Week 2

Multivariate distributions

4.1.1/5.1.1 (a) The probability of the event $A = \{X \leq 2, Y \leq 3\}$ is directly be given by the joint CDF for $x = 2$ and $y = 3$:

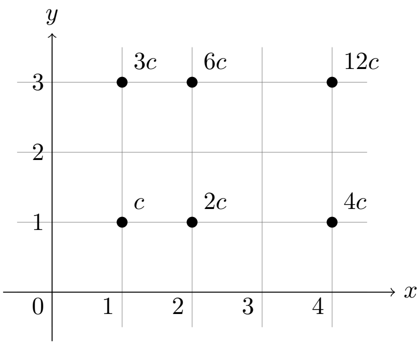
$$P[X \leq 2, Y \leq 3] = F_{X,Y}(x = 2, y = 3) = (1 - e^{-2})(1 - e^{-3}) \tag{2.1}$$

(b) The marginal cdf can be found by filling in $y = \infty$:

$$F_X(x) = F_{X,Y}(x, y = \infty) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

(c) This marginal cdf can be found by filling in $x = \infty$:

$$F_Y(y) = F_{X,Y}(x = \infty, y) = \begin{cases} 1 - e^{-y} & y > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$



4.2.1/5.2.1 First make a picture:

(a) The c should be chosen such, that the PMF adds up to one:

$$c + 2c + 4c + 3c + 6c + 12c = 28c = 1 \quad \rightarrow \ c = \frac{1}{28} \tag{2.4}$$

(b) In the event $\{Y < X\}$ there are the outcomes $(2, 1)$, $(4, 1)$ and $(4, 3)$, so

$$P[Y < X] = P_{X,Y}(2, 1) + P_{X,Y}(4, 1) + P_{X,Y}(4, 3) = 2c + 4c + 12c = 18/28 \tag{2.5}$$

(c) In the event $\{Y > X\}$ there are the outcomes $(1, 3)$ and $(2, 3)$, so

$$P[Y > X] = P_{X,Y}(1, 3) + P_{X,Y}(2, 3) = 3c + 6c = 9/28 \tag{2.6}$$

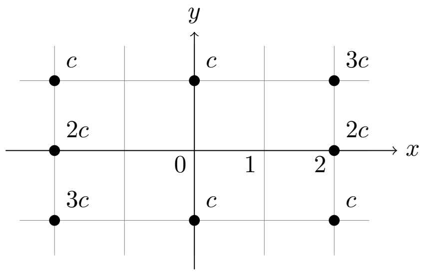
(d) In the event $\{Y = X\}$ there is only the outcome $(1, 1)$ so

$$P[Y = X] = P_{X,Y}(1, 1) = c = 1/28 \tag{2.7}$$

(e) In the event $\{Y = 3\}$ there are the outcomes $(1, 3)$, $(2, 3)$ and $(4, 3)$, so

$$P[Y = 3] = P_{X,Y}(1, 3) + P_{X,Y}(2, 3) + P_{X,Y}(4, 3) = 3c + 6c + 12c = 21/28 \tag{2.8}$$

4.2.2/5.2.2 It is always a good idea to make a picture:



(a) The constant c is found by summing the PMF over all values for X and Y, and equating it to 1.

$$\sum_x \sum_y P_{X,Y}(x, y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c|x + y| = 6c + 2c + 6c = 14c = 1 \tag{2.9}$$

Therefore $c = 1/14$.

(b) Simply:

$$P[Y < X] = P_{X,Y}(0, -1) + P_{X,Y}(2, -1) + P_{X,Y}(2, 0) + P_{X,Y}(2, 1) = 1/2 \tag{2.10}$$

(c) Surprisingly:

$$P[Y > X] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, 1) = 1/2 \tag{2.11}$$

(d) There is no outcome with $X = Y$ so $P[X = Y] = 0$.

(e)

$$P[X < 1] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(0, -1) + P_{X,Y}(0, 1) + PX, Y(-2, 1) = 8/14 \tag{2.12}$$

4.3.2/5.3.2 (a) Use the definition of the marginal probability:

$$P_X(x) = \sum_y P_{X,Y}(x, y) = \sum_{y=-1,0,1} P_{X,Y}(x, y) = \begin{cases} 6/14 & x = -2, 2 \\ 2/14 & x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

Similarly for $P_Y(y)$:

$$P_Y(y) = \sum_x P_{X,Y}(x, y) = \sum_{x=-2,0,2} P_{X,Y}(x, y) = \begin{cases} 5/14 & y = -1, 1 \\ 4/14 & y = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

(b) Given the marginal pdf's it is easy to compute $E[X]$:

$$E[X] = \sum_x xP_X(x) = \sum_{x=-2,0,2} xP_X(x) = -2 \cdot 6/14 + 2 \cdot 6/14 = 0 \tag{2.15}$$

$$E[Y] = \sum_y y P_Y(y) = \sum_{y=-1,0,1} y P_Y(y) = -1 \cdot 5/14 + 1 \cdot 5/14 = 0 \tag{2.16}$$

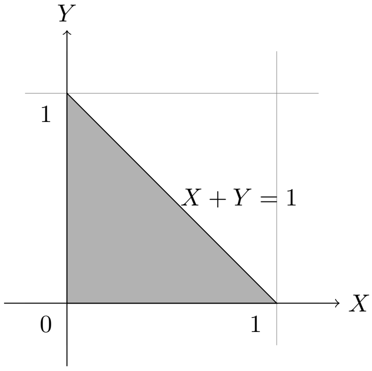
(c) For both variances $Var(X) = E[X^2] - (E[X])^2 = E[X^2]$:

$$Var(X) \quad = \quad E[X^2] = \sum_x x^2 P_X(x) = (-2)^2 \cdot 6/14 + 2^2 \cdot 6/14 = 24/7 \tag{2.17}$$

$$Var(Y) \quad = \quad E[Y^2] = \sum_y y^2 P_Y(y) = (-1)^2 \cdot 5/14 + 1^2 \cdot 5/14 = 5/7 \tag{2.18}$$

So $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.

4.4.1/5.4.1 Let's first make a picture:



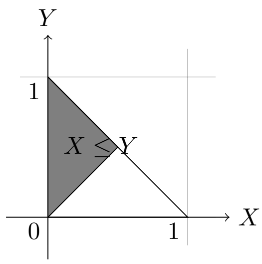
With: $f_{X,Y} = \begin{cases} c & x + y \leq 1,\ x \leq 0,\ y \leq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) The integral over the shaded area should become one:

$$\int f_{X,Y}(x,y)dxdy = \int_0^1 \int_0^{1-x} c dy dx = \int_0^1 [cy]_0^{1-x} dx = c \int_0^1 (1-x) dx = c \left[x - \frac{1}{2}x^2\right]_0^1 = c/2 = 1 \tag{2.19}$$

and therefore $c = 2$.

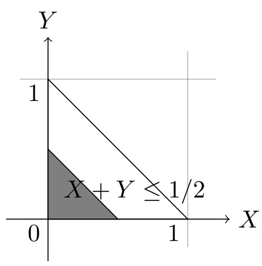
(b) For $P[X \leq Y]$ we have to integrate over the following area:



That means (first integrating over y, then over x):

$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} c dy dx = c \int_0^{1/2} [cy]_x^{1-x} dx = c \int_0^{1/2} (1-2x) dx = c \left[x - x^2\right]_0^{1/2} = 1/2$$

(c) For $P[X+Y \leq 1/2]$ we have to integrate over the following area:



That means (first integrating over y, then over x):

$$P[X \leq Y] = \int_0^{1/2} \int_0^{1/2-x} c dy dx = c \int_0^{1/2} [cy]_0^{1/2-x} dx = c \int_0^{1/2} (1/2-x) dx = c \left[x/2 - x^2/2\right]_0^{1/2} = 1/4$$

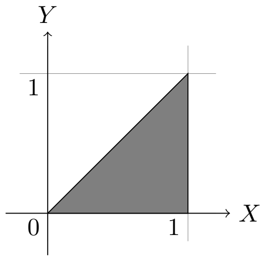
4.5.2/5.5.3 (a) This joint pdf is the same as in question 4.4.1. The marginals are computed using the definition:

$$f_X(x) = \int_y f_{X,Y}(x,y)dy = \int_0^{1-x} 2dy = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

(b) And...

$$f_Y(y) = \int_x f_{X,Y}(x,y)dx = \int_0^{1-y} 2dx = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.21}$$

4.5.6/5.5.9 (a) As always, first make a picture:

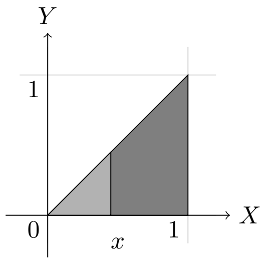


(b) The constant c is determined by the fact that the pdf should integrate to 1. I decide to first integrate over y for a given value of x. For a given value of x variable y runs between 0 and x. The second integral is then over x, between 0 and 1:

$$\int_0^1 \int_0^x cydydx = \int_0^1 \frac{1}{2}cx^2dx = \left[\frac{1}{6}cx^3\right]_0^1 = \frac{1}{6}c = 1 \tag{2.22}$$

And therefore $c = 6$.

(c) To compute the cdf $F_X(x) = P[X \le x]$ we have to integrate over the nonzero probability region left of the vertical line at x:



When we choose $x < 0$ this integral is 0, and if we choose $x > 1$ then this integral is 1. We now only have to worry for situations where $0 \le x \le 1$.

I introduce two integration variables u (over X) and v (over Y), and the integral becomes:

$$F_X(x) = \int_0^x \int_0^u cv\,dv\,du \tag{2.23}$$

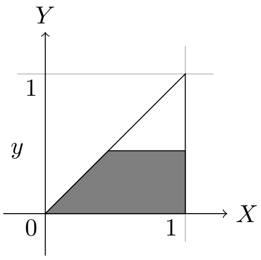
$$= \int_0^x \frac{1}{2}cu^2du \tag{2.24}$$

$$= \frac{1}{6}cx^3 = x^3 \tag{2.25}$$

So, in total the cdf becomes:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{2.26}$$

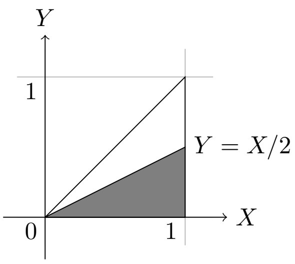
(d) To compute the cdf $F_Y(y)$, we have to integrate over the nonzero probability region below the horizontal line at y. When we choose $y < 0$ this integration becomes 0, and when we choose $y > 1$, this integration becomes 1. When we have $0 \leq y \leq 1$ we have to make the integration over the following region:



Again, I introduce two integration variables u (over X) and v (over Y). I decide to integrate first in the Y direction, then over the X direction:

$$
\begin{aligned}
F_Y(y) &= \int_0^y \int_v^1 cv\, du\, dv && (2.27)\\
&= \int_0^y [cvu]_v^1\, dv && (2.28)\\
&= \int_0^y cv(1-v)dv && (2.29)\\
&= c\left[\frac{1}{2}v^2 - \frac{1}{3}v^3\right]_0^y = c\left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right) && (2.30)
\end{aligned}
$$

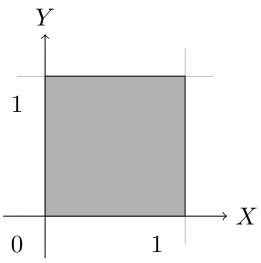
(e) To compute $P(Y \leq X/2)$ we first have to find the event. It is indicated in the figure:



I integrate first over Y, then over X:

$$
P[Y \leq X/2] = \int_0^1 \int_0^{x/2} cy\, dy\, dx = \int_0^1 \left[\frac{1}{2}cy^2\right]_0^{x/2} dx = \int_0^1 \frac{1}{8}cx^2\, dx = \left[\frac{1}{24}cx^3\right]_0^1 = \frac{1}{24}c \tag{2.31}
$$

4.7.9/5.7.12 Here we go again. First a picture (well, a bit overdone, it is quite simple):



(a) For the computation of $E[X] = \int x f_X(x)dx$ and $Var[X]$ we need the marginal distribution, so let's do that first:

$$f_X(x) = \int f_{X,Y}(x,y)dy = \int_0^1 4xydy = \left[2xy^2\right]_0^1 = 2x \quad \text{for } 0 \leq x \leq 1$$

The expected value for X is

$$E[X] = \int x f_X(x)dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3} \tag{2.32}$$

and

$$E[X^2] = \int x^2 f_X(x)dx = \int_0^1 2x^3 dx = \left[2x^4\right]_0^1 = \frac{1}{2} \tag{2.33}$$

The variance therefore becomes

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} \tag{2.34}$$

(b) When we compute the marginal pdf of Y:

$$f_Y(y) = \int f_{X,Y}(x,y)dx = \int_0^1 4xydx = \left[2yx^2\right]_0^1 = 2y \quad \text{for } 0 \leq y \leq 1$$

we see that it is exactly the same as the one for X. So also the expected value and variance is the same! The expected value for Y is

$$E[Y] = \frac{2}{3}, \qquad Var[Y] = \frac{1}{18} \tag{2.35}$$

(c) The covariance is defined as $Cov[X,Y] = E[XY] - E[X]E[Y]$, so we need the correlation first:

$$\begin{aligned} E[XY] &= \int xy f_{X,Y}(x,y)dxdy = \int_0^1 \int_0^1 4x^2y^2 dydx && (2.36)\\ &= \int_0^1 4y^2 \left[\frac{1}{3}x^3\right]_0^1 dy && (2.37)\\ &= \frac{4}{3}\left[\frac{1}{3}y^3\right]_0^1 = \frac{4}{9} && (2.38) \end{aligned}$$

so the covariance:

$$Cov[X,Y] = \frac{4}{9} - \frac{2}{3} \cdot \frac{2}{3} = 0 \tag{2.39}$$

(d) Well, this is simply $E[X+Y] = E[X] + E[Y] = 4/3$.

(e) Use Theorem 4.15,

$$Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y] = \frac{1}{18} + \frac{1}{18} + 0 = \frac{1}{9} \tag{2.40}$$