Week 6

Statistical estimation

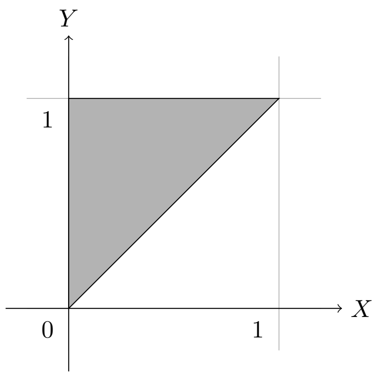
6.1.5/- First note that $C_X[m,k] = R_X[m,k] - E[X_m]E[X_{m+k}] = R_X[m,k]$. Then:

$$E[Y_n] = E[X_n/3 + X_{n-1}/3 + X_{n-2}/3] = 0 \tag{6.1}$$

For the variance:

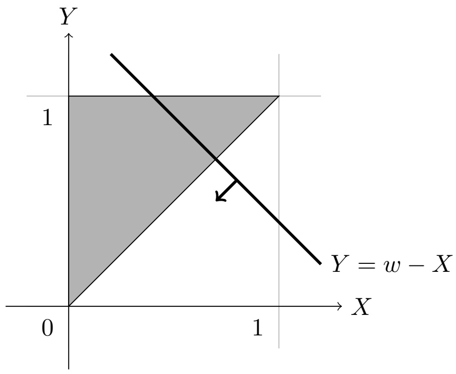
$$\begin{aligned}
Var[Y_n] &= E[Y_n^2] - E[Y_n]^2 = E[Y_n^2] \\
&= \frac{1}{9}E[(X_n + X_{n-1} + X_{n-2})^2] \\
&= \frac{1}{9}E[X_n^2 + X_{n-1}^2 + X_{n-2}^2 + 2X_nX_{n-1} + 2X_nX_{n-2} + 2X_{n-1}X_{n-2}] \\
&= \frac{1}{9}[3R_X(0) + 2R_X(1) + 2R_X(2) + 2R_X(1)] \\
&= \frac{3 + 2/4 + 2/4}{9} = \frac{4}{9}
\end{aligned} \tag{6.2}$$

6.2.1/- First a picture of the region where the pdf is defined:

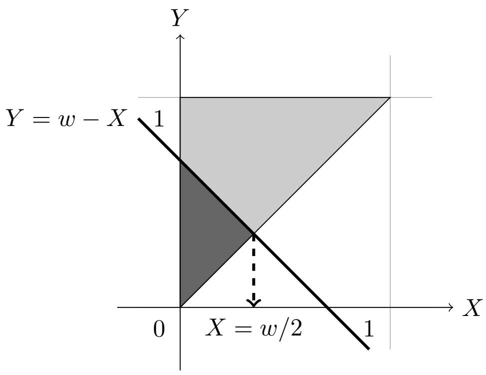


The standard procedure is now to derive the cumulative distribution function $F_W(w)$ for W, and then take the derivative. To find the CDF, we have to find the integration area.

Ok, what values $W = X + Y$ can take? The minimum value for $W = 0$ at $X = 0, Y = 0$. The maximum value is $W = 2$ for $X = Y = 1$. On the diagonal line $Y = 1 - X$ we get $W = 1$. So it seems we have to integrate over the region below the diagonal line $Y = w - X$:



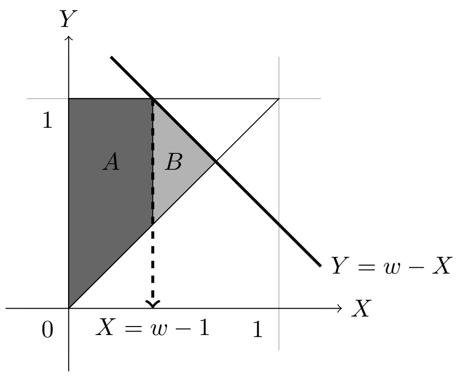
The integration therefore falls apart in two parts, one for $0 \leq w \leq 1$ and another one for $1 < w \leq 2$.



Starting with $0 \leq w \leq 1$, we first integrate over Y, and then over X. The integration of Y runs between x and $w - x$. The integration over X runs until we reach the point that $Y = w - X = X$, or when $x = w/2$:

$$\begin{aligned}
F_W(w) &= \int_0^{w/2} \int_x^{w-x} 2dydx \\
&= \in_0^{w/2} [2y]_x^{w-x}\, dx = \int_0^{w/2} (2w - 4x)dx \\
&= \left[2wx - 2x^2\right]_0^{w/2} = w^2 - w^2/2 = w^2/2 \qquad (6.3)
\end{aligned}$$

The second integration is harder, and we have to split the region in two parts, A and B:



Integrating first over Y and then over X:

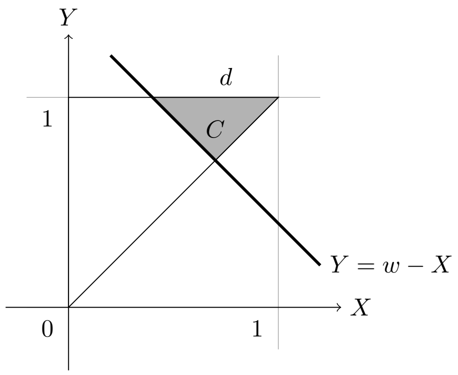
$$\begin{aligned}
F_W(w) &= A + B = \int_0^{w-1}\int_x^1 2dydx + \int_{w-1}^{w/2}\int_x^{w-x} 2dydx \\
&= \int_0^{w-1} [2y]_x^1 \, dx + \int_{w-1}^{w/2} [2y]_x^{w-x} \, dx \\
&= 2\int_0^{w-1} (1-x)dx + 2\int_{w-1}^{w/2} (w-2x)dx \\
&= 2\left[x - \frac{1}{2}x^2\right]_0^{w-1} + 2\left[wx - x^2\right]_{w-1}^{w/2} \\
&= 2(w-1-\frac{1}{2}(w-1)^2) + 2(w\cdot w/2 - (w/2)^2) - 2(w(w-1) - (w-1)^2) \\
&= -\frac{1}{2}w^2 + 2w - 1 \qquad (6.4)
\end{aligned}$$

So we finally found that:

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{2}w^2 & 0 \leq w < 1 \\ -\frac{1}{2}w^2 + 2w - 1 & 1 \leq w < 2 \\ 1 & w \geq 2 \end{cases} \qquad (6.5)$$

To find the pdf, we have to take the derivative with respect to w:

$$f_W(w) = \begin{cases} 0 & w < 0 \\ w & 0 \leq w < 1 \\ -w + 2 & 1 \leq w < 2 \\ 0 & w \geq 2 \end{cases} \qquad (6.6)$$



Note: from symmetry reasons, you could also have argued that

$$F_W(w) = 1 - \text{area}\, C \qquad (6.7)$$

And area C we computed above, that was $\frac{1}{2}w^2$. But beware, that instead of w, we should have used distance d! You can express d in terms of w by $d = 2 - w$, and therefore:

$$F_W(w) = 1 - \frac{1}{2}d^2 = 1 - (2-w)^2 = -1 + 2w - \frac{1}{2}w^2 \qquad (6.8)$$

6.6.1/9.4.1 We first have to read the question carefully. We have a random variable $X = W+R = W+3$, and on top of that $A = 12X = 12W + 36$.

(a) $E[X] = E[W + R] = E[W + 3] = 3 + E[W] = 3 + 5 = 8$.

(b) $Var[X] = Var[W + 3] = Var[W] = \frac{(10-0)^2}{12} = 100/12$.

(c) $E[A] = E[\sum_{i-1}^{12} X_i] = 12E[X] = 12 \cdot 8 = 96$.

(d) $\sigma_A = \sqrt{Var[A]} = \sqrt{12Var[X]} = \sqrt{12 \cdot 100/12} = 10$ because

$$Var[A] = Var[\sum_{i=1}^{12} X_i] = \sum_{i=1}^{12} Var[X_i] = 12Var[X] \tag{6.9}$$

(e) With a bit of goodwill, we may say that 12 repetitions are sufficient to use the central limit theorem:

$$P[A > 116] = 1 - \Phi(\frac{116 - 96}{10}) = 0.0228 \tag{6.10}$$

(where we have used the table on page 123 in the book to find the values for Φ).

(f)

$$P[A < 86] = \Phi(\frac{86 - 96}{10}) = 0.1587 \tag{6.11}$$

7.1.2/10.1.2 (a) It is given that X_i are uniform random variables. For the uniform random variable, you can find in the tables in the Appendix A in the book that:

$$\mu_X = E[X] = \frac{a + b}{2} = 7 \tag{6.12}$$

and

$$Var[X] = \frac{(b - a)^2}{12} = 3 \tag{6.13}$$

Now we have two equations, with two unknowns. Solving gives:

$$a = 4 \qquad b = 10 \tag{6.14}$$

Note that the pdf should integrate to one, and the constant should therefore be $1/(10 - 4)$. The pdf becomes:

$$f_X(x) = \begin{cases} \frac{1}{6} & 4 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{6.15}$$

(b) Realizing that we are looking at a sum of 16 random variables, we can use Theorem 7.1, and:

$$Var[M_{16}(X)] = \frac{Var[X]}{16} = \frac{3}{16} \tag{6.16}$$

(c)

$$P[X_1 \geq 9] = \int_9^\infty f_{X_1}(x)dx = \int_9^{10} \frac{1}{6}dx = \frac{1}{6} \tag{6.17}$$

(d) From question (b) we see that the variance of M_{16} is smaller than the variance of X_1. When the variance is smaller, it is less likely that realizations appear that are larger than 9. So I would expect $P[M_{16}(X) > 9]$ to be smaller than $P[X_1 > 9]$.

So, 16 looks like a large number, so we apply the central limit theorem. We have to rephrase the problem in terms of sums of random variables, and use Definition 6.2, pg 259.

$$P[M_{16} > 9] = P[(X_1+...+X_{16})/16 > 9] = P[X_1+...+X_{16} > 144] = 1-P[X_1+...+X_{16} \leq 144] \tag{6.18}$$

Then we apply

$$P[M_{16} > 9] = 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) = 1 - \Phi(4.61) = 2.01 \cdot 10^{-6} \tag{6.19}$$

So indeed, this is much smaller than we obtained in (c).

7.2.4/10.2.5 Let us first consider the number of dice rolls that we need to obtain the *first* snake eyes. At each roll of the two dice, we have a probability of $p = 1/6 \cdot 1/6$ of obtaining snake eyes. The number of rolls we need for the first snake eyes has therefore a geometric distribution, with a change of success of 1/36:

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{6.20}$$

From the appendix in the book we find that $E[X] = 1/p$ and $Var[X] = (1-p)/p^2$. Now we want to wait for the *third* occurrance of snake eyes. Because the dice are thrown independently, we can sum the number of rolls of three independent geometrical random variables:

$$R = X + Y + Z \tag{6.21}$$

where X, Y and Z are each random variables that indicate the number of rolls of finding snake eyes for the first time. Therefore:

$$E[R] = E[X] + E[Y] + E[Z] = 3 \cdot 1/p = 108 \tag{6.22}$$
$$Var[R] = Var[X] + Var[Y] + Var[Z] = 3 \cdot (1-p)/p^2 = 3680 \tag{6.23}$$

(a) The Markov inequality states:

$$P\left[X \geq c^2\right] \leq \frac{E[X]}{c^2} \tag{6.24}$$

For our situation, it means:

$$P\left[R \geq 250\right] \leq \frac{E[X]}{250} = \frac{108}{250} = 0.43 \tag{6.25}$$

(b) The Chebychev inequality states:

$$P[|Y - \mu_Y| \geq c] \leq \frac{Var[Y]}{c^2} \tag{6.26}$$

which means for us:

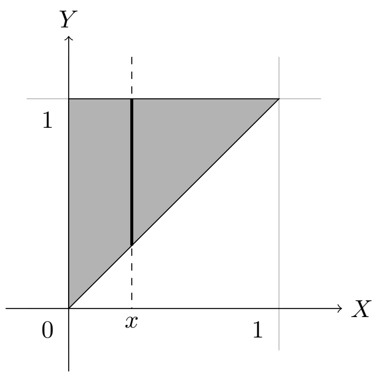
$$P[P \geq 250] \leq P[|R - 108| \geq 250 - 108] \leq \frac{3780}{142^2} = 0.19 \tag{6.27}$$

(c) The true distribution requires the convolution of the geometric distributions:

$$P_R(r) = P_X(x) * P_Y(y) * P_Z(z) \tag{6.28}$$

This may be a bit too much for a homework exercise, so we skip this...

9.1.2/12.1.2 (a) First we make a picture:



Then we use the definition, and we integrate over y:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_x^1 6(y-x)dy \\ &= \left[3y^2 - 6xy\right]_x^1 = 3x^2 - 6x + 3 \end{aligned} \tag{6.29}$$

(b)

$$\hat{x}_B = E[X] = \int_0^1 x(3x^2 - 6x + 3)dx = \left[\frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2\right]_0^1 = \frac{1}{4} \tag{6.30}$$

(c) Define event $A : X < 0.5$. The probability of this event is:

$$P[X < 0.5] = \int_0^{0.5} (3x^2 - 6x + 3)dx = [x^3 - 3x^2 + 3x]_0^{0.5} = \frac{7}{8} \tag{6.31}$$

Then the minimum MSE estimate becomes:

$$\hat{X}_A = E[X|A] = \frac{8}{7}\int_0^{0.5} x(3x^2 - 6x + 3)dx = \frac{8}{7}\left[\frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2\right]_0^{0.5} = \frac{11}{56} \tag{6.32}$$

(d) Again the definition, but now integrate over x:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} = \begin{cases} \int_0^y 6(y-x)dx, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \left[6xy - 3x^2\right]_0^y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 3y^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{6.33}$$

(e) Again:

$$\hat{y}_B = E[Y] = \int_0^1 y \cdot 3y^2 dy = \left[\frac{3}{4}y^4\right]_0^1 = \frac{3}{4} \tag{6.34}$$

(f) Finally, the event $Y > 0.5$ has probability:

$$P[Y > 0.5] = \int_{0.5}^1 3y^2 dy = [y^3]_{0.5}^1 = 1 - 0.5^3 = \frac{7}{8} \tag{6.35}$$

This gives for our minimum MSE:

$$E[Y|Y > 0.5] = \int_{0.5}^1 y \cdot f_{Y|Y>0.5}(y)dy \tag{6.36}$$

$$= \int_{0.5}^1 y \cdot \frac{8}{7}3y^2\, dy \tag{6.37}$$

$$= \frac{24}{7}\int_{0.5}^1 4y^3 dy = \frac{24}{7}[y^4]_{0.5}^1 = \frac{6}{7}(1 - (0.5)^4) = \frac{3}{56} \tag{6.38}$$