Week 1

Refresher random variables

1.4.1/1.5.1 For the probability $P[H_0]$ that a phone makes no hand-offs, we have to find all (disjoint) events that contain H_0. These are LH_0 and BH_0, so:

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5 \tag{1.1}$$

Similarly, we have for a brief call

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6 \tag{1.2}$$

Finally, the probability that a call is long or makes at least two hand-offs:

$$\begin{aligned} P[L \cup H_2] &= P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] & (1.3) \\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5 & (1.4) \end{aligned}$$

1.5.1/1.4.1 (a) The probability for a brief call is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6 \tag{1.5}$$

Now the probability that a brief call has no hand-offs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3} \tag{1.6}$$

(b) The probability of one hand-off is

$$P[H_1] = P[H_1B] + P[H_1L] = 0.2 \tag{1.7}$$

The probability that a call with one handoff is long, is:

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2} \tag{1.8}$$

(c) The probability of a long call is:

$$P[L] = P[H_0L] + P[H_1L] + P[H_2L] = 0.4 \tag{1.9}$$

The probability that a long call has one or more handoffs is:

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4} \tag{1.10}$$

1.5.5/1.4.7 First, the sample space is $\{234, 243, 324, 342, 423, 432\}$. Each of the outcomes are equally likely, i.e. 1/6. The events are:

$$E_1 = \{234, 243, 423, 432\} \quad E_2 = \{243, 324, 342, 423\} \quad E_3 = \{234, 324, 342, 432\} \tag{1.11}$$
$$O_1 = \{324, 342\} \quad O_2 = \{234, 432\} \quad O_3 = \{243, 423\} \tag{1.12}$$

Then the rest becomes straighforward:

(a) The cond. prob. that the second card is even, given the first is even:

$$P[E_2|E_1] = \frac{P[E_1E_2]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = \frac{1}{2} \tag{1.13}$$

(b) The prob. that the first two cards are even, given the third is even:

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = \frac{0}{\text{something}} = 0 \tag{1.14}$$

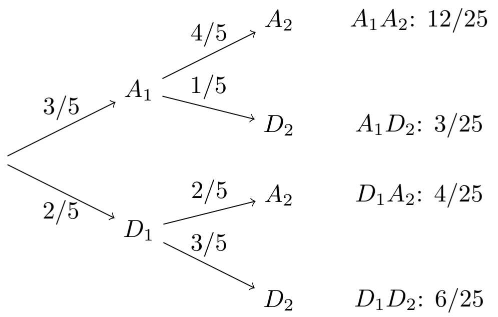
(c) The cond. prob. that the second card is even given the first card is odd:

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1 \tag{1.15}$$

(d) The cond. prob. that the second card is odd given the first card is odd:

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0 \tag{1.16}$$

1.7.6/2.1.6 A photodetector can be acceptable (A) or defective (D). Because the two detectors are tested in sequence, we can make the following tree:



(a) To find the probability of one acceptable photodetector, we look at the tree and find:

$$P[E_1] = P[A_1D_2] + P[D_1A_2] = 3/25 + 4/25 = 7/25 \tag{1.17}$$

(b) The probability that both photodetectors are defective is $P[D_1d_2] = 6/25$.

2.4.3/3.4.3 (a) The CDF looks like

(b) The pmf looks like

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

2.5.5/3.5.8 When we have the PMF

$$P_x(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise.} \end{cases} \tag{1.19}$$

we can compute the expected value using the definition:

$$E[X] = \sum_x x P_x(x) = -3 \cdot 0.4 + 5 \cdot 0.4 + 7 \cdot 0.2 = 2.2 \tag{1.20}$$

2.8.4/3.8.4 Again, we have the PMF as in (1.18). The expected value was computed in question 2.5.5: $E[X] = 2.2$. The expected value of X^2 is

$$E[X^2] = \sum_x x^2 P_X(x) = (-3)^2 \cdot 0.4 + 5^2 \cdot 0.4 + 7^2 \cdot 0.2 = 23.4 \tag{1.21}$$

Therefore the variance becomes:

$$Var[X] = E[X^2] - E[X]^2 = 23.4 - 2.2^2 = 18.6 \tag{1.22}$$

2.9.3/- To compute $E[X|B]$ we need the probability for event B:

$$P[B] = P[X > 0] = P_X(5) + P_X(7) = 0.6 \tag{1.23}$$

Using the definition of the conditional probability:

$$P_{X|B} = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1.24}$$

To find the variance, we first have to compute $E[X^2|B]$:

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = 5^2 \cdot 2/3 + 7^2 \cdot 1/3 = 33 \tag{1.25}$$

and $E[X|B]$:

$$E[X|B] = \sum_x x P_{X|B}(x) = 5 \cdot 2/3 + 7 \cdot 1/3 = 17/3 \tag{1.26}$$

therefore

$$Var[X|B] = E[X^2|B] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \tag{1.27}$$

3.4.5/4.5.10 (a) Using the definition from Appendix A from the book:

$$f_X(x) = \begin{cases} 1/10 & -5 < x < 5 \\ 0 & \text{otherwise} \end{cases} \tag{1.28}$$

(b) To find the CDF, we have to integrate. For $x \leq -5$ $F_X(x) = 0$, and for $x \geq 5$ we get that $F_X(x) = 1$. For $-5 < x < 5$ we obtain:

$$F_X(x) = \int_{-5}^{x} f_X(u)du = \frac{x+5}{10} \tag{1.29}$$

The CDF is therefore:

$$F_X(x) = \begin{cases} 0 & x \leq -5 \\ (x+5)/10 & -5 < x < 5 \\ 1 & x \geq 5 \end{cases} \tag{1.30}$$

(c) To compute the expected value:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-5}^{5} \frac{x}{10}dx = \left[\frac{x^2}{20}\right]_{-5}^{5} = 0 \tag{1.31}$$

(d) Similarly,

$$E[X^5] = \int_{-\infty}^{\infty} x^5 f_X(x)dx = \int_{-5}^{5} \frac{x^5}{10}dx = \left[\frac{x^6}{60}\right]_{-5}^{5} = 0 \tag{1.32}$$

(e) Similarly,

$$E[e^X] = \int_{-\infty}^{\infty} e^x f_X(x)dx = \int_{-5}^{5} \frac{e^x}{10}dx = \left[\frac{e^x}{10}\right]_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.8 \tag{1.33}$$