Exam Random Processing IN4309

Friday April 9th 2010

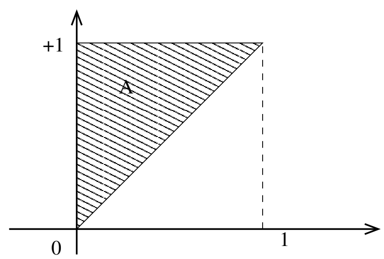
Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1

We consider the following joint-pdf of the random variables X and Y.

$$f_{XY}(x,y) = \begin{cases} cx & (x,y) \in A \\ 0 & \text{otherwise,} \end{cases}$$

where region A is defined as the shaded area:



(2 p) (a) Determine the value of c.

(4 p) (b) Determine the (complete) expression of the CDF $F_{X,Y}(x,y)$.

(2 p) (c) Calculate the conditional probability $f_{X,Y|Y\leq\frac{1}{2}}(x,y)$.

(2 p) (d) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(2 p) (e) Calculate $E[X+Y]$.

Question 2 - Autocorrelation functions

Let $R(k)$ be an auto-correlation function of a random process.

(2 p) (a) Proof that for a wide sense stationary random process it holds that

$$R(k) = R(-k).$$

Let $S[n]$ be a wide-sense stationary random process with autocorrelation function

$$R_S(k) = \frac{1}{4}^{|k|} + 4.$$

(2 p) (b) What are the expected value and variance of process $S[n]$?

Let $x[n]$ be a random process defined as

$$X[n] = S[n] + n.$$

(2 p) (c) Determine the autocorrelation function of the process $X[n]$ in terms of the autocorrelation function $R_S(k)$ and expected value $E[S[n]]$.

(1 p) (d) Explain why / why not the property $R(k) = R(-k)$ holds for process $X[n]$.

Let the impulse response of a filter be given by $h[n] = \delta[n] - \delta[n-1]$.

(2 p) (e) Determine the autocorrelation function of the output $Y[n]$ of this filter in terms of $R_S(k)$, for the situation that process $X[n]$ is the input of this filter.

(1 p) (f) Is the output $Y[n]$ a wide-sense stationary process?

Question 3 - Filtering and autocorrelation functions

For this question you can make use of the Table of Fourier transform pairs in Appendix A.

The impulse response of a linear and time-invariant filter with input $X(t)$ and output $Y(t)$ is given by

$$h(t) = \begin{cases} e^{-t/4} & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

Let the input $X(t)$ of this filter be a wide sense stationary process with $E[X(t)] = 2$ and autocorrelation function $R_X(\tau) = \sigma^2\delta(\tau) + 4$.

(2 p) (a) Is the output $Y(t)$ also wide sense stationary? Why?

(2 p) (b) Compute the expected value of process $Y(t)$ using the impulse response.

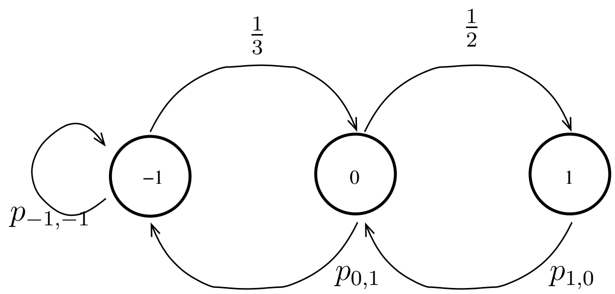
(2 p) (c) Compute the transfer function $H(f)$ of the above given system and use this to compute the expected value of process $Y(t)$.

(2 p) (d) Compute the cross-autocorrelation function $R_{XY}(\tau)$ between the input and the output.

(2 p) (e) Compute the autocorrelation function $R_Y(k)$ of the output of the above given system.

Question 4 - Markov Chains

A time-discrete amplitude-discrete random process X_n is modeled as a Markov chain with the following state transition diagram:



(1 p) (a) Which of the following series are *not* sample functions (realizations) that can be generated by the given Markov chain? Explain your answer.

I.	0	-1	-1	0	0	1	0	1	0	1	0	-1
II.	-1	-1	-1	0	-1	0	-1	0	-1	0	-1	-1
III.	-1	-1	0	1	1	0	-1	-1	-1	0	-1	0
IV.	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1
V.	0	1	0	1	0	1	0	1	0	1	0	1
VI.	-1	0	1	0	-1	0	1	0	-1	0	1	0

(1 p) (b) Explain why X_n is WSS.

(2 p) (c) Show by calculation that the limiting state probabilities are given by $P[X_n = -1] = \frac{1}{2}$ and $P[X_n = 0] = 2P[X_n = 1]$.

(2 p) (d) Calculate $E[X_n]$.

(2 p) (e) Calculate $R_X(k)$ for $k = 0$, $k = 1$, and $k = -1$.

A Table of Fourier Transform Pairs

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau - \tau_0)$	$e^{-j2\pi f \tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$e^{j2\pi f_0 \tau}$	$\delta(f - f_0)$
$\cos 2\pi f_0 \tau$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin 2\pi f_0 \tau$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a + j2\pi f}$
$ae^{-a\|\tau\|}$	$\frac{2a^2}{a^2 + (2\pi f)^2}$
$ae^{-\pi a^2 \tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\,\mathrm{rect}(\frac{f}{2W})$
$g(\tau - \tau_0)$	$G(f)e^{-j2\pi f \tau_0}$
$g(\tau)e^{j2\pi f_0 \tau}$	$G(f - f_0)$
$g(-\tau)$	$G^*(f)$
$\frac{dg(\tau)}{d\tau}$	$j2\pi f G(f)$
$\int_{-\infty}^{\tau} g(v)\,dv$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$
$\int_{-\infty}^{\infty} h(v)g(\tau - v)\,dv$	$G(f)H(f)$
$g(t)h(t)$	$\int_{-\infty}^{\infty} H(f')G(f - f')\,df'$

Note that a is a positive constant and that the rectangle and sinc functions are defined as

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < 1/2, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Table 11.1 Fourier transform pairs and properties.