Week 7

Markov chains

12.1.1/1.3 This is not so hard, all data is available:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \tag{7.1}$$

12.2.2/2.2 To find the n-step probabilities, we have to compute P^n. The easiest is to do an eigenvalue decomposition of P: $P = SDS^{-1}$, so that we can easily compute $P = SD^nS^{-1}$. For the eigenvalue decomposition, we need the eigenvalues and eigenvectors. First we find the eigenvalues by solving:

$$\det(P - \lambda\mathcal{I}) = 0 \tag{7.2}$$

So:

$$\begin{aligned} \det\left(\begin{bmatrix} 1/2-\lambda & 1/2 & 0 \\ 1/2 & 1/2-\lambda & 0 \\ 1/4 & 1/4 & 1/2-\lambda \end{bmatrix}\right) &= 0 = (0.5-\lambda)^3 - 0.5\cdot 0.5(0.5-\lambda)) \\ &= (0.5-\lambda)\lambda(1-\lambda) \end{aligned} \tag{7.3}$$

The eigenvalues are therefore $\lambda_1 = 1$, $\lambda_2 = 0.5$ and $\lambda_3 = 0$. Now we have to find the corresponding eigenvectors:

$$\begin{aligned} (P - \lambda_1\mathcal{I})s_1 &= \begin{bmatrix} 1/2-\lambda & 1/2 & 0 \\ 1/2 & 1/2-\lambda & 0 \\ 1/4 & 1/4 & 1/2-\lambda \end{bmatrix} s_1 \\ &= \begin{bmatrix} 1/2-1 & 1/2 & 0 \\ 1/2 & 1/2-1 & 0 \\ 1/4 & 1/4 & 1/2-1 \end{bmatrix} s_1 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1/4 & 1/4 & -1/2 \end{bmatrix} s_1 = 0 \end{aligned} \tag{7.4}$$

Call the individual components of s: $s_1 = (a, b, c)$, then we have to solve the following equations:

$$-0.5a + 0.5b = 0 \tag{7.5}$$

$$0.5a - 0.5b = 0 \tag{7.6}$$

$$0.25a + 0.25b - 0.5c = 0 \tag{7.7}$$

This gives the solution $a = b = c = 1$, so

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7.8}$$

In a similar way we obtain:

$$s_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad s_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{7.9}$$

Combining everything gives:

$$P = SDS^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \tag{7.10}$$

and thus:

$$\begin{aligned} P^n = SD^nS^{-1} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (0.5)^n \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & (0.5)^n \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5-(0.5)^{n+1} & 0.5-(0.5)^{n+1} & (0.5)^n \end{bmatrix} \end{aligned} \tag{7.11}$$

12.5.2/5.2 Now we know the transition matrix P, and we have to solve

$$\pi = \pi P \tag{7.12}$$

Because P is a transition matrix (each row adds up to one), this matrix equation defines 2 independent equations. To solve three variables, we need the last independent equation:

$$\pi_0 + \pi_1 + \pi_2 = 1 \tag{7.13}$$

The set of equations becomes:

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.5\pi_1 + 0.25\pi_2 \\ \pi_2 &= 0.5\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

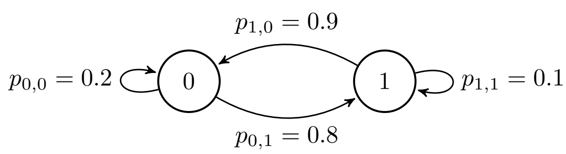
From the second equation, we see that $\pi_2 = 0$, so we are left with:

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.5\pi_1 \\ 1 &= \pi_0 + \pi_1 \end{aligned}$$

Solving this gives $\pi_0 = \pi_1 = 0.5$, and therefore:

$$\pi = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix} \tag{7.14}$$

12.1.2/1.4 The Markov chain becomes:



and the state transition matrix becomes:

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix} \tag{7.15}$$

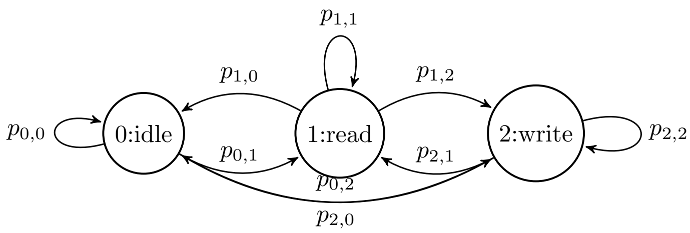
12.2.1/2.1 Again, for the n-step probabilities we need an eigenvalue decomposition. From example 12.6, pg.449, we can directly see that the eigenvalues become $\lambda_1 = 1$ and $\lambda_2 = -0.7$. Also the eigenvectors can directly be obtained:

$$P = SDS^{-1} = \begin{bmatrix} 1 & \frac{-0.8}{1.7} \\ 1 & \frac{0.9}{1.7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.7 \end{bmatrix} \begin{bmatrix} \frac{0.9}{1.7} & \frac{0.8}{1.7} \\ -1 & 1 \end{bmatrix}$$

and thus

$$\begin{aligned} P^n &= \begin{bmatrix} 1 & \frac{-0.8}{1.7} \\ 1 & \frac{0.9}{1.7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-0.7)^n \end{bmatrix} \begin{bmatrix} \frac{0.9}{1.7} & \frac{0.8}{1.7} \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{1.7} \left(\begin{bmatrix} 0.9 & 0.8 \\ 0.9 & 0.8 \end{bmatrix} + (0.7)^n \begin{bmatrix} 0.8 & -0.8 \\ 0.9 & -0.8 \end{bmatrix} \right) \end{aligned} \tag{7.16}$$

12.1.5/1.5 We start with example 12.3, page 446. We have a Markov chain like this:



where the states are 0: idle, 1: read, 2: write.

Now the most tricky part comes. What does this mean: "... each read or write operation reads or writes an entire file and that files contain a geometric number of sectors with mean 50." This means that the length N of a file has geometric distribution:

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{7.17}$$

Furthermore, it is given that the average length of the file is 50. For a geometric distribution holds that $E[N] = 1/p$, and therefore we can conclude that $p = 1/50$.

This can be easily modeled in a Markov chain. When we happen to be in a Read state, at each time step with probability $(1-p)$ we keep reading, and with probability p we reached the end of the file, and we jump out of the state Read. The same holds for the Write state. With this we can conclude:

$$p_{1,1} = 1 - 1/50 = \frac{980}{1000}, \quad p_{2,2} = 1 - 1/50 = \frac{980}{1000} \tag{7.18}$$

For the Idle state, something very similar holds, but here it is given that the average time is 500, and therefore

$$p_{0,0} = 1 - 1/500 = \frac{998}{1000} \tag{7.19}$$

Next, what does it mean that: "After an idle period, the system is equally likely to read or write a file". This means that after the system has left Idle, the probability to go to Read or to go to Write are the same. So $p_{0,1} = p_{0,2}$. And because the sum of the probabilities should be one $p_{0,0} + p_{0,1} + p_{0,2} = (1 - 1/500) + p_{0,1} + p_{0,2} = 1$, we can solve and find:

$$p_{0,1} = p_{0,2} = \frac{1}{2} \cdot \frac{1}{500} = \frac{1}{1000} \tag{7.20}$$

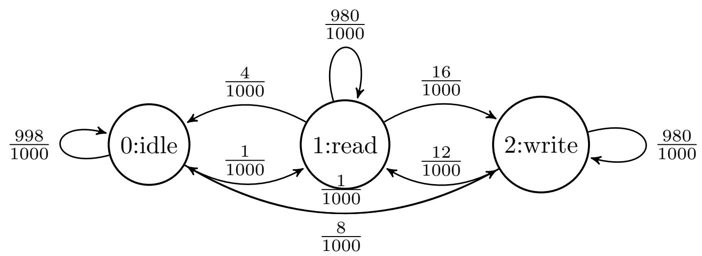
The next sentence says "Following the completion of a read, a write follows with probability 0.8.". That means that after the read state has been left (which was happening with probability 1/50), with probability of 0.8 it will jump to the write state. And therefore it has to jump with probability 0.2 to the idle state:

$$p_{1,2} = \frac{1}{50} \cdot 0.8 = \frac{16}{1000} \quad p_{1,0} = \frac{1}{50} \cdot 0.2 = \frac{4}{1000} \tag{7.21}$$

Finally, the sentence "On completion of a write operation, a read operation follows with probability 0.6" indicates that

$$p_{2,1} = \frac{1}{50} \cdot 0.6 = \frac{12}{1000} \quad p_{2,0} = \frac{1}{50} \cdot 0.4 = \frac{8}{1000} \tag{7.22}$$

So, the resulting Markov chain therefore becomes:



To be complete, this ends up in the following transition matrix:

$$P = \frac{1}{1000} \begin{bmatrix} 998 & 1 & 1 \\ 4 & 980 & 16 \\ 8 & 12 & 980 \end{bmatrix} \tag{7.23}$$

12.5.1/5.1 We have to solve:

$$\pi = \pi \begin{bmatrix} 0.998 & 0.001 & 0.001 \\ 0.004 & 0.98 & 0.016 \\ 0.008 & 0.012 & 0.98 \end{bmatrix} = \pi \frac{1}{1000} \begin{bmatrix} 998 & 1 & 1 \\ 4 & 980 & 6 \\ 8 & 12 & 980 \end{bmatrix} \tag{7.24}$$

with the extra equation that $\pi_0 + \pi_1 + \pi_2 = 1$. So then we solve:

$$\begin{cases} 1000\pi_0 = 998\pi_0 + 4\pi_1 + 8\pi_2 \\ 1000\pi_1 = \pi_0 + 980\pi_1 + 12\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \tag{7.25}$$

Rewrite the first two equations:

$$\begin{cases} \pi_0 = 2\pi_1 + 4\pi_2 \\ 20\pi_1 = \pi_0 + 12\pi_2 \end{cases} \tag{7.26}$$

Substitute the first into the second:

$$20\pi_1 = 2\pi_1 + 4\pi_2 + 12\pi_2 \rightarrow \pi_1 = \frac{8}{9}\pi_2 \tag{7.27}$$

Substitute this into the last equation of (7.25):

$$2 \cdot \frac{8}{9}\pi_2 + 4\pi_2 + \frac{8}{9}\pi_2 + \pi_2 = 1 \quad (7.28)$$

$$(\frac{24}{9} + 5)\pi_2 = \frac{69}{9}\pi_2 = 1 \quad (7.29)$$

$$\pi_2 = \frac{9}{69} \quad (7.30)$$

This gives for $\pi_1 = \frac{8}{69}$ and $\pi_0 = \frac{52}{69}$. So:

$$\pi = \left[\frac{52}{69}, \frac{8}{69}, \frac{9}{69}\right] \quad (7.31)$$

12.5.6/5.12 In this story, you have to realize that in order to jump from one state to the other, you have to consider *two* cars at the same time. Let us first define the states:

state 0 : front teller busy, rear teller idle

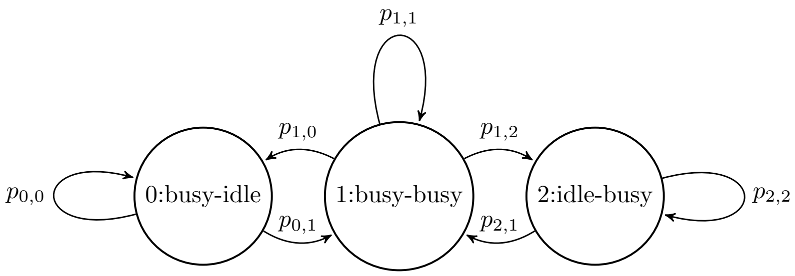
state 1 : front teller busy, rear teller busy

state 2 : front teller idle, rear teller busy

state 3 : front teller idle, rear teller idle

The first remark is, that when both tellers are idle (state 3), immediately two cars drive up to the empty tellers. So actually, this state is not used, and can be removed from the analysis.

The resulting Markov chain will therefore look like this:



Now realize that the probability that we change from state 1 to state 2, is the probability that the first car finishes his business, times the probability that the second car is still not finished (stays busy).

What is the probability that a car finishes its business? It is given that the service time has a geometric distribution with a mean of 120 seconds. Like in question 12.1.5, it means that with probability $p = 1/120$ a car leaves its busy state, and stays busy with probability $1 - p = 119/120$.

So the probability that we change from state 1 to state 2, is

$$p_{1,2} = p(1-p) = \frac{1}{120} \cdot \frac{119}{120} \quad (7.32)$$

The same holds for changing from state 1 to state 0:

$$p_{1,0} = p(1-p) = \frac{1}{120} \cdot \frac{119}{120} \quad (7.33)$$

The probability of staying in state 1, is therefore:

$$p_{1,1} = 1 - p_{1,0} - p_{1,2} \tag{7.34}$$

Next, when we are in state 0, we can stay in this state, or we can go back to state 1 (but not to 2!). What is the probability of staying in state 0? We stay in state 0 only when the (front) car is still being served, which had a probability of $p_{0,0} = 1 - p = 119/120$. The probability of leaving state 0 is therefore $p_{0,1} = p$. The same holds for state 2: $p_{2,2} = 119/120$ and $p_{2,1} = p$.

The overall state transition matrix therefore becomes:

$$P = \begin{bmatrix} 1-p & p & 0 \\ p(1-p) & 1-2p(1-p) & p(1-p) \\ 0 & p & 1-p \end{bmatrix} \quad \text{where} \quad p = \frac{1}{120} \tag{7.35}$$

To find the stationary probabilities, we have to solve:

$$\begin{aligned} \pi P &= \pi && (7.36) \\ \pi_0 + \pi_1 + \pi_2 &= 1 && (7.37) \end{aligned}$$

From the first matrix equality I choose the first and last equation:

$$\begin{aligned} \pi_0(1-p) + \pi_1 p(1-p) + 0 &= \pi_0 && (7.38) \\ 0 + \pi_1 p(1-p) + \pi_2 &= \pi_2 && (7.39) \\ \pi_0 + \pi_1 + \pi_2 &= 1 && (7.40) \end{aligned}$$

The first equation gives us $\pi_0(1-p) + \pi_1 p(1-p) = \pi_0 \rightarrow \pi_0 \cdot -p + \pi_1 p(1-p) = 0$ and therefore $\pi_0 = \pi_1(1-p)$. Similarly, the second equation gives $\pi_2 = \pi_1(1-p)$. Combining these two results gives that $\pi_0 = \pi_2$.

Finally, we use equation (7.40):

$$\begin{aligned} \pi_1(1-p) + \pi_1 + \pi_1(1-p) &= 1 \\ (2(1-p)+1)\pi_1 &= 1 \\ \pi_1 = \frac{1}{2(1-p)+1} &= \frac{1}{3-2p} && (7.41) \end{aligned}$$

Combining everything gives: $\pi = (\frac{1-p}{3-2p}, \frac{1}{3-2p}, \frac{1-p}{3-2p})$, and the stationary probability that two tellers are busy is $\frac{1}{3-2p}$.