CS4070 – PART 2
EXERCISES RELATED TO LECTURE 1

(1) Review some concepts from linear algebra such as: eigenvalue decomposition of a matrix, best approximation theorem, null/column space, spanned subspace of a set of vectors, invertibility of a matrix, trace of a matrix.

(2) Suppose the square matrix A has eigenvalue decomposition $A = V\Lambda V^T$, where Λ is a diagonal matrix containing the eigenvalues and V is an orthogonal matrix containing eigenvectors as columns. Show that $\mathrm{tr}A = \mathrm{tr}\Lambda$ and hence that the trace of A is the sum of its eigenvalues.

(3) Show that $\mathcal{N}(A) = \mathcal{N}(A^T A)$, where $\mathcal{N}(A) = \{x : Ax = 0\}$.

(4) If

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

then the mean-vector and covariance matrix are defined by

$$\mathbb{E}Y = \begin{bmatrix} \mathbb{E}Y_1 \\ \vdots \\ \mathbb{E}Y_n \end{bmatrix}$$

and

$$\mathrm{Cov}Y = \mathbb{E}[(Y - \mathbb{E}Y)(Y - \mathbb{E}Y)^T]$$

respectively.

(a) Check that element $[i, j]$ of the matrix $\mathrm{Cov}(Y)$ is given by $\mathrm{Cov}(Y_i, Y_j)$.

(b) Suppose that A is a $k \times n$ matrix. Verify that $\mathbb{E}[AY] = A\mathbb{E}Y$. *Hint: note that for $j \in \{1, \ldots, k\}$, the j-th element of the vector AY satisfies $(AY)[j] = \sum_{i=1}^{n} A_{ji} Y_i$.*

(c) Verify that

$$\mathrm{Cov}(AY) = A\,(\mathrm{Cov}Y)\,A^T.$$

(5) Suppose U and V are independent random variables, each with the $N(0, 1)$-distribution. for $\rho \in [-1, 1]$, define

$$\begin{aligned} X_1 &= \sqrt{1-\rho^2}U + \rho V \\ X_2 &= \phantom{\sqrt{1-\rho^2}U + \rho} V \end{aligned}$$

Verify that the correlation between X_1 and X_2 equals ρ.

(6) Consider the linear model

$$\boldsymbol{y} = X\boldsymbol{\theta} + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \sim N(0, \Sigma).$$

Assume Σ is nonsingular and known.

Date: November 11, 2023.

(a) Derive an expression for the maximum-likelihood estimator for $\boldsymbol{\theta}$, if it exists. Also explain when the maximum-likelihood estimator is unique.
(b) Derive an expression for the Hessian matrix of the loglikelihood (this is the matrix containing the second-order derivatives of the loglikelihood).