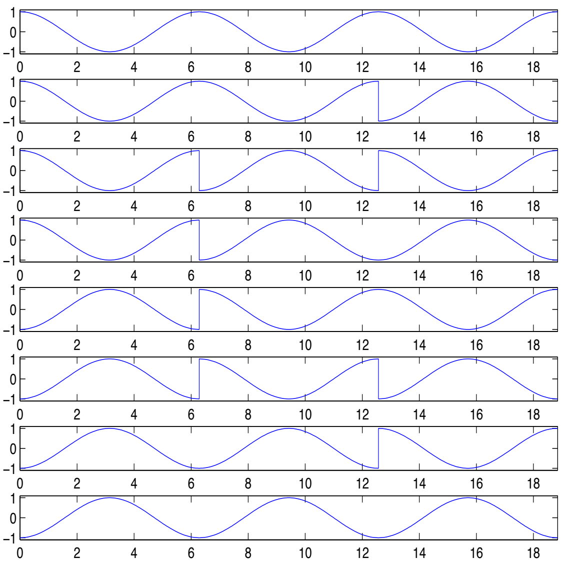
Week 3

Random processes

10.2.3 Each of the sample functions should encode one of the sequences $\{(0,0,0),(1,0,0),...,(1,1,1)\}$.



(For these pictures, I have chosen: $T = 2\pi$, and $f_0 = \frac{1}{T}$. But feel free to choose other values for T and f_0 if you like.)

10.10.3 To find the autocorrelation function of the random process $W(t) = X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)$ (with uncorrelated RV's X and Y) we have to compute:

$$
\begin{aligned}
R_W(t,\tau) &= E[W(t)W(t+\tau)] && (3.1)\\
&= E[(X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau)))]\\
&= E[X\cos(2\pi f_0 t)X\cos(2\pi f_0(t+\tau)) + X\cos(2\pi f_0 t)Y\sin(2\pi f_0(t+\tau)) +\\
&\quad Y\sin(2\pi f_0 t)X\cos(2\pi f_0(t+\tau))) + Y\sin(2\pi f_0 t)Y\sin(2\pi f_0(t+\tau)))]\\
&= E[X^2]\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + E[XY]\cos(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) +\\
&\quad E[XY]\sin(2\pi f_0 t)\cos(2\pi f_0(t+\tau))) + E[Y^2]\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau)))]\\
& && (3.2)
\end{aligned}
$$

Because X and Y are uncorrelated $Cov[X,Y] = E[XY] - E[X]E[Y] = 0$, and because the expected value $E[X] = E[Y] = 0$, we know $E[XY] = 0$. Furthermore is given that

$Var[X] = E[X^2] - E[X]^2 = \sigma^2$, and therefore $E[X^2] = E[Y^2] = \sigma^2$. Combining gives:

$$R_W(t,\tau) = \sigma^2 \left(\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + \sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau))\right)$$

From Math Fact B.2 we know:

$$\begin{aligned}\cos(A)\cos(B) &= \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]\\ \sin(A)\sin(B) &= \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\end{aligned}$$

Substituting $A = 2\pi f_0 t$ and $B = 2\pi f_0(t+\tau)$, rewriting, and noticing that the terms $\cos(A+B)$ cancel, we obtain:

$$R_W(t,\tau) = \sigma^2\cos(-2\pi f_0\tau) = \sigma^2\cos(2\pi f_0\tau) \tag{3.3}$$

and we see that the autocovariance function is *independent* on t.

10.4.1 For Y_k to be iid, it should have identical distribution for different k, and it should be independent. Each Y_k is the sum of two identical independent Gaussian random variables, so each Y_k has the same pdf. Next, Y_k is independent of Y_l when $l \neq k$ because they do not share any samples of X_k.

10.4.2 Again, each W_k is the sum of two identical independent Gaussian random variables, so they have the same pdf. But variables W_k and W_{k-1} share the sample X_{k-1}, so W_k and W_{k-1} are *not* independent.

4.11.1 Looking at the joint pdf:

$$F_{X,Y}(x,y) = ce^{-x^2/8-y^2/18} \tag{3.4}$$

it looks suspiciously like a Gaussian distribution. Looking at Definition 4.17, pg 191 of the book, we see that this definition contains μ_X, μ_Y, σ_1, σ_2 and ρ.

When we can identify σ_1, σ_2 and ρ, the constant c can be computed as:

$$c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \tag{3.5}$$

To find σ_1 and $E[X] = \mu_1$, we have to solve:

$$\frac{1}{2}\left(\frac{x-E[X]}{\sigma_1}\right)^2 = \frac{x^2}{8} \quad \rightarrow E[X] = 0, \quad \text{and} \quad \sigma_1 = \sqrt{4} = 2 \tag{3.6}$$

$$\frac{1}{2}\left(\frac{y-E[Y]}{\sigma_2}\right)^2 = \frac{y^2}{18} \quad \rightarrow E[Y] = 0, \quad \text{and} \quad \sigma_2 = \sqrt{9} = 3 \tag{3.7}$$

Because there is no cross term with $x \cdot y$, we have to conclude that $\rho = 0$. Solving c gives:

$$c = \frac{1}{2\pi\sqrt{8}\sqrt{18}\sqrt{1-0}} = \frac{1}{12\pi} \tag{3.8}$$

And because $\rho = 0$ the variables X and Y are uncorrelated, which means for a Gaussian that they are also independent.

4.11.4 (a) When the two random variables X and Y are iid continuous uniform between -50 and 50, it can actually happen that the archer misses the circular target completely! (For instance, when $x = 49$ and $y = 49$.) Because X and Y are independent, we can easily give the joint pdf:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases}10^{-4} & -50 \leq x \leq 50,\ -50 \leq y \leq 50\\ 0 & \text{otherwise}\end{cases} \tag{3.9}$$

(both X and Y are uniformly distributed between -50 and 50). Therefore the probability of bullseye is:

$$P[A] = P[X^2 + Y^2 \leq 2^2] = 10^{-4} \cdot \pi 2^2 = 0.0013 \tag{3.10}$$

(b) When $f_{X,Y}(x,y)$ is uniform over the circular area (and X and Y are not independent anymore!), the density becomes the inverse of the area:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi 50^2} & x^2 + y^2 \leq 50^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

Then the probability of bullseye becomes:

$$P[A] = P[X^2 + Y^2 \leq 2^2] = \frac{\pi 2^2}{\pi 50^2} = 0.0016 \tag{3.12}$$

(c) When X and Y are independent Gaussian distributions with mean 0 and variance σ^2, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}, \tag{3.13}$$

and the joint probability density becomes:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}, \tag{3.14}$$

For the probability of bullseye we have to compute an interesting integral:

$$\begin{aligned} P[A] &= P[X^2 + Y^2 \leq 2^2] && (3.15) \\ &= \int_{x^2+y^2 \leq 2^2} f_{X,Y}(x,y) dxdy && (3.16) \\ &= \frac{1}{2\pi\sigma^2} \int_{x^2+y^2 \leq 2^2} e^{-(x^2+y^2)/2\sigma^2} dxdy && (3.17) \end{aligned}$$

Now we have to do a trick, a coordinate transform in polar coordinates: $r^2 = x^2 + y^2$ and $dxdy = rdrd\theta$, and we integrate:

$$\begin{aligned} P[A] &= \frac{1}{2\pi\sigma} \int_0^2 \int_0^{2\pi} e^{-r^2/2\sigma^2} rdrd\theta && (3.18) \\ &= \frac{1}{\sigma^2} \int_0^2 re^{-r^2/2\sigma^2} dr && (3.19) \\ &= \left[-e^{-r^2/2\sigma^2}\right]_0^2 = 1 - e^{-4/200} = 0.020 && (3.20) \end{aligned}$$

10.5.1 The arrivals of new telephone calls can be modelled by a Poisson process. The rate $\lambda = 4$ is given, and therefore our PRM is defined:

$$P_{N(T)}(n) = \begin{cases} (4T)^n e^{-4T}/n! & n = 0, 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

(a) Now we only have to fill in:

$$P_{N(1)}(0) = (41)^0 e^{-4 \cdot 1}/0! = e^{-4} \tag{3.22}$$

(b) and (c) Similarly:

$$\begin{aligned} P_{N(1)}(4) &= (4)^4 e^{-4}/4! = 10.67e^{-4} && (3.23) \\ P_{N(2)}(2) &= (8)^2 e^{-8}/2! = 32e^{-8} && (3.24) \end{aligned}$$

10.5.6 It is given that the response time T is an exponential random variable with mean 8. That means that the pdf is:

$$f_T(t) = \begin{cases} \frac{1}{8}e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.25}$$

(a) The probability of a response time larger than 4:

$$\begin{aligned} P[T \geq 4] &= 1 - P[T < 4] = 1 - \int_{-\infty}^{4} f_T(t)dt && (3.26) \\ &= 1 - \int_0^4 \frac{1}{8}e^{-t/8}dt = 1 + \left[e^{-t/8}\right]_0^4 && (3.27) \\ &= 1 + e^{-4/8} - 1 = e^{-1/2} && (3.28) \end{aligned}$$

(b) The conditional probability is (because when $T \geq 13$ then T is also always larger than 5):

$$P[T \geq 13|T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} = \frac{P[T \geq 13]}{P[T \geq 5]} \tag{3.29}$$

Now we can compute:

$$\begin{aligned} P[T \geq 13] &= 1 - \int_0^{13} \frac{1}{8}e^{-t/8}dt = e^{-13/8} && (3.30) \\ P[T \geq 5] &= 1 - \int_0^{5} \frac{1}{8}e^{-t/8}dt = e^{-5/8} && (3.31) \\ & && (3.32) \end{aligned}$$

so therefore:

$$P[T \geq 13|T \geq 5] = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \tag{3.33}$$

(c) This seems simple: we have a sequence of arrivals, and their interarrival time is exponential. So the $N(t)$ should be a Poisson process:

$$P[N(t) = n] = \begin{cases} \left(\frac{t}{8}\right)^n e^{-t/8}/n! & n = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.34}$$

But there is one tricky detail here: it is given that the first query is made at time zero. So the Poisson process does not start with $N(t) = 0$ counts, but with $N(t) = 1$ counts. We have to shift the Poisson distribution by one count:

$$P[N(t) = n] = \begin{cases} \left(\frac{t}{8}\right)^{n-1} e^{-t/8}/(n-1)! & n = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.35}$$

Note that for $t > 0$ we will always have that $N(t) \geq 1$.