Week 4

The autocorrelation function

10.8.1/13.7.1 First, write the definition:

$$\begin{aligned} C_X[m,k] &= E[(X_m-\mu_X)(X_{m+k}-\mu_X)] && (4.1)\\ &= E[X_mX_{m+k}-X_m\mu_X-\mu_XX_{m+k}+\mu_X^2] && (4.2)\\ &= E[X_mX_{m+k}]-E[X_m]\mu_X-\mu_XE[X_{m+k}]+\mu_X^2 && (4.3) \end{aligned}$$

Now is given that X_n is iid, with mean μ_X and variance σ_X^2, we can simplify it further. For $k \neq 0$:

$$\begin{aligned} C_X[m,k] &= E[X_mX_{m+k}]-E[X_m]\mu_X-\mu_XE[X_{m+k}]+\mu_X^2 && (4.4)\\ &= E[X_m]E[X_{m+k}]-\mu_X\mu_X-\mu_X\mu_X+\mu_X^2=0 && (4.5) \end{aligned}$$

For $k=0$ we get

$$\begin{aligned} C_X[m,0] &= E[X_mX_m]-E[X_m]\mu_X-\mu_XE[X_m]+\mu_X^2 && (4.6)\\ &= E[X_m^2]-\mu_X^2=\sigma^2 && (4.7) \end{aligned}$$

So, in total:

$$C_X[m,k]=\begin{cases}\sigma^2 & k=0\\ 0 & \text{otherwise}\end{cases} \qquad (4.8)$$

10.8.3/13.7.3 (a) To compute the expected value, use the definition, and the fact that X_n is iid with zero mean:

$$E[C_n]=E\left[16\left(1-\cos\frac{2\pi n}{365}\right)+4X_n\right]=16E\left[1-\cos\frac{2\pi n}{365}\right]+4E[X_n]=16(1-\cos\frac{2\pi n}{365}) \qquad (4.9)$$

(b) Again, fill in the definition:

$$\begin{aligned} C_C[m,k] &= E[(C_m-\mu_C(m))(C_{m+k}-\mu_C(m+k))] && (4.10)\\ &= E[C_mC_{m+k}]-E[C_m]E[C_{m+k}] && (4.11)\\ &= E[\left(16\left(1-\cos\frac{2\pi m}{365}\right)+4X_m\right)\left(16\left(1-\cos\frac{2\pi(m+k)}{365}\right)+4X_{m+k}\right)] \\ &\quad -16^2(1-\cos\frac{2\pi m}{365})(1-\cos\frac{2\pi(m+k)}{365}) && (4.12)\\ &= E[4X_m\cdot 4X_{m+k}]=16E[X_mX_{m+k}] && (4.13) \end{aligned}$$

Now is given that X_n is iid, and using the result of question 10.8.1, we get:

$$C_C[m,k] = \begin{cases} 16 & k=0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

(c) It captures the overall seasonal changes, but it does not capture the correlations between consecutive days.

10.9.1/13.8.1 So, we assume that for an arbitrary a, we have $Y(t) = X(t+a)$. Further it is given that $X(t)$ is stationary. If X is stationary, then holds that:

$$f_{X(t_1+\Delta t),\ldots,X(t_k+\Delta t)} = f_{X(t_1),\ldots,X(t_k)} \tag{4.15}$$

If we want to check if process Y is also stationary, we have to check if this holds:

$$f_{Y(t_1+\Delta t),\ldots,Y(t_k+\Delta t)} = f_{Y(t_1),\ldots,Y(t_k)} \tag{4.16}$$

So, we fill in for Y:

$$\begin{aligned} f_{Y(t_1+\Delta t),\ldots,Y(t_k+\Delta t)} &= f_{X(t_1+a+\Delta t),\ldots,X(t_k+a+\Delta t)} && (4.17)\\ &= f_{X(t_1+a),\ldots,X(t_k+a)} = f_{Y(t_1),\ldots,Y(t_k)} && (4.18)\end{aligned}$$

So, Y should also be stationary.

10.10.4/13.9.9 (a) When $X(t)$ is WSS with average power 1, then by definition $E[X^2(t)] = 1$.

(b) Given that Θ is a uniform random variable over $[0, 2\pi]$, we can compute:

$$\begin{aligned} E[\cos(2\pi f_c t + \Theta)] &= = \int \cos(2\pi f_c t + u) f_\Theta(u) du && (4.19)\\ &= \int_0^{2\pi} \cos(2\pi f_c t + u)\frac{1}{2\pi} du && (4.20)\\ &= \frac{1}{2\pi}\left[\sin(2\pi f_c t + u)\right]_0^{2\pi} = 0 && (4.21)\end{aligned}$$

(c) We are saved by the fact that X and Θ are independent, because then:

$$E[Y(t)] = E[X(t)\cos(2\pi f_c t + \Theta)] = E[X(t)]E[\cos(2\pi f_c t + \Theta)] = 0 \tag{4.22}$$

(d) Finally, the average power becomes:

$$\begin{aligned} E[Y^2(t)] &= E[X^2(t)\cos^2(2\pi f_c t + \Theta))] = E[X^2(t)]E[\cos^2(2\pi f_c t + \Theta))] && (4.23)\\ &= E[\cos^2(2\pi f_c t + \Theta))] && (4.24)\end{aligned}$$

Now we have to be creative to integrate the $\cos^2()$. We can use Math Fact B.2 and derive:

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) \tag{4.25}$$

so now we can integrate:

$$\begin{aligned} E[Y^2(t)] &= E[\cos^2(2\pi f_c t + \Theta))] && (4.26)\\ &= E[\frac{1}{2}(1 + \cos(4\pi f_c t + 2\Theta))] && (4.27)\\ &= E[\frac{1}{2}] + E[\cos(4\pi f_c t + 2\Theta))] && (4.28)\\ &= \frac{1}{2} + 0 = \frac{1}{2} && (4.29)\end{aligned}$$

10.10.1/13.9.1 R_1 and R_2 are valid autocorrelation functions, because it is symmetric around $\tau = 0$ and $R_X(0) \geq R_X(\tau)$. R_3 is not valid, because it is not symmetric around 0, and R_4 is not valid because for $\tau \to \infty$ the autocorrelation function does not converge to μ^2 (it converges to a negative number!).

11.1.1/1.1 To compute $R_Y(t, \tau)$ we rewrite the definition:

$$\begin{aligned} R_Y(t,\tau) = &= E[Y(t)Y(t+\tau)] = E[(2+X(t))(2+X(t+\tau))] \\ &= E[4+2X(t)+2X(t+\tau)+X(t)X(t+\tau)] \\ &= 4+2E[X(t)]+2E[X(t+\tau)]+E[X(t)X(t+\tau)] \\ &= 4+0+0+R_X(t,\tau) = R_X(t,\tau)+4 \qquad (4.30) \end{aligned}$$

Because X is WSS, $E[X]$ and R_X do not depend on t and therefore also Y is WSS.

11.1.3/1.3 This is very simple, when we use Theorem 11.2 on page 396:

$$\begin{aligned} \mu_Y = 1 &= \mu_X \int h(u)du \\ &= 4\int_0^\infty e^{-u/a}du = \left[-4ae^{-t/a}\right]_0^\infty = 4a \qquad (4.31) \end{aligned}$$

Solving gives $a = 1/4$.

E1 The expected value can directly be computed using the definition (and using that the expected value of X_n is 0):

$$E[W_n] = E[\frac{1}{2}(X_n + X_{n+1})] = \frac{1}{2}(E[X_n] + E[X_{n+1}]) = 0 \qquad (4.32)$$

Next, the variance (using that the variance of X_n is 1):

$$\begin{aligned} Var[W_n] &= E[W_n^2] - E[W_n]^2 = E[W_n^2] \qquad (4.33) \\ &= \frac{1}{4}E\left[X_n^2 + 2X_nX_{n-1} + X_{n-1}^2\right] \qquad (4.34) \\ &= \frac{1}{4}E[X_n^2] + \frac{1}{2}E[X_nX_{n-1}] + \frac{1}{4}E[X_{n-1}^2] \qquad (4.35) \\ &= 1/4 + 0 + 1/4 = 1/2 \qquad (4.36) \end{aligned}$$

Then the covariance:

$$\begin{aligned} Cov[W_{n+1}, W_n] &= E[W_{n+1}W_n] - E[W_{n-1}]E[W_n] = E[W_{n+1}W_n] \qquad (4.37) \\ &= \frac{1}{4}E[(X_{n+1} + X_{n+2})(X_n + X_{n+1})] \qquad (4.38) \\ &= \frac{1}{4}E[X_{n+1}X_n + X_{n+2}X_n + X_{n+1}X_{n+1} + X_{n+2}X_{n+1}] \qquad (4.39) \\ &= \frac{1}{4}(R_X(1) + R_X(-2) + R_X(0) + R_X(-1)) \qquad (4.40) \end{aligned}$$

Now use that X_n is iid:

$$Cov[W_{n+1}, W_n] = \frac{1}{4}(0+0+1+0) = \frac{1}{4} \qquad (4.41)$$

Finally,

$$\rho_{W_{n+1},W_n} = \frac{Cov[W_{n+1}W_n]}{\sqrt{Var[W_{n+1}]Var[W_n]}} = \frac{1/4}{1/2} = \frac{1}{2} \qquad (4.42)$$