Metropolis-Hastings algorithm

Goal sample from π

We choose a Markov chain with transition probabilities q. Define a new Markov chain which evolves from $\theta = \theta_n$ to θ_{n+1} as follows

1) Propose θ° from $q(\theta, \ )$
2) Compute $\alpha(\theta, \theta^\circ) = \min\left(1, \dfrac{\pi(\theta^\circ)\, q(\theta^\circ, \theta)}{\pi(\theta)\, q(\theta, \theta^\circ)}\right)$

3) Set $\theta_{n+1} = \begin{cases} \theta^{\circ} & \text{with prob } \alpha(\theta, \theta^{\circ}) \\ \theta & \text{with prob } 1-\alpha(\theta, \theta^{\circ}) \end{cases}$

Only need to know π up to multiplicative constant

Special case Gibbs sampler

Let θ_{-i} be θ with i-th component removed

Suppose we can sample from $\pi(\theta_i \mid \theta_{-i})$ for all $i \in \{1, \dots, n\}$

Let $\theta^0 = (\theta_1, \theta_2, \dots, \theta_{i-1}, \theta_i^0, \theta_{i+1}, \dots, \theta_n)$

Then $\theta^0_{-i} = \theta_{-i}$

Set $q(\theta, \theta^\circ) = \pi(\theta^\circ_{\cdot} \mid \theta_{-\cdot})$

$q(\theta^\circ, \theta) = \pi(\theta_{\cdot} \mid \theta^\circ_{-\cdot}) = \pi(\theta_{\cdot} \mid \theta_{-\cdot})$

$$\frac{\pi(\theta^\circ)\, q(\theta^\circ, \theta)}{\pi(\theta)\, q(\theta, \theta^\circ)} = \frac{\pi(\theta^\circ)\, \pi(\theta_{\cdot} \mid \theta_{-\cdot})}{\pi(\theta)\, \pi(\theta^\circ_{\cdot} \mid \theta_{-\cdot})}$$

$$= \frac{\pi(\theta^\circ)\, \pi(\theta_{\cdot} \mid \theta_{-\cdot})\, \pi(\theta_{-\cdot})}{\pi(\theta)\, \pi(\theta^\circ_{\cdot} \mid \theta_{-\cdot})\, \pi(\theta_{-\cdot})} = \frac{\pi(\theta^\circ)\, \pi(\theta)}{\pi(\theta)\, \pi(\theta^\circ)} = 1$$

and so $\alpha(\theta, \theta^\circ) = 1$ We always accept

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B)$$

partially conjugate prior

A prior is partially conjugate if the conditional prior and conditional posterior belong to the same family of distributions

$$\underbrace{p(y \mid \theta, \tau)}_{p(y \mid \theta)} \; p(\theta \mid \tau)\, p(\tau) \sim p(\theta \mid \tau)\, p(\tau)$$

const wrt τ