/3$ and $q_{20}/\nu_2 = 1/3$ and that $\nu_2 = 1/2$. Hence $q_{21} = 1/3$ and $q_{20} = 1/6$. The complete Markov chain is



In these examples, we have seen that a Markov chain is characterized by the set $\{q_{ij}\}$ of transition rates. Self transitions from state i immediately back to state i are trivial simply because nothing would actually change in a self transition. Hence, $q_{ii} = 0$ for every state i. When the continuous-time Markov chain has a finite state space $\{0, 1, \ldots, K\}$, we can represent the Markov chain by the state transition matrix $\mathbf{Q}$, which has i, jth entry q_{ij}. It follows that the main diagonal of $\mathbf{Q}$ is always zero.

For our subsequent calculations of probabilities, it will be useful to define a rate matrix $\mathbf{R}$ with i, jth entry

$$r_{ij} = \begin{cases} q_{ij} & i \neq j, \\ -\nu_i & i = j. \end{cases} \tag{72}$$

Recall that $\nu_i = \sum_{j \neq i} q_{ij}$ is the departure rate from state i. Off the diagonal, matrices $\mathbf{R}$ and $\mathbf{Q}$ are identical; on the diagonal, $q_{ii} = 0$ while $r_{ii} = -\nu_i$.

Just as we did for discrete-time Markov chains, we would like to know how to calculate the probability that the system is in a state j. For this probability, we will use the notation

$$p_j(t) = \mathrm{P}\left[X(t) = j\right]. \tag{73}$$

When the Markov chain has a finite set of states $\{0, \ldots, K\}$, the state probabilities can be written as the vector $\mathbf{p}(t) = \begin{bmatrix} p_0(t) & \cdots & p_K(t) \end{bmatrix}'$. We want to compute $p_j(t)$ both for a particular time instant t as well as in the limiting case when t approaches infinity. Because the continuous-time Markov chain has events occurring in infinitesimal intervals of length Δ, transitions in the discrete-time Chapman-Kolmogorov equations are replaced by continuous-time differential equations that we derive by considering a time step of size Δ. For a finite chain, these differential equations can be written as a first-order vector differential equation for $\mathbf{p}(t)$.

In the next two theorems, we write the relevant equations in two forms: on the left using the notation of individual state probabilities, and on the right in the concise notation of the state probability vector.

Theorem 21

For a continuous-time Markov chain, the state probabilities $p_j(t)$ evolve according to the differential equations

$$\frac{dp_j(t)}{dt} = \sum_i r_{ij} p_i(t), \quad j = 0, 1, 2, \ldots,$$

or, in vector form,

$$\frac{d\mathbf{p}'(t)}{dt} = \mathbf{p}'(t)\mathbf{R}.$$

Proof Given the state probabilities $p_j(t)$ at time t, we can calculate the state probabilities at time $t + \Delta$ using Definition 15:

$$p_j(t + \Delta) = [1 - (\nu_j \Delta)] p_j(t) + \sum_{i \neq j} (q_{ij} \Delta) p_i(t). \tag{74}$$

Subtracting $p_j(t)$ from both sides, we have

$$p_j(t + \Delta) - p_j(t) = -(\nu_j \Delta) p_j(t) + \sum_{i \neq j} (q_{ij} \Delta) p_i(t). \tag{75}$$

Dividing through by Δ and expressing Equation (75) in terms of the rates r_{ij} yields

$$\frac{p_j(t+\Delta) - p_j(t)}{\Delta} = -\nu_j p_j(t) + \sum_{i \neq j} q_{ij} p_i(t) = \sum_i r_{ij} p_i(t). \tag{76}$$

As Δ approaches zero, we obtain the desired differential equation.

Students familiar with linear systems theory will recognize that these equations are equivalent to the differential equations that describe the natural response of a dynamic system. For a system with two states, these equations have the same form as the coupled equations for the capacitor voltage and inductor current in an RLC circuit. Further, for the finite Markov chain, it is well known that the solution to the vector differential equation for $\mathbf{p}(t)$ is

$$\mathbf{p}'(t) = \mathbf{p}'(0) e^{\mathbf{R}t} \tag{77}$$

where the matrix $e^{\mathbf{R}t}$, known as the matrix exponential, is defined as

$$e^{\mathbf{R}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{R}t)^k}{k!}. \tag{78}$$

Our primary interest will be systems in which the state probabilities will converge to constant values, much like circuits or dynamic systems that converge to a steady-state response for a constant input. By analogy, it is said that the state probabilities converge to a *steady-state*. We caution the reader that this analogy can be misleading. If we observe a circuit, the state variables such as inductor currents or capacitor voltages will converge to constants. In a Markov chain, if we inspect the system after a very long time, then the "steady-state" probabilities describe the likelihood of the states. However, if we observe the Markov chain system, the actual state of the system typically is always changing even though the state probabilities $p_j(t)$ may have converged.

The state probabilities converge when the state probabilities stop changing, that is, when $dp_j(t)/dt = 0$ for all j. In this case, we say that the state probabilities have reached a *limiting state distribution*. Just as for discrete-time Markov chains, another name for the limiting state distribution is the *stationary distribution* because if $p_j(t) = p_j$ for all states j, then $dp_j(t)/dt = 0$ and $p_j(t)$ never changes.

Theorem 22

For an irreducible, positive recurrent continuous-time Markov chain, the state probabilities satisfy

$$\lim_{t\to\infty} p_j(t) = p_j, \qquad \text{or, in vector form,} \qquad \lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}$$

where the limiting state probabilities are the unique solution to

$$\sum_i r_{ij} p_i = 0, \qquad \text{or, in vector form,} \quad \mathbf{p}'\mathbf{R} = \mathbf{0}',$$

$$\sum_j p_j = 1, \qquad \text{or, in vector form,} \quad \mathbf{p}'\mathbf{1} = 1.$$

Just as for the discrete-time chain, the limiting state probability p_j is the fraction of time the system spends in state j over the sample path of the process. Since $r_{jj} = -\nu_j$, and $r_{ij} = q_{ij}$, Theorem 22 has a nice interpretation when we write

$$p_j \nu_j = \sum_{i \neq j} p_i q_{ij}. \tag{79}$$

On the left side, we have the product of p_j, the fraction of time spent in state j, and ν_j, the transition rate out of state j. That is, the left side is the average rate of transitions out of state j. Similarly, on the right side, $p_i q_{ij}$ is the average rate of transitions from state i into state j so that $\sum_{i\neq j} p_i q_{ij}$ is the average rate of transitions into state j. In short, the limiting state probabilities balance the average transition rate into state j against the average transition rate out of state j. Because this is a balance of rates, p_i depends on both the transition probabilities P_{ij} as well as on the expected time $1/\nu_i$ that the system stays in state i before the transition.

Example 28

Calculate the limiting state probabilities for the ON/OFF system of Example 26.

The stationary probabilities satisfy $p_0\lambda = p_1\mu$ and $p_0 + p_1 = 1$. The solution is

$$p_0 = \frac{\mu}{\lambda+\mu}, \qquad p_1 = \frac{\lambda}{\lambda+\mu}. \tag{80}$$

Increasing λ, the departure rate from state 0, decreases the time spent in state 0, and correspondingly, increases the probability of state 1.

Example 29

Find the stationary distribution for the Markov chain describing the air conditioning system of Example 27.

. . . .

The stationary probabilities satisfy

$$\frac{1}{3}p_0 = \frac{1}{4}p_1 + \frac{1}{6}p_2, \quad \frac{1}{2}p_1 = \frac{1}{3}p_0 + \frac{1}{3}p_2, \quad \frac{1}{2}p_2 = \frac{1}{4}p_1. \tag{81}$$

Although we have three equations and three unknowns, these equations do not have a unique solution. We can conclude only that $p_1 = p_0$ and $p_2 = p_0/2$. Finally, the requirement that $p_0 + p_1 + p_2 = 1$ yields $p_0 + p_0 + p_0/2 = 1$. Hence, the limiting state probabilities are

$$p_0 = 2/5, \qquad p_1 = 2/5, \qquad p_2 = 1/5. \tag{82}$$

Quiz 8

A processor in a parallel processing computer can work on up to four tasks at once. When the processor is working on one or more tasks, the task completion rate is three tasks per millisecond. When there are three or fewer tasks assigned to the processor, tasks are assigned at the rate of two tasks per millisecond. The processor is unreliable in the sense that any time the processor is working, it may reboot and discard all of its tasks. In any state $i \neq 0$, reboots occur at a rate of 0.01 per millisecond. Sketch the continuous-time Markov chain and find the stationary probabilities.

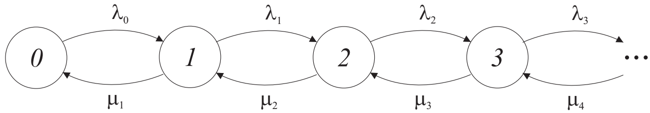


Figure 5 The birth-death model of a queue

9 Birth-Death Processes and Queueing Systems

A birth-death process has states $\{0, 1, 2\, ldots\}$ that track the size of a population. From state i, a transition to state $i+1$ is a birth and to state $i-1$ is a death. Births and deaths always occur one at a time.

A simple yet important form of continuous-time Markov chain is the birth-death process.

Definition 20 Birth-Death Process

A continuous-time Markov chain is a ***birth-death process*** *if the transition rates satisfy $q_{ij} = 0$ for $|i - j| > 1$.*

As depicted in Figure 5, a birth-death process in state i can make transitions only to states $i - 1$ or $i + 1$. Birth-death processes earn their name because the state can represent the number in a population. A transition from i to $i + 1$ is a birth since the population increases by one. A transition from i to $i - 1$ is a death in the population.

Queueing systems are often modeled as birth-death processes in which the population consists of the customers in the system. A queue can represent any service facility in which customers arrive, possibly wait, and depart after being served. In a Markov model of a queue, the state represents the number of customers in the queueing system. For a Markov chain that represents a queue, we make use of some new terminology and notation. Specifically, the transition probability $q_{i,i-1}$ is denoted by μ_i and is called the *service rate* in state i since the transition from i to $i-1$ occurs only if a customer completes service and leaves the system. Similarly, $\lambda_i = q_{i,i+1}$ is called the *arrival rate* *in state* i since a transition from state i to $i+1$ corresponds to the arrival of a customer.

The continuous-time birth-death process representing a queue always resembles the chain shown in Figure 5. Since any birth-death process can be described in terms of the transition rates λ_i and μ_i, we will use this notation in our subsequent development, whether or not the birth-death process represents a queue. We will also assume that $\mu_i > 0$ for all states i that are reachable from state 0. This ensures that we have an irreducible chain.

For birth-death processes, the limiting state probabilities are easy to compute.

Theorem 23

For a birth-death queue with arrival rates λ_i and service rates μ_i, the stationary probabilities p_i satisfy

$$p_{i-1}\lambda_{i-1} = p_i\mu_i, \qquad \sum_{i=0}^{\infty} p_i = 1.$$

Proof We prove by induction on i that $p_{i-1}\lambda_{i-1} = p_i\mu_i$. For $i = 1$, Theorem 22 implies that $p_0\lambda_0 = p_1\mu_1$. Assuming $p_{i-1}\lambda_{i-1} = p_i\mu_i$, we observe that Theorem 22 requires

$$p_i(\lambda_i + \mu_i) = p_{i-1}\lambda_{i-1} + p_{i+1}\mu_{i+1}. \tag{83}$$

From this equation, the assumption that $p_{i-1}\lambda_{i-1} = p_i\mu_i$ implies $p_i\lambda_i = p_{i+1}\mu_{i+1}$, completing the induction.

For birth-death processes, Theorem 23 can be viewed as analagous to Theorem 13 for discrete-time queues in that it says that the average rate of transitions from state $i-1$ to state i must equal the average rate of transitions from state i to state $i-1$. It follows from Theorem 23 that the stationary probabilities of the birth-death queue have a particularly simple form.

Theorem 24

For a birth-death queue with arrival rates λ_i and service rates μ_i, let $\rho_i =$

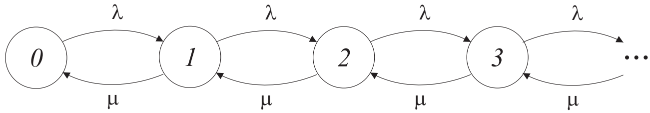


Figure 6 The Markov chain of an M/M/1 queue.

λ_i/μ_{i+1}. *The limiting state probabilities, if they exist, are*

$$p_i = \frac{\prod_{j=0}^{i-1} \rho_j}{1 + \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} \rho_j}.$$

Whether the stationary probabilities exist depends on the actual arrival and service rates. Just as in the discrete-time case, the states may be null recurrent or even transient. For the birth-death process, this depends on whether the sum $\sum_{k=1}^{\infty} \prod_{j=1}^{k-1} \rho_j$ converges.

In the following sections, we describe several common queue models. Queueing theorists use a naming convention of the form A/S/n/m for common types of queues. In this notation, 'A' describes the arrival process, 'S' the service times, n the number of servers, and m the number of customers that can be in the queue. For example, A = M says that the arrival process is *Memoryless* in that the arrivals are a Poisson process. A second possibility is that A = D for a *Deterministic* arrival process in which the inter-arrival times are constant. Another possibility is that A = G corresponding to a *General* arrival process. In all cases, a common assumption is that the arrival process is independent of the service requirements of the customers. Similarly, S = M corresponds to memoryless (exponential) service times, S = D is for deterministic service times, and S = G denotes a general service time distribution. When the number of customers in the system is less than the number of servers n, then an arriving customer is immediately assigned to a server. When m is finite, new arrivals are blocked (i.e., discarded) when the queue has m customers. Also, if m is unspecified, then it is assumed to be infinite.

Using the birth-death Markov chain, we can model a large variety of queues with memoryless arrival processes and service times.

The M/M/1 Queue

In an *M/M/1 queue*, the arrivals are a Poisson process of rate λ, independent of the service requirements of the customers. The service time of a customer is an exponential (μ) random variable, independent of the system state. Since the queue has only one server, the departure rate from any state $i > 0$ is $\mu_i = \mu$. Thus μ is often called the service rate of the system. The Markov chain for number of customers in the M/M/1 queue is shown in Figure 6. The simple structure of the queue makes calculation of the limiting state probabilities quite simple.

Theorem 25

The M/M/1 queue with arrival rate $\lambda > 0$ and service rate μ, $\mu > \lambda$, has limiting state probabilities

$$p_n = (1-\rho)\rho^n, \qquad n = 0, 1, 2, \ldots$$

where $\rho = \lambda/\mu$.

Proof By Theorem 23, the limiting state probabilities satisfy $p_{i-1}\lambda = p_i\mu$, implying $p_i = \rho p_{i-1}$. Thus $p_i = \rho^i p_0$. Applying $\sum_{j=0}^{\infty} p_j = 1$ yields

$$p_0\left(1 + \rho + \rho^2 + \cdots\right) = 1. \tag{84}$$

If $\rho < 1$, we obtain $p_0 = 1 - \rho$ and the limiting state probabilities exist.

Note that if $\lambda > \mu$, then new customers arrive faster than customers depart. In this case, all states of the Markov chain are transient and the queue backlog grows without bound. We note that it is a typical property of queueing systems that the system is stable (i.e., the queue has positive recurrent Markov chain and the limiting state probabilities exist) as long as the system service rate is greater than the arrival rate when the system is busy.

Example 30

Cars arrive at an isolated toll booth as a Poisson process with arrival rate $\lambda = 0.6$ cars per minute. The service required by a customer is an exponential random

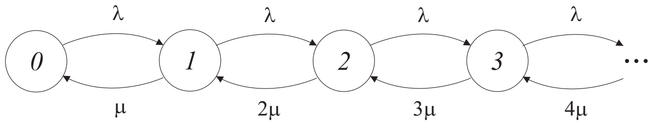


Figure 7 The Markov chain for the M/M/∞ queue.

variable with expected value $1/\mu = 0.3$ minutes. What are the limiting state probabilities for N, the number of cars at the toll booth? What is the probability that the toll booth has zero cars some time in the distant future?

The toll booth is an M/M/1 queue with arrival rate λ and service rate μ. The offered load is $\rho = \lambda/\mu = 0.18$, so the limiting state probabilities are

$$p_n = (0.82)(0.18)^n, \qquad n = 0, 1, 2, \ldots. \tag{85}$$

The probability that the toll booth is idle is $p_0 = 0.82$.

The M/M/∞ Queue

In an M/M/∞ queue, the arrivals are a Poisson process of rate λ, independent of the state of the queue. The service time of a customer is an exponential random variable with parameter μ, independent of the system state. These facts are the same as for the M/M/1 queue. The difference is that with an infinite number of servers, each arriving customer is immediately served without waiting. When n customers are in the system, all n customers are in service and the system departure rate is $n\mu$. Although we still refer to μ as the service rate of the M/M/∞ queue, we must keep in mind that μ is only the service rate of each individual customer. The Markov chain describing this queue is shown in Figure 7.

Theorem 26

The M/M/∞ queue with arrival rate $\lambda > 0$ and service rate $\mu > 0$ has limiting *state probabilities*

$$p_n = \begin{cases} \rho^n e^{-\rho}/n! & n = 0, 1, 2, \ldots, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\rho = \lambda/\mu$.

Proof Theorem 23 implies that the limiting state probabilities satisfy

$$p_n = (\rho/n)p_{n-1}, \tag{86}$$

where $\rho = \lambda/\mu$. This implies $p_n = p_0(\rho^n/n!)$. The requirement that $\sum_{n=0}^{\infty} p_n = 1$ yields

$$p_0 \left(1 + \rho + \frac{\rho}{2!} + \frac{\rho^3}{3!} + \cdots \right) = p_0 e^{\rho} = 1. \tag{87}$$

Hence, $p_0 = e^{-\rho}$ and the theorem follows.

Unlike the M/M/1 queue, the condition $\lambda < \mu$ is unnecessary for the M/M/∞ queue. The reason is that even if μ is very small, a sufficiently large backlog of n customers will yield a system service rate $n\mu$ greater than the arrival rate λ.

Example 31

At a beach in the summer, swimmers venture into the ocean as a Poisson process of rate 300 swimmers per hour. The time a swimmer spends in the ocean is an exponential random variable with expected value of 20 minutes. Find the limiting state probabilities of the number of swimmers in the ocean.

. . . .

We model the ocean as an M/M/∞ queue. The arrival rate is $\lambda = 300$ swimmers per hour. Since 20 minutes is $1/3$ hour, the expected service time of a customer is $1/\mu = 1/3$ hours. Thus the ocean is an M/M/∞ queue with $\rho = \lambda/\mu = 100$. By Theorem 26, the limiting state probabilities are

$$p_n = \begin{cases} 100^n e^{-100}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{88}$$

The expected number of swimmers in the ocean at a random time is $\sum_{n=0}^{\infty} np_n =$ 100 swimmers.

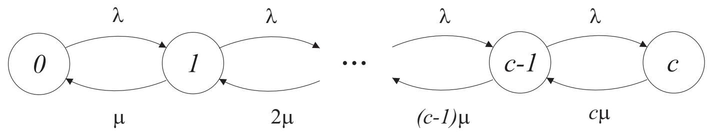


Figure 8 The Markov chain for the M/M/c/c queue.

The M/M/c/c Queue

The M/M/c/c queue has c servers and a capacity for c customers in the system. Customers arrive as a Poisson process of rate λ. When the system has $c-1$ or fewer customers in service, a new arrival immediately goes into service. When there are c customers in the system, new arrivals are blocked and never enter the system. A customer admitted to the system has an exponential (μ) service time. The Markov chain for the M/M/c/c queue is essentially the same as that of the M/M/∞ queue except that the state space is truncated at c customers. The Markov chain is shown in Figure 8.

Theorem 27

For the M/M/c/c queue with arrival rate λ and service rate μ, the limiting state probabilities satisfy

$$p_n = \begin{cases} \dfrac{\rho^n/n!}{\sum_{j=0}^{c} \rho^j/j!} & j = 0, 1, \ldots, c, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\rho = \lambda/\mu$.

Proof By Theorem 23, for $1 \leq n \leq c$, $p_n n\mu = p_{n-1}\lambda$. This implies $p_n = (\rho^n/n!)p_0$. The requirement that $\sum_{n=0}^{c} p_n = 1$ yields

$$p_0 \left(1 + \rho + \rho^2/2! + \rho^3/3! + \cdots + \rho^c/c!\right) = 1. \tag{89}$$

After a very long time, the number N in the queue will be modeled by the stationary probabilities p_n. That is, $P_N(n) = p_n$ for $n = 0, 1, \ldots$. The probability that a customer is blocked is the probability that a new arrival finds the queue has c customers. Since the arrival at time t is independent of the current state of the queue, a new arrival is blocked with probability

$$P_N(c) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!}. \tag{90}$$

This result is called the *Erlang-B formula* and has many applications.

Example 32

A rural telephone switch has 100 circuits available to carry 100 calls. A new call is blocked if all circuits are busy at the time the call is placed. Calls have exponential durations with an expected length of 2 minutes. If calls arrive as a Poisson process of rate 40 calls per minute, what is the probability that a call is blocked?

We can model the switch as an M/M/100/100 queue with arrival rate $\lambda = 40$, service rate $\mu = 1/2$, and load $\rho = \lambda/\mu = 80$. The probability that a new call is blocked is

$$P_N(100) = \frac{80^{100}/100!}{\sum_{k=0}^{100} 80^k/k!} = 0.0040. \tag{91}$$

Example 33

One cell in a cellular phone system has 50 radio channels available to carry cell-phone calls. Calls arrive at a Poisson rate of 40 calls per minute and have an exponential duration lasting 1 minute on average. What is the probability that a call is blocked?

We can model the cell as an M/M/50/50 queue with load $\rho = 40$. The probability that a call is blocked is

$$P_N(50) = \frac{40^{50}/50!}{\sum_{k=0}^{50} 40^k/k!} = 0.0187. \tag{92}$$

More about Queues

In both the M/M/1 and M/M/∞ queues, the ratio $\rho = \lambda/\mu$ of the customer arrival rate to the service rate μ completely characterizes the limiting state probabilities. This is typical of almost all queues in which customers arrive as a Poisson process of rate λ and a customer in service is served at rate μ. Consequently, ρ is called the load on the queue. In the case of the M/M/1 queue, the limiting state probabilities fail to exist if $\rho \geq 1$. In this case, the queue will grow infinitely long because customers arrive faster than they are served.

For a queue, the limiting state probabilities are significant because they describe the performance of the service facility. For a queue that has been operating for a very long time, an arbitrary arrival will see a random number N of customers already in the system. Since the queue has been functioning for a long time, the random variable N has a PMF that is given by the limiting state probabilities for the queue. That is, $P_N(n) = p_n$ for $n = 0, 1, \ldots$. Furthermore, we can use the properties of the random variable N to calculate performance measures such as the average time a customer spends in the system.

Example 34

For the M/M/1 queue with offered load $\rho = \lambda/\mu$, find the PMF of N, the number of customers in the queue. What is the expected number of customers $\mathrm{E}[N]$? What is the average system time of a customer?

. . . .

For the M/M/1 queue, the stationary probability of state n is $p_n = (1-\rho)\rho^n$. Hence the PMF of N is

$$P_N(n) = \begin{cases} (1-\rho)\rho^n & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{93}$$

The expected number in the queue is

$$\mathrm{E}[N] = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \rho \sum_{n=1}^{\infty} n(1-\rho)\rho^{n-1} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}. \tag{94}$$

When an arrival finds N customers in the system, the arrival must wait for each of the N queued customers to be served. After that, the new arrival must have its own service requirement satisfied. Using Y_i to denote the service requirement of the ith queued customer, and Y to denote the service needed by the new arrival, the system time of the new arrival is

$$T = Y_1 + \cdots + Y_N + Y. \tag{95}$$

We see that T is a random sum of iid random variables. Since the service times are exponential (μ) random variables, $\mathrm{E}[Y_i] = \mathrm{E}[Y] = 1/\mu$. From Theorem 9.11,

$$\mathrm{E}\left[T\right] = \mathrm{E}\left[Y\right]\mathrm{E}\left[N\right] + \mathrm{E}\left[Y\right] = \frac{\mathrm{E}\left[N\right]+1}{\mu} = \frac{1}{\mu-\lambda}. \tag{96}$$

Quiz 9

The M/M/c/∞ queue has c servers but infinite waiting room capacity. Arrivals occur as a Poisson process of rate λ arrivals per second and service times measured in seconds are exponential (μ) random variables. A new arrival waits for service only if all c servers are busy at the time of arrival. Find the PMF of N, the number of customers in the queue after a long period of operation.

10 MATLAB

MATLAB makes it easy to simulate Markov chains. A collection of MATLAB functions appears in Table 1.

MATLAB can be applied to Markov chains for calculation of probabilities such as the n-step transition matrix or the stationary distribution, and also for simulation of systems described by a Markov chain. In the following two subsections, we consider discrete-time Markov chains and continuous-time chains separately.

Discrete-Time Markov Chains

We start by calculating an n-step transition matrix.

Example 35

Suppose in the disk drive of Example 3 that an IDLE system stays IDLE with probability 0.95, goes to READ with probability 0.04, or goes to WRITE with probability 0.01. From READ, the next state is READ with probability 0.9, otherwise the system is equally likely to go to IDLE OR WRITE. Similarly, from WRITE, the next state is WRITE with probability 0.9, otherwise the system is equally likely to go to IDLE OR READ. Write a MATLAB function `markovdisk(n)` to calculate the n-step transition matrix. Calculate the 10-step transition matrix $\mathbf{P}(10)$.

From the problem description, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.04 & 0.01 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.05 & 0.90 \end{bmatrix}. \tag{97}$$

The program `markovdisk.m` simply embeds this matrix and calculates $\mathbf{P}^n$. Executing `markovdisk(10)` produces the output as shown:

```
function M = markkovdisk(n)
P= [0.95 0.04 0.01; ...
        0.05 0.90 0.05; ...
        0.05 0.05 0.90];
M=P^n;
```

```
>> markovdisk(10)
ans =
   0.6743    0.2258    0.0999
   0.3257    0.4530    0.2213
   0.3257    0.2561    0.4182
```

Another natural application of MATLAB is the calculation of the stationary distribution for a finite Markov chain with state transition matrix $\mathbf{P}$.

```
function pv = dmcstatprob(P)
n=size(P,1);
A=(eye(n)-P);
A(:,1)=ones(n,1);
pv=([1 zeros(1,n-1)]*A^(-1))';
```

From Theorem 11, we need to find the vector $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi}'(\mathbf{I}-\mathbf{P}) = \mathbf{0}$ and $\boldsymbol{\pi}'\mathbf{1} = 1$. In `dmcstatprob(P)`, the matrix $\mathbf{A}$ is $\mathbf{I}-\mathbf{P}$, except we replace the first column by the vector $\mathbf{1}$. The solution is $\boldsymbol{\pi}' = \mathbf{e}'\mathbf{A}^{-1}$ where $\mathbf{e} = \begin{bmatrix}1 & 0 \cdots & 0\end{bmatrix}'$.

```
function x=simdmc(P,p0,n)
K=size(P,1)-1;                %highest no. state
sx=0:K;                       %state space
x=zeros(n+1,1);               %initialization
if (length(p0)==1)            %convert integer p0 to prob vector
    p0=((0:K)==p0);
end
x(1)=finiterv(sx,p0,1);       %x(m)= state at time m-1
for m=1:n,
  x(m+1)=finiterv(sx,P(x(m)+1,:),1);
end
```

Figure 9 The function `simdmc.m` for simulating n steps of a discrete-time Markov chain with state transition probability matrix `P` and initial state probabilities `p0`.

Example 36

Find the stationary probabilities for the disk drive of Example 35.

```
>> P
P =
    0.9500    0.0400    0.0100
    0.0500    0.9000    0.0500
    0.0500    0.0500    0.9000
>> dmcstatprob(P)'
ans =
    0.5000    0.3000    0.2000
>>
```

As shown, it is a trivial exercise for MATLAB to find the stationary probabilities of the simple 3-state chain of the disk drive. It should be apparent that MATLAB can easily solve far more complicated systems.

We can also use the state transition matrix $\mathbf{P}$ to simulate a discrete-time Markov chain. For an n-step simulation, the output will be the random sequence $X_0, \ldots, X_n$. Before proceeding, it will be helpful to clarify some issues related to indexing vectors and matrices. In MATLAB, we use the matrix `P` for the transition matrix $\mathbf{P}$. We have chosen to label the states of the Markov chain $0, 1, \ldots, K$ because in a variety of systems, most notably queues, it is natural to use state 0 for an empty system. On the other hand,

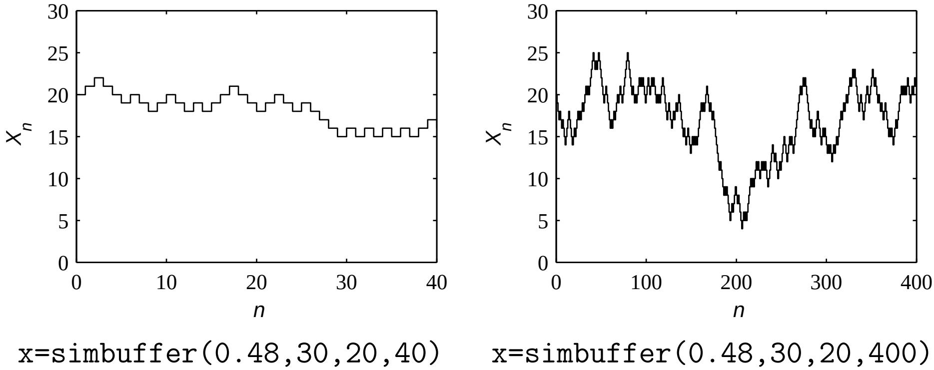


Figure 10 Two simulation traces from Example 37.

the MATLAB convention is to use `x(1)` for the first element in a vector `x`. As a result, P_{00}, the $0,0$th element of $\mathbf{P}$, is represented in MATLAB by `P(1,1)`. Similarly, $\begin{bmatrix} P_{i0} & P_{i1} & \cdots & P_{iK} \end{bmatrix}$, the ith row of $\mathbf{P}$, holds the conditional probabilities $P_{X_{n+1}|X_n}(j|i)$. However, these same conditional probabilities appear in `P(i+1,:)`, which is row $i+1$ of `P`. Despite these indexing offsets, it is fairly simple to implement a Markov chain simulation in MATLAB.

The MATLAB command `x=simdmc(P,p0,n)` simulates n steps of a discrete-time Markov chain with state transition matrix `P`. The starting state is specified in `p0`. If `p0` is simply an integer i, then the system starts in state i at time 0; otherwise, if `p0` is a state probability vector for the chain, then the initial state at time 0 is chosen according to the probabilities of `p0`. The output `x` is an $N+1$-element vector that holds a sample path $X_0, \ldots, X_N$ of the Markov chain.

In MATLAB, it is generally preferable to generate vectors using vector operations. However, in simulating a Markov chain, we cannot generate X_{n+1} until X_n is known and so we must proceed sequentially. The primary step occurs in the use of `finitepmf( )`. We recall from Section 3.9 that `finiterv(sx,px,1)` returns a sample value of a discrete random variable which takes on value `sx(i)` with probability `px(i)`. In `simdmc.m`, `P(x(n)+1,:)`, which is row `x(n)+1` of `P`, holds the conditional transition probabilities for state `x(n+1)` given that the current state is `x(n)`. Proceeding sequentially, we generate `x(n+1)` using the conditional pmf of `x(n+1)` given `x(n)`.

Example 37

Simulate the router buffer of Example 18 for $p = 0.48$, buffer capacity $c = 30$ packets, and $x_0 = 20$ packets initially buffered. Perform simulation runs for 40 and 400 time steps.

```
function x=simbuffer(p,c,x0,N)
P=zeros(c+1,c+1);
P(1,1)=1-p;
for i=1:c,
    P(i,i+1)=p; P(i+1,i)=1-p;
end
P(c+1,c+1)=p;
x=simdmc(P,x0,N);
```

Based on the Markov chain in Figure 3, almost all of the `simbuffer.m` code is to set up the transition matrix `P`. For starting state `x0` and `N` steps, the actual simulation requires only the command `simdmc(P,x0,N)`. Figure 10 shows two sample paths.

Continuous-Time Chains

```
function pv = cmcprob(Q,p0,t)
%Q has zero diagonal rates
%initial state probabilities p0
K=size(Q,1)-1; %max no. state
%check for integer p0
if (length(p0)==1)
    p0=((0:K)==p0);
end
R=Q-diag(sum(Q,2));
pv= (p0(:)'*expm(R*t))';
```

For a continuous-time Markov chain, `cmcprob.m` calculates the state probabilities $\mathbf{p}(t)$ by a direct implementation of matrix exponential solution of Equation (77).

Example 38

Assuming the air conditioner of Example 27 is off at time $t = 0$, calculate the state probabilities at time $t = 3.3$ minutes.

The program `aircondprob.m` performs the calculation for arbitrary initial state (or state probabilities) `p0` and time t. For the specified conditions, here is the output:

```
function pv=aircondprob(p0,t)
Q=[ 0  1/3   0 ; ...
   1/4  0  1/4; ...
   1/6 1/3  0];
pv=cmcprob(Q,p0,t);
```

```
>> aircondprob(0,3.3)'
ans =
   0.5024  0.3744  0.1232
>>
```

Finding $p_j(t)$ for an arbitrary time t may not be particularly instructive. Often, more can be learned using Theorem 22 to find the limiting state probability vector $\mathbf{p}$. We implement the MATLAB function `cmcstatprob(Q)` for this purpose.

```
function pv = cmcstatprob(Q)
%Q has zero diagonal rates
R=Q-diag(sum(Q,2));
n=size(Q,1);
R(:,1)=ones(n,1);
pv=([1 zeros(1,n-1)]*R^(-1))';
```

As we did in the discrete Markov chain, we replace the first column of `R` with $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}'$ to impose the constraint that the state probabilities sum to 1.

Example 39

Find the stationary probabilities of the air conditioning system of Example 27.

```
>> Q=[0  1/3   0 ; ...
     1/4  0   1/4; ...
     1/6 1/3   0];
>> cmcstatprob(Q)'
ans =
    0.4000  0.4000  0.2000
```

Although this problem is easily solved by hand, it is also an easy problem for MATLAB. Of course, the value of MATLAB is the ability to solve much larger problems.

Finally, we wish to simulate continuous-time Markov chains. This is more complicated than for discrete-time chains. For discrete-time systems, a sample path is completely specified by the random sequence of states. For a continuous-time chain, this is insufficient. For example, consider the 2 state

ON/OFF continuous-time Markov chain. Starting in state 0 at time $t = 0$, the state sequence is always $0, 1, 0, 1, 0 \ldots$; what distinguishes one sample path from another is how long the system spends in each state. Thus a complete characterization of a sample path specifies

- the sequence of states $X_0, X_1, X_2, \ldots, X_N$,
- the sequence of state visit times $T_0, T_1, T_2, \ldots, T_N$.

We generate these random sequences with the functions

`S=simcmcstep(Q,p0,n)` and `S=simcmc(Q,p0,T)`

shown in Figure 11. The function `simcmcstep.m` produces n steps of a continuous-time Markov chain with rate transition matrix `Q`. Using the function `simcmcstep.m` as a building block, `simcmc(Q,p0,T)` produces a sample path with a sufficient number, N, of state transitions to ensure that the simulation runs for time T. Note that N, the number of state transitions in a simulation of duration T, is a random variable. For repeated simulation experiments, `simcmc(Q,p0,T)` is preferable because each simulation run has the same time duration and comparing results from different runs is more straightforward.

As in the discrete-time simulation `simdmc.m`, if parameter `p0` is an integer, then it is the starting state of the simulation; otherwise, `p0` must be a state probability vector for the initial state. The output `S` is a $(N+1) \times 2$ matrix. The first column, `S(:,1)`, is the vector of states $\begin{bmatrix} X_0 & X_1 & \cdots & X_N \end{bmatrix}'$. The second column, `S(:,2)`, is the vector of visit times $\begin{bmatrix} T_0 & T_1 & \cdots & T_N \end{bmatrix}'$.

The code for `simcmc.m` is somewhat ugly in that it tries to guess a number n such that n transitions are sufficient for the simulation to run for time T. The guess n is based on the stationary probabilities and a calculated average state transition rate. If the n-step simulation, `simcmcstep(Q,p0,n)`, runs past time T, the output is truncated to time T. If n transitions are not enough, an additional $n' = \lceil n/2 \rceil$ are simulated. Additional segments of n' simulation steps are appended until a simulation of duration T is assembled. Note that care is taken so that state transitions across the boundaries of the simulation segments have the correct transition probabilities. The program

```
function ST=simcmc(Q,p0,T);
K=size(Q,1)-1; %max no. state
%calc  average trans. rate
ps=cmcstatprob(Q);
v=sum(Q,2); R=ps'*v;
n=ceil(0.6*T/R);
ST=simcmcstep(Q,p0,2*n);
while (sum(ST(:,2))<T),
    s=ST(size(ST,1),1);
    p00=Q(1+s,:)/v(1+s);
    S=simcmcstep(Q,p00,n);
    ST=[ST;S];
end
n=1+sum(cumsum(ST(:,2))<T);
ST=ST(1:n,:);
%truncate last holding time
ST(n,2)=T-sum(ST(1:n-1,2));
```

```
function S=simcmcstep(Q,p0,n);
%S=simcmcstep(Q,p0,n)
%  Simulate n steps of a cts
%  Markov Chain, rate matrix Q,
%  init. state probabilities p0
K=size(Q,1)-1; %max no. state
S=zeros(n+1,2);%init allocation
%check for integer p0
if (length(p0)==1)
    p0=((0:K)==p0);
end
v=sum(Q,2); %state dep. rates
t=1./v;
P=diag(t)*Q;
S(:,1)=simdmc(P,p0,n);
S(:,2)=t(1+S(:,1)) ...
    .*exponentialrv(1,n+1);
```

Figure 11 The MATLAB functions `simcmcstep` and `simcmc` for simulation of continuous-time Markov chains.

`simcmc.m` is not optimized to minimize its run time. We encourage you to examine and improve the code if you wish.

The real work of `simcmc.m` occurs in `simcmcstep.m`, which first generates the vectors $\begin{bmatrix}\nu_0 & \cdots & \nu_K\end{bmatrix}'$ of state departure rates and $\begin{bmatrix}t_0 & \cdots & t_K\end{bmatrix}'$ where $t_i = 1/\nu_i$ is the average time the system spends in a visit to state i. We recall that q_{ij}/ν_i is the conditional probability that the next system state is j given the current state is i. Thus we create a discrete-time state transition matrix $\mathbf{P}$ by dividing the ith row of $\mathbf{Q}$ by ν_i. We then use $\mathbf{P}$ in a discrete-time simulation to produce the state sequence $\begin{bmatrix}X_0 & X_1 & \cdots & X_N\end{bmatrix}'$, stored in the column `S(:,1)`. To generate the visit times, we first create the vector $\begin{bmatrix}t_{X_0} & t_{X_1} & \cdots & t_{X_N}\end{bmatrix}'$, stored in the MATLAB vector `t(1+S(:,1))`, where the component t_{X_i} is the conditional average duration of visit i, given that visit i was in state X_i. Lastly, we recall that if Y is an exponential (1) random variable, then $W = tY$ is an exponential $(1/t)$ random variable. To take advantage of this, we generate the vector $\begin{bmatrix}Y_0 & \cdots & Y_N\end{bmatrix}'$ of $N+1$ iid exponential (1) random variables. Finally, the vector $\begin{bmatrix}T_0 & \cdots & T_N\end{bmatrix}'$ with

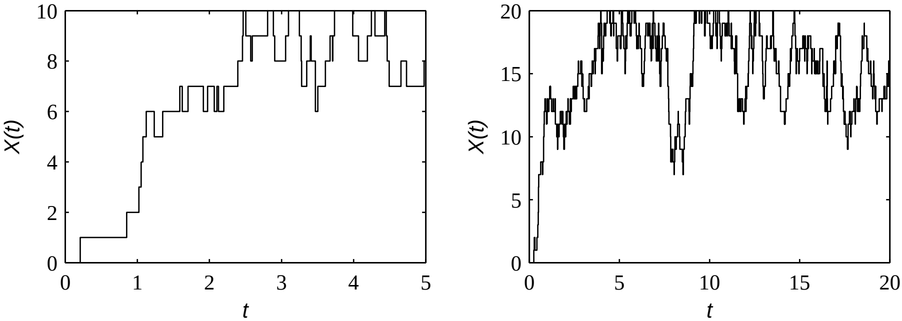
Matlab Function	**Output/Explanation**
`p = dmcstatprob(P)`	stationary probability vector of a discrete Markov chain
`x = simdmc(P,p0,n)`	`n` step simulation of a discrete Markov chain
`p = cmcprob(Q,p0,t)`	state probability vector at time `t` for a continuous MC
`p = cmcstatprobQ`	stationary probability vector for a continuous Markov chain
`S = simcmcstep(Q,p0,n)`	`n` step simulation of a continuous-time Markov chain
`S = simcmc(Q,p0,T)`	simulation of a continuous Markov chain for time `T`
`simplot(x,xlabel,ylabel)`	stairs plot for discrete-time state sequence `x`
`simplot(S,xlabel,ylabel)`	stairs plot for `simcmc` output `S`

In these MATLAB functions, `P` is a transition probability matrix for a discrete-time Markov chain, `p0` is an initial state probability vector, `p` is a stationary state probability vector, and `Q` is a transition rate matrix for a continuous-time Markov chain.

Table 1 MATLAB functions for Markov chains.

$T_n = t_{X_n} Y_n$ has the exponential visit times with the proper parameters. In particular, if $X_n = i$, then T_n is an exponential (ν_i) random variable. In terms of MATLAB, the vector $\begin{bmatrix} T_0 & \cdots & T_N \end{bmatrix}'$ is stored in the column `S(:,2)`.

To display the result `S` of a simulation generated by either `simcmc` or `simcmcstep`, we use the function `simplot(S,xlabel,ylabel)` which uses the `stairs` function to plot the state changes as a function of time. The optional arguments `xlabel` and `ylabel` label the x and y plot axes.



$\lambda = 8, \mu = 1, c = 10$
(a)

$\lambda = 16, \mu = 1, c = 20$
(b)

Figure 12 Simulation runs of the M/M/c/c queue for Example 40.

Example 40

Simulate the M/M/c/c blocking queue with the following system parameters:

(a) $\lambda = 8$ arrivals/minute, $c = 10$ servers, $\mu = 1$ min^{-1}, $T = 5$ minutes.

(b) $\lambda = 16$ arrivals/minute, $c = 20$ servers, $\mu = 1$, $T = 20$ minutes

. . . .

The function `simmmcc` implements a simulation of the M/M/c/c queue.

```
function ...
   S=simmmcc(lam,mu,c,p0,T);
%Simulate M/M/c/c queue, time T.
%lam=arr. rate, mu=svc. rate
%p0=init. state distribution
%c= number of servers
Q=zeros(c+1,c+1);
for i=1:c,
    Q(i,i+1)=lam;
    Q(i+1,i)=(i-1)*mu;
end
S=simcmc(Q,p0,T);
```

The program calculates the rate transition matrix $\mathbf{Q}$ and calls `S=simcmc(Q,0,20)` to perform the simulation for 20 time units. Sample simulation runs for the M/M/c/c queue appear in Figure 12. The output of Figure 12(a) is generated with the commands:

```
lam=8;mu=1.0;c=10;T=5;
S=simmmcc(lam,mu,c,0,T);
simplot(S,'t', 'X(t)');
```

The simulation programs `simdmc` and `simcmc` can be quite useful because they simulate a system given a state transition matrix $\mathbf{P}$ or $\mathbf{Q}$ that one is likely to have coded in order to calculate the stationary probabilities. However, these simulation programs do suffer from serious limitations. In particular, for a system with $K+1$ states, complete enumeration of all elements of a $K+1 \times K+1$ state transition matrix is needed. This can become a problem because K can be *very* large for practical problems. In this case, complete enumeration of the states becomes impossible.

Quiz 10

Simulate the M/M/1/c blocking queue for 40 minutes with the following system parameters: $\lambda = 0.8$ arrivals/minute, $c = 5$ servers, $\mu = 1$ min^{-1}.

Further Reading: Markov chains and queuing theory comprise their own branches of mathematics. For several theorems in this supplement, the result may be intuitive, but proof was beyond the scope of this text. References [Ros03], [Gal13] and [Kle75] are entry points to these subjects for students interested in going further.

Gal13. R. G. Gallager. *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.

Kle75. L. Kleinrock. *Queueing Systems Volume 1: Theory*. John Wiley & Sons, 1975.

Ros03. S. M. Ross. *Introduction to Probability Models*. Academic Press, 8th edition, 2003.

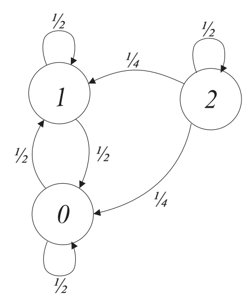
Problems

Difficulty: ● Easy ■ Moderate ♦ Difficult ♦♦ Experts Only

1.1● The packet voice model of Example 2 can be enhanced by examining speech in mini-slots of 100 microseconds duration. On this finer timescale, we observe that the ON periods are interrupted by mini-OFF periods. In the OFF state, the system goes to ON with probability 1/14,000. In the ON state, the system goes to OFF with probability 0.0001, or goes to mini-OFF with probability 0.1; otherwise, the system remains ON. In the mini-OFF state, the system goes to OFF with probability 0.0001 or goes to ON with probability 0.3; otherwise, it stays in the mini-OFF state. Sketch the Markov chain with states (0) OFF, (1) ON, and (2) mini-OFF. Find the state transition matrix $\mathbf{P}$.

1.2● Each second, a laptop computer's wireless LAN card reports the state of the radio channel to an access point. The channel may be (0) poor, (1) fair, (2) good, or (3) excellent. In the poor state, the next state is equally likely to be poor or fair. In states 1, 2, and 3, there is a probability 0.9 that the next system state will be unchanged from the previous state and a probability 0.04 that the next system state will be poor. In states 1 and 2, there is a probability 0.06 that the next state is one step up in quality. When the channel is excellent, the next state is either good with probability 0.04 or fair with probability 0.02. Sketch the Markov chain and find the state transition matrix $\mathbf{P}$.

1.3● Find the state transition matrix $\mathbf{P}$ for the Markov chain:



1.4● In a two-state discrete-time Markov chain, state changes can occur each second. Once the system is OFF, the system stays off for another second with probability 0.2. Once the system is ON, it stays on with probability 0.1. Sketch the Markov chain and find the state transition matrix $\mathbf{P}$.

1.5■ For Example 3, suppose each read or write operation reads or writes an entire file and that files contain a geometric number of sectors with mean 50. Further, suppose idle periods last for a geometric time with mean 500. After an idle period, the system is equally likely to read or write a file. Following the completion of a read, a write follows with probability 0.8. However, on completion of a write operation, a read operation follows with probability 0.6. Label the transition probabilities for the Markov chain in Example 3.

1.6■ The state of a discrete-time Markov chain with transition matrix $\mathbf{P}$ can change once each second; X_n denotes the system state after n seconds. An observer examines the system state every m seconds, producing the observation sequence $\hat{X}_0, \hat{X}_1, \ldots$ where $\hat{X}_n = X_{mn}$. Is $\hat{X}_0, \hat{X}_1, \ldots$ a Markov chain? If so, find the state transition matrix $\hat{\mathbf{P}}$.

1.7♦ The state of a discrete-time Markov chain with transition matrix $\mathbf{P}$ can change once each second; $X_n \in \{0, 1, \ldots, K\}$ denotes the system state after n seconds. An observer examines the system state at a set of random times $T_0, T_1, \ldots$. Given an iid random sequence $K_0, K_1, \ldots$ with PMF $P_K(k)$, the random inspection times are given by $T_0 = 0$ and $T_m = T_{m-1} + K_{m-1}$ for $m \geq 1$. Is the observation sequence $Y_n = X_{T_n}$ a Markov chain? If so, find the transition matrix $\hat{\mathbf{P}}$.

1.8♦♦ Continuing Problem 1.7, suppose the observer waits a random time that depends on the most recent state observation until the next inspection. The incremental time K_n until inspection $n+1$ depends on the state Y_n; however, given Y_n, K_n is conditionally independent of $K_0, \ldots, K_{n-1}$. In particular, assume that $P_{K_n|Y_n}(k|y) = g_y(k)$, where each $g_y(k)$ is a valid PMF satisfying $g_y(k) \geq 0$ and $\sum_{k=0}^{K} g_y(k) = 1$. Is $Y_0, Y_1, \ldots$ a Markov chain?

2.1● Find the n-step transition matrix $\mathbf{P}(n)$ for the Markov chain of Problem 1.4.

2.2■ Find the n-step transition matrix $\mathbf{P}^n$ for the Markov chain of Problem 1.3.

3.1● Consider the packet voice system in Example 8. If the speaker is silent at time 0, how long does it take until all components $p_j(n)$ of $\mathbf{p}(n)$ are within 1% of the stationary probabilities $\boldsymbol{\pi}_j$.

3.2■ A Markov chain with transition probabilities P_{ij} has an unusual state k such that $P_{ik} = q$ for every state i. Prove that the probability of state k at any time $n \geq 1$ is $p_k(n) = q$.

3.3■ A wireless packet communications channel suffers from clustered errors. That is, whenever a packet has an error, the next packet will have an error with probability 0.9. Whenever a packet is error-free, the next packet is error-free with probability 0.99. In steady-state, what is the probability that a packet has an error?

4.1● For the Markov chain in Problem 2.2, find all the ways that we can replace a transition P_{ij} with a new transition $P_{ij'} = P_{ij}$ to create an aperiodic irreducible Markov chain.

4.2● What is the minimum number of transitions $P_{ij} > 0$ that must be added to the Markov chain in Example 11 to create an irreducible Markov chain?

4.3■ Prove that if states i and j are positive recurrent and belong to the same communicating class, then $\mathrm{E}[T_{ij}] < \infty$.

5.1● For the transition probabilities found in Problem 1.5, find the stationary distribution $\boldsymbol{\pi}$.

5.2● Find the stationary probability vector $\boldsymbol{\pi}$ for the Markov chain of Problem 1.3.

5.3● Consider a variation on the five-position random walk of Example 12 such that:

- In state 0, the system goes to state 1 with probability $P_{01} = p$ or stays in state 0 with probability $P_{00} = 1 - p$.
- In state 4, the system with stays there with probability p or goes to state 3 with probability $1 - p$.

Sketch the Markov chain and find the stationary probability vector.

5.4● Each morning on your way to school, you go to a coffee shop with probability p and get a cup of coffee. With each cup purchased, you get your "club card" punched. After 7 punches, you redeem your club card on your next visit for one free cup of coffee, and then receive a new unpunched card. Let X_n denote the number of punches on your card when you wake up on day n. What is $\pi_k = \lim_{n\to\infty} \mathrm{P}[X_n = k]$?

5.5● Suppose you are a student who is always in one of three possible states:

(0) sleeping in the dorm,
(1) eating in the cafeteria, or
(2) reading in the library.

Suppose a location teleportation technology enables state changes to occur instantly on the change of the hour, i.e. at exactly 11AM, 12 noon, 1PM and so on. We will model the student state by a discrete-time Markov chain with state transition matrix $\mathbf{P}$.

(a) For each of the following facts, indicate what elements P_{ij} of $\mathbf{P}$ are revealed.

  (A) If the student is eating, the student continues to eat for another hour with probability 0.1; otherwise, the student begins to read.

  (B) Sleeping is always followed by eating. Each time the student goes to sleep (takes a nap), the average nap time is 4 hours.

  (C) After an hour of reading, another hour of reading is four times more probable than an hour of eating. However, the probability of going to sleep equals the sum of the probabilities of reading or eating.

(b) After a long time, a friend goes looking for the student. To maximize the probability of finding the student, should the friend look in the dorm room, the cafeteria or the library?

5.6● In the following problem, time passes in slots indexed by the integer $t = 1, 2, \ldots$ In each slot t, a visitor passes by a dog Charm with probability α, independent of any other slot. If Charm is idle, then she begs for attention so that each visitor is willing to give Charm a random number R of "rubs behind her ears." When $R = r$, rubs are given in slots $t+1, \ldots, t+r$ (since each rub lasts for one slot) and the visitor departs at the end of slot $t + r$. Charm will look for attention from the next visitor. If a visitor arrives while Charm is getting rubbed by a prior visitor, the new visitor is ignored by Charm and immediately departs. As it happens, the number of rubs offered by each visitor has PMF

$$P_R(r) = \begin{cases} 1/5 & r = 0, 1, 2, 3, 4, \\ 0 & \text{otherwise}, \end{cases}$$

independent of the number of rubs offered by any visitor. Let A_t denote the event that Charm is rubbed in slot t. Find $\lim_{t\to\infty} \mathrm{P}[A_t]$. Hint: Use a discrete time Markov chain in which the state in slot t is k if Charm is receiving her kth rub from the current visitor.

5.7● An electronic signboard in a subway station displays W, the number of minutes until the next train arrives. Note that the signboard only displays integers: if the next train arrival is in $t = 3.02$ minutes, the signboard displays $W = \lceil t \rceil = 4$. The instant after a train stops, the signboard displays the time until the next train, even if the train the station takes a few seconds to depart. Curiously, the trains follow a synchronized schedule: interarrival times are independent and are exactly K minutes long where K is a discrete uniform $(1, 5)$ random variable. That is, $P_K(k) = 1/5$ for $k = 1, 2, 3, 4, 5$. If you arrive at a random time, what is the probability π_3 that the signboard says $W = 3$?

5.8● An electronic signboard in a subway station displays W, the time (in whole minutes) since the last train arrived. Note that the signboard only displays integers: if the last train arrived $t = 3.02$ minutes ago, the signboard displays $W = \lfloor t \rfloor = 3$. The instant a train arrives, the signboard resets to $W = 0$. Curiously, the trains follow a synchronized schedule: interarrival times are independent and are exactly K minutes long such that K is a discrete uniform $(2, 5)$ random variable. That is, $P_K(k) = 1/4$ for $k = 2, 3, 4, 5$. If you arrive at a random time, what is the probability π_3 that the signboard says $W = 3$?

5.9■ In this problem, we extend the random walk of Problem 5.3 to have positions $0, 1, \ldots, K$. In particular, the state transitions are

$$P_{ij} = \begin{cases} 1-p & i = j = 0, \\ p & j = i+1; \\ & i = 0, \ldots, K-1, \\ p & i = j = K, \\ 1-p & j = i-1; i = 1, \ldots, K, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the Markov chain and calculate the stationary probabilities.

5.10■ A checkout counter has a "Take-a-Penny-Leave-a-Penny" bin.. The kth customer's charge (in pennies) is a random variable X_k such that $Y_k = X_k \bmod 5$ is a discrete uniform $(0, 4)$ random variable, independent of any other transaction. When customer k checks out, the penny bin goes unused if $Y_k \in \{0, 2, 3\}$. If $Y_k = 1$ and the penny bin has at least one penny, the customer takes a penny from the bin to pay the clerk. If $Y_k = 4$, the customer receives a penny in change from the clerk and deposits this in the penny bin. If this is the fourth penny in the bin, the clerk puts the four pennies in his pocket immediately after the customer departs, leaving the penny bin empty. (The clerk views these four pennies as a reward for providing the penny bin.)

(a) Let N_k denote the number of pennies in the bin when the kth customer is served. Construct a discrete time Markov chain for N_k and find the stationary probabilities $\pi_n = \lim_{k\to\infty} \mathrm{P}[N_k = n]$.

(b) Let R_k denote the reward received by the clerk after serving the kth customer. What is $\lim_{k\to\infty} \mathrm{E}[R_k]$?

5.11■ A circular game board has K spaces numbered $0, 1, \ldots, K-1$. Starting at space 0 at time $n = 0$, a player rolls a fair six-sided die to move a token. Given the current token position X_n, the next token position is $X_{n+1} = (X_n + R_n) \bmod K$ where R_n is the result of the player's nth roll. Find the stationary probability vector $\boldsymbol{\pi} = [\pi_0 \quad \cdots \quad \pi_{K-1}]'$.

5.12■ A very busy bank has two drive-thru teller windows in series served by a single line. When there is a backlog of waiting cars, two cars begin service simultaneously. The front customer can leave if she completes service before the rear customer. However, if the rear customer finishes first, he cannot leave until the front customer finishes. Consequently, the teller at each window will sometimes be idle if their customer completes service before the customer at the other window. Assume there is an infinite backlog of waiting cars and that service requirements of the cars (measured in seconds) are geometric random variables with a mean of 120 seconds. Draw a Markov chain that describes whether each teller is busy. What is the stationary probability that both tellers are busy?

5.13 Repeat Problem 5.12 under the assumption that each service time is equally likely to last either exactly one minute or exactly two minutes.

5.14 Let N be an integer-valued positive random variable with range $S_N = \{1, \ldots, K+1\}$. We use N to generate a Markov chain with state space $\{0, \ldots, K\}$ in the following way. In state 0, a transition back to state 0 occurs with probability $P_N(1)$. When the system is in state $i \geq 0$, either a transition to state $i+1$ occurs with probability

$$P_{i,i+1} = \mathrm{P}\left[N > i+1/N > i\right],$$

or a transition to state 0 occurs with probability

$$P_{i,0} = \mathrm{P}\left[N = i+1/N > i\right].$$

Find the stationary probabilities. Compare your answer to the solution of Quiz 5.

5.15 A wireless communication link transmits fixed-length packets. The transmission of a packet requires exactly one unit of time, called a "time slot." The wireless link is well designed so that a transmitted packet is always received correctly. We say the link is in the idle state in slot t if it has no packets to transmit in that slot. Here are some additional facts regarding the link:

- In each time slot t, a packet arrives with probability p, independent of the event of an arrival in any other slot and independent of the state of the system prior to its arrival.
- A packet arriving in slot t can be transmitted as early as slot $t+1$ if the link was busy in slot t. However, if the transmitter is idle in slot t, the new arriving packet must be queued while slots $t+1$ and $t+2$ are used for a link initialization procedure.
- Any additional packets that arrive during the initialization procedure are also queued until the initialization procedure is done.

Using 0 to denote the idle state, construct a discrete-time Markov chain for this system. Define (in words) what your system states represent. Calculate the stationary probability π_0 that the link is idle.

5.16 $K_1, K_2, \ldots$ is an iid sequence such that K_i is a key chosen equiprobably from the set $\{1000, 1001, \ldots, 9999\}$. Let X_n denote the number of *unique* keys in the set $\{K_1, K_2, \ldots, K_n\}$. Obviously $X_1 = 1$. Let $U_n = 1$ if code K_n is different from all previously chosen keys $K_1, \ldots, K_{n-1}$; otherwise $U_n = 0$.

(a) Find the conditional PMF $P_{U_n|X_{n-1}}(u|x)$.

(b) Using the fact that $X_n = X_{n-1} + U_n$, find the conditional expected values $\mathrm{E}[X_n|X_{n-1} = x]$ and $\mathrm{E}[X_n|X_{n-1}]$.

(c) Find $\mathrm{E}[X_n]$. Hint: Let $\mu_n = \mathrm{E}[X_n]$ and find a recursion for μ_n.

(d) What is $\lim_{n\to\infty} \mathrm{E}[X_n]$? Explain your answer.

(e) Suppose we define $X_0 = 0$. Is the random sequence $\{X_n | n = 0, 1, 2, \ldots\}$ a Markov chain? If not, explain why. If so, explain why, find and sketch the Markov chain, and find the limiting state probabilities (or explain why they don't exist.)

5.17■ Starting infinitely long ago in the past, a new customer arrives each minute at a bank, exactly at the start of each minute. Each arriving customer is immediately served by a teller. (There are always as many tellers as needed to serve all customers in the bank.) After each minute of service, a customer departs with probability $1-p$, independent of the departures of all other customers. Departures at the end of minute $t-1$ occur the instant before the new customer arrives at the start of minute t. We say that a customer is in service at minute t, if the customer arrived at a minute $s \le t$ and did not depart prior to the end of minute t.

(a) Let $X_{t,k}$ denote an indicator random variable such that $X_{t,k} = 1$ if the customer that arrived at the start of minute t is still in service at minute $t+k$; otherwise $X_{t,k} = 0$. What is the probability mass function (PMF) $P_{X_{t,k}}(x)$ of $X_{t,k}$?

(b) Let Y_t denote the number of customers in service at minute t (i.e. the instant after the new arrival at the start of minute t.) Find the expected value $\mathrm{E}[Y_t]$ and variance $\mathrm{Var}[Y_t]$. Hint: $Y_t = X_{t,0} + X_{t-1,1} + \cdots$.

(c) Suppose that the customers are afraid of crowded places. Upon arrival, a customer instantly departs if the bank already has two customers. Model the process Y_t as a discrete-time Markov chain. Either find the limiting state probabilities or explain why they do not exist.

5.18♦♦ Prove that for an aperiodic, irreducible finite Markov chain there exists a constant $\delta > 0$ and a time step τ such that

$$\min_{i,j} P_{ij}(\tau) = \delta.$$

5.19♦♦ To prove Theorem 11, complete the following steps.

(a) Define

$$m_j(n) = \min_i P_{ij}(n),$$
$$M_j(n) = \max_i P_{ij}(n).$$

Show that $m_j(n) \le m_j(n+1)$ and $M_j(n+1) \le M_j(n)$.

(b) To complete the proof, we need to show that $\Delta_j(n) = M_j(n) - m_j(n)$ goes to zero. First, show that

$$\Delta_j(n+\tau) = \max_{\alpha,\beta} \sum_k \left(P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right) P_{kj}(n).$$

(c) Define $Q = \{k | P_{\alpha k}(\tau) \ge P_{\beta k}(\tau)\}$ and show that

$$\sum_k (P_{\alpha k}(\tau) - P_{\beta k}(\tau)) P_{kj}(n) \le \Delta_j(n) \sum_{k\in Q} (P_{\alpha k}(\tau) - P_{\beta k}(\tau)).$$

(d) Show that step (c) combined with the result of Problem 5.18 implies

$$\Delta_j(n+\tau) \le (1-\delta)\Delta_j(n).$$

(e) Conclude that $\lim_{n\to\infty} P_{ij}(n) = \pi_j$ for all i.

6.1■ Consider an irreducible Markov chain. Prove that if the chain is periodic, then $P_{ii} = 0$ for all states i. Is the converse also true? If $P_{ii} = 0$ for all i, is the irreducible chain periodic?

6.2■ A particular discrete-time finite Markov chain has the property that the set of states can be partioned into classes $\{C_0, C_2, \ldots, C_{L-1}\}$ such that for all states $i \in C_l$, $P_{ij} = 0$ for all $j \notin C_{l+1 \mod L}$. Prove that all states have period $d = L$.

6.3■ For the periodic chain of Problem 6.2, either prove that the chain has a single recurrent communicating class or find a counterexample to show this is not always the case.

6.4♦ Squares are labeled 1 through n consecutively from left to right. A player starts by placing a token on square k where $1 < k < n$. On each turn, a six-sided die is rolled. With a roll of 1 or 2, the token moves one square to the left. With a roll of 3 or higher, the token moves one square to the right. The player wins if the token reaches square 1. The game is a loss if the token reaches square n. When the game ends, the token stays on its final square (1 or n).

(a) Sketch a Markov chain that describes the position of the token and find the state transition matrix.

(b) Identify any recurrent communicating classes, the set of transient states (if any).

(c) Identify the set of all possible stationary distributions (if any exist).

(d) Let W_k denote the event that you win the game starting from position k. Outline a calculation procedure to find $P[W_k]$ for a given value of k.

(e) For the special case of $n = 4$, find $P[W_2]$ and $P[W_3]$. Hint: Don't use the calculation procedure from part (d).

6.5♦ In this problem, we prove a converse to the claim of Problem 6.2. An irreducible Markov chain has period d. Prove that the set of states can be partioned into classes $\{C_0, C_2, \ldots, C_{d-1}\}$ such that for all states $i \in C_l$, $P_{ij} = 0$ for all $j \notin C_{l+1 \mod d}$.

7.1● Consider a discrete random walk with state space $\{0, 1, 2, \ldots\}$ similar to Example 4 except there is a barrier at the origin so that in state 0, the system can remain in state 0 with probability $1 - p$ or go to state 1 with probability p. In states $i > 0$, the system can go to state $i - 1$ with probability $1 - p$ or to state $i + 1$ with probability p. Sketch the Markov chain and find the stationary probabilities.

7.2■ At an airport information kiosk, customers wait in line for help. The customer at the front of the line who is actually receiving assistance is called the customer in service. Other customers wait in line for their turns. The queue evolves under the following rules.

- If there is a customer in service at the start of the one-second interval, that customer completes service (by receiving the information she needs) and departs with probability q, independent of the number of past seconds of service she has received; otherwise that customer stays in service for the next second.

- In each one-second interval, a new customer arrives with probability p; otherwise no new customer arrives. Whether a customer arrives is independent of both the number of customers already in the queue and the amount of service already received by the customer in service.

Using the number of customers in the system as a system state, construct a Markov chain for this system. Under what conditions does a stationary distribution exist? Under those conditions, find the stationary probabilities.

7.3■ Verify Equation (61) that the expected number of return visits to state i over all time is given by

$$\mathrm{E}\left[N_{ii}\right] = \sum_{n=1}^{\infty} P_{ii}(n).$$

Hint: Use the Bernoulli indicator random variable $I_{ii}(n)$ such that $I_{ii}(n) = 1$ if $X_n = i$; otherwise $I_{ii}(n) = 0$.

7.4♦ For the random walk in Example 4, use Equation (61) to find those values of p for which the state 0 is recurrent. Hint: Keep in mind that the system can return to state 0 after a even number $2n$ of transitions and use Stirling's approximation $n! \approx \sqrt{2n\pi}n^n e^{-n}$ to simplify $P_{00}(2n)$.

7.5♦ In a variation on the Markov chain for Example 22, a system with states $\{0, 1, 2, \ldots\}$ has transition probabilities

$$P_{ij} = \begin{cases} 1 & i = 0, j = 1, \\ [i/(i+1)]^{\alpha} & i > 0, j = i+1, \\ 1 - [i/(i+1)]^{\alpha} & i > 0, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha > 0$ is an unspecified parameter. Sketch the Markov chain and identify those values of α for which the states are positive recurrent, null recurrent, and transient.

8.1● A tiger is always in one of three states: (0) sleeping, (1) hunting, and (2) eating. A tiger's life is fairly monotonous, and it always goes from sleeping to hunting to eating and back to sleeping. On average, the tiger sleeps for 3 hours, hunts for 2 hours, and eats for 30 minutes. Assuming the tiger remains in each state for an exponential time, model the tiger's life by a continuous-time Markov chain. What are the stationary probabilities?

8.2● At a bus stop, buses arrive as a Poisson process of rate μ and riders arrive as a Poisson process of rate λ. Arriving riders always wait for the next bus to arrive. When a bus arrives, all waiting riders board the bus and depart.

(a) Sketch a continuous-time Markov chain for the process $N(t)$, the number of riders waiting at the bus stop at time t.

(b) Find the limiting state probabilities $p_n = \lim_{t\to\infty} \mathrm{P}[N(t) = n]$.

(c) Identify any necessary conditions needed to ensure the existence of the limiting state probabilities. Does your answer depend on some (perhaps unusual) modeling assumptions?

8.3● In a continuous-time model for the two-state packet voice system, talkspurts (active periods) and silent periods have exponential durations. The average silent period lasts 1.4 seconds and the average talkspurt lasts 1 second. What are the limiting state probabilities? Compare your answer to the limiting state probabilities in the discrete-time packet voice system of Example 8.

8.4■ A bank has two drive-thru teller windows in series served by a single line. When there is a backlog of waiting cars, two cars begin service simultaneously. The front customer can leave if she completes service before the rear customer. However, if the rear customer finishes first, he cannot leave until the front customer finishes. Consequently, the teller at each window will sometimes be idle if their customer completes service before the customer at the other window. Assume there is an infinite backlog of waiting cars. The two tellers worked at different speeds. For teller 1, the average service time is 1 minute, but for teller 2 the average service time is 2 minutes. The service times of the customers are independent exponential random variables. Should teller 1 serve the front customer or the rear customer?

8.5● Consider a continuous-time Markov chain with states $\{1, \ldots, k\}$. From every state i, the transition rate to any state j is $q_{ij} = 1$. What are the limiting state probabilities?

8.6● Let $N_0(t)$ and $N_1(t)$ be independent Poisson processes with rates λ_0 and λ_1. Construct a Markov chain that tracks whether the most recent arrival was type 0 or type 1. Assume the system starts at time $t = 0$ in state 0 since there were no previous arrivals. In the distant future, what is the probability that the system is in state 1?

8.7■ $N(t)$ is a rate λ Poisson process.

(a) For $s < t$, find the conditional PMF

$$P_{N(s)|N(t)}(k|n) = \mathrm{P}\left[N(s) = k|N(t) = n\right].$$

(b) For $s > t$, find the conditional PMF

$$P_{N(s)|N(t)}(k|n) = \mathrm{P}\left[N(s) = k|N(t) = n\right].$$

8.8■ A remote sensor transmits measurement packets as a Poisson process of rate λ packets/sec to a data collection receiver through a random radio channel. The packets are very short so that we can assume the sensor always completes the transmission of a packet well before a new packet is created. The radio channel alternates between good and bad states G and B. Good channel periods have an exponential duration with a mean value of $1/\gamma = 0.8$ sec while the duration of a bad period is exponential with mean $1/\mu = 0.4$ sec. The lengths of good and bad periods are all independent. At time 0, the system has been running for a long time and we start to count $R(t)$ the number of succesfully received packets in the interval $[0, t]$, Please complete the following parts:

(a) Sketch a two state continuous time Markov chain for the radio channel. In steady state, what is the probability P_G that the channel is good at an arbitrary random time?

(b) The sensor transmits measurement packets without examining the channel state. A packet will be received succesfully (without error) only if the entire transmission of the packet occurs during a period when the channel is good. Packets received in error are simply discarded by the receiver and no packets are ever retransmitted. If a packet has a deterministic transmission time t_0, what is the probability $\mathrm{P}[R_d]$ that a measurement packet transmitted at a random time is received successfully?

(c) Suppose now that the packet transmission time is an exponential random variable T with mean value of $1/\alpha$, now what is the probability $\mathrm{P}[R_e]$ that a packet is received successfully?

(d) Assuming again that the packet transmssion times are deterministic and short, is $R(t)$ a Poisson process? If so, justify your answer. If not, explain under what circumstances a Poisson model might be appropriate.

8.9♦ A wedding ring store only serves couples. Couples arrive at the store as a Poisson process of rate λ couples/hour. The store has only a single clerk and each person has an independent exponential (μ) service requirement. However, persons in a couple are served in succession; when the first person in a couple completes service, then that person's partner is served next. Of course the partner who has completed service waits for his/her partner to complete service so they can depart the store together. Sketch a continuous Markov chain for the backlog of customers in the store. Calculate the stationary probabilities. Under what conditions is the Markov chain ergodic? If $N(t)$ is the number of customers in the store at time t, calculate $\lim_{t\to\infty} \mathrm{P}[N(t) = n]$.

8.10♦ Packets arrive at a forwarding node as a Poisson process of rate 1 per millisecond (ms). The forwarder simply forwards (i.e. transmits) packets stored in its infinite capacity buffer. When the node is working, arriving packets are queued in the buffer and packet transmission times are independent exponential random variables with expected service time of $1/\mu = 0.5$ ms. However, the forwarding node takes a break after the completion of a packet transmission. This break has an exponential duration with expected value $1/\beta$ ms, independent of the arrival process and packet transmission times. During the break, the forwarder discards all arriving packets. Following the break, the node goes back to work by transmitting a previously buffered packet.

(a) Sketch a continuous time Markov chain for this system. Hint: the forwarder may be *W*orking or on *B*reak when there are there are n buffered packets. For what values of β is the Markov chain irreducible?

(b) Find the limiting state probabilities when the chain is irreducible.

(c) After the system has been running for a long time, what is the probability $\mathrm{P}[D]$ that an arriving packet is discarded?

9.1● For the M/M/c/c queue with $c = 2$ servers, what is the maximum normalized load $\rho = \lambda/\mu$ such that the blocking probability is no more than 0.1?

9.2● For the telephone switch in Example 32, suppose we double the number of circuits to 200 in order to serve 80 calls per minute. Assuming the average call duration remains at 2 minutes, what is the probability that a call is blocked?

9.3● Find the limiting state distribution of the $M/M/1/c$ queue that has one server and capacity c.

9.4■ A set of c toll booths at the entrance to a highway can be modeled as a c server queue with infinite capacity. Assuming the service times are independent exponential random variables with mean $\mu = 1$ second, sketch a continuous-time Markov chain for the system. Find the limiting state probabilities. What is the maximum arrival rate such that the limiting state probabilities exist?

9.5■ Consider a grocery store with two queues. At either queue, a customer has an exponential service time with an expected value of 3 minutes. Customers arrive at the two queues as a Poisson process of rate λ customers per minute. Consider the following possibilities:

(a) Customers choose a queue at random so each queue has a Poisson arrival process of rate $\lambda/2$.

(b) Customers wait in a combined line. When a customer completes service at either queue, the customer at the front of the line goes into service.

For each system, calculate the limiting state probabilities. Under which system is the average system time smaller?

9.6■ In a last come first served (LCFS) queue, the most recent arrival is put at the front of the queue and given service. If a customer was in service when an arrival occurs, that customer's service is discarded when the new arrival goes into service. Find the limiting state probabilities for this queue when the arrivals are Poisson with rate λ and service times are exponential with mean $1/\mu$.

9.7■ Customers arrive at the checkout counter of a store as a Poisson process of rate $\lambda = 0.5$ per minute. When the checkout clerk provides service, the service time of a customer is an exponential random variable with expected value $1/\mu = 1$ minute. However, the clerk is also an addicted gamer. Whenever the checkout is idle, the clerk sneaks off to the backroom to play *World of Warcraft (WoW)*. The clerk starts working at the checkout once there are two customers waiting in line. The clerk then serves customers until the checkout queue is empty.

(a) Construct a continuous-time Markov chain for this system and find the stationary probabilities.

(b) Let W_t denote the event that the clerk is playing WoW at time t. Find $\lim_{t\to\infty} \mathrm{P}[W(t)]$.

(c) Find $\overline{T}$, the average time a customer spends in the checkout.

9.8♦ In a cellular phone system, each cell must handle new call attempts as well as handoff calls that originated in other cells. Calls in the process of handoff from one cell to another cell may suffer forced termination if all the radio channels in the new cell are in use. Since dropping, which is another name for forced termination, is considered very undesirable, a number r of radio channels are reserved for handoff calls. That is, in a cell with c radio channels, when the number of busy circuits exceeds $c-r$, new calls are blocked and only handoff calls are admitted. Both new calls and handoff calls occupy a radio channel for an exponential duration lasting 1 minute on average. Assuming new calls and handoff calls arrive at the cell as independent Poisson processes with rates λ and h calls/minute, what is the probability $\mathrm{P}[H]$ that a handoff call is dropped?

10.1■ In a self-fulfilling prophecy, it has come to pass that for a superstitious basketball player, his past free throws influence the probability of success of his next attempt in the following curious way. After n consecutive successes, the probability of success on the next free throw is

$$\alpha_n = 0.5 + 0.1 \min(n, 4).$$

Similarly, after n consecutive failures, the probability of failure is also $\alpha_n = 0.5 + 0.1 \min(n, 4)$. At the start of the game, the player has no past history and so $n = 0$. Identify a Markov chain for the system using state space $\{-4, -3, \ldots, 4\}$ where state $n > 0$ denotes n consecutive successes and state $n < 0$ denotes $|n|$ consecutive misses. With a few seconds left in a game, the player has already attempted 10 free throws in the game. What is the probability that his eleventh will be successful?

10.2♦ A mobile terminal can send a packet in a 1ms timeslot over a wireless channel. The channel is time varying:

- If the channel is good in a timeslot, then it remains good in the next timeslot with probability $q = 0.95$; otherwise the channel goes bad with probability $1 - q = 0.05$.
- If the channel is bad in a timeslot, then it remains bad in the next timeslot with probability $p = 0.9$; otherwise the channel becomes good with probability $1 - p = 0.1$.

When the channel is good, a transmitted packet is received in error with probability $\gamma = 10^{-2}$. When the channel is bad, a packet is received in error with probability $\beta = 1/2$.

Following each transmitted packet, the sender immediately receives an acknowledgment indicating whether the packet was received successfully or in error. Whenever two consecutive packets are received in error, the mobile enters a timeout state in which no packets are sent. During a timeout, the mobile performs an independent Bernoulli trial with success probability h in every slot. When a success occurs, the mobile starts sending in the next slot as though no packets had been in error.

Construct a Markov chain model of the system in order to answer the following questions.

(a) Let S_t denote the event that a packet is sent (transmitted) in timeslot t. Find

$$\mathrm{P}[S] = \lim_{t\to\infty} \mathrm{P}[S_t].$$

What value of h maximizes $\mathrm{P}[S]$?

(b) Let R_t denote the event that a packet is received successfully in timeslot t. Find

$$\mathrm{P}[R] = \lim_{t\to\infty} \mathrm{P}[R_t].$$

What value of h maximizes $\mathrm{P}[R]$.

(c) The efficiency of the transmitter is measured by

$$\eta = \frac{\mathrm{P}[R]}{\mathrm{P}[S]}.$$

What value of h maximizes η?

Since calculating $\mathrm{P}[S]$, $\mathrm{P}[R]$ and η as a function of h may be too much of an algebra challenge, plot these quantities as a function of h and interpret your results.

10.3◆ A store employs a checkout clerk and a manager. Customers arrive at the checkout counter as a Poisson process of rate λ and have independent exponential service times with a mean of 1 minute. As long as the number of checkout customers stays below five, the clerk handles the checkout. However, as soon as there are five customers in the checkout, the manager opens a new checkout counter. At that point, both clerk and manager serve the customers until the checkout counters have just a single customer. Let N denote the number of customers in the queue in steady-state. For each $\lambda \in \{0.5, 1, 1.5\}$, answer the following questions regarding the steady-state behavior of the checkout queue:

(a) What is $\mathrm{E}[N]$?

(b) What is the probability, $\mathrm{P}[W]$, that the manager is working the checkout?

Hint: Although the chain is countably infinite, a bit of analysis and solving a system of nine equations is sufficient to find the stationary distribution.

10.4◆ At an autoparts store, let X_n denote how many brake pads are in stock at the start of day n. Each day, the number of orders for brake pads is a Poisson random variable K with mean $\mathrm{E}[K] = 50$. If $K \leq X_n$, then all orders are fulfilled and K pads are sold during the day. If $K > X_n$, then X_n pads are sold but $K - X_n$ orders are lost. If at the end of the day, the number of pads left in stock is less than 60, then 50 additional brake pads are delivered overnight. Identify the transition probabilities for the Markov chain X_n. Find the stationary probabilities. Let Y denote the number of pads sold in a day. What is $\mathrm{E}[Y]$?

10.5◆ The Veryfast Bank has a pair of drive-thru teller windows in parallel. Each car requires an independent exponential service time with a mean of 1 minute. Cars wait in a common line such that the car at the head of the waiting line enters service with the next available teller. Cars arrive as a Poisson process of rate 0.75 car per minute, independent of the state of the queue. However, if an arriving car sees that there are six cars waiting (in addition to the cars in service), then the arriving customer becomes discouraged and immediately departs. Identify a Markov chain for this system, find the stationary probabilities, and calculate the average number of cars in the system.

10.6◆ In the game of *Risk*, adjacent countries may attack each other. If the attacking county has a armies, the attacker rolls $\min(a-1, 3)$ dice. If the defending country has d armies, the defender rolls $\min(d-1, 2)$ dice. The highest rolls of the attacker and the defender are compared. If the attacker's roll is strictly greater, then the defender loses 1 army; otherwise the attacker loses 1 army. In the event that $a > 1$ and $d > 1$, the attacker and defender compare their second highest rolls. Once again, if the attacker's roll is strictly higher, the defender loses 1 army; otherwise the attacker loses 1 army. Suppose the battle ends when either the defender has 0 armies or the attacker is reduced to 1 army. Given that the attacker starts with a_0 armies and the defender starts with d_0 armies, what is the probability that the attacker wins? Find the answer for $a_0 = 50$ and $d_0 \in \{10, 20, 30, 40, 50, 60\}$

10.7♦ The Veryfast Bank has a pair of drive-thru teller windows in parallel. Each car requires an independent exponential service time with a mean of 1 minute. Because of a series of concrete lane dividers, an arriving car must choose a waiting line. In particular, a car always chooses the shortest waiting line upon arrival. Cars arrive as a Poisson process of rate 0.75 car per minute, independent of the state of the queue. However, if an arriving car sees that *each* teller has at least 3 waiting cars, then the arriving customer becomes discouraged and immediately departs. Identify a Markov chain for this system, find the stationary probabilities, and calculate the average number of cars in the system. Hint: The state must track the number of cars in each line.

10.8♦♦ Consider the following discrete-time model for a traffic jam. A traffic lane is a service facility consisting of a sequence of L spaces numbered $0, 1, \ldots, L-1$. Each of the L spaces can be empty or hold one car. One unit of time, a "slot," is the time required for a car to move ahead one space. Before space 0, cars may be waiting (in an "external" queue) to enter the service facility (the traffic lane). Cars in the queue and cars occupying the spaces follow these rules in the transition from time n to $n+1$:

- If space $l+1$ is empty, a car in space l moves to space $l+1$ with probability $q > 0$.
- If space $l+1$ is occupied, then a car in space l cannot move ahead.
- A car at space $L-1$ departs the system with probability q.
- If space 0 is empty, a car waiting at the head of the external queue moves to space 0.
- In each slot, an arrival (another car) occurs with probability p, independent of the state of the system. If the external queue is empty at the time of an arrival, the new car moves immediately into space 0; otherwise the new arrival joins the external queue. If the external queue already has c customers, the new arrival is discarded (which can be viewed as an immediate departure).

Find the stationary distribution of the system for $c = 30$ and $q = 0.9$. Find and plot the average number $\mathrm{E}[N]$ of cars in the system in steady-state as a function of the arrival rate p for $L = 1, 2, \ldots$.

Hints: Note that the state of the system will need to track both the positions of the cars in the L spaces as well as the number of cars in the external queue. A state description vector would be $(k, y_0, \ldots, y_{L-1})$ where k is the number of cars in the external queue and $y_i \in \{0, 1\}$ is a binary indicator for whether space i holds a car. To reduce this descriptor to a state index, we suggest that $(k, y_0, \ldots, y_{L-1})$ correspond to state

$$\begin{aligned} i &= i(k, y_0, y_1, \ldots, y_{L-1}) \\ &= k2^L + \sum_{i=0}^{L-1} y_i 2^i. \end{aligned}$$

10.9♦♦ The traffic jam of Problem 10.8 has the following continuous analogue. As in the discrete-time system, the system state is still specified by $(k, y_0, \ldots, y_{L-1})$ where k is the number in the external queue and y_i indicates the occupancy of space i. Each of the L spaces can be empty or hold one car. In the continuous-time system, cars in the queue and cars occupying the spaces follow these rules:

- If space $l + 1$ is empty, a car in space l moves to space $l+1$ at rate μ. Stated another way, at any instant that space $l+1$ is empty, the residual time a car spends in space i is exponential with mean $1/\mu$.
- If space $l + 1$ is occupied, then a car in space l cannot move ahead.
- A car at space $L - 1$ departs the system with rate μ.
- If space 0 is empty, a car waiting at the head of the external queue moves to space 0.
- Arrivals of cars occur as an independent (of the system state) Poisson process of rate λ If the external queue is empty at the time of an arrival, the new car moves immediately into space 0; otherwise the new arrival joins the external queue. If the external queue already has c customers, the new arrival is discarded.

Find the stationary distribution of the system for $c = 30$ and $\mu = 1.0$. Pind and plot the average number $\mathrm{E}[K]$ of cars in the external queue in steady-state as a function of the arrival rate λ for $L = 1, 2, \ldots$. The same hints apply as for the discrete case.

10.10♦♦ The game of *Monopoly* has 40 spaces. It is of some interest to *Monopoly* players to know which spaces are the most popular. For our purposes, we will assume these spaces are numbered 0 (GO) through 39 (Boardwalk). A player starts at space 0. The sum K of two independent dice throws, each a discrete uniform $(1, 6)$ random variable, determines how many spaces to advance. In particular, the position X_n after n turns obeys

$$X_{n+1} = (X_n + K) \mod 40.$$

However, there are several complicating factors.

- After rolling "doubles" three times in a row, the player is sent directly to space 10 (Jail). Or, if the player lands on space 30 (Go to Jail), the player is immediately sent to Jail.
- Once in Jail, the player has several options to continue from space 10.
  - Pay a fine, and roll the dice to advance.
  - Roll the dice and see if the result is doubles and then advance the amount of the doubles. However, if the roll is not doubles, then the

player must remain in Jail for the turn. After three failed attempts at rolling doubles, the player must pay the fine and simply roll the dice to advance.

Note that player who lands on space 10 via an ordinary roll is "Just Visiting" and the special rules of Jail do not apply.

- Spaces 7, 22, and 36 are labeled "Chance." When landing on Chance, the player draws 1 of 15 cards, including 10 cards that specify a new location. Among these 10 cards, 6 cards are in the form "Go to n" where $n \in \{0, 5, 6, 19, 10, 39\}$. Note that the rule "Go to 10" sends the player to Jail where those special rules take effect. The remaining 4 cards implement the following rules.
  - Go back three spaces
  - Go to nearest utility: from 7 or 36, go to 12; from 22 go to 28.
  - Go to nearest railroad: from 7 go to 15; from 22 go to 25; from 36 go to 5.

  Note that there are two copies of the "Go to nearest Railroad" card.
- Spaces 2, 17 and 33 are labeled "Community Chest." Once again, 1 of 15 cards is drawn. Two cards specify new locations:
  - Go to 10 (Jail)
  - Go to 0 (GO)

Find the stationary probabilities of a player's position X_n. To simplify the Markov chain, suppose that when you land on Chance or Community Chest, you independently draw a random card. Consider two possible strategies for Jail: (a) immediately pay to get out, and (b) stay in jail as long as possible. Does the choice of Jail strategy make a difference?

Quiz Solutions

Quiz 1 Solution

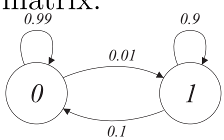
The system has two states depending on whether the previous packet was received in error. From the problem statement, we are given the conditional probabilities

$$\mathrm{P}\left[X_{n+1}=0|X_n=0\right]=0.99, \qquad \mathrm{P}\left[X_{n+1}=1|X_n=1\right]=0.9. \tag{1}$$

Since each X_n must be either 0 or 1, we can conclude that

$$\mathrm{P}\left[X_{n+1}=1|X_n=0\right]=0.01, \qquad \mathrm{P}\left[X_{n+1}=0|X_n=1\right]=0.1. \tag{2}$$

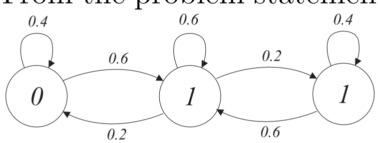
These conditional probabilities correspond to the Markov chain and transition matrix:



$$\mathbf{P}=\begin{bmatrix}0.99 & 0.01\\ 0.10 & 0.90\end{bmatrix}. \tag{3}$$

Quiz 2 Solution

From the problem statement, the Markov chain and the transition matrix are



$$\mathbf{P}=\begin{bmatrix}0.4 & 0.6 & 0\\ 0.2 & 0.6 & 0.2\\ 0 & 0.6 & 0.4\end{bmatrix}. \tag{1}$$

The eigenvalues of $\mathbf{P}$ are

$$\lambda_1=0, \qquad \lambda_2=0.4, \qquad \lambda_3=1. \tag{2}$$

We can diagonalize $\mathbf{P}$ into

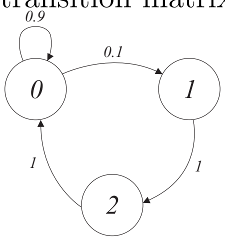
$$\mathbf{P}=\mathbf{S}^{-1}\mathbf{D}\mathbf{S}=\begin{bmatrix}-0.6 & 0.5 & 1\\ 0.4 & 0 & 1\\ -0.6 & -0.5 & 1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}-0.5 & 1 & -0.5\\ 1 & 0 & -1\\ 0.2 & 0.6 & 0.2\end{bmatrix}. \tag{3}$$

where $\mathbf{s}_i$, the ith row of $\mathbf{S}$, is the left eigenvector of $\mathbf{P}$ satisfying $\mathbf{s}_i\mathbf{P} = \lambda_i\mathbf{s}_i$. Algebra will verify that the n-step transition matrix is

$$\mathbf{P}^n = \mathbf{S}^{-1}\mathbf{D}^n\mathbf{S} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} + (0.4)^n \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}. \tag{4}$$

Quiz 3 Solution

The Markov chain describing the factory status and the corresponding state transition matrix are



$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

With $\boldsymbol{\pi} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}'$, the system of equations $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ yields $\pi_1 = 0.1\pi_0$ and $\pi_2 = \pi_1$. This implies

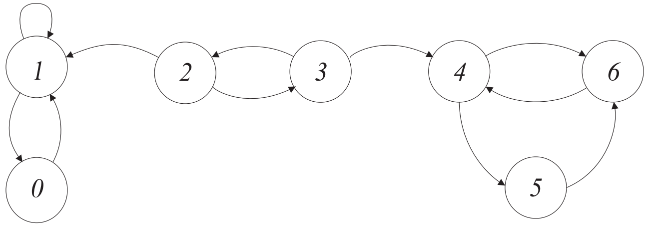
$$\pi_0 + \pi_1 + \pi_2 = \pi_0(1 + 0.1 + 0.1) = 1. \tag{2}$$

It follows that the limiting state probabilities are

$$\pi_0 = 5/6, \qquad \pi_1 = 1/12, \qquad \pi_2 = 1/12. \tag{3}$$

Quiz 4 Solution

In Markov chains of the form



the communicating classes are

$$C_1 = \{0, 1\}, \qquad C_2 = \{2, 3\}, \qquad C_3 = \{4, 5, 6\}. \tag{1}$$

The states in C_1 and C_3 are aperiodic. The states in C_2 have period 2. Once the system enters a state in C_1, the class C_1 is never left. Thus the states in C_1 are recurrent. That is, C_1 is a recurrent class. Similarly, the states in C_3 are recurrent. On the other hand, the states in C_2 are transient. Once the system exits C_2, the states in C_2 are never reentered.

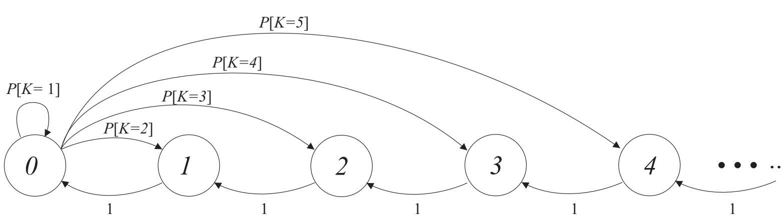
Quiz 5 Solution

At any time t, the state n can take on the values $0, 1, 2, \ldots$. The state transition probabilities are

$$P_{n-1,n} = \mathrm{P}[K > n | K > n-1] = \frac{\mathrm{P}[K > n]}{\mathrm{P}[K > n-1]}, \tag{1}$$

$$P_{n-1,0} = \mathrm{P}[K = n | K > n-1] = \frac{\mathrm{P}[K = n]}{.}\mathrm{P}[K > n-1] \tag{2}$$

The Markov chain resembles



The stationary probabilities satisfy

$$\begin{aligned}
\pi_0 &= \pi_0 \, \mathrm{P}[K=1] + \pi_1, & (3)\\
\pi_1 &= \pi_0 \, \mathrm{P}[K=2] + \pi_2, & (4)\\
&\ \vdots \\
\pi_{k-1} &= \pi_0 \, \mathrm{P}[K=k] + \pi_k, \qquad k = 1, 2, \ldots. & (5)
\end{aligned}$$

From Equation (3), we obtain

$$\pi_1 = \pi_0 \left(1 - \mathrm{P}[K=1]\right) = \pi_0 \, \mathrm{P}[K>1]. \tag{6}$$

Similarly, Equation (4) implies

$$\begin{aligned}
\pi_2 &= \pi_1 - \pi_0 \, \mathrm{P}[K=2]\\
&= \pi_0 \left(\mathrm{P}[K>1] - \mathrm{P}[K=2]\right) = \pi_0 \, \mathrm{P}[K>2]. & (7)
\end{aligned}$$

This suggests that $\pi_k = \pi_0 \, \mathrm{P}[K>k]$. We verify this pattern by showing that $\pi_k = \pi_0 \, \mathrm{P}[K>k]$ satisfies Equation (5):

$$\pi_0 \, \mathrm{P}[K>k-1] = \pi_0 \, \mathrm{P}[K=k] + \pi_0 \, \mathrm{P}[K>k]. \tag{8}$$

When we apply $\sum_{k=0}^{\infty} \pi_k = 1$, we obtain $\pi_0 \sum_{n=0}^{\infty} \mathrm{P}[K>k] = 1$. From Problem 3.5.20, we recall that $\sum_{k=0}^{\infty} \mathrm{P}[K>k] = \mathrm{E}[K]$. This implies

$$\pi_n = \frac{\mathrm{P}[K>n]}{\mathrm{E}[K]}. \tag{9}$$

This Markov chain models repeated random countdowns. The system state is the time until the counter expires. When the counter expires, the system is in state 0, and we randomly reset the counter to a new value $K = k$ and then we count down k units of time. Since we spend one unit of time in each state, including state 0, we have $k-1$ units of time left after the state 0 counter reset. If we have a random variable W such that the PMF of W satisfies $P_W(n) = \pi_n$, then W has a discrete PMF representing the remaining time of the counter at a time in the distant future.

Quiz 6 Solution

(a) By inspection, the number of transitions need to return to state 0 is always a multiple of 2. Thus the period of state 0 is $d = 2$.

(b) To find the stationary probabilities, we solve the system of equations $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and $\sum_{i=0}^{3} \pi_i = 1$:

$$\pi_0 = (3/4)\pi_1 + (1/4)\pi_3 \tag{1}$$
$$\pi_1 = (1/4)\pi_0 + (1/4)\pi_2 \tag{2}$$
$$\pi_2 = (1/4)\pi_1 + (3/4)\pi_3 \tag{3}$$
$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 \tag{4}$$

Solving the second and third equations for π_2 and π_3 yields

$$\pi_2 = 4\pi_1 - \pi_0, \tag{5}$$
$$\pi_3 = (4/3)\pi_2 - (1/3)\pi_1 = 5\pi_1 - (4/3)\pi_0. \tag{6}$$

Substituting π_3 back into the first equation yields

$$\begin{aligned}\pi_0 &= (3/4)\pi_1 + (1/4)\pi_3 \\ &= (3/4)\pi_1 + (5/4)\pi_1 - (1/3)\pi_0. \end{aligned} \tag{7}$$

This implies $\pi_1 = (2/3)\pi_0$. It follows from the first and second equations that $\pi_2 = (5/3)\pi_0$ and $\pi_3 = 2\pi_0$. Lastly, we choose π_0 so the state probabilities sum to 1:

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 = \pi_0 \left(1 + \frac{2}{3} + \frac{5}{3} + 2\right) = \frac{16}{3}\pi_0. \tag{8}$$

It follows that the state probabilities are

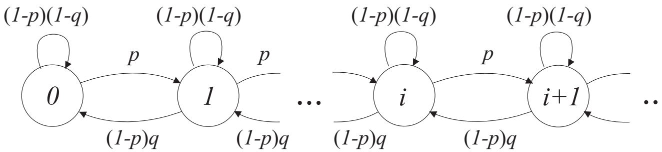
$$\pi_0 = \frac{3}{16}, \quad \pi_1 = \frac{2}{16}, \quad \pi_2 = \frac{5}{16}, \quad \pi_3 = \frac{6}{16}. \tag{9}$$

(c) Since the system starts in state 0 at time 0, we can use Theorem 14 to find the limiting probability that the system is in state 0 at time nd:

$$\lim_{n\to\infty} P_{00}(nd) = d\pi_0 = \frac{3}{8}. \tag{10}$$

Quiz 7 Solution

The number of customers in the ”friendly” store is given by the Markov chain



In the above chain, we note that $(1-p)q$ is the probability that no new customer arrives, an existing customer gets one unit of service and then departs the store.

By applying Theorem 13 with the state space partition

$$S = \{0, 1, \ldots, i\}, \qquad S' = \{i+1, i+2, \ldots\}, \tag{1}$$

we see that for any state $i \geq 0$,

$$\pi_i p = \pi_{i+1}(1-p)q. \tag{2}$$

This implies

$$\pi_{i+1} = \frac{p}{(1-p)q}\pi_i. \tag{3}$$

Since Equation (3) holds for $i = 0, 1, \ldots$, we have that $\pi_i = \pi_0 \alpha^i$ where

$$\alpha = \frac{p}{(1-p)q}. \tag{4}$$

Requiring the state probabilities to sum to 1, we have that for $\alpha < 1$,

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \alpha^i = \frac{\pi_0}{1-\alpha} = 1. \tag{5}$$

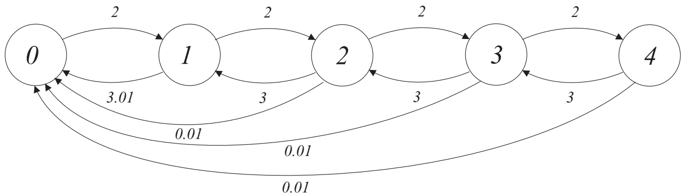
Thus for $\alpha < 1$, the limiting state probabilities are

$$\pi_i = (1-\alpha)\alpha^i, \qquad i = 0, 1, 2, \ldots \tag{6}$$

In addition, for $\alpha \geq 1$ or, equivalently, $p \geq q/(1-q)$, the limiting state probabilities do not exist.

Quiz 8 Solution

The continuous time Markov chain describing the processor is



Note that $q_{10} = 3.1$ since the task completes at rate 3 per msec and the processor reboots at rate 0.1 per msec and the rate to state 0 is the sum of those two rates. From the Markov chain, we obtain the following useful equations for the stationary distribution.

$$5.01p_1 = 2p_0 + 3p_2, \qquad 5.01p_2 = 2p_1 + 3p_3,$$
$$5.01p_3 = 2p_2 + 3p_4, \qquad 3.01p_4 = 2p_3.$$

We can solve these equations by working backward and solving for p_4 in terms of p_3, p_3 in terms of p_2 and so on, yielding

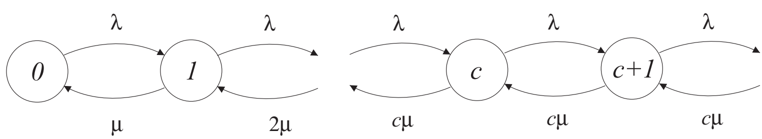
$$p_4 = \frac{20}{31}p_3, \quad p_3 = \frac{620}{981}p_2, \quad p_2 = \frac{19620}{31431}p_1, \quad p_1 = \frac{628,620}{1,014,381}p_0. \tag{1}$$

Applying $p_0 + p_1 + p_2 + p_3 + p_4 = 1$ yields $p_0 = 1,014,381/2,443,401$ and the stationary probabilities are

$$p_0 = 0.4151, \quad p_1 = 0.2573, \quad p_2 = 0.1606, \quad p_3 = 0.1015, \quad p_4 = 0.0655. \tag{2}$$

Quiz 9 Solution

The $M/M/c/\infty$ queue has Markov chain



From the Markov chain, the stationary probabilities must satisfy

$$p_n = \begin{cases} (\rho/n)p_{n-1} & n = 1, 2, \ldots, c \\ (\rho/c)p_{n-1} & n = c+1, c+2, \ldots \end{cases} \tag{1}$$

It is straightforward to show that this implies

$$p_n = \begin{cases} p_0\rho^n/n! & n = 1, 2, \ldots, c \\ p_0\,(\rho/c)^{n-c}\,\rho^c/c! & n = c+1, c+2, \ldots \end{cases} \tag{2}$$

The requirement that $\sum_{n=0}^{\infty} p_n = 1$ yields

$$p_0 = \left(\sum_{n=0}^{c} \rho^n/n! + \frac{\rho^c}{c!} \frac{\rho/c}{1-\rho/c} \right)^{-1}. \tag{3}$$

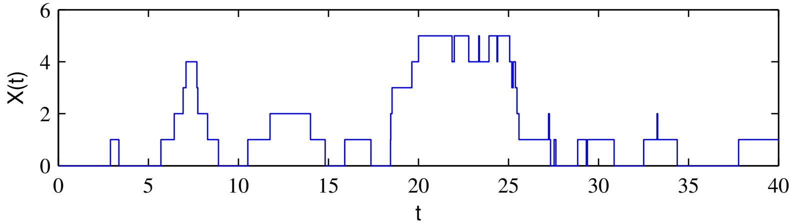
Quiz 10 Solution

The function `simmm1c` implements a simulation of the M/M/1/c queue. It is the same program as the M/M/c/c simulation except for adjusting the service rate.

```
function ...
   S=simmm1c(lam,mu,c,p0,T);
%Simulate M/M/1/c queue, time T.
%lam=arr. rate, mu=svc. rate
%p0=init. state distribution
%c= number of servers
Q=zeros(c+1,c+1);
for i=1:c,
    Q(i,i+1)=lam;
    Q(i+1,i)=mu;
end
S=simcmc(Q,p0,T);
```

The program calculates the rate transition matrix **Q** and calls `S=simcmc(Q,0,20)` to perform the simulation for 20 time units.

Here is a simulation run for the M/M/1/c queue:



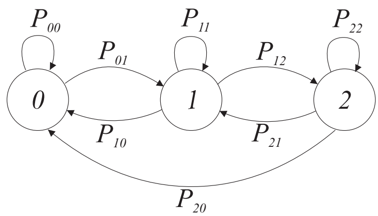
This output was generated with the commands:

```
lam=0.8;mu=1;c=5;T=40;
S=simmm1c(lam,mu,c,0,T);
simplot(S,'t','X(t)');
```

Problem Solutions

Problem 1.1 Solution

In addition to the normal OFF and ON states for packetized voice, we add state 2, the "mini-OFF" state. The Markov chain is



The only difference between this chain and an arbitrary 3 state chain is that transitions from 0, the OFF state, to state 2, the mini-OFF state, are not allowed. From the problem statement, the corresponding Markov chain is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.999929 & 0.000071 & 0 \\ 0.000100 & 0.899900 & 0.1 \\ 0.000100 & 0.699900 & 0.3 \end{bmatrix}. \tag{1}$$

Problem 1.3 Solution

From the given Markov chain, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}. \tag{1}$$

Problem 1.5 Solution

In this problem, it is helpful to go fact by fact to identify the information given.

- *... each read or write operation reads or writes an entire file and that files contain a geometric number of sectors with mean* 50.

  This statement says that the length L of a file has PMF

$$P_L(l) = \begin{cases} (1-p)^{l-1}p & l = 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

  with $p = 1/50 = 0.02$. This says that when we write a sector, we will write another sector with probability $49/50 = 0.98$. In terms of our Markov chain, if we are in the write state, we write another sector and stay in the write state with probability $P_{22} = 0.98$. This fact also implies $P_{20} + P_{21} = 0.02$.

  Also, since files that are read obey the same length distribution,

$$P_{11} = 0.98, \qquad P_{10} + P_{12} = 0.02. \tag{2}$$

- *Further, suppose idle periods last for a geometric time with mean* 500.

  This statement simply says that given the system is idle, it remains idle for another unit of time with probability $P_{00} = 499/500 = 0.998$. This also says that $P_{01} + P_{02} = 0.002$.

- *After an idle period, the system is equally likely to read or write a file.*

  Given that at time n, $X_n = 0$, this statement says that the conditional probability that

$$\mathrm{P}\left[X_{n+1} = 1 | X_n = 0, X_{n+1} \neq 0\right] = \frac{P_{01}}{P_{01} + P_{02}} = 0.5. \tag{3}$$

  Combined with the earlier fact that $P_{01} + P_{02} = 0.002$, we learn that

$$P_{01} = P_{02} = 0.001. \tag{4}$$

- *Following the completion of a read, a write follows with probability* 0.8.

  Here we learn that given that at time n, $X_n = 1$, the conditional probability that

$$\mathrm{P}\left[X_{n+1} = 2 | X_n = 1, X_{n+1} \neq 1\right] = \frac{P_{12}}{P_{10} + P_{12}} = 0.8. \tag{5}$$

Combined with the earlier fact that $P_{10} + P_{12} = 0.02$, we learn that

$$P_{10} = 0.004, \qquad P_{12} = 0.016. \tag{6}$$

- *However, on completion of a write operation, a read operation follows with probability* 0.6.

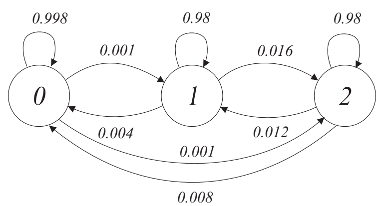
  Now we find that given that at time n, $X_n = 2$, the conditional probability that

$$\mathrm{P}\left[X_{n+1} = 1 | X_n = 2, X_{n+1} \neq 2\right] = \frac{P_{21}}{P_{20} + P_{21}} = 0.6. \tag{7}$$

  Combined with the earlier fact that $P_{20} + P_{21} = 0.02$, we learn that

$$P_{20} = 0.008, \qquad P_{21} = 0.012. \tag{8}$$

The complete Markov chain is



Problem 1.7 Solution

Let

$$\mathbf{Y} = \begin{bmatrix} Y_{n-1} & Y_{n-2} & \cdots & Y_0 \end{bmatrix}' = \begin{bmatrix} X_{T_{n-1}} & X_{T_{n-2}} & \cdots & X_0 \end{bmatrix}' \tag{1}$$

denote the past history of the process. In the conditional space where $Y_n = i$ and $\mathbf{Y} = \mathbf{y}$, we can use the law of total probability to write

$$\begin{aligned} &\mathrm{P}\left[Y_{n+1} = j | Y_n = i, \mathbf{Y} = \mathbf{y}\right] \\ &\quad = \sum_k \mathrm{P}\left[Y_{n+1} = j, | Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k\right] \mathrm{P}\left[K_n = k | Y_n = i, \mathbf{Y} = \mathbf{y}\right]. \end{aligned} \tag{2}$$

Since K_n is independent of Y_n and the past history $\mathbf{Y}$,

$$\mathrm{P}[K_n = k|Y_n = i, \mathbf{Y} = \mathbf{y}] = \mathrm{P}[K_n = k]. \tag{3}$$

Next we observe that

$$\begin{aligned}\mathrm{P}\left[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k\right] &= \mathrm{P}\left[X_{T_n+k} = j|X_{T_n} = i, K_n = k, \mathbf{Y} = \mathbf{y}\right] \\ &= \mathrm{P}\left[X_{T_n+k} = j|X_{T_n} = i, K_n = k\right]. \end{aligned} \tag{4}$$

because the state X_{T_n+k} is independent of the past history $\mathbf{Y}$ given the most recent state X_{T_n}. Moreover, by time invariance of the Markov chain,

$$\mathrm{P}\left[X_{T_n+k} = j|X_{T_n} = i, K_n = k\right] = \mathrm{P}\left[X_{T_n+k} = j|X_{T_n} = i\right] = [\mathbf{P}^k]_{ij}. \tag{5}$$

Equations (4) and (5) imply

$$\mathrm{P}\left[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k\right] = [\mathbf{P}^k]_{ij}. \tag{6}$$

It then follows from Equation (2) that

$$\begin{aligned}&\mathrm{P}\left[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}\right] \\ &\quad = \sum_k \mathrm{P}\left[Y_{n+1} = j, |Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k\right] \mathrm{P}\left[K_n = k\right] \\ &\quad = \sum_k [\mathbf{P}^k]_{ij} \mathrm{P}\left[K_n = k\right]. \end{aligned} \tag{7}$$

Thus $\mathrm{P}[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}]$ depends on i and j and is independent of the past history $\mathbf{Y}$. Thus we conclude that Y_n is a Markov chain.

Problem 2.1 Solution

From Example 1, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix}. \tag{1}$$

This chain is a special case of the two-state chain in Example 5 and Example 6 with $p = 0.8$ and $q = 0.9$. You may wish to derive the eigenvalues and eigenvectors of $\mathbf{P}$ in order to diagonalize and then find $\mathbf{P}^n$. Or, you may wish just to refer to Example 6 which showed that the chain has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1-(p+q) = -0.7$ and that the n-step transition matrix is

$$\mathbf{P}^n = \begin{bmatrix} P_{00}(n) & P_{01}(n) \\ P_{10}(n) & P_{11}(n) \end{bmatrix} = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{\lambda_2^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$
$$= \frac{1}{1.7}\begin{bmatrix} 0.9 & 0.8 \\ 0.9 & 0.8 \end{bmatrix} + \frac{(-0.7)^n}{1.7}\begin{bmatrix} 0.8 & -0.8 \\ -0.9 & 0.9 \end{bmatrix}. \qquad (2)$$

Problem 3.1 Solution

From Example 8, the state probabilities at time n are

$$\mathbf{p}(n) = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} + \lambda_2^n \begin{bmatrix} \frac{5}{12}p_0 - \frac{7}{12}p_1 & \frac{-5}{12}p_0 + \frac{7}{12}p_1 \end{bmatrix} \qquad (1)$$

with

$$\lambda_2 = 1-(p+q) = 344/350. \qquad (2)$$

With initial state probabilities $\begin{bmatrix} p_0 & p_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

$$\mathbf{p}(n) = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} + \lambda_2^n \begin{bmatrix} \frac{5}{12} & \frac{-5}{12} \end{bmatrix}. \qquad (3)$$

The limiting state probabilities are

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix}. \qquad (4)$$

Note that $p_j(n)$ is within 1% of π_j if

$$|\pi_j - p_j(n)| \le 0.01\pi_j. \qquad (5)$$

These requirements become

$$\lambda_2^n \le 0.01\frac{7/12}{5/12}, \qquad \lambda_2^n \le 0.01. \qquad (6)$$

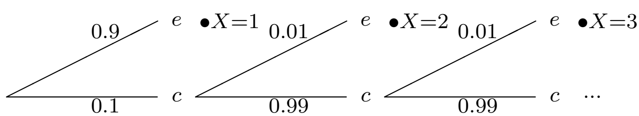
The minimum value n that meets both requirements is

$$n = \left\lceil \frac{\ln 0.01}{\ln \lambda_2} \right\rceil = 267. \tag{7}$$

Hence, after 267 time steps, the state probabilities are all within one percent of the limiting state probability vector. Note that in the packet voice system, the time step corresponded to a 10 ms time slot. Hence, 2.67 seconds are required.

Problem 3.3 Solution

In this problem, the arrivals are the occurrences of packets in error. It would seem that $N(t)$ cannot be a renewal process because the interarrival times seem to depend on the previous interarrival times. However, following a packet error, the sequence of packets that are correct (c) or in error (e) up to and including the next error is given by the tree



Assuming that sending a packet takes one unit of time, the time X until the next packet error has the PMF

$$P_X(x) = \begin{cases} 0.9 & x = 1 \\ 0.001(0.99)^{x-2} & x = 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, following an error, the time until the next error always has the same PMF. Moreover, this time is independent of previous interarrival times since it depends only on the Bernoulli trials following a packet error. It would appear that $N(t)$ is a renewal process; however, there is one additional complication. At time 0, we need to know the probability p of an error for the first packet.

If $p = 0.9$, then X_1, the time until the first error, has the same PMF as X above and the process is a renewal process. If $p \neq 0.9$, then the time until the first error is different from subsequent renewal times. In this case, the process is a delayed renewal process.

Problem 4.1 Solution

The hardest part of this problem is that we are asked to find *all* ways of replacing a branch. The primary problem with the Markov chain in Problem 1.3 is that state 2 is a transient state. We can get rid of the transient behavior by making a nonzero branch probability P_{12} or P_{02}. The possible ways to do this are:

- Replace $P_{00} = 1/2$ with $P_{02} = 1/2$.
- Replace $P_{01} = 1/2$ with $P_{02} = 1/2$.
- Replace $P_{11} = 1/2$ with $P_{12} = 1/2$.
- Replace $P_{10} = 1/2$ with $P_{12} = 1/2$.

Keep in mind that even if we make one of these replacements, there will be at least one self transition probability, either P_{00} or P_{11}, that will be nonzero. This will guarantee that the resulting Markov chain will be aperiodic.

Problem 4.3 Solution

The idea behind this claim is that if states j and i communicate, then sometimes when we go from state j back to state j, we will pass through state i. If $E[T_{ij}] = \infty$, then on those occasions we pass through i, the expected time to go to back to j will be infinite. This would suggest $E[T_{jj}] = \infty$ and thus state j would not be positive recurrent. Using a math to prove this requires a little bit of care.

Suppose $E[T_{ij}] = \infty$. Since i and j communicate, we can find n, the smallest nonnegative integer such that $P_{ji}(n) > 0$. Given we start in state j, let G_i denote the event that we go through state i on our way back to j. By conditioning on G_j,

$$\mathrm{E}\left[T_{jj}\right] = \mathrm{E}\left[T_{jj}|G_i\right]\mathrm{P}\left[G_i\right] + \mathrm{E}\left[T_{jj}|G_i^c\right]\mathrm{P}\left[G_i^c\right]. \tag{1}$$

Since $\mathrm{E}[T_{jj}|G_i^c]\,\mathrm{P}[G_i^c] \geq 0$,

$$\mathrm{E}\left[T_{jj}\right] \geq \mathrm{E}\left[T_{jj}|G_i\right]\mathrm{P}\left[G_i\right]. \tag{2}$$

Given the event G_i, $T_{jj} = T_{ji} + T_{ij}$. This implies

$$\mathrm{E}\left[T_{jj}|G_i\right] = \mathrm{E}\left[T_{ji}|G_i\right] + \mathrm{E}\left[T_{ij}|G_i\right] \geq \mathrm{E}\left[T_{ij}|G_i\right]. \tag{3}$$

Since the random variable T_{ij} assumes that we start in state i, $\mathrm{E}[T_{ij}|G_i] = \mathrm{E}[T_{ij}]$. Thus $\mathrm{E}[T_{jj}|G_i] \geq \mathrm{E}[T_{ij}]$. In addition, $\mathrm{P}[G_i] \geq P_{ji}(n)$ since there may be paths with more than n hops that take the system from state j to i. These facts imply

$$\mathrm{E}\left[T_{jj}\right] \geq \mathrm{E}\left[T_{jj}|G_i\right]\mathrm{P}\left[G_i\right] \geq \mathrm{E}\left[T_{ij}\right] P_{ji}(n) = \infty. \tag{4}$$

Thus, state j is not positive recurrent, which is a contradiction. Hence, it must be that $\mathrm{E}[T_{ij}] < \infty$.

Problem 5.1 Solution

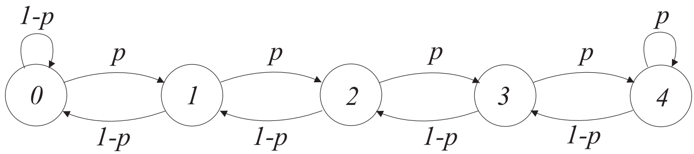
In the solution to Problem 1.5, we found that the state transition matrix was

$$\mathbf{P} = \begin{bmatrix} 0.998 & 0.001 & 0.001 \\ 0.004 & 0.98 & 0.016 \\ 0.008 & 0.012 & 0.98 \end{bmatrix}. \tag{1}$$

We can find the stationary probability vector $\boldsymbol{\pi} = \begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}'$ by solving $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ along with $\pi_0 + \pi_1 + \pi_2 = 1$. It's possible to find the solution by hand but its easier to use MATLAB or a similar tool. The solution is $\boldsymbol{\pi} = [0.7536\ 0.1159\ 0.1304]$.

Problem 5.3 Solution

From the problem statement, the Markov chain is



The self-transitions in state 0 and state 4 guarantee that the Markov chain is aperiodic. Since the chain is also irreducible, we can find the stationary probabilities by solving $\boldsymbol{\pi} = \boldsymbol{\pi}'\mathbf{P}$; however, in this problem it is simpler to apply Theorem 13. In particular, by partitioning the chain between states i and $i+1$, we obtain

$$\pi_i p = \pi_{i+1}(1-p). \tag{1}$$

This implies $\pi_{i+1} = \alpha\pi_i$ where $\alpha = p/(1-p)$. It follows that $\pi_i = \alpha^i \pi_0$. REquiring the stationary probabilities to sum to 1 yields

$$\sum_{i=0}^{4} \pi_i = \pi_0(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 1. \tag{2}$$

This implies

$$\pi_0 = \frac{1-\alpha^5}{1-\alpha}. \tag{3}$$

Thus, for $i = 0, 1, \ldots, 4$,

$$\pi_i = \frac{1-\alpha^5}{1-\alpha}\alpha^i = \frac{1-\left(\frac{p}{1-p}\right)^5}{1-\left(\frac{p}{1-p}\right)}\left(\frac{p}{1-p}\right)^i. \tag{4}$$

Problem 5.5 Solution

(a) Each item reveals some facts about $\mathbf{P}$:

(A) From state 1, $P_{11} = 0.1$ and $P_{12} = 0.9$. Since the sum of the transition probabilities from state 1 must sum to 1, $P_{10} = 0$.

(B) After waking up, the student cannot start reading; hence $P_{02} = 0$ and $P_{01} = 1 - P_{00}$. When the student goes to sleep, he then wakes up each hour with probability $1 - P_{00}$. That is, the nap length is a geometric $(1 - P_{00})$ random variable. The average sleep time is $1/(1 - P_{00}) = 4$. Thus $P_{00} = 3/4$ and $P_{01} = 1/4$.

(C) After an hour of reading, another hour of reading is four times more probable than an hour of eating tells us that $P_{22} = 4P_{21}$. The probability of going to sleep equals the sum of the probabilities of reading or eating tell us that $P_{20} = P_{21} + P_{22}$. These two equations along with $P_{20} + P_{21} + P_{22} = 1$ yields $P_{20} = 0.5$, $P_{21} = 0.1$ and $P_{22} = 0.4$.

(b) If you sketch the Markov chain, you will see it is ergodic. Thus the correct answer is to go to the location corresponding to the state that has the highest stationary probability. As far as I can tell, you need to solve for the stationary probabilities π_0, π_1, π_2 to find out which state is most probable. Balancing the rate of transitions into and out of state 0, we obtain $(1/4)\pi_0 = 0.5\pi_2$. Balancing the rate of transitions into and out of state 2 yields $(0.5 + 0.1)\pi_2 = 0.9\pi_1$. These two equations along with $\pi_0 + \pi_1 + \pi_2 = 1$ yields

$$\pi_0 = 6/11, \qquad \pi_1 = 2/11, \qquad \pi_2 = 3/11. \tag{1}$$

Since the student is most likely sleeping, the most likely place to find the student is in the dorm.

Problem 5.7 Solution

The states of the chain are $\{1, 2, 3, 4, 5\}$. Note that the signboard never says 0 since a train arrives at time 0 and the signboard then switches to $W > 0$ for the next train arrival. From state 1, the next state is i with probability

$P_{1i} = 1/5$ for $i = 1, 2, \ldots, 5$. In state $i > 1$, the next state is $i - 1$ with probability 1. That is, $P_{i,i-1} = 1$ for $i > 1$. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1}$$

You should be able to draw the chain. To solve for the limiting state probabilities, we have

$$\pi_5 = \pi_1/5, \tag{2}$$
$$\pi_4 = \pi_1/5 + \pi_5 = 2\pi_1/5, \tag{3}$$
$$\pi_3 = \pi_1/5 + \pi_4 = 3\pi_1/5, \tag{4}$$
$$\pi_2 = \pi_1/5 + \pi_3 = 4\pi_1/5. \tag{5}$$

Making the state probabilities add up to 1, we get

$$\sum_{i=1}^{5} \pi_i = \pi_1 \left(1 + \frac{4}{5} + \frac{3}{5} + \frac{2}{5} + \frac{1}{5}\right) = 3\pi_1 = 1. \tag{6}$$

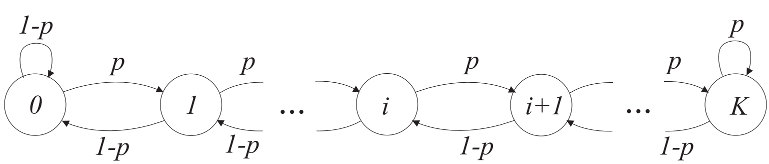
Thus $\pi_1 = 1/3$ and

$$\begin{bmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5\end{bmatrix} = \begin{bmatrix}5/15 & 4/15 & 3/15 & 2/15 & 1/15\end{bmatrix}. \tag{7}$$

The probability that the sign saw $W = 3$ is $\pi_3 = 1/5$.

Problem 5.9 Solution

From the problem statement, the Markov chain is



The self-transitions in state 0 and state K guarantee that the Markov chain is aperiodic. Since the chain is also irreducible, we can find the stationary probabilities by solving $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$; however, in this problem it is simpler to apply Theorem 13. In particular, by partitioning the chain between states i and $i+1$, we obtain

$$\pi_i p = \pi_{i+1}(1-p). \tag{1}$$

This implies $\pi_{i+1} = \alpha\pi_i$ where $\alpha = p/(1-p)$. It follows that $\pi_i = \alpha^i \pi_0$. REquiring the stationary probabilities to sum to 1 yields

$$\sum_{i=0}^{K} \pi_i = \pi_0(1 + \alpha + \alpha^2 + \cdots + \alpha^K) = 1. \tag{2}$$

This implies

$$\pi_0 = \frac{1-\alpha^{K+1}}{1-\alpha}. \tag{3}$$

Thus, for $i = 0, 1, \ldots, K$,

$$\pi_i = \frac{1-\alpha^{K+1}}{1-\alpha}\alpha^i = \frac{1-\left(\frac{p}{1-p}\right)^{K+1}}{1-\left(\frac{p}{1-p}\right)}\left(\frac{p}{1-p}\right)^i. \tag{4}$$

Problem 5.11 Solution

For this system, it's hard to draw the entire Markov chain since from each state n there are six branches, each with probability $1/6$ to states $n+1, n+2, \ldots, n+6$. (Of course, if $n+k > K-1$, then the transition is to state $n+k \bmod K$.) Nevertheless, finding the stationary probabilities is not very hard. In particular, the nth equation of $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ yields

$$\pi_n = \frac{1}{6}\left(\pi_{n-6} + \pi_{n-5} + \pi_{n-4} + \pi_{n-3} + \pi_{n-2} + \pi_{n-1}\right). \tag{1}$$

Rather than try to solve these equations algebraically, it's easier to guess that the solution is

$$\boldsymbol{\pi} = \begin{bmatrix} 1/K & 1/K & \cdots & 1/K \end{bmatrix}'. \tag{2}$$

It's easy to check that $1/K = (1/6) \cdot 6 \cdot (1/K)$.

Problem 5.13 Solution

In this case, we will examine the system each minute. For each customer in service, we need to keep track of how soon the customer will depart. For the state of the system, we will use (i, j), the remaining service requirements of the two customers, To reduce the number of states, we will order the requirements so that $i \leq j$. For example, when two new customers start service each requiring two minutes of service, the system state will be $(2, 2)$. Since the system assumes there is always a backlog of cars waiting to enter service, the set of states is

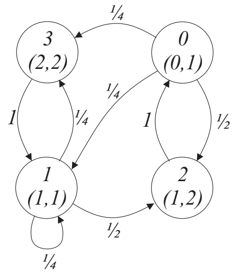
0: $(0, 1)$ One teller is idle, the other teller has a customer requiring one more minute of service

1: $(1, 1)$ Each teller has a customer requiring one more minute of service.

2: $(1, 2)$ One teller has a customer requring one minute of service. The other teller has a customer requiring two minutes of service.

3: $(2, 2)$ Each teller has a customer requiring two minutes of service.

The resulting Markov chain is shown on the right. Note that when we departing from either state $(0,1)$ or $(1,1)$ corresponds to both custoemrs finishing service and two new customers entering service. The state transiton probabilities reflect the fact that both customer will have two minute service requirements with probability $1/4$, or both customers will hae one minute service requirements with probability $1/4$, or one customer will need one minute of service and the other will need two minutes of service with probability $1/2$.



Writing the stationary probability equations for states 0, 2, and 3 and adding the constraint $\sum_j \pi_j = 1$ yields the following equations:

$$\pi_0 = \pi_2, \tag{1}$$
$$\pi_2 = (1/2)\pi_0 + (1/2)\pi_1, \tag{2}$$
$$\pi_3 = (1/4)\pi_0 + (1/4)\pi_1, \tag{3}$$
$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3. \tag{4}$$

Substituting $\pi_2 = \pi_0$ in the second equation yields $\pi_1 = \pi_0$. Substituting that result in the third equation yields $\pi_3 = \pi_0/2$. Making sure the probabilities add up to 1 yields

$$\boldsymbol{\pi} = \begin{bmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{bmatrix}' = \begin{bmatrix}2/7 & 2/7 & 2/7 & 1/7\end{bmatrix}'. \tag{5}$$

Both tellers are busy unless the system is in state 0. The stationary probability both tellers are busy is $1 - \pi_0 = 5/7$.

Problem 5.15 Solution

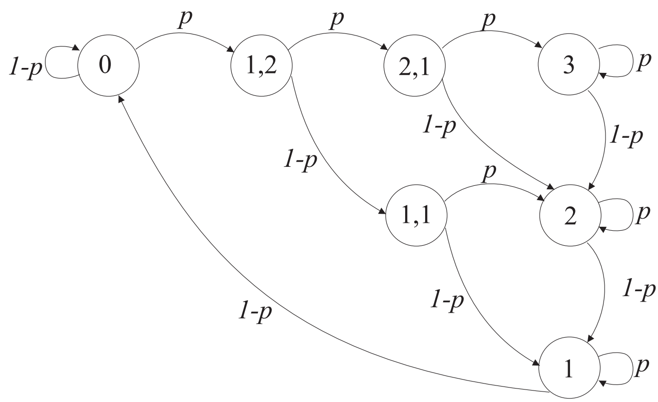
For the Markov chain, we use $n \in \{0, 1, 2, 3\}$ to denote the state in which the system has n packets buffered and, if $n > 0$, is able to transmit a packet in the current slot. The system needs three additional states:

$1, 2$ The transmitter has 1 buffered packet but must wait 2 slots before transmitting.

1, 1 The transmitter has 1 buffered packet but must wait 1 slot before transmitting.

2, 1 The transmitter has 2 buffered packets but must wait 1 slot before transmitting.

The resulting Markov chain is



The equations for the stationary probabilities are surprisingly simple. By omitting the equation for π_2 which has the most complicated set of incoming transitions, we obtain

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \quad \Rightarrow \quad \pi_1 = \frac{p}{1-p}\pi_0, \tag{1}$$

$$\pi_{1,2} = p\pi_0, \tag{2}$$

$$\pi_{2,1} = p\pi_{1,2} \quad \Rightarrow \quad \pi_{2,1} = p^2\pi_0, \tag{3}$$

$$\pi_{1,1} = (1-p)\pi_{1,2} \quad \Rightarrow \quad \pi_{1,1} = p(1-p)\pi_0, \tag{4}$$

$$\pi_3 = p\pi_{2,1} + p\pi_3 \quad \Rightarrow \quad \pi_3 = \frac{p}{1-p}\pi_{2,1} = \frac{p^3}{1-p}\pi_0, \tag{5}$$

$$\pi_1 = p\pi_1 + (1-p)(\pi_{1,1} + \pi_2) \quad \Rightarrow \quad \pi_2 = \pi_1 - \pi_{1,1} = \frac{p^2(2-p)}{1-p}\pi_0. \tag{6}$$

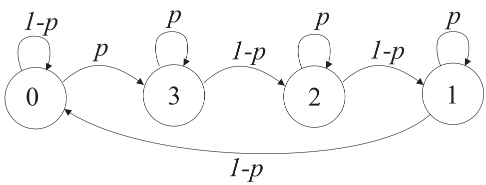
Applying the condition

$$\pi_0 + \pi_{1,2} + \pi_{2,1} + \pi_{1,1} + \pi_1 + \pi_2 + \pi_3 = 1 \tag{7}$$

yields

$$\pi_0 = \frac{1-p}{1-2p}. \tag{8}$$

In fact, there is a reason π_0 is so simple. There is a much simpler way to model the system state. The idea is that we can model the initialization period by requiring the system to transmit a dummy packet in each initialization slot. In this case, the state of the system is captured by the total number of buffered packets, including both real packets and dummy packets. The system is empty only when the real packets and the dummy packets are sent. The simplified Markov chain is



The equations for this Markov chain are also much simpler.

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \qquad \Rightarrow \qquad \pi_1 = \frac{p}{1-p}\pi_0, \tag{9}$$

$$\pi_3 = p\pi_0 + p\pi_3 \qquad \Rightarrow \qquad \pi_3 = \frac{p}{1-p}\pi_0, \tag{10}$$

$$\pi_2 = p\pi_2 + (1-p)\pi_3 \qquad \Rightarrow \qquad \pi_2 = \pi_3 = \frac{p}{1-p}\pi_0. \tag{11}$$

Applying the condition

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \tag{12}$$

yields $\pi_0 = (1-p)/(1-2p)$.

Problem 5.17 Solution

(a) The customer that arrived at the start of minute t stays in service at the end of each minute with probability p. This customer is in service at time $t+k$ with probability p^k. In this case $X_{t,k}=1$; otherwise $X_{t,k}=0$. The Bernoulli PMF is

$$P_{X_{t,k}}(x) = \begin{cases} p^k & x=1, \\ 1-p^k & x=0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) We will use the fairly huge hint. Since $\mathrm{E}[X_{t,k}] = \mathrm{P}[X_{t,k}=1] = p^k$,

$$\mathrm{E}[Y_t] = \mathrm{E}\left[\sum_{k=0}^{\infty} X_{t-k,k}\right] = \sum_{k=0}^{\infty} \mathrm{E}[X_{t-k,k}] = \sum_{k=0}^{\infty} p^k = \frac{1}{1-p}. \tag{2}$$

Since a Bernoulli (p) random variable has variance $p(1-p)$, $\mathrm{Var}[X_{t,k}] = p^k(1-p^k)$. Next we observe for $k \neq k'$ that $X_{t-k,k}$ and $X_{t-k',k'}$ are independent since the service time of a customer arriving at time $t-k$ is independent of the service time of the customer arriving at time $t-k'$. Since the variance of a sum of independent random variables equals the sum of the variances,

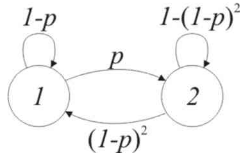
$$\begin{aligned} \mathrm{Var}[Y_t] = \sum_{k=0}^{\infty} \mathrm{Var}[X_{t-k,k}] &= \sum_{k=0}^{\infty} p^k(1-p^k) \\ &= \sum_{k=0}^{\infty} p^k - \sum_{k=0}^{\infty} (p^2)^k \\ &= \frac{1}{1-p} - \frac{1}{1-p^2} = \frac{p}{1-p^2}. \end{aligned} \tag{3}$$

(c) Since Y_t is the number of customers just after the new arrival, we know that $Y_t \geq 1$. Since the customers don't like to be crowded, $Y_t \leq 2$. Thus the Markov chain describing Y_t has state space $\{1,2\}$. The state transitions can be found from the following observations:

- If $Y_t = 1$, then $Y_{t+1} = 1$ if and only if the customer in service departs, which occurs with probability $p_{11} = 1-p$.

- If $Y_t = 2$, then $Y_{t+1} = 1$ if and only if both customers in service depart. This occurs with probability $p_{21} = (1-p)^2$.

The two-state Markov chain is



The chain is (obviously) irreducible and positive recurrent. The stationary probabilities satisfy $\pi_1 p = \pi_2(1-p)^2$. This implies

$$\pi_1 = \frac{(1-p)^2}{(1-p)^2 + p}, \qquad \pi_2 = \frac{p}{(1-p)^2 + p}. \tag{4}$$

Problem 5.19 Solution

(a) By the Chapman-Kolmogorov equations,

$$P_{ij}(n+1) = \sum_k P_{ik} P_{kj}(n). \tag{1}$$

Since $P_{kj}(n) \geq \min_{k'} P_{k'j}(n)$ and P_{ik} is nonnegative,

$$\begin{aligned} P_{ij}(n+1) &\geq \sum_k P_{ik} \min_{k'} P_{kj}(n) \\ &= \left(\min_{k'} P_{kj}(n)\right) \sum_k P_{ik} \\ &= \min_{k'} P_{kj}(n) = m_j(n). \end{aligned} \tag{2}$$

Since (2) holds for for every i, it must hold for the i that minimizes $P_{ij}(n+1)$. Thus

$$m_j(n+1) = \min_i P_{ij}(n+1) \geq m_j(n). \tag{3}$$

Similarly for $M_j(n)$, (1) permits us to write

$$\begin{aligned} P_{ij}(n+1) &\le \sum_k P_{ik} \max_{k'} P_{kj}(n) \\ &= \left(\max_{k'} P_{kj}(n)\right) \sum_k P_{ik} \\ &= \max_{k'} P_{kj}(n) = M_j(n). \end{aligned} \tag{4}$$

Since (4) holds for for every i, it must hold for the i that maximizes $P_{ij}(n+1)$. Thus

$$M_j(n+1) = \max_i P_{ij}(n+1) \le M_j(n). \tag{5}$$

(b) From the definition of $\Delta_j(n)$,

$$\begin{aligned} \Delta_j(n+\tau) &= M_j(n+\tau) - m_j(n+\tau) \\ &= \max_\alpha P_{\alpha j}(n+\tau) - \min_\beta P_{\beta j}(n+\tau) \\ &= \max_\alpha \left(P_{\alpha j}(n+\tau) - \min_\beta P_{\beta j}(n+\tau) \right) \\ &= \max_\alpha \max_\beta \left(P_{\alpha j}(n+\tau) - P_{\beta j}(n+\tau) \right). \end{aligned} \tag{6}$$

Using the Chapman-Kolmogorov equations again,

$$P_{\alpha j}(n+\tau) = \sum_k P_{\alpha k}(\tau) P_{kj}(n), \tag{7}$$

$$P_{\beta j}(n+\tau) = \sum_k P_{\beta k}(\tau) P_{kj}(n). \tag{8}$$

Combining Equations (6), (7) and (8), we obtain

$$\begin{aligned} \Delta_j(n+\tau) &= \max_{\alpha,\beta} \left(\sum_k P_{\alpha k}(\tau) P_{kj}(n) - \sum_k P_{\beta k}(\tau) P_{kj}(n) \right) \\ &= \max_{\alpha,\beta} \sum_k \left[P_{\alpha k}(\tau) P_{kj}(n) - P_{\beta k}(\tau) P_{kj}(n) \right] \\ &= \max_{\alpha,\beta} \sum_k \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau) \right] P_{kj}(n). \end{aligned} \tag{9}$$

(c) Given Q, the complement is $Q^c = \{k | P_{\alpha k}(\tau) < P_{\beta k}(\tau)\}$. With these definitions,

$$\begin{aligned}\sum_k &\left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] P_{kj}(n) \\ &= \sum_{k \in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] P_{kj}(n) + \sum_{k \in Q^c} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] P_{kj}(n). \quad (10)\end{aligned}$$

Since $P_{\alpha k}(\tau) - P_{\beta k}(\tau)$ is nonnegative for each term in Q but strictly negative for each term in Q^c, we have

$$\begin{aligned}\sum_k &\left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] P_{kj}(n) \\ &\le \sum_{k \in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \max_{k'} P_{k'j}(n) \\ &\quad + \sum_{k \in Q^c} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \min_{k'} P_{k'j}(n). \quad (11)\end{aligned}$$

Now we observe that

$$\sum_{k \in Q^c} P_{\alpha k}(\tau) = 1 - \sum_{k \in Q} P_{\alpha k}(\tau), \quad (12)$$

$$\sum_{k \in Q^c} P_{\beta k}(\tau) = 1 - \sum_{k \in Q} P_{\beta k}(\tau). \quad (13)$$

These equations imply

$$\sum_{k \in Q^c} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] = -\sum_{k \in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right]. \quad (14)$$

Combining Equations (11) and (14), we obtain

$$\begin{aligned}
&\sum_k \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] P_{kj}(n) \\
&\qquad \le \sum_{k\in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \max_{k'} P_{k'j}(n) \\
&\qquad\qquad - \sum_{k\in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \min_{k'} P_{k'j}(n) \\
&\qquad = \sum_{k\in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \left[\max_{k'} P_{k'j}(n) - \min_{k'} P_{k'j}(n)\right] \\
&\qquad = \sum_{k\in Q} \left[P_{\alpha k}(\tau) - P_{\beta k}(\tau)\right] \Delta_j(n). \qquad (15)
\end{aligned}$$

(d) Let $\alpha = \alpha^*$ and $\beta = \beta^*$ maximize $\sum_k [P_{\alpha k}(\tau) - P_{\beta k}(\tau)] P_{kj}(n)$. Since Equation (15) holds for all α and β, Equations (6) and (15) imply

$$\begin{aligned}
\Delta_j(n+\tau) &= \sum_k \left[P_{\alpha^* k}(\tau) - P_{\beta^* k}(\tau)\right] P_{kj}(n) \\
&\le \Delta_j(n) \sum_{k\in Q} \left[P_{\alpha^* k}(\tau) - P_{\beta^* k}(\tau)\right] \\
&\le \Delta_j(n) \left(1 - \sum_{k\in Q} P_{\beta^* k}(\tau)\right). \qquad (16)
\end{aligned}$$

By Problem 5.18, we can choose τ so that $P_{ij}(\tau) \ge \delta > 0$ for all i, j. This implies

$$\sum_{k\in Q} P_{\beta^* k}(\tau) \ge \sum_{k\in Q} \delta = |Q|\delta \ge \delta. \qquad (17)$$

Note that $|Q| \ge 1$; otherwise if $|Q| = 0$, then $P_{\alpha^* k}(\tau) < P_{\beta^* k}(\tau)$ for all k, which would yield the contradiction

$$\sum_k P_{\alpha^* k}(\tau) < \sum_k P_{\beta^* k}(\tau) = 1. \qquad (18)$$

Thus, Equations (16) and (17) imply

$$\Delta_j(n+\tau) \le (1-\delta)\Delta_j(n). \tag{19}$$

(e) Since $\Delta_j(0) = \max_i Pij - \min_i P_{ij} \le 1$, it follows from (19) with $n = 0$ that

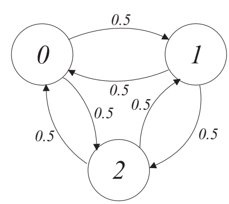
$$\Delta_j(k\tau) \le (1-\delta)^k \Delta_j(0) = (1-\delta)^k. \tag{20}$$

We conclude that $\lim_{k\to\infty} \Delta_j(k\tau) = 0$. Moreover, since we showed in part (a) that $\Delta_j(n)$ is a monotone decreasing sequence, we have that $\lim_{n\to\infty} \Delta_j(n) = 0$. Finally, this implies that all elements $P_{ij}(n)$ in column j of $\mathbf{P}^n$ converge to the same value, which we define to be π_j.

Problem 6.1 Solution

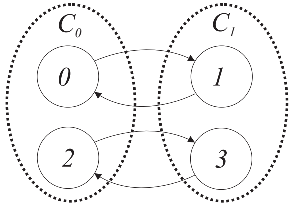
Equivalently, we can prove that if $P_{ii} \ne 0$ for some i, then the chain cannot be periodic. So, suppose for state i, $P_{ii} > 0$. Since $P_{ii} = P_{ii}(1)$, we see that the largest d that divides n for all n such that $P_{ii}(n) > 0$ is $d = 1$. Hence, state i is aperiodic and thus the chain is aperiodic.

The converse that $P_{ii} = 0$ for all i implies the chain is periodic is false. As a counterexample, consider the simple chain on the right with $P_{ii} = 0$ for each i. Note that $P_{00}(2) > 0$ and $P_{00}(3) > 0$. The largest d that divides both 2 and 3 is $d = 1$. Hence, state 0 is aperiodic. Since the chain has one communicating class, the chain is also aperiodic.



Problem 6.3 Solution

This problem is easy as long as you don't try to prove that the chain must have a single communicating class. A counterexample is easy to find. Here is a 4 state Markov chain with two recurrent communicating classes and each state having period 2. The two classes C_0 and C_1 are shown.



From each state $i \in C_0$, all transitions are to states $j \in C_1$. Similarly, from each state $i \in C_1$, only transitions to states $j \in C_0$ are permitted. The sets $\{0, 1\}$ and $\{2, 3\}$ are each communicating classes. However, each state has period 2.

Problem 6.5 Solution

Since the Markov chain has period d, there exists a state i_0 such that $P_{i_0 i_0}(d) > 0$. For this state i_0, let $C_n(i_0)$ denote the set of states that reachable from i_0 in n hops. We make the following claim:

- If $j \in C_n(i_0)$ and $j \in C_{n'}(i_0)$ with $n' > n$, then $n' = n + kd$ for some integer k.

To prove this claim, we observe that irreducibility of the Markov chain implies there is a sequence of m hops from state j to state i_0. Since there is an n hop path from i_0 to j and an m hop path from j back to i_0, $n + m = kd$ for some integer k. Similarly, since there is an n' hop path from i_0 to j, $n' + m = k'd$ for some integer k'. Thus

$$n' - n = (n' + m) - (n + m) = (k' - k)d. \tag{1}$$

Now we define

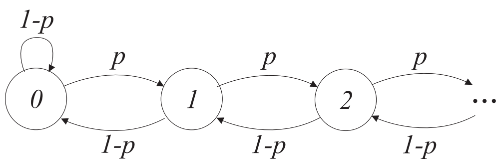
$$C_n = \bigcup_{k=0}^{\infty} C_{n+kd}(i_0), \qquad n = 0, 1, \ldots, d - 1. \tag{2}$$

Because the chain is irreducible, any state i belongs to some set $C_n(i_0)$, and thus any state i belongs to at least one set C_n. By our earlier claim, each node i belongs to exactly one set C_n. Hence, $\{C_0, \ldots, C_{d-1}\}$ is a partition of the states of the states of the Markov chain. By construction of the set $C_n(i_0)$, there exists states $i \in C_n(i_0)$ and $j \in C_{n+1}(i_0)$ such that $P_{ij} > 0$. Hence there exists $i \in C_n$ and $j \in C_{n+1}$ such that $P_{ij} > 0$.

Now suppose there exists states $i \in C_n$ and $j \in C_{n+m}$ such that $P_{ij} > 0$ and $m > 1$. In this case, the sequence of $n + kd$ hops from i_0 to i followed by one hop to state j is an $n + 1 + kd$ hop path from i_0 to j, implying $j \in C_{n+1}$. This contradicts the fact that j cannot belong to both C_{n+1} and C_{n+m}. Hence no such transition from $i \in C_n$ to $j \in C_{n+m}$ is possible.

Problem 7.1 Solution

A careful reader of the text will observe that the Markov chain in this problem is identical to the Markov chain introduced in Example 21:



The stationary probabilities were found in Example 25 to be

$$\pi_i = (1-\alpha)\alpha^i, \qquad i = 0, 1, 2, \ldots \tag{1}$$

where $\alpha = p/(1-p)$. Note that the stationary probabilities do not exist if $\alpha \geq 1$, or equivalently, $p \geq 1/2$.

Problem 7.3 Solution

Given that the starting state $X_0 = i$, we count whether an arrival occurred at each time step n to find the number of returns to state i. Using the hint, we write

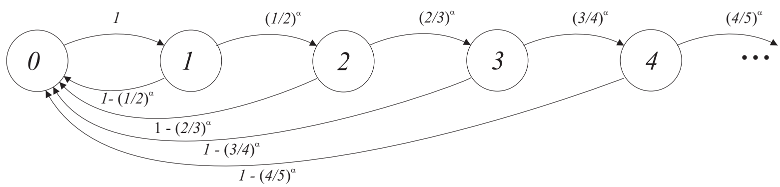
$$N_{ii} = \sum_{n=1}^{\infty} I_{ii}(n). \tag{1}$$

Since $I_{ii}(n)$ is a Bernoulli indicator, $\mathrm{E}[I_{ii}(n)] = \mathrm{P}[X_n = i|X_0 = i] = P_{ii}(n)$. Taking the expectation of Equation (1), we have

$$\mathrm{E}\left[N_{ii}\right] = \sum_{n=1}^{\infty} \mathrm{E}\left[I_{ii}(n)\right] = \sum_{n=1}^{\infty} P_{ii}(n). \tag{2}$$

Problem 7.5 Solution

The Markov chain has the same structure as that in Example 22. The only difference is the modified transition rates:



The event $T_{00} > n$ occurs if the system reaches state n before returning to state 0, which occurs with probability

$$\mathrm{P}\left[T_{00} > n\right] = 1 \times \left(\frac{1}{2}\right)^{\alpha} \times \left(\frac{2}{3}\right)^{\alpha} \times \cdots \times \left(\frac{n-1}{n}\right)^{\alpha} = \left(\frac{1}{n}\right)^{\alpha}. \tag{1}$$

Thus the CDF of T_{00} satisfies $F_{T_{00}}(n) = 1 - \mathrm{P}[T_{00} > n] = 1 - 1/n^{\alpha}$. To determine whether state 0 is recurrent, we observe that for all $\alpha > 0$

$$\mathrm{P}\left[V_{00}\right] = \lim_{n\to\infty} F_{T_{00}}(n) = \lim_{n\to\infty} 1 - \frac{1}{n^{\alpha}} = 1. \tag{2}$$

Thus state 0 is recurrent for all $\alpha > 0$. Since the chain has only one communicating class, all states are recurrent. (We also note that if $\alpha = 0$, then all states are transient.)

To determine whether the chain is null recurrent or positive recurrent, we need to calculate $E[T_{00}]$. In Example 23, we did this by deriving the PMF $P_{T_{00}}(n)$. In this problem, it will be simpler to use the result of Problem 3.5.20 which says that $\sum_{k=0}^{\infty} P[K > k] = E[K]$ for any non-negative integer-valued random variable K. Applying this result, the expected time to return to state 0 is

$$E\left[T_{00}\right] = \sum_{n=0}^{\infty} P\left[T_{00} > n\right] = 1 + \sum_{n=1}^{\infty} \frac{1}{n^\alpha}. \tag{3}$$

For $0 < \alpha \leq 1$, $1/n^\alpha \geq 1/n$ and it follows that

$$E\left[T_{00}\right] \geq 1 + \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{4}$$

We conclude that the Markov chain is null recurrent for $0 < \alpha \leq 1$. On the other hand, for $\alpha > 1$,

$$E\left[T_{00}\right] = 2 + \sum_{n=2}^{\infty} \frac{1}{n^\alpha}. \tag{5}$$

Note that for all $n \geq 2$

$$\frac{1}{n^\alpha} \leq \int_{n-1}^{n} \frac{dx}{x^\alpha}. \tag{6}$$

This implies

$$\begin{aligned} E\left[T_{00}\right] &\leq 2 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{dx}{x^\alpha} \\ &= 2 + \int_{1}^{\infty} \frac{dx}{x^\alpha} \\ &= 2 + \left. \frac{x^{-\alpha+1}}{-\alpha+1} \right|_{1}^{\infty} = 2 + \frac{1}{\alpha - 1} < \infty. \end{aligned} \tag{7}$$

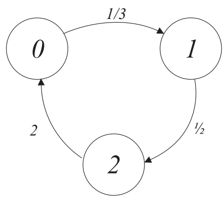
Thus for all $\alpha > 1$, the Markov chain is positive recurrent.

Problem 8.1 Solution

From the problem statement, we learn that in each state i, the tiger spends an exponential time with parameter λ_i. When we measure time in hours,

$$\lambda_0 = q_{01} = 1/3, \qquad \lambda_1 = q_{12} = 1/2, \qquad \lambda_2 = q_{20} = 2. \tag{1}$$

The corresponding continous time Markov chain is shown below:



The state probabilities satisfy

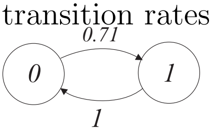
$$\frac{1}{3}p_0 = 2p_2, \qquad \frac{1}{2}p_1 = \frac{1}{3}p_0, \qquad p_0 + p_1 + p_2 = 1. \tag{2}$$

The solution is

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 6/11 & 4/11 & 1/11 \end{bmatrix}. \tag{3}$$

Problem 8.3 Solution

In the continuous time chain, we have states 0 (silent) and 1 (active). The transition rates and the chain are



$$q_{01} = \frac{1}{1.4} = 0.7143, \qquad q_{10} = 1.0. \tag{1}$$

The stationary probabilities satisfy $(1/1.4)p_0 = p_1$. Since $p_0 + p_1 = 1$, the stationary probabilities are

$$p_0 = \frac{1.4}{2.4} = \frac{7}{12}, \qquad p_1 = \frac{1}{2.4} = \frac{5}{12}. \tag{2}$$

In this case, the continuous time chain and the discrete time chain have the exact same state probabilities. In this problem, this is not surprising since we could use a renewal-reward process to calculate the fraction of time spent in state 0. From the renewal-reward process, we know that the fraction of time spent in state 0 depends only on the expected time in each state. Since in both the discrete time and continuous time chains, the expected time in each state is the same, the stationary probabilties must be the same. It is always possible to approximate a continuous time chain with a discrete time chain in which the unit of time is chosen to be very small. In general however, the stationary probabilities of the two chains will be close though not identical.

Problem 8.5 Solution

From each state i, there are transitions of rate $q_{ij} = 1$ to each of the other $k-1$ states. Thus each state i has departure rate $\nu_i = k-1$. Thus, the stationary probabilities satisfy

$$p_j(k-1) = \sum_{i \neq j} p_j, \qquad j = 1, 2, \ldots, k. \tag{1}$$

It is easy to verify that the solution to these equations is

$$p_j = \frac{1}{k}, \qquad j = 1, 2, \ldots, k, \tag{2}$$

Problem 8.7 Solution

(a) The key is in using the independent increments property. For $s < t$, $N(s) \leq N(t)$. Thus we find the conditional PMF for $k \leq n$,

$$
\begin{aligned}
P_{N(s)|N(t)}(k|n) &= \mathrm{P}\left[N(s)=k|N(t)=n\right] \\
&= \frac{\mathrm{P}\left[N(s)=k, N(t)=n\right]}{\mathrm{P}\left[N(t)=n\right]} \\
&= \frac{\mathrm{P}\left[N(s)=k, N(t)-N(s)=n-k\right]}{\mathrm{P}\left[N(t)=n\right]}. \qquad (1)
\end{aligned}
$$

Since $N(t)-N(s)$, the number of arrivals in the interval $(s,t]$, is a Poisson $(\lambda(t-s)$ random variable that is independent of $N(s)$,

$$
\begin{aligned}
P_{N(s)|N(t)}(k|n) &= \frac{\mathrm{P}\left[N(s)=k\right]\mathrm{P}\left[N(t)-N(s)=n-k\right]}{\mathrm{P}\left[N(t)=n\right]} \\
&= \frac{\dfrac{[\lambda s]^k e^{-\lambda s}}{k!}\dfrac{[\lambda(t-s)]^{n-k}e^{-\lambda(t-s)}}{(n-k)!}}{\dfrac{[\lambda t]^n e^{-\lambda t}}{n!}} \\
&= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}. \qquad (2)
\end{aligned}
$$

That is, given there were n arrivals in $[0,t]$, the number of arrivals in $[0,s]$ has a binomial $(n, p=s/t)$ distribution. The interpretation of this is that the collection $S_1,\ldots,S_n$ of n arrival times of the Poisson process over $[0,t]$ is equivalent to an ordered sequence of n continuous uniform $(0,t)$ random varaibles. In short, the n arrivals can be viewed as being equally likely to be anywhere in the interval and thus each arrival is in the $[0,s]$ interval with probability s/t.

(b) For $s>t$,

$$
\begin{aligned}
P_{N(s)|N(t)}(k|n) &= \mathrm{P}\left[N(s)=k|N(t)=n\right] \\
&= \frac{\mathrm{P}\left[N(s)=k, N(t)=n\right]}{\mathrm{P}\left[N(t)=n\right]} \\
&= \frac{\mathrm{P}\left[N(t)=n, N(s)-N(t)=k-n\right]}{\mathrm{P}\left[N(t)=n\right]}. \qquad (3)
\end{aligned}
$$

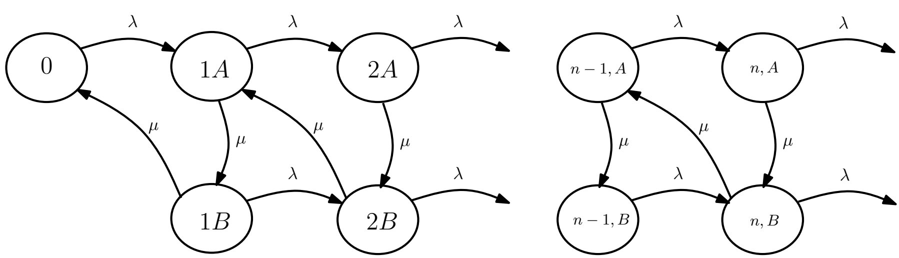
In this case, $N(s) - N(t)$, the number of arrivals in the interval $(t, s]$, is a Poisson $(\lambda(s-t))$ random variable, independent of $N(t)$. For $k \geq n$,

$$\begin{aligned} P_{N(s)|N(t)}(k|n) &= \frac{\mathrm{P}[N(t)=n]\,\mathrm{P}[N(s)-N(t)=k-n]}{\mathrm{P}[N(t)=n]} \\ &= \mathrm{P}[N(s)-N(t)=k-n] \\ &= \frac{[\lambda(s-t)]^{k-n}e^{-\lambda(s-t)}}{(k-n)!}. \end{aligned} \tag{4}$$

In this case, given $N(t) = n$, $N(s) = n + M$ where M is the Poisson number of arrivals in the interval $(t, s]$.

Problem 8.9 Solution

The main idea in this problem is that you need to keep track of both how many couples are in the store and whether the clerk is serving the first or second person in a couple. To do this, we say each couple consists of person A who is served first and person B who is served second and define a set of states $\{0, 1A, 1B, 2A, 2B, \ldots\}$ where states nA and nB indicate n couples in the store and which person is being served. What makes the problem a little tricky is that new couples can arrives while either A or B is in service. The continuous time Markov chain is



Calculating the stationary probabilities is a little tricky. Defining $\rho = \lambda/\mu$ and partitioning the space at state 0 yields

$$p_0\lambda = p_{1B}\mu \quad \Longrightarrow \quad p_{1B} = \rho p_0. \tag{1}$$

Partitioning at state $1B$ yields

$$(\lambda+\mu)p_{1B} = \mu p_{1A} \qquad \Longrightarrow \qquad p_{1A} = (1+\rho)p_{1B} = \rho(1+\rho)p_0. \tag{2}$$

For the rest of the states, we observe that

$$(p_{n-1,A} + p_{n-1,B})\lambda = p_{n,B}\mu \qquad \Longrightarrow \qquad p_{n,B} = \rho p_{n-1,A} + \rho p_{n-1,B} \tag{3}$$

and that

$$p_{n,B}(\lambda+\mu) = p_{n,A}\mu + p_{n-1,B}\lambda. \tag{4}$$

This implies

$$\begin{aligned} p_{n,A} &= (1+\rho)p_{n,B} - \rho p_{n-1,B} \\ &= (1+\rho)(\rho p_{n-1,A} + \rho p_{n-1,B}) - \rho p_{n-1,B} \\ &= \rho(1+\rho)p_{n-1,A} + \rho^2 p_{n-1,B}. \end{aligned} \tag{5}$$

By defining the vector $\mathbf{p}_n = \begin{bmatrix} p_{n,A} & p_{n,B} \end{bmatrix}'$, we can combine these facts as

$$\mathbf{p}_1 = \begin{bmatrix} \rho+\rho^2 \\ \rho \end{bmatrix} p_0, \tag{6}$$

$$\mathbf{p}_n = \rho \underbrace{\begin{bmatrix} 1+\rho & \rho \\ 1 & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{p}_{n-1}, \qquad n = 2, 3, \ldots. \tag{7}$$

To go further, we need to define the diagonalization $\mathbf{A} = \mathbf{SDS}^{-1}$, with $\mathbf{D} = \mathrm{diag}[d_1, d_2]$ denoting the eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{S}$ as the right eigenvectors of $\mathbf{A}$, so that

$$\mathbf{p}_n = \rho^{n-1}\mathbf{A}^{n-1}\mathbf{p}_1 = \mathbf{S}\rho^{n-1}\mathbf{D}^{n-1}\mathbf{S}^{-1}\mathbf{p}_1. \tag{8}$$

In terms of $\mathbf{D}$ and $\mathbf{S}$, we can solve the problem. At the end, we will go back and find $\mathbf{D}$ and $\mathbf{S}$ in terms of λ and μ. At this point is also convenient to define

$$p_n = \mathbf{1}'\mathbf{p}_n = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{p}_n = p_{n,A} + p_{n,B} \tag{9}$$

as the probability that n couples are in the system. With this notation, we can write the normalization of state probabilities as

$$1 = \sum_{n=0}^{\infty} p_n = p_0 + \sum_{n=1}^{\infty} \mathbf{1}'\mathbf{S}\rho^{n-1}\mathbf{D}^{n-1}\mathbf{S}^{-1}\mathbf{p}_1$$
$$= p_0 + \mathbf{1}'\mathbf{S}\left(\sum_{n=1}^{\infty} \rho^{n-1}\mathbf{D}^{n-1}\right)\mathbf{S}^{-1}\mathbf{p}_1 \tag{10}$$

Since $\mathbf{D}$ is a diagonal matrix,

$$1 = \sum_{n=0}^{\infty} p_n = p_0 + \mathbf{1}'\mathbf{S}\begin{bmatrix} \sum_{n=1}^{\infty} \rho^{n-1}d_1^{n-1} & 0 \\ 0 & \sum_{n=1}^{\infty} \rho^{n-1}d_2^{n-1} \end{bmatrix}\mathbf{S}^{-1}\mathbf{p}_1$$
$$= p_0 + \mathbf{1}'\mathbf{S}\begin{bmatrix} \frac{1}{1-\rho d_1} & 0 \\ 0 & \frac{1}{1-\rho d_2} \end{bmatrix}\mathbf{S}^{-1}\mathbf{p}_1$$
$$= p_0\left(1 + \mathbf{1}'\mathbf{S}\begin{bmatrix} \frac{1}{1-\rho d_1} & 0 \\ 0 & \frac{1}{1-\rho d_2} \end{bmatrix}\mathbf{S}^{-1}\begin{bmatrix} \rho+\rho^2 \\ \rho \end{bmatrix}\right). \tag{11}$$

These steps require that $|\rho d_1| < 1$ and $|\rho d_2| < 1$. as the Markov chain has one communicating class, these are the conditions for ergodicity. We will simplify these later once we find $\mathbf{S}$ and $\mathbf{D}$. For now, we observe that

$$p_0 = \left(1 + \mathbf{1}'\mathbf{S}\begin{bmatrix} \frac{1}{1-\rho d_1} & 0 \\ 0 & \frac{1}{1-\rho d_2} \end{bmatrix}\mathbf{S}^{-1}\begin{bmatrix} \rho+\rho^2 \\ \rho \end{bmatrix}\right)^{-1}, \tag{12}$$
$$p_n = p_0\mathbf{1}'\mathbf{S}\begin{bmatrix} (\rho d_1)^{n-1} & 0 \\ 0 & (\rho d)^{n-1} \end{bmatrix}\mathbf{S}^{-1}\begin{bmatrix} \rho+\rho^2 \\ \rho \end{bmatrix}. \tag{13}$$

It follows for even integers n that n people corresponds to $n/2$ couples so that

$$\lim_{t\to\infty} \mathrm{P}\left[N(t) = n\right] = p_{n/2}. \tag{14}$$

All that remains is to simplify p_n. Solving $(\mathbf{A} - d\mathbf{I})\mathbf{s} = \mathbf{0}$ yields

$$d = 1 + \frac{\rho}{2} \pm \frac{1}{2}\sqrt{4\rho+\rho^2}. \tag{15}$$

Defining $\alpha = \sqrt{4\rho + \rho^2}$, we have that

$$d = d_1 = 1 + \frac{\rho + \alpha}{2}, \qquad \mathbf{s}_1 = \begin{bmatrix} \frac{\rho+\alpha}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 - 1 \\ 1 \end{bmatrix}, \tag{16}$$

$$d = d_2 = 1 + \frac{\rho - \alpha}{2}, \qquad \mathbf{s}_2 = \begin{bmatrix} \frac{\rho-\alpha}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} d_2 - 1 \\ 1 \end{bmatrix}. \tag{17}$$

This implies

$$\mathbf{S} = \begin{bmatrix} d_1 - 1 & d_2 - 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{S}^{-1} = \frac{1}{\alpha} \begin{bmatrix} 1 & 1 - d_2 \\ -1 & d_1 - 1 \end{bmatrix}. \tag{18}$$

Note that the eigenvalues have some nice properties that simplify the algebra to come:

$$d_1 + d_2 = 2 + \rho, \tag{19}$$

$$d_1 - d_2 = \alpha, \tag{20}$$

$$d_1 d_2 = 1. \tag{21}$$

Now to calculate p_0, we observe that $\mathbf{1}'\mathbf{S} = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$, implying

$$\begin{aligned} p_0 &= \left(1 + \begin{bmatrix} \frac{d_1}{1-\rho d_1} & \frac{d_2}{1-\rho d_2} \end{bmatrix} \frac{1}{\alpha} \begin{bmatrix} 1 & 1 - d_2 \\ -1 & d_1 - 1 \end{bmatrix} \begin{bmatrix} \rho + \rho^2 \\ \rho \end{bmatrix}\right)^{-1} \\ &= \left(1 + \frac{\rho}{\alpha} \begin{bmatrix} \frac{d_1}{1-\rho d_1} & \frac{d_2}{1-\rho d_2} \end{bmatrix} \begin{bmatrix} 2 + \rho - d_2 \\ -(2 + \rho - d_1) \end{bmatrix}\right)^{-1} \\ &= \left(1 + \frac{\rho}{\alpha} \begin{bmatrix} \frac{d_1}{1-\rho d_1} & \frac{d_2}{1-\rho d_2} \end{bmatrix} \begin{bmatrix} d_1 \\ -d_2 \end{bmatrix}\right)^{-1} \\ &= \left(1 + \frac{2\rho}{1 - 2\rho}\right)^{-1} \\ &= 1 - 2\rho. \end{aligned} \tag{22}$$

It appears that whether the clerk is busy is insensitive to whether the customers arrive in pairs at rate λ or individually at rate 2λ. A little more algebra will show that

$$\mathbf{p}_n = \frac{(1 - 2\rho)\rho^n}{\alpha} \begin{bmatrix} (d_1 - 1)d_1^n - (d_2 - 1)d_2^n \\ d_1^n - d_2^n \end{bmatrix}, \tag{23}$$

but this doesn't appear to simplify in a nice way. Finally we note that the ergodicity conditions were $|\rho d_1| < 1$ and $|\rho d_2| < 1$. The condition for d_1 holds iff $\rho < 1/2$ while the condition on d_2 is implied by that for d_1.

Problem 9.1 Solution

In Equation (90), we found that the blocking probability of the $M/M/c/c$ queue was given by the Erlang-B formula

$$\mathrm{P}\left[B\right] = P_N(c) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!}. \tag{1}$$

The parameter $\rho = \lambda/\mu$ is the normalized load. When $c = 2$, the blocking probability is

$$\mathrm{P}\left[B\right] = \frac{\rho^2/2}{1 + \rho + \rho^2/2}. \tag{2}$$

Setting $\mathrm{P}[B] = 0.1$ yields the quadratic equation

$$\rho^2 - \frac{2}{9}\rho - \frac{2}{9} = 0. \tag{3}$$

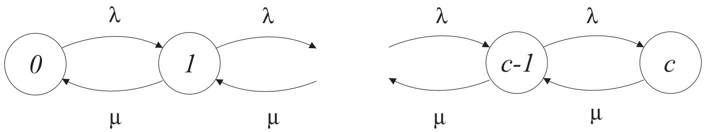
The solutions to this quadratic are

$$\rho = \frac{1 \pm \sqrt{19}}{9}. \tag{4}$$

The meaningful nonnegative solution is $\rho = (1 + \sqrt{19})/9 = 0.5954$.

Problem 9.3 Solution

In the $M/M/1/c$ queue, there is one server and the system has capacity c. That is, in addition to the server, there is a waiting room that can hold $c - 1$ customers. With arival rate λ and service rate μ, the Markov chain for this queue is



By Theorem 23, the stationary probabilities satisfy $p_{i-1}\lambda = p_i\mu$. By defining $\rho = \lambda/mu$, we have $p_i = \rho p_{i-1}$, which implies

$$p_n = \rho^n p_0, \qquad n = 0, 1, \ldots, c. \tag{1}$$

applying the requirement that the stationary probabilities sum to 1 yields

$$\sum_{i=0}^{c} p_i = p_0 \left[1 + \rho + \rho^2 + \cdots + \rho^c\right] = 1. \tag{2}$$

This implies

$$p_0 = \frac{1-\rho}{1-\rho^{c+1}}. \tag{3}$$

The stationary probabilities are

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{c+1}}, \qquad n = 0, 1, \ldots, c. \tag{4}$$

Problem 9.5 Solution

(a) In this case, we have two $M/M/1$ queues, each with an arrival rate of $\lambda/2$. By defining $\rho = \lambda/\mu$, each queue has a stationary distribution

$$p_n = (1-\rho/2)\,(\rho/2)^n \qquad n = 0, 1, \ldots \tag{1}$$

Note that in this case, the expected number in queue i is

$$\mathrm{E}\left[N_i\right] = \sum_{n=0}^{\infty} n p_n = \frac{\rho/2}{1-\rho/2}. \tag{2}$$

The expected number in the system is

$$\mathrm{E}[N_1] + \mathrm{E}[N_2] = \frac{\rho}{1-\rho/2}. \tag{3}$$

(b) The combined queue is an $M/M/2/\infty$ queue. As in the solution to Quiz 9, the stationary probabilities satisfy

$$p_n = \begin{cases} p_0\rho^n/n! & n = 1, 2, \\ p_0\rho^{n-2}\rho^2/2 & n = 3, 4, \ldots. \end{cases} \tag{4}$$

The requirement that $\sum_{n=0}^{\infty} p_n = 1$ yields

$$p_0 = \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^2}{2}\frac{\rho/2}{1-\rho/2}\right)^{-1} = \frac{1-\rho/2}{1+\rho/2}. \tag{5}$$

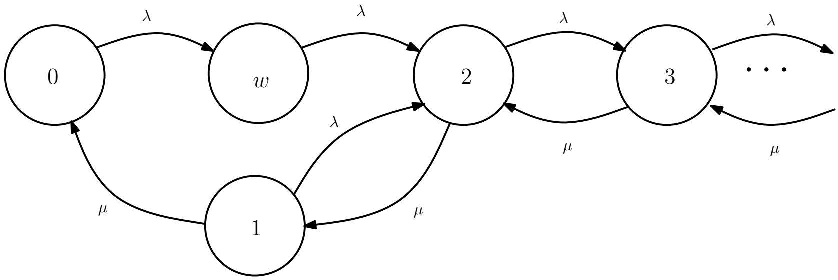
The expected number in the system is $\mathrm{E}[N] = \sum_{n=1}^{\infty} np_n$. Some algebra will show that

$$\mathrm{E}[N] = \frac{\rho}{1-(\rho/2)^2}. \tag{6}$$

We see that the average number in the combined queue is lower than in the system with individual queues. The reason for this is that in the system with individual queues, there is a possibility that one of the queues becomes empty while there is more than one person in the other queue.

Problem 9.7 Solution

(a) The main idea is that the state of the system has to indicate both the number of customers in the system as well as whether the clerk is actually working. In particular, when there is one person in the system, the system will be in state 1 when that person is receiving service and in state w when the clerk is playing WoW. The Markov chain is



for the stationary probabilities, we observe that for $n \geq 3$ that $p_n\mu = p_{n-1}\mu$, implying $p_n = \rho p_{n-1}$ where $\rho = \lambda/\mu$. It follows that

$$p_n = \rho^{n-2} p_2, \qquad (n \geq 2). \tag{1}$$

In the following we use S to denote a partition (S, S') of the state space in which S' is the complement of S.

- With $S = \{w\}$, we see that $p_0\lambda = p_w\lambda$, implying $p_w = p_0$.
- With $S = \{0\}$, $p_0\lambda = p_1\mu$, implying $p_1 = \rho p_0$.
- $S = \{0, w, 1\}$ yields $(p_w + p_1)\lambda = \mu p_2$. This implies

$$p_2 = \rho(p_0 + \rho p_0) = \rho(1 + \rho)p_0. \tag{2}$$

With normalization, we obtain

$$\begin{aligned} 1 &= p_0 + p_w + p_1 + \sum_{n=2}^{\infty} p_n \\ &= p_0 \left(1 + \rho + 1 + \sum_{n=2}^{\infty} \rho(1+\rho)\rho^{n-2} \right) \\ &= p_0 \left(\frac{2}{1-\rho} \right). \end{aligned} \tag{3}$$

Thus $p_0 = p_w = (1-\rho)/2$, $p_1 = \rho(1-\rho)/2$ and for $n \geq 2$,

$$p_n = p_0 \rho (1+\rho)\rho^{n-2} = (1-\rho^2)\rho^{n-1}. \tag{4}$$

(b) The Clerk is playing WoW in states 0 and w. Thus

$$\lim_{t\to\infty} \mathrm{P}\left[W(t)\right] = p_0 + p_w = 1 - \rho. \tag{5}$$

Note that $1-\rho$ is exactly the fraction of time the server is idle in an ordinary M/M/1 queue in which the server doesn't wait until two people are in the queue before beginning service. That is, there is no way for the server to extract more time for playing WoW.

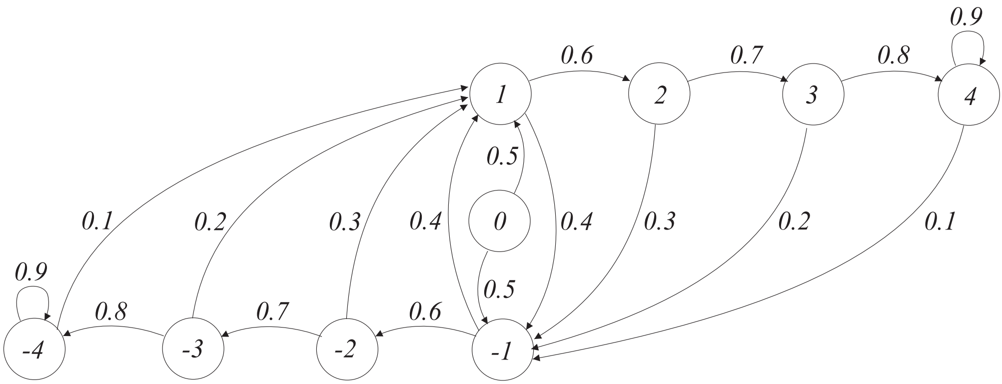
(c) Little's law says that $\overline{T} = \mathrm{E}[N]/\lambda = 2\,\mathrm{E}[N]$ where

$$\begin{aligned}
\mathrm{E}\left[N\right] &= 0\cdot p_0 + 1\cdot(p_w + p_1) + \sum_{n=2}^{\infty} n p_n \\
&= (1-\rho^2)/2 + (1-\rho^2)\sum_{n=2}^{\infty} n\rho^{n-1} \\
&= (1-\rho^2)/2 + (1+\rho)\left(-(1-\rho) + \sum_{n=1}^{\infty} n(1-\rho)\rho^{n-1}\right) \\
&= (1-\rho)^2/2 + (1+\rho)\left(-(1-\rho) + \frac{1}{1-\rho}\right) \\
&= \frac{1+\rho}{1-\rho} - \frac{1-\rho^2}{2} = 2.625.
\end{aligned} \tag{6}$$

The average system time for a customer is $\overline{T} = \mathrm{E}[N]/\lambda = 5.25$ minutes.

Problem 10.1 Solution

Here is the Markov chain describing the free throws.



Note that state 4 corresponds to "4 or more consecutive successes" while state -4 corresponds to "4 or more consecutive misses." We denote the stationary probabilities by the vector

$$\boldsymbol{\pi} = \begin{bmatrix}\pi_{-4} & \pi_{-3} & \pi_{-2} & \pi_{-1} & \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4\end{bmatrix}'. \tag{1}$$

For this vector $\boldsymbol{\pi}$, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}. \tag{2}$$

To solve the problem at hand, we divide the work into two functions:

```
function Pn=freethrowmat(n);
P=[0.9 0   0   0   0  0.1 0   0   0;...
   0.8 0   0   0   0  0.2 0   0   0;...
   0   0.7 0   0   0  0.3 0   0   0;...
   0   0   0.6 0   0  0.4 0   0   0;...
   0   0   0   0.5 0  0.5 0   0   0;...
   0   0   0   0.4 0   0  0.6 0   0;...
   0   0   0   0.3 0   0  0   0.7 0;...
   0   0   0   0.2 0   0  0   0   0.8;...
   0   0   0   0.1 0   0  0   0   0.9];
Pn=P^n;
```

returns the `n` step transition matrix and

```
function ps=freethrowp(n);
PP=freethrowmat(n-1);
p0=[zeros(1,4) 1 ...
        zeros(1,4)];
ps=p0*PP*0.1*(1:9)';
```

that calculates the probability of a success on the free throw `n`.

In `freethrowp.m`, `p0` is the initial state probability row vector $\boldsymbol{\pi}'(0)$. Thus `p0*PP` is the state probability row vector after $n-1$ free throws. Finally, `p0*PP*0.1*(1:9)'` multiplies the state probability vector by the conditional probability of successful free throw given the current state. The answer to our problem is simply

```
>> freethrowp(11)
ans =
    0.5000
>>
```

In retrospect, the calculations are unnecessary! Because the system starts in state 0, symmetry of the Markov chain dictates that states $-k$ and k will have the same probability at every time step. Because state $-k$ has success probability $0.5-0.1k$ while state k has success probability $0.5+0.1k$, the conditional success probability given the system is in state $-k$ or k is 0.5. Averaged over $k=1,2,3,4$, the average success probability is still 0.5.

Comment: Perhaps finding the stationary distribution is more interesting. This is done fairly easily:

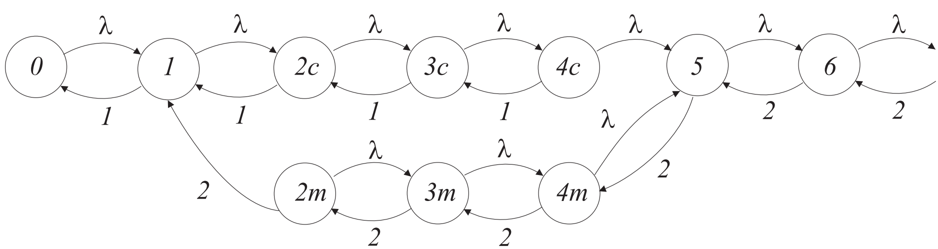
```
>> p=dmcstatprob(freethrowmat(1));
>> p'
ans =
  0.3123 0.0390 0.0558 0.0929 0 0.0929 0.0558 0.0390 0.3123
```

About sixty precent of the time the shooter has either made four or more consecutive free throws or missed four or more free throws. On the other hand, one can argue that in a basketball game, a shooter rarely gets to take more than a half dozen (or so) free throws, so perhaps the stationary distribution isn't all that interesting.

Problem 10.3 Solution

In this problem, we model the system as a continuous time Markov chain. The clerk and the manager each represent a "server." The state describes the number of customers in the queue and the number of active servers. The Markov chain issomewhat complicated because when the number of customers in the store is 2, 3, or 4, the number of servers may be 1 or may be 2, depending on whether the manager became an active server.

When just the clerk is serving, the service rate is 1 customer per minute. When the manager and clerk are both serving, the rate is 2 customers per minute. Here is the Markov chain:



In states $2c$, $3c$ and $4c$, only the clerk is working. In states $2m$, $3m$ and $4m$, the manager is also working. The state space $\{0, 1, 2c, 3c, 4c, 2m, 3m, 4m, 5, 6, \ldots\}$ is countably infinite. Finding the state probabilities is a little bit complicated because the are enough states that we would like to use MATLAB; however, MATLAB can only handle a finite state space. Fortunately, we can use MATLAB because the state space for states $n \geq 5$ has a simple structure.

We observe for $n \geq 5$ that the average rate of transitions from state n to state $n+1$ must equal the average rate of transitions from state $n+1$ to state n, implying

$$\lambda p_n = 2p_{n+1}, \qquad n = 5, 6, \ldots \tag{1}$$

It follows that $p_{n+1} = (\lambda/2)p_n$ and that

$$p_n = \alpha^{n-5} p_5, \qquad n = 5, 6, \ldots, \tag{2}$$

where $\alpha = \lambda < 2 < 1$. The requirement that the stationary probabilities sum to 1 implies

$$\begin{aligned}
1 &= p_0 + p_1 + \sum_{j=2}^{4} (p_{jc} + p_{jm}) + \sum_{n=5}^{\infty} p_n \\
&= p_0 + p_1 + \sum_{j=2}^{4} (p_{jc} + p_{jm}) + p_5 \sum_{n=5}^{\infty} \alpha^{n-5} \\
&= p_0 + p_1 + \sum_{j=2}^{4} (p_{jc} + p_{jm}) + \frac{p_5}{1-\alpha}. \qquad (3)
\end{aligned}$$

This is convenient because for each state $j < 5$, we can solve for the staitonary probabilities. In particular, we use Theorem 22 to write $\sum \sum_i r_{ij} p_i = 0$. This leads to a set of matrix equations for the state probability vector

$$\mathbf{p} = \begin{bmatrix} p_0 & p_1 & p_{2c} & p_{3c} & p_{3c} & p_{4c} & p_{2m} & p_{3m} & p_{4m} & p_5 \end{bmatrix}'. \tag{4}$$

The rate transition matrix associated with $\mathbf{p}$ is

$$\mathbf{Q} = \begin{array}{c} \begin{matrix} p_0 & p_1 & p_{2c} & p_{3c} & p_{4c} & p_{2m} & p_{3m} & p_{4m} & p_5 \end{matrix} \\ \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 2 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \end{array}, \tag{5}$$

where the first row just shows the correspondence of the state probabilities and the matrix columns.For each state i, excepting state 5, the departure rate ν_i from that state equals the sum of entries of the corresponding row of $\mathbf{Q}$. To find the stationary probabilities, our normal procedure is to use Theorem 22 and solve $\mathbf{p}'\mathbf{R} = \mathbf{0}$ and $\mathbf{p}'\mathbf{1} = 1$, where $\mathbf{R}$ is the same as $\mathbf{Q}$ except the zero diagonal entries are replaced by $-\nu_i$. The equation $\mathbf{p}'\mathbf{1} = 1$ replaces one column of the set of matrix equations. This is the approach of `cmcstatprob.m`.

In this problem, we follow almost the same procedure. We form the matrix $\mathbf{R}$ by replacing the diagonal entries of $\mathbf{Q}$. However, instead of replacing an arbitrary column with the equation $\mathbf{p}'\mathbf{1} = 1$, we replace the column corresponding to p_5 with the equation

$$p_0 + p_1 + p_{2c} + p_{3c} + p_{4c} + p_{2m} + p_{3m} + p_{4m} + \frac{p_5}{1-\alpha} = 1. \tag{6}$$

That is, we solve

$$\mathbf{p}'\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}'. \tag{7}$$

where, with the definitions $\lambda_1 = -1 - \lambda$ and $\lambda_2 = -2 - \lambda$,

$$\mathbf{R} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \lambda_1 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & \lambda_1 & \lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \lambda_1 & \lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \lambda_1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & \lambda_2 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & \lambda_2 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \frac{1}{1-\alpha} \end{bmatrix}. \tag{8}$$

Given the stationary distribution, we can now find $\mathrm{E}[N]$ and $\mathrm{P}[W]$.

Recall that N is the number of customers in the system at a time in the distant future. Defining

$$p_n = p_{nc} + p_{nm}, \qquad n = 2, 3, 4, \tag{9}$$

we can write

$$\mathrm{E}[N] = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{4} np_n + \sum_{n=5}^{\infty} np_5\alpha^{n-5}. \tag{10}$$

The substitution $k = n - 5$ yields

$$\begin{aligned} \mathrm{E}[N] &= \sum_{n=0}^{4} np_n + p_5 \sum_{k=0}^{\infty} (k+5)\alpha^k \\ &= \sum_{n=0}^{4} np_n + p_5 \frac{5}{1-\alpha} + p_5 \sum_{k=0}^{\infty} k\alpha^k. \end{aligned} \tag{11}$$

From Math Fact B.7, we conclude that

$$\begin{aligned} \mathrm{E}[N] &= \sum_{n=0}^{4} np_n + p_5 \left(\frac{5}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} \right) \\ &= \sum_{n=0}^{4} np_n + p_5 \frac{5-4\alpha}{(1-\alpha)^2}. \end{aligned} \tag{12}$$

Furthermore, the manager is working unless the system is in state 0, 1, $2c$, $3c$, or $4c$. Thus

$$\mathrm{P}[W] = 1 - (p_0 + p_1 + p_{2c} + p_{3c} + p_{4c}). \tag{13}$$

We implement these equations in the following program, alongside the corresponding output.

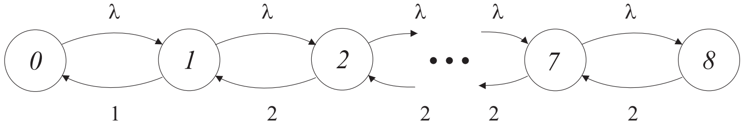
```
function [EN,PW]=clerks(lam);
Q=diag(lam*[1 1 1 1 0 1 1 1],1);
Q=Q+diag([1 1 1 1 0 2 2 2],-1);
Q(6,2)=2; Q(5,9)=lam;
R=Q-diag(sum(Q,2));
n=size(Q,1);
a=lam/2;
R(:,n)=[ones(1,n-1) 1/(1-a)]';
pv=([zeros(1,n-1) 1]*R^(-1));
EN=pv*[0;1;2;3;4;2;3;4; ...
        (5-4*a)/(1-a)^2];
PW=1-sum(pv(1:5));
```

```
>> [en05,pw05]=clerks(0.5)
en05 =
    0.8217
pw05 =
    0.0233
>> [en10,pw10]=clerks(1.0)
en10 =
    2.1111
pw10 =
    0.2222
>> [en15,pw15]=clerks(1.5)
en15 =
    4.5036
pw15 =
    0.5772
>>
```

We see that in going from an arrival rate of 0.5 customers per minute to 1.5 customers per minute, the average number of customers goes from 0.82 to 4.5 customers. Similarly, the probability the manager is working rises from 0.02 to 0.57.

Problem 10.5 Solution

This problem is actually very easy. The state of the system is given by X, the number of cars in the system.When $X = 0$, both tellers are idle. When $X = 1$, one teller is busy, however, we do not need to keep track of which teller is busy. When $X = n \geq 2$, both tellers are busy and there are $n - 2$ cars waiting. Here is the Markov chain:



Since this is a birth death process, we could easily solve this problem using analysis. However, as this problem is in the MATLAB section of this chapter, we might as well construct a MATLAB solution:

```
function [p,en]=veryfast2(lambda);
c=2*[0,eye(1,8)]';
r=lambda*[0,eye(1,8)];
Q=toeplitz(c,r);
Q(2,1)=1;
p=cmcstatprob(Q);
en=(0:8)*p;
```

The code solves the stationary distribution and the expected number of cars in the system for an arbitrary arrival rate λ.

Here is the output:

```
>> [p,en]=veryfast2(0.75);
>> p'
ans =
 0.4546 0.3410 0.1279 0.0480 0.0180 0.0067 0.0025 0.0009 0.0004
>> en
en =
    0.8709
>>
```

Problem 10.7 Solution

If you work out a solution and submit it to the authors, you will likely see it acknowledged here in the next edition of this supplement.

Problem 10.9 Solution

If you work out a solution and submit it to the authors, you will likely see it acknowledged here in the next edition of this supplement.