Exam Random Processing IN4309 - Answers

Friday April 9th 2010

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 - Answer

(2 p) (a) The pdf should integrate to one:

$$\begin{aligned}
\int_A f_{XY}(x,y)dxdy &= \int_0^1 \int_x^1 ccxdydx \\
&= \int_0^1 cx\,[y]_x^1\,dx \\
&= c\int_0^1 x(1-x)dx \\
&= c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 \\
&= c[\frac{1}{2} - \frac{1}{3}] = c\frac{1}{6} = 1
\end{aligned}$$

Therefore $c = 6$.

(4 p) (b) The CDF consists of several parts:

- if $x < 0$ or $y < 0$: $F_{XY} = 0$,
- if $x > 1$ and $y > 1$: $F_{XY} = 1$,
- if $y > 1, 0 \leq x \leq 1$:

$$F_{XY}(a,b) = \int_0^a \int_x^1 cxdydx = c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^a = 6\left(\frac{1}{2}a^2 - \frac{1}{3}a^3\right)$$

- if $x > y, 0 \leq y \leq 1$:

$$F_{XY}(a,b) = \int_0^b \int_0^y cxdxdy = c\int_0^b \left[\frac{1}{2}x^2\right]_0^y dy = c\int_0^b \frac{1}{2}y^2dy = 3\left[\frac{1}{3}y^3\right]_0^b = b^3$$

- if $(x,y) \in A$:

$$\begin{aligned}
F_{XY}(a,b) &= \int_0^a \int_x^b cxdydx = c\int_0^a x\,[y]_x^b\,dy = 6\int_0^a x(b-x)dy \\
&= 6\int_0^a (bx - x^2)dy = 6\left[\frac{1}{2}bx^2 - \frac{1}{3}x^3\right]_0^a = 3ba^2 - 2a^3
\end{aligned}$$

(2 p) (c) Per definition: $f_{XY|Y<1/2}(x,y) = \frac{f_{XY}(x,y)}{P[Y<\frac{1}{2}]}$. Therefore:

$$P\left[Y < \frac{1}{2}\right] = \int_0^{1/2} \int_0^y cx dx dy = \int_0^{1/2} \left[3x^2\right]_0^y = \int_0^{1/2} 3y^2 dy = \left[y^3\right]_0^{1/2} = \frac{1}{8}$$

and:

$$f_{XY|Y<1/2} = \begin{cases} 8f_{XY}(x,y) & \text{if} \quad Y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(2 p) (d) Use the definitions:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy = \int_x^1 cxdy = 6x\,[y]_x^1 = 6x(1-x), \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dx = \int_0^y cxdx = 6\left[\frac{1}{2}x^2\right]_0^y = 3y^2, \quad 0 \leq y \leq 1$$

(2 p) (e) Again, just use definitions, and the results of question (d):

$$\begin{aligned} E[X+Y] &= E[X] + E[Y] = \int_0^1 x f_X(x)dx + \int_0^1 y f_Y(y)dy \\ &= \int_0^1 6x^2(1-x)dx + \int_0^1 3y^3 dy = \left[2x^3 - \frac{6}{4}x^4\right]_0^1 + \left[\frac{3}{4}y^4\right]_0^1 \\ &= 2 - \frac{6}{4} + \frac{3}{4} = \frac{5}{4} \end{aligned}$$

Question 2 - Autocorrelation functions

Let $R(k)$ be an auto-correlation function of a random process.

(2 p) (a) Proof that for a wide sense stationary random process it holds that

$$R(k) = R(-k).$$

(2 p) (b)

$$\lim_{k\to\infty} R_S(k) = 4 = E[S[n]]^2.$$

The expected value is thus either 2 or -2. The variance is $E[S[n]^2] - E[S[n]]^2 = R_S(0) - E[S[n]]^2 = 1$

(2 p) (c) $R_X(k) = R_S(k) + (2n+k)E[S[n]]$+n(n+k).

(1 p) (d) The property does not hold, because $X[n]$ is not WSS.

Let the impulse response of a filter be given by $h[n] = \delta[n] - \delta[n-1]$.

(2 p) (e) The output is: $Y[n] = S[n] - S[n-1] + 1$. The auto-correlation function is then: $R_Y(k) = 2R_S(k) - R_S(k-1) - R_S(k+1) + 1$

(1 p) (f) Yes, $R_Y(k)$ is only a function of k, i.e., the time difference, and not dependent on the actual time indices.

Question 3 - Answer

The impulse response of a linear and time-invariant filter with input $X(t)$ and output $Y(t)$ is given by

$$h(t) = \begin{cases} e^{-t/4} & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

Let the input $X(t)$ of this filter be a wide sense stationary process with $E[X(t)] = 2$ and autocorrelation function $R_X(\tau) = \sigma^2\delta(\tau) + 4$.

(2 p) (a) Yes, the input is WSS, and the filter is linear and time invariant.

(2 p) (b) $E[Y(t)] = E[X(t)] \int_0^\infty h(\tau)d\tau = 8$

(2 p) (c) $H(f) = \frac{1}{1/4+j2\pi f}$ and $E[Y(t)] = E[X(t)]H(0) = 8$.

(2 p) (d) $R_{XY}(\tau) = \sigma^2 e^{-\tau/4}u(\tau) + 16$

(2 p) (e) $|H(f)|^2 = \frac{1}{1/16+4\pi^2 f^2}$. Therefore, $S_Y = |H(f)|^2 S_X(f) = \frac{\sigma^2+4\delta(f)}{1/16+4\pi^2 f^2}$. Then $R_Y(\tau) = 2\sigma^2 e^{-|\tau|/4} + 64$

Question 4 - Answer

(1 p) (a) I cannot (no transition from 0 to 0), II is OK, III cannot (no transition from 1 to 1), IV cannot (no transition from -1 to +1), V and VI are OK.

(1 p) (b) Both the expected value and the autocorrelation function do not depend on time.

(2 p) (c) First we have to find the state transition matrix (and using the fact that all rows should add up to 1):

$$P = \begin{pmatrix} p_{-1,-1} & 1/3 & 0 \\ p_{0,-1} & 0 & 1/2 \\ 0 & p_{1,0} & 0 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Now solve $\pi P = \pi$, and I decide to use the first and the third equation:

$$\begin{aligned} 2/3\pi_1 + 1/2\pi_2 &= \pi_1 \quad \rightarrow \quad 3/2\pi_2 = \pi_1 && (1) \\ 1/2\pi_2 &= \pi_3 \quad \rightarrow \quad \pi_2 = 2\pi_3 && (2) \\ \pi_1 + \pi_2 + \pi_3 &= 1 && (3) \end{aligned}$$

So $P[X_n = 0] = 2P[X_n = 1]$ comes from (2).

Fill (1) and (2) into (3), and obtain:

$$3/2\pi_2 + \pi_2 + 1/2\pi_2 = 1 = 3\pi_2 \qquad (4)$$

so $\pi_2 = 1/3$ and $3/2\pi_2 = 1/2 = \pi_1$.

(2 p) (d) To calculate $E[X_n] = 1/2 \cdot -1 + 1/3 \cdot 0 + 1/6 \cdot 1 = -1/3$.

(2 p) (e) $R_X(k) = E[X(n)X(n+k)]$. First, $k = 0$: $R_X(0) = E[X(n)^2] = (-1)^2 \cdot 1/2 + 0 + 1^2 \cdot 1/6 = 2/3$.

Next $k = 1$, $R_X(1) = E[X(n)X(n+1)] = 1/2 \cdot -1(2/3 \cdot -1 + 1/3 \cdot 0) + 1/6 \cdot 1(1 \cdot 0) = 1/3$.

Next $k = -1$, $R_X(-1) = E[X(n)X(n-1)] = 1/3$.