Week 5

Using the autocorrelation function

11.2.1/2.1 (a) The impuls response becomes obvious when you rewrite it:

$$Y_n = \frac{X_{n+1} + X_n + X_{n-1}}{3} = \frac{X_{n+1}}{3} + \frac{X_n}{3} + \frac{X_{n-1}}{3} \tag{5.1}$$

So:

$$h_n = \begin{cases} \frac{1}{3} & n = -1, 0, 1, \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

(b) Be strong!:

$$\begin{aligned}
R_Y[n,k] &= E[Y(n)Y(n+k)] && (5.3)\\
&= E[(\frac{1}{3}X_{n+1} + \frac{1}{3}X_n + \frac{1}{3}X_{n-1})(\frac{1}{3}X_{n+k+1} + \frac{1}{3}X_{n+k} + \frac{1}{3}X_{n+k-1})] && (5.4)\\
&= E[\frac{1}{9}X_{n+1}X_{n+k+1} + \frac{1}{9}X_{n+1}X_{n+k} + \frac{1}{9}X_{n+1}X_{n+k-1}] \\
&\quad +E[\frac{1}{9}X_nX_{n+k+1} + \frac{1}{9}X_nX_{n+k} + \frac{1}{9}X_nX_{n+k-1}] \\
&\quad +E[\frac{1}{9}X_{n-1}X_{n+k+1} + \frac{1}{9}X_{n-1}X_{n+k} + \frac{1}{9}X_{n-1}X_{n+k-1}] && (5.5)\\
&= \frac{1}{9}(E[X_nX_{n+k}] + E[X_nX_{n+k-1}] + E[X_nX_{n+k-2}] \\
&\quad +E[X_nX_{n+k+1}] + E[X_nX_{n+k}] + E[X_nX_{n+k-1}] \\
&\quad +E[X_nX_{n+k+2}] + E[X_nX_{n+k+1}] + E[X_nX_{n+k}]) && (5.6)\\
&= \frac{1}{9}(R_X(k) + R_X(k-1) + R_X(k-2) \\
&\quad +R_X(k+1) + R_X(k) + R_X(k-1) \\
&\quad +R_X(k+2) + R_X(k+1) + R_X(k) && (5.7)\\
&= \frac{1}{9}\left(3R_X(k) + 2R_X(k-1) + 2R_X(k+1) + R_X(k-2) + R_X(k+2)\right) \\
&= \begin{cases} \frac{1}{3} & k = 0 \\ \frac{2}{9} & |k| = 1 \\ \frac{1}{9} & |k| = 2 \\ 0 & \text{otherwise} \end{cases} && (5.8)
\end{aligned}$$

11.2.3/2.3 Note that there is an error/inconsistency in the question! It is given that $\mu_Y = 1$, but that contradicts $\lim_{n\to\infty} R_Y[n] = \mu_Y^2$. The function that is defined is *not* the autocorrelation, but the autocovariance function:

$$C_Y[n] = \begin{cases} 3 & n=0 \\ 2 & |n|=1 \\ 0.5 & |n|=2 \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

Therefore the autocorrelation function becomes $C_Y[n] = R_Y[n] - \mu_Y^2$:

$$R_Y[n] = \begin{cases} 4 & n=0 \\ 3 & |n|=1 \\ 1.5 & |n|=2 \\ 1 & \text{otherwise} \end{cases} \tag{5.10}$$

(a) Using Theorem 11.5:

$$\mu_W = \mu_Y \sum_n h_n = 2\mu_Y = 2 \tag{5.11}$$

(b) The autocorrelation function of a filtered signal is:

$$\begin{aligned} R_W[n] &= \sum_i \sum_j h_i h_j R_Y[n+i-j] && (5.12) \\ &= \sum_{i=0}^{1} \sum_{j=0}^{1} R_y[n+i-j] && (5.13) \\ &= R_Y[n-1] + R_Y[n] + R_Y[n] + R_Y[n+1] && (5.14) \end{aligned}$$

Now we try for different values of n:

$$R_W[n] = \begin{cases} R_Y[-1] + 2R_Y[0] + R_Y[1], & n=0 \\ R_Y[0] + 2R_Y[1] + R_Y[2], & n=1 \\ R_Y[-2] + 2R_Y[-1] + R_Y[0], & n=-1 \\ R_Y[1] + 2R_Y[2] + R_Y[3], & n=2 \\ R_Y[-3] + 2R_Y[-2] + R_Y[-1], & n=-2 \\ R_Y[2] + 2R_Y[3] + R_Y[4], & n=3 \\ R_Y[-4] + 2R_Y[-3] + R_Y[-2], & n=-3 \\ R_Y[3] + 2R_Y[4] + R_Y[5], & n=4 \\ \ldots \end{cases} = \begin{cases} 14, & n=0 \\ 11.5, & n=1 \\ 11.5, & n=-1 \\ 7, & n=2 \\ 7, & n=-2 \\ 4.5, & n=3 \\ 4.5, & n=-3 \\ 4, & n=4 \\ \ldots \end{cases} = \begin{cases} 14 & n=0 \\ 11.5 & |n|=1 \\ 7 & |n|=2 \\ 4.5 & |n|=3 \\ 4 & \text{otherwise} \end{cases} \tag{5.15}$$

(c) When we have the autocorrelation function, the variance is just:

$$Var[W_n] = E[W_n^2] - E[W_n]^2 = 14 - 2^2 = 10 \tag{5.16}$$

11.2.8/2.8 Note that in the definition of $Y_n = a(X_n + Y_{n-1})$ there appears also a Y_{n-1} on the right side of the equation. We have a recursive definition. Using the definition of Y_n we expand:

$$\begin{aligned} Y_n &= aX_n + aY_{n-1} \\ &= aX_n + a(aX_{n-1} + aY_{n_2}) \\ &= aX_n + a^2X_{n-1} + a^2(aX_{n-2} + aY_{n-3}) \\ &= \sum_{i=0}^{n} a^{i+1}X_{n-i} + a^nY_0 \\ &= \sum_{i=0}^{n} a^{i+1}X_{n-i} && (5.17) \end{aligned}$$

Because we are looking at standard normal distributed X_n we know that $E[X_n] = 0$.

$$E[Y_n] = E[\sum_{i=0}^{n} a^{i+1} X_{n-i}] = \sum_{i=0}^{n} a^{i+1} E[X_{n-i}] = 0 \tag{5.18}$$

To find the autocorrelation function

$$R_Y[m,k] = E\left[(\sum_{i=0}^{m} a^{i+1} X_{m-i})(\sum_{j=0}^{m+k} a^{j+1} X_{m+k-j}) \right] \tag{5.19}$$

we first note that X_n is iid, with a variance of 1, so

$$E[X_i X_j] = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \tag{5.20}$$

So only the terms in (5.19) survive for which the indices in X_{m-i} and X_{m+k-j} are equal. In the current notation it is not so easy, so I re-index:

$$i' = m - i, \qquad \text{and therefore} \quad i + 1 = m + 1 - i' \tag{5.21}$$

$$j' = m + k - j, \qquad \text{and therefore} \quad j + 1 = m + k + 1 - j' \tag{5.22}$$

Using these indices we get (where we assumed $k \geq 0$ for now):

$$\begin{aligned} R_Y[m,k] &= E\left[(\sum_{i'=0}^{m} a^{m+1-i'} X_{i'})(\sum_{j'=0}^{m+k} a^{m+k+1-j'} X_{j'}) \right] \\ &= \sum_{i'=0}^{m} \sum_{j'=0}^{m+k} a^{m+1-i'} a^{m+k+1-j'} E[X_{i'} X_{j'}] \\ &= \sum_{i'=0}^{m} a^{m+1-i'} a^{m+k+1-i'} E[X_{i'}^2] \qquad (5.23) \\ &= \sum_{i=0}^{m} a^{m+1-i} a^{m+k+1-i} \\ &= \sum_{i=0}^{m} a^{2m+2+k-2i} = a^{2m+2+k} \sum_{i=0}^{m} a^{-2i} = a^{2m+2+k} \sum_{i=0}^{m} (a^{-2})^i \qquad (5.24) \end{aligned}$$

Now we have to use Math Fact B.4, from which we can conclude that:

$$\sum_{i=0}^{m} (a^{-2})^i = \frac{1 - (a^{-2})^{m+1}}{1 - a^{-2}} \tag{5.25}$$

So therefore we found:

$$R_Y[m,k] = a^{2m+2+k} \frac{1 - (a^{-2})^{m+1}}{1 - a^{-2}}, \quad \text{for} \quad k \geq 0 \tag{5.26}$$

For $k < 0$ a very similar derivation can be given, only that the sum in (5.23) does not run to m, but just up to $m + k$ (which is smaller than m because $k < 0$. In this situation we obtain:

$$R_Y[m,k] = a^{2m+2+k} \frac{1 - (a^{-2})^{m+k+1}}{1 - a^{-2}}, \quad \text{for} \quad k < 0 \tag{5.27}$$

All in all, from (5.26) and (5.27) we see that R_Y actually depends on m, so Y_n is *not* WSS.

E2 Because X_n is iid, we already know that $E[X_m] = E[X_{m+k}]$.

Now to find the autocovariance function $C_Y[m,k]$, we use the defition again:

$$C_Y[m,k] = E[Y_m Y_{m+k}] - E[Y_m]E[Y_{m+k}] \tag{5.28}$$

First we compute:

$$E[Y_n] = E[X_{n+1}] + E[X_n] + E[X_{n-1}] = 0 \tag{5.29}$$

and second

$$\begin{aligned}
C_Y[m,k] &= E[Y_m Y_{m+k}] - E[Y_m]E[Y_{m+k}] = E[Y_m Y_{m+k}] && (5.30)\\
&= E[(X_{m+1} + X_m + X_{m-1})(X_{m+1+k} + X_{m+k} + X_{m-1+k})] && (5.31)\\
&= E[X_{m+1}X_{m+1+k}] + E[X_{m+1}X_{m+k}] + E[X_{m+1}X_{m+k-1}]\\
&\quad +E[X_m X_{m+1+k}] + E[X_m X_{m+k}] + E[X_m X_{m+k-1}]\\
&\quad +E[X_{m-1}X_{m+1+k}] + E[X_{m-1}X_{m+k}] + E[X_{m-1}X_{m+k-1}]\\
&= R_X(m+1,k) + R_X(m+1,k-1) + R_X(m+1,k-2)\\
&\quad +R_X(m,k+1) + R_X(m,k) + R_X(m,k-1)\\
&\quad +R_X(m-1,k+2) + R_X(m-1,k+1) + R_X(m-1,k)
\end{aligned}$$

Because X_n is stationary, it is WSS and $C_X[m,k]$ does not depend on m. Furthermore, because $E[X_m] = 0$ we also have that $C_X[k] = R_X[k]$. So we get:

$$\begin{aligned}
C_Y[m,k] &= R_X(k) + R_X(k-1) + R_X(k-2)\\
&\quad +R_X(k+1) + R_X(k) + R_X(k-1)\\
&\quad +R_X(k+2) + R_X(k+1) + R_X(k)\\
&= 3R_X(k) + 2R_X(k-1) + R_X(k-2) + 2R_X(k+1) + R_X(k+2) && (5.32)
\end{aligned}$$

Now we fill in various values for k:

$$\begin{aligned}
k = -3 &\rightarrow C_Y(-3) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 0\\
k = -2 &\rightarrow C_Y(-2) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 1 = 1\\
k = -1 &\rightarrow C_Y(-1) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 1 + 0 = 2\\
k = 0 &\rightarrow C_Y(0) = 3\cdot 1 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 3\\
k = 1 &\rightarrow C_Y(1) = 3\cdot 0 + 2\cdot 1 + 0 + 2\cdot 0 + 0 = 2\\
k = 2 &\rightarrow C_Y(2) = 3\cdot 0 + 2\cdot 0 + 1 + 2\cdot 0 + 0 = 1\\
k = 3 &\rightarrow C_Y(2) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 0
\end{aligned}$$

So in total:

$$C_Y[m,k] = \begin{cases} 3 & k = 0\\ 2 & |k| = 1\\ 1 & |k| = 2\\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

E3 Again, because X_n is iid, it is stationary and therefore also WSS. That means that $E[X_n] = E[X_{n+k}] = 0$ (given) and that

$$C_X[m,k] = C_X[k] = R_X[k] - E[X_m]E[X_{m+k}] = R_X[k] = \begin{cases} \sigma^2 & k = 0\\ 0 & \text{otherwise} \end{cases} \tag{5.34}$$

(a) Just fill in:

$$E[Y_n] = \frac{1}{2}E[X_n + Y_{n-1}] = \frac{1}{2}E[X_n] + \frac{1}{2}E[Y_{n-1}] = 0 + \frac{1}{2}E[Y_{n-1}] \tag{5.35}$$

Now we have $E[Y]$ on both sides of the equation. Realise that X_n is a wide-sense stationary process (it is actually iid!), and that the filter is a linear, time-invariant filter. So the output Y_n should also be WSS, and $E[Y_n]$ should not depend on n. The only solution is

$$E[Y_n] = 0 \tag{5.36}$$

(b) Because X_n are iid, it is most efficient to rewrite Var using Theorem 4.15, pg 173:

$$\begin{aligned}
Var[Y_n] &= Var[\frac{1}{2}(X_n + Y_{n-1})] \\
&= Var[\frac{1}{2}(X_n + \frac{1}{2}X_{n-1} + \frac{1}{2}Y_{n-2})] \\
&= Var[\frac{1}{2}(X_n + \frac{1}{2}X_{n-1} + \frac{1}{4}X_{n-2} + \frac{1}{4}Y_{n-3})] \\
&= Var[\frac{1}{2}X_n + \frac{1}{4}X_{n-1} + \frac{1}{8}X_{n-2} + ...] \\
&= \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + ...\right] Var[X_n] \qquad (5.37)\\
&= \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i \sigma^2 = \left[\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i - 1\right] \sigma^2 \qquad (5.38)
\end{aligned}$$

This series we can find in the book. Using Math Fact B.5

$$\left[\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i - 1\right] \sigma^2 = \left(\frac{1}{1-1/4} - 1\right) \sigma^2 = \left(\frac{4}{3} - 1\right) \sigma^2 = \sigma^2/3 \tag{5.39}$$

(c) For the covariance, we expand Y_n similarly as in (b). In the second step, we use that X_n is iid, and therefore $E[X_n X_{n+k}] = 0$ for $k \neq 0$:

$$\begin{aligned}
Cov[Y_{n+1}, Y_n] &= E[(\frac{1}{2}X_n + \frac{1}{4}X_{n-1} + \frac{1}{8}X_{n-2} + ...)(\frac{1}{2}X_{n-1} + \frac{1}{4}X_{n-2} + \frac{1}{8}X_{n-3} + ...)] \\
&= \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \frac{1}{2^i} E[X_i^2] = \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \frac{1}{2^{i+1}} E[X_i^2] \qquad (5.40)\\
&= \frac{1}{8} \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{1}{2^i} E[X_i^2] = \frac{1}{8} \sum_{i=0}^{\infty} \frac{1}{4^i} E[X_i^2] \qquad (5.41)\\
&= \frac{1}{8} \frac{1}{1-1/4} E[X_i^2] = \sigma^2/6 \qquad (5.42)
\end{aligned}$$

(d) We now have to combine previous results:

$$\rho_{Y_{n+1},Y_n} = \frac{Cov[Y_{n+1}, Y_n]}{\sqrt{Var[Y_{n+1}]Var[Y_n]}} = \frac{\sigma^2/6}{\sigma^2/3} = \frac{1}{2} \tag{5.43}$$

11.5.1/5.1 With the use of Table 11.1, pg 413:

$$\begin{aligned}
S_X(f) &= \int R_X(\tau) e^{-j2\pi f\tau} d\tau \\
&= 10 \int \frac{\sin 2000\pi\tau}{2000\pi\tau} e^{-j2\pi f\tau} d\tau + \frac{10}{2} \int \frac{\sin 1000\pi\tau}{1000\pi\tau} e^{-j2\pi f\tau} d\tau \\
&= 10 \frac{1}{2000} \text{rect}\left(\frac{f}{2000}\right) + 5 \frac{1}{1000} \text{rect}\left(\frac{f}{1000}\right) \qquad (5.44)
\end{aligned}$$

11.8.2/8.2 (a) Using Table 11.1, pg 413, we see that the inverse Fourier transform of $S_W(f) = 1$ is $R_W(\tau) = \delta(\tau)$.

(b) Now we can take advantage of the Fourier transform:

$$S_Y(f) = |H(f)|^2 S_W(f) = |H(f)|^2 \tag{5.45}$$

(where $H(f)$ is given in the exercise).

(c) Use the definition:

$$E[Y^2(t)] = \int S_Y(f)df = \int_{-B/2}^{B/2} df = B \tag{5.46}$$

(d)

$$E[Y(t)] = E[W(t)]H(0) = 0 \tag{5.47}$$

11.8.5/8.5 (a) The power of a signal can directly be computed using Theorem 11.13:

$$E[X^2(t)] = \int S_X(f)df = \int_{-100}^{100} 1 \cdot 10^{-4} df = 0.02 \tag{5.48}$$

(b) Because we now have everything in the Fourier domain:

$$S_{XY}(f) = H(f)S_X(f) = \begin{cases} \frac{10^{-4}}{100\pi j 2\pi f} & |f| \leq 100 \\ 0 & \text{otherwise} \end{cases} \tag{5.49}$$

(c) Swapping X and Y in $S_{XY}(f)$ means that you swap the X and Y in $R_{XY}(\tau)$. From Theorem 10.14, pg 382, we see that $R_{XY}(\tau) = R_{YX}(-\tau)$. So when we fill this in, in the definition of S_{YX}

$$S_{YX}(f) = \int R_{YX}(\tau)e^{-j2\pi f\tau}d\tau = \int R_{XY}(-\tau)e^{-j2\pi f\tau}d\tau = S^*_{XY}(f) \tag{5.50}$$

where in the last step Table 11.1, pg 413 is used (to find the transform of $g(-\tau)$).

(d)

$$S_Y(f) = H^*(f)S_{XY}(f) = |H(f)|^2 S_X(f) = \begin{cases} \frac{10^4}{10^4\pi^2 + (2\pi f)^2} & |f| \leq 100 \\ 0 & \text{otherwise} \end{cases} \tag{5.51}$$

(e) Just compute the integral:

$$\begin{aligned}
E[Y^2(t)] &= \int S_Y(f)df = \int_{-100}^{100} \frac{10^{-4}}{10^4\pi^2 + 4\pi^2 f^2} df \\
&= \frac{10^{-4}}{\pi^2} \int_{-100}^{100} \frac{1}{10^4 + 4f^2} df \\
&= \frac{10^{-8}}{\pi^2} \int_{-100}^{100} \frac{1}{1 + (0.02f)^2} df \\
&= \frac{10^{-8}}{0.02\pi^2} \int_{-100}^{100} \frac{1}{1 + (0.02f)^2} d(0.02f) \\
&= \frac{10^{-8}}{0.02\pi^2} \left(\tan^{-1}(0.02 \cdot 100) - \tan^{-1}(0.02 \cdot -100)\right) \\
&= \frac{10^{-8}}{0.02\pi^2} 2\tan^{-1}(2) = 1.12 \cdot 10^{-5}
\end{aligned}$$