Financial Derivatives

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Preamble

The course Financial Derivatives delves into the intricacies of derivative instruments, focusing on both fixed and conditional contracts, including forwards, futures, swaps, and options. It explores the utilization of derivatives within financial markets, particularly for hedging against various risks. Students will gain a comprehensive understanding of the essence and valuation of financial derivatives, along with detailed insights into the exchange trading mechanisms for futures and options.

Upon completing this course, the student will be able to:

- define financial derivatives and understand their specific characteristics
- explain the differences between fixed future contracts and options
- price financial derivatives
- suggest their use

References:

- HULL, John. Options, futures, and other derivatives. Global edition. Harlow: Pearson, 2018. ISBN 9781292212890.
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. xix, 597. ISBN 9781119381815.
- BLÜMKE, Andreas. How to invest in structured products : a guide for investors and investment advisors. Chichester: Wiley, 2009. xvi, 374. ISBN 9780470746790.

1 Introduction to Financial Derivatives

ℹ References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Define and explain the concept of derivatives in financial markets.
- Analyze the importance and role of derivatives in finance.
- Distinguish between derivative markets, including their structure and function.
- Compare and contrast forward contracts with contingent claims, highlighting their differences and uses.
- Identify the various types of derivatives.
- Discuss the applications of derivatives in hedging, speculation, and arbitrage.
- Evaluate criticisms and potential misuses of derivatives in financial markets.

1.1 What Are Derivatives?

1.1.1 Understanding Derivatives

- A derivative is a financial contract whose value depends on the performance of **an underlying asset**.
- This underlying asset, referred to as the underlying, may be a stock, currency, interest rate, commodity, or any other marketable item. Its market price is known as the cash or spot price.

- The range of underlying assets encompasses a wide array, including stocks, currencies, commodities, debt instruments, and even non-traditional metrics like weather conditions or insurance claims.

1.1.2 The Significance of Derivatives

Derivatives are pivotal in the financial ecosystem due to their ability to:

- Facilitate risk management by transferring risks between parties.
- Craft unique investment strategies and returns not feasible with direct investments.
- Serve as a source of market insight and future price expectations.
- Offer cost efficiencies in transactions.
- Minimize required capital investment.
- Simplify the process of taking short positions compared to dealing with the underlying directly.
- Enhance the liquidity and operational efficiency of the primary markets they relate to.

1.2 Derivative Markets

Derivatives are traded in two main venues: **organized exchanges** and **over-the-counter (OTC)** markets.

1.2.1 Exchange-Traded Derivatives

These derivatives are **standardized** contracts with fixed features such as contract size, expiration date, and underlying assets. Their trading is **guaranteed by a clearing house**, which requires participants to post margins. This arrangement ensures transparency and regulation, safeguarding the integrity of the market.

1.2.2 Over-The-Counter (OTC) Derivatives

In contrast, OTC derivatives offer **customization** in terms of contract size, asset specification, and expiration dates. This flexibility, however, comes with a higher risk of default (credit risk) for the parties involved.

1.2.2.1 The OTC Market Evolution

- **Prior to 2008**: The OTC market was largely unregulated, with banks serving as the primary market makers. Transactions were governed by master agreements, and some were cleared through central counterparties (CCPs), similar to clearing houses.
- **Since 2008**: Significant regulatory changes have been implemented to reduce systemic risk and enhance transparency. Standardized OTC transactions must now be cleared through CCPs, and all trades are required to be reported to a central repository. These measures aim to bolster market stability and integrity.

> ℹ The Lehman Bankruptcy
>
> The bankruptcy of Lehman Brothers on September 15, 2008, stands as the largest in U.S. history. Lehman Brothers was heavily involved in the over-the-counter (OTC) derivatives market, engaging in high-risk financial activities. The firm's inability to refinance its short-term debt ultimately led to its downfall. At the time of its bankruptcy, Lehman Brothers had an extensive network of transactions, with hundreds of thousands outstanding across approximately 8,000 counterparties. The process of unwinding these transactions has posed significant challenges for both the Lehman liquidators and the involved counterparties, illustrating the complex and interconnected nature of modern financial markets.

1.3 Forward Contracts vs. Contingent Claims

Both forward contracts and contingent claims are essential financial instruments that derive their value from the performance of an underlying asset, playing pivotal roles in the global financial markets for hedging, speculation, and arbitrage.

1.3.1 Forward Commitments

Forward commitments are agreements to buy or sell an asset at a predetermined future date and price. They include:

- **Forward Contracts**: Private, ***non-standardized*** agreements between two parties to buy or sell an asset at a specified future time at a price agreed upon at the contract's initiation.
- **Futures Contracts**: ***Standardized*** forward contracts traded on organized exchanges that require the posting of a margin. Futures contracts are marked to market daily, which mitigates credit risk.

- **Swaps**: Agreements between two parties to exchange sequences of cash flows for a set period based on a specified principal amount. Swaps often involve the exchange of a fixed interest rate for a floating rate, or vice versa.

1.3.2 Contingent Claims

Contingent claims are financial instruments that ***offer the holder the right, but not the obligation***, to buy or sell an asset at a predetermined price within a specified timeframe. The primary form of contingent claims is:

- **Options**: Contracts that give the buyer the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a specified strike price on or before a certain date. Options are of two types: *European options*, which can be exercised only at expiration, and *American options*, which can be exercised at any time before expiration.

1.3.3 Key Differences and Risks

- **Obligation vs. Right**: A forward contract represents an obligation to carry out the transaction, potentially leading to large losses depending on the market's movement. In contrast, a contingent claim like an option provides a right, limiting the buyer's potential loss to the premium paid for acquiring this right.
- **Risk Profile**: The potential loss on a long forward contract can equal the full contract price, while for a short position, the loss can be theoretically infinite due to the asset's price potential to rise indefinitely. Conversely, the risk of a contingent claim is limited to the premium paid, providing a risk-averse strategy for speculators and hedgers.
- **Uses and Applications**: Forward contracts and futures are commonly used for hedging against price movements in commodities, currencies, and interest rates. Swaps are utilized for managing interest rate risk and currency exposure. Options are employed for hedging, speculative trading, and income generation through premium collection.

Incorporating these instruments into investment and risk management strategies requires an understanding of their mechanics, risks, and market behavior. Their usage reflects the financial goals, risk tolerance, and market outlook of the participants, illustrating the complexity and diversity of modern financial markets.

1.4 Types of Derivatives

1.4.1 Forward Contract

Definition

A **forward contract** is a customized, over-the-counter derivative agreement between two parties, where the buyer agrees to purchase, and the seller agrees to sell, an underlying asset at a predetermined future date and price established at the contract's inception.

- **Long Position**: The party committing to purchase the asset.
- **Short Position**: The party committing to sell the asset.

Characteristics of Forward Contracts

- **Underlying Asset**: Specifies the type and quantity of the asset to be traded.
- **Settlement Method**: Describes how the contract will be executed or settled upon expiration.
- **Forward Price**: The agreed-upon price for the underlying asset exchange, designed to make the contract's initial value zero.

Key Points

- **Popularity in Foreign Exchange**: Forward contracts are frequently used for hedging in the foreign exchange markets.
- **OTC Markets**: Typically involves at least one financial institution, allowing for customization but lacking centralized regulation.
- **Flexibility and Risk Management**: Forward contracts offer tailored solutions for specific hedging needs, particularly in markets lacking standardized instruments. This customization enables precise risk management tailored to the parties' unique requirements.
- **Market Dynamics and Pricing**: The determination of forward prices is influenced by various factors, including the underlying asset's current price, interest rates, and the asset's expected future price volatility. This dynamic pricing mechanism reflects the market's consensus on future price movements, adjusted for the time value of money.

1.4.2 Futures Contract

- A futures contract is a **standardized** derivative traded on **futures exchanges**, like the CME Group or Intercontinental Exchange, facilitating the buying and selling of underlying assets at future dates.
- **Futures and Liquidity**: Futures contracts provide liquidity and price discovery in a regulated environment, offering a transparent and efficient means for market participants to hedge against price volatility or speculate on future price movements.

Forwards vs. Futures

Forwards	Futures
Customized terms, traded over-the-counter.	Standardized terms, traded on regulated exchanges.
Counterparty risk, with less regulatory oversight.	Mitigated counterparty risk through clearinghouses.
Settlement occurs at contract maturity.	Daily mark-to-market settlement.

1.4.3 Swap Contract

- A **swap** is an OTC derivative where two parties exchange cash flow series, addressing multi-period risks. Unlike a forward contract focusing on a single-period risk, swaps often manage interest rate, currency, or commodity exposure over extended periods.
- Swaps can be structured in numerous ways to suit different types of risk management strategies. For example, interest rate swaps exchange fixed for floating interest rate payments to manage interest rate risk, while currency swaps exchange cash flows in different currencies to hedge against currency risk.

1.4.4 Options

Definition

Options are versatile financial derivatives allowing the holder to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specific timeframe. The buyer of the option pays a premium to the seller or writer for this right, without the obligation to execute the transaction.

Types of Options

- **Call Options**: Grant the holder the right to **purchase** the underlying asset at the strike price. Investors buy calls when they anticipate the underlying asset's price will increase.
- **Put Options**: Provide the holder the right to **sell** the underlying asset at the strike price. Puts are purchased when an investor expects the underlying asset's price to decline.

Exercise Styles

- **American Options**: These can be exercised at any point up to and including the expiration date, offering maximum flexibility to the holder.
- **European Options**: Can only be exercised on the expiration date itself, limiting the timing of execution to this single moment.

1.5 Applications of Derivatives

1.5.1 Hedging

Hedging is a strategic financial practice aimed at reducing potential risks and mitigating financial exposure. Its core purpose is to **minimize risk**, though it does not assure a more favorable outcome. This strategy employs various financial instruments, such as forward and option contracts, each with distinct characteristics and uses.

Forward contracts are pivotal in risk management, enabling the hedger to lock in a future transaction price for an underlying asset, thereby neutralizing the risk of adverse price fluctuations. This fixed-price agreement ensures predictability in financial planning.

Option contracts, on the other hand, serve as a form of financial insurance. They grant investors the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specific timeframe. This mechanism allows investors to safeguard against potential adverse price movements while preserving the opportunity to capitalize on favorable price changes. Unlike forward contracts, acquiring an option requires paying an upfront premium, which represents the cost of the protection offered.

Examples to Illustrate Hedging

1. Consider a U.S. company that anticipates a payment of GBP 10 million for imports from Britain in three months. To hedge against the risk of GBP appreciating against USD, the company can enter into a long position in a forward contract. By doing so, it secures a fixed exchange rate for the future payment, thus avoiding the uncertainty of currency fluctuations.

2. An investor holding 1,000 shares of Microsoft, valued at \$28 per share, faces the risk of a potential decrease in share value. To mitigate this risk, the investor can purchase a put option, which provides the right to sell the shares at a strike price of \$27.50. For example, if a two-month put option costs \$1 per share, the investor would buy 10 contracts (each contract typically covers 100 shares), paying a total premium of \$1,000. This strategy protects the investor's portfolio from a decline below the strike price, while still allowing participation in any upward price movements.

1.5.2 Arbitrage

Arbitrage represents a foundational concept in financial markets, involving the exploitation of price differentials across different markets or forms of an asset to secure risk-free profits. This practice underscores the **Law of One Price**, which posits that in efficient markets with minimal transaction costs and unrestricted information flow, identical assets should uniformly price.

When disparities arise, arbitrageurs can purchase the undervalued asset in one market and sell it at a higher price in another, leveraging these discrepancies to generate a risk-free return. This activity not only capitalizes on the price differentials but also plays a crucial role in driving the prices of identical assets towards convergence, thus contributing to market efficiency.

Arbitrage serves as a relative valuation method, providing insights into the correct pricing of one asset or derivative in relation to another. It operates under the principle that no two identical assets should exist with different pricing or that equivalent asset combinations yielding the same returns should not vary in price.

The presence and subsequent elimination of arbitrage opportunities are indicative of market efficiency, where markets compensate investors for risks appropriately. However, when arbitrage allows for returns above the risk-free rate without exposure to risk, it challenges the notions of market efficiency by presenting an anomaly of abnormal returns.

Example to Illustrate Arbitrage

Imagine a scenario where a stock is priced at GBP 100 in London and USD 150 in New York, with the current exchange rate being 1.5300 GBP/USD. Purchase the stock for USD 150 in

New York, then sell it in London for GBP 100. Convert GBP 100 to USD 153 to secure a profit of USD 3.

1.5.3 Speculation

Speculation in financial markets is driven by the intention to profit from anticipated market movements. Speculators engage in market positions with the expectation that prices will either rise or fall, aiming to capitalize on these predicted changes.

When engaging in futures contracts, speculators face significant risk and reward potential. Futures contracts obligate the purchase or sale of an asset at a predetermined future date and price, exposing the speculator to potentially unlimited losses or gains, depending on market movements.

Conversely, options contracts offer a different risk profile. By purchasing options, a speculator gains the right, but not the obligation, to buy (call option) or sell (put option) an asset at a specified price within a certain period. The maximum loss in this scenario is limited to the premium paid for the options, providing a safety net against adverse market movements.

Examples to Illustrate Speculation

Trading strategies for an investor with $2,000 anticipating a stock price increase:

1. **Buying Shares Outright:** If the investor is confident in their prediction and prefers direct exposure, they could use their $2,000 to buy shares of the stock directly. This approach offers unlimited upside potential if the stock price increases but also exposes the investor to potential losses if the price falls.
2. **Buying Call Options:** The investor could purchase call options on the stock they believe will increase in value. This strategy allows the investor to control a larger amount of stock with a smaller investment (the premium paid for the options), with the potential for significant gains if the stock price rises as anticipated. The risk is limited to the premium paid.
3. **Futures Contracts:** If the investor has access to futures trading and is knowledgeable about its risks, they could engage in a futures contract that bets on the stock price increasing. This strategy requires careful risk management due to the potential for large losses.

1.6 Criticisms and Misuses of Derivatives

Derivatives, while fundamental to risk management and price discovery in financial markets, have faced criticisms and concerns regarding their potential for misuse and the risks they can introduce to the financial system. These criticisms encompass several key areas:

1. **Speculation and Gambling:** Derivatives are often utilized for speculative purposes, where investors bet on the direction of market prices with the aim of generating profits. This speculative use can resemble gambling when trades are made based on predictions rather than informed decisions, potentially leading to significant losses.

2. **Systemic Risk and Destabilization:** Derivatives can add systemic risk to the financial system, especially when used in large volumes or in complex combinations that are difficult to understand and manage. The interconnectedness of market participants through derivatives can lead to contagion effects, where the failure of one entity can trigger a chain reaction affecting others.

3. **Complexity:** The inherent complexity of many derivative products makes them difficult to value and understand, even for sophisticated investors. This complexity can obscure the real risks involved and lead to mispricing or inappropriate use of these instruments.

4. **Role Shifting:** Traders and institutions may shift roles between being hedgers, who use derivatives to manage risk, to speculators, who seek to profit from market movements, or from being arbitrageurs, who exploit price discrepancies, to speculators. This shifting can blur the lines between risk management and speculative activities, increasing the potential for losses and systemic risks.

5. **Need for Controls:** Given the potential for misuse and the complex nature of derivatives, it is crucial to establish comprehensive controls and regulatory frameworks. These controls should ensure that derivatives are used for their intended purposes, such as hedging risk, rather than for unchecked speculation. Proper oversight, transparent reporting requirements, and clear guidelines for risk assessment and management are essential components of a robust regulatory framework.

To address these criticisms and mitigate the risks associated with derivatives, financial markets and regulatory bodies have implemented measures such as:

- **Central Clearing Counterparties (CCPs):** These entities act as intermediaries for derivative transactions, reducing counterparty risk and increasing transparency.

- **Margin Requirements:** These requirements ensure that parties in derivative contracts have sufficient capital to cover potential losses, reducing the risk of default.

- **Regulatory Reforms:** Post-2008 financial crisis, reforms like the Dodd-Frank Act in the United States and EMIR in Europe have increased oversight and regulation of derivative markets, aiming to improve transparency, reduce systemic risk, and ensure that derivatives fulfill their role in financial markets responsibly.

1.7 Practice Questions and Problems

1.7.1 Time Value of Money

1. A bank quotes an interest rate of 7% per annum with quarterly compounding. How much you will earn from \$100 investment after (a) 1 year and (b) 3 years? What is the equivalent rate with (a) continuous compounding and (b) annual compounding? Verify your results.

1.7.2 Theoretical Foundations of Derivatives

1. Explain carefully the difference between hedging, speculation, and arbitrage.
2. What is the difference between the over-the-counter market and the exchange-traded market?
3. "Options and futures are zero-sum games." What do you think is meant by this statement?
4. What is the difference between a long forward position and a short forward position?
5. What is the difference between entering into a long forward contract when the forward price is \$50 and taking a long position in a call option with a strike price of \$50?
6. Explain carefully the difference between selling a call option and buying a put option.
7. When first issued, a stock provides funds for a company. Is the same true of an exchange-traded stock option? Discuss.

1.7.3 Practical Applications of Forwards and Futures

1. A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?
2. An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.5000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.4900 and (b) 1.5200?

1.7.4 Practical Applications of Options

1. A trader buys a call option with a strike price of \$30 for \$3. Does the trader ever exercise the option and lose money on the trade. Explain.
2. Suppose that you write a put contract with a strike price of \$40 and an expiration date in three months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?
3. Suppose you own 5,000 shares that are worth \$25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?
4. You would like to speculate on a rise in the price of a certain stock. The current stock price is \$29, and a three-month call with a strike of \$30 costs \$2.90. You have \$5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

1.7.5 Hedging Strategies

1. Explain why a futures contract can be used for either speculation or hedging.
2. A US company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract and (b) an option.
3. The CME Group offers a futures contract on long-term Treasury bonds. Characterize the investors likely to use this contract.

2 Forwards and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 2 - Mechanics of futures markets
  - Chapter 3 - Hedging Strategies Using Futures
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Understand forwards and futures, including their characteristics, payoff structures, and key differences.
- Gain knowledge of exchange and over-the-counter (OTC) markets, focusing on their functionalities and distinctions.
- Learn the basics of hedging with futures, managing basis risk, and applying cross hedging techniques.
- Explore the role of stock index futures in portfolio risk management and speculation.

2.1 Forwards and Futures Characteristics

Definition

Forwards and futures are derivative contracts **obligating** two parties to exchange an asset at a predetermined future date and price.

- **Forward/Futures Price:** This is the agreed-upon price at which the underlying asset will be exchanged in the future. It's important to note that this price is determined at the contract's inception and may vary across contracts with different expiration dates, reflecting the market's expectations of future price movements.

- **Positioning:**
  - A **long position** signifies the buyer's **commitment to purchase** the underlying asset. The buyer stands to benefit from a rise in the asset's price over time but also bears the risk of a potential decrease.
  - A **short position** represents the seller's **obligation to sell** the asset. While the seller can profit from a decline in the asset's price, there is also the risk of unlimited loss if the asset's price increases substantially.

- **Risk Exposure:**
  - The potential loss for the holder of a **long position** can extend up to the full contract price, emphasizing the risk of a total loss if the asset's value drops to zero.
  - Conversely, the **short position** holder faces potentially unlimited loss, as there is no upper limit to how high an asset's price can climb.

- **Contract Specifications:**
  - **Deliverable Assets:** Clearly defines the asset or assets that can be delivered under the contract, including any standards or grades if applicable.
  - **Delivery Location:** Specifies where the asset will be delivered, which can significantly impact logistics and costs.
  - **Delivery Time:** Outlines when delivery of the asset is expected, providing a timeframe within which the contract must be settled.

- **Settlement:** Futures contracts often settle daily based on market price changes, a process known as "marking to market." Forwards, however, usually settle at the end of the contract term, with the final payment reflecting the difference between the forward price and the underlying asset's price at maturity.

```python
import plotly.graph_objects as go
import numpy as np

# Parameters
F = 100  # Futures price
Q = 1  # Quantity of the asset
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices

# Payoff calculations
long_payoff = (spot_prices - F) * Q
short_payoff = (F - spot_prices) * Q

# Create the figure
fig = go.Figure()
```

```
# Add traces for long and short positions
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_payoff,
        mode="lines",
        name="Long Position",
        line=dict(width=3),
        hovertemplate="Long Position<br>Spot Price: %{x:.0f}<br>Payoff: %{y:.0f}<extra></ext
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_payoff,
        mode="lines",
        name="Short Position",
        line=dict(width=3),
        hovertemplate="Short Position<br>Spot Price: %{x:.0f}<br>Payoff: %{y:.0f}<extra></ex
    )
)
fig.add_hline(y=0, line_dash="solid", line_color="black", line=dict(width=0.7))

# Layout
fig.update_layout(
    title="Payoff from a Futures Contract",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Payoff",
    legend_title="Position",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

```
Unable to display output for mime type(s): text/html
```

2.2 Payoff From a Forward Contract

When discussing forward contracts, it's essential to understand the setup:

- **Initiation Time:** The contract begins at $t = 0$
- **Expiration Time:** The contract ends at $t = T$
- **Initial Spot Price:** The price of the underlying asset at start, denoted as S_0
- **Final Spot Price:** The price of the underlying asset at expiration, S_T
- **Forward Price:** Agreed upon price for the asset, represented as F_0

Forward contracts involve two parties with contrasting positions:

- The **long position** commits to purchasing the asset at F_0, seeking to profit from an increase in S_T.
- The **short position** commits to selling the asset at F_0, benefiting if S_T decreases.

Payoff from a forward contract is then calculated as:

- **Long Party Payoff at Expiration:** $S_T - F_0$
- **Short Party Payoff at Expiration:** $-[S_T - F_0]$

Example: Gold Purchase Agreement

Barbara Nix agreed to buy 1 kilogram of gold at a price of $38,000 per kilo from Metals Inc. in 90 days. After 90 days, the spot price of gold reached $38,500 per kilo. What is the payoff for each party?

- **For Nix (Long Party):** The gain is $500, calculated as

$$S_T - F_0 = \$38,500 - \$38,000 = \$500$$

- **For PM Metals Inc. (Short Party):** The loss is $500

2.3 Exchange Markets

Exchange markets facilitate the trading of futures contracts, allowing participants to hedge against price changes or speculate on market movements. Key characteristics include the standardized nature of contracts and the regulatory oversight provided to ensure market integrity.

- **Regulation:** In the United States, the Commodity Futures Trading Commission (CFTC) oversees the futures markets, aiming to protect market participants and maintain market integrity.

- **Futures Contract Specifications:** Futures contracts are highly standardized, with detailed specifications, e.g., CME Group's Gold Futures.

2.3.1 Delivery Mechanisms

- **Contract Settlement:** Most futures contracts are closed out before maturity, avoiding physical delivery. Instead, positions are settled by entering into an offsetting transaction.
- **Delivery Options:** For contracts requiring physical delivery, specifications outline the deliverable commodities, delivery locations, and timelines. The short position holder usually determines the delivery specifics.
- **Cash Settlement:** Certain contracts, such as those for stock indices, settle in cash instead of physical delivery.

2.3.2 Understanding Market Quotes

- **Settlement Price:** This is the closing price used for the daily settlement process.
- **Open Interest:** Represents the total number of open contracts, equal to either the number of long or short positions.
- **Trading Volume:** Indicates the total number of contracts traded in a day, which can exceed the open interest.

Example of Open Interest and Trading Volume

Imagine a futures market for wheat. At the start of the trading day, there are 1,000 open contracts for December Wheat futures. Throughout the day, traders conduct the following transactions:

1. **Transaction 1:** A new buyer and a new seller enter the market, agreeing on a contract. This transaction increases the open interest by 1 contract, bringing the total open interest to 1,001.
2. **Transaction 2:** An existing holder of a long position sells to a new participant. This transaction does not change the open interest since one new party replaces another in the open position count, keeping the open interest at 1,001.
3. **Transaction 3:** Two new participants (a buyer and a seller) enter the market with 5 new contracts. This increases the open interest by 5, making it 1,006.
4. **Transaction 4:** An existing buyer and an existing seller close out their position by entering into an offsetting contract. This transaction decreases the open interest by 1, bringing it down to 1,005.

By the end of the day, the open interest in December Wheat futures is 1,005, indicating the total number of active contracts.

The **trading volume** for the day is calculated by adding up all the new contracts traded, regardless of whether they increase, decrease, or leave open interest unchanged. For the day's transactions:

- **Transaction 1:** Adds 1 to the trading volume.
- **Transaction 2:** Adds 1 to the trading volume (although it doesn't affect open interest).
- **Transaction 3:** Adds 5 to the trading volume.
- **Transaction 4:** Adds 1 to the trading volume (as it involves the exchange of contracts, despite decreasing open interest).

Therefore, the total trading volume for the day is 8 contracts.
This example illustrates how open interest and trading volume provide different insights into market activity. Open interest shows the total number of outstanding contracts, indicating market liquidity and potential future activity. Trading volume measures the number of contracts traded within a day, offering insights into the market's current activity level.

2.3.3 Daily Settlement Process and Margins

The clearinghouse calculates the settlement price daily, adjusting traders' margin accounts based on market movements:

- **Mark-to-Market:** Daily gains or losses are reflected in the margin account, ensuring that the account balance accurately represents the market value of the position.

Margins are crucial for managing credit risk in futures trading:

- **Initial Margin:** The upfront deposit required to open a position.
- **Maintenance Margin:** A lower threshold that must be maintained; falling below triggers a margin call, requiring the account to be topped up to the initial margin level again.
- **Margin Calls:** If an account falls below the maintenance margin, additional funds must be deposited to meet the initial margin requirement.

Example of Margin, Maintenance Margin, and Margin Call

Let's consider an investor, Alex, who wants to enter into a futures contract for crude oil. The price of one contract (representing 1,000 barrels) is $60 per barrel, making the total value of the contract $60,000.

2.3.3.1 Initial Margin

The exchange requires an **initial margin** of 5% to open a position. Therefore, Alex must deposit:

$$5\% \times \$60,000 = \$3,000$$

This deposit is a security against potential losses on the position.

2.3.3.2 Maintenance Margin

The exchange sets a **maintenance margin** at 3% of the contract value, which is the minimum amount that must be maintained in the margin account. For Alex, this is:

$$3\% \times \$60,000 = \$1,800$$

2.3.3.3 Margin Call

Assume the price of crude oil drops to $58 per barrel, decreasing the value of Alex's futures contract to:

$$1,000 \text{ barrels} \times \$58 = \$58,000$$

This decrease results in a loss of $2,000 on Alex's position, reducing his margin account to $1,000 ($3,000 initial margin - $2,000 loss), which is below the maintenance margin of $1,800. As a result, Alex receives a **margin call** from his broker, requiring him to deposit additional funds to bring his margin account back up to the initial margin level of $3,000.

To meet the margin call, Alex must deposit an additional $2,000 into his margin account. If Alex fails to meet the margin call, his position may be closed out by the broker to limit further losses.

This example illustrates the concepts of initial margin, maintenance margin, and margin calls within the context of futures trading. These mechanisms protect both the investor and the broker from the volatility and potential losses associated with leveraged positions in the futures market.

Some Useful Links

- Margin: Know What's Needed
- The Benefits of Futures Margins
- Understanding Futures Margin

2.4 OTC Markets

Over-the-Counter (OTC) markets facilitate the trading of forward contracts and other financial instruments directly between two parties without the intermediation of exchanges. These markets offer flexibility and customization in the contracts traded but historically lacked the transparency and regulation of their exchange-traded counterparts.

2.4.1 Regulatory Evolution

- **Pre-Crisis Era:** Prior to the 2007–2008 financial crisis, the OTC market operated with minimal regulatory oversight. This lack of regulation contributed to systemic risks, as evidenced by the crisis.
- **Post-Crisis Reforms:** In response to the financial crisis, significant regulatory measures were introduced globally to increase transparency, reduce systemic risk, and improve market stability. These include the Dodd-Frank Act in the United States and the European Market Infrastructure Regulation (EMIR) in the EU, mandating the reporting of OTC transactions and the use of central clearing parties for certain classes of derivatives.

2.4.2 Collateral Requirements

To mitigate counterparty risk, collateral requirements were introduced, including:

- **Initial Margin:** An upfront deposit required to enter into a contract, intended to cover potential future exposure in the period immediately following a default.
- **Variational Margin (or Variation Margin):** Additional funds that must be deposited to cover adverse price movements, ensuring the value of the collateral matches the exposure at the end of each trading day.

2.4.3 Clearing Mechanisms

- **Bilateral Clearing:** Traditionally, OTC markets relied on bilateral clearing, where each transaction is a separate agreement between two parties. This method allows for customization but lacks centralized risk management, making it harder to monitor and mitigate systemic risk.

- **Central Clearing:** The introduction of a **Central Counterparty (CCP)** for clearing OTC trades represents a significant shift. The CCP acts as a buyer to every seller and a seller to every buyer, reducing counterparty risk and increasing market transparency. Central clearing houses also enforce margin requirements and perform regular mark-to-market adjustments to manage risk. While OTC contracts are customized, CCPs have facilitated a move towards some level of standardization, especially in the processes for managing risk and settling trades.

2.5 Forward vs. Futures Contracts Summary

Characteristics	**Forward Contracts**	**Futures Contracts**
Market Type	Over-the-Counter (OTC)	Exchange-Traded
Standardization	Customized features to suit parties' needs	Standardized in terms of contract size and expiration
Delivery Date	Typically one specified date	Range of specified dates allowing flexibility
Settlement of Gains and Losses	At contract maturity	Daily, via marking to market
Settlement Method	Physical delivery or cash settlement	Often closed out before maturity, usually cash settled
Credit Risk	Present, due to lack of central clearing	Mitigated by the clearinghouse's guarantee
Regulation and Transparency	Limited, due to private nature	High, due to regulatory oversight and transparency
Liquidity and Market Depth	Varies, often lower due to customization	Higher, facilitated by standardization and exchange

2.6 Introduction to Hedging with Futures

Hedging using futures contracts is a risk management strategy that allows individuals and corporations to stabilize the price of assets they intend to purchase or sell in the future. This technique is critical for managing the volatility associated with commodity prices, foreign exchange rates, interest rates, and other financial variables.

- **Long Futures Hedge:** Used when anticipating the purchase of an asset in the future. A long hedge allows the buyer to lock in a purchase price, mitigating the risk of rising prices.
- **Short Futures Hedge:** Employed when planning to sell an asset in the future. It enables the seller to secure a selling price, protecting against falling prices.

2.6.0.1 Arguments in Favor of Hedging

- **Business Focus:** Hedging allows companies to concentrate on their core business activities by reducing exposure to financial market risks.
- **Cost Predictability:** It provides predictability in costs and revenues, which is beneficial for budgeting and financial planning.
- **Risk Management:** Effectively manages risks related to fluctuations in interest rates, exchange rates, and commodity prices.

2.6.0.2 Arguments Against Hedging

- **Shareholder Autonomy:** Critics argue that shareholders can diversify their portfolios independently and undertake personal hedging strategies if desired.
- **Competitive Risk:** There's a belief that hedging could introduce additional risk, especially if competitors choose not to hedge, potentially affecting market dynamics and competitive positioning.
- **Complexity and Perception:** Hedging can be complex to implement and explain, especially in scenarios where the hedge results in a loss while the underlying asset gains in value, which might confuse stakeholders.

2.7 Hedging and Basis Risk

Basis risk is a critical concept in the realm of financial hedging, referring to the risk that the difference between the spot price and the futures price (the basis) will not behave as anticipated when the hedge is unwound. This discrepancy can lead to less effective hedges and unexpected financial outcomes.

Basis risk arises due to several factors:

1. **Asset Mismatch:** Often, the asset being hedged does not perfectly match the asset underlying the futures contract, introducing a discrepancy in price movements.
2. **Timing Uncertainty:** The exact timing of when an asset will be bought or sold can add uncertainty, affecting the effectiveness of the hedge.
3. **Early Contract Closure:** Sometimes, it is necessary to close out a futures contract before its delivery month, which can influence the basis and the outcome of the hedge.

To mitigate basis risk, selecting an appropriate futures contract is vital:

- **Delivery Month:** Opt for a contract with a delivery month as close as possible to but after the anticipated end of the hedge to minimize the time gap's impact on the basis.

- **Cross Hedging:** When direct futures contracts for the asset are unavailable, choose a contract with a price highly correlated to the asset's price. This approach, known as cross hedging, can help manage the risk despite the asset mismatch.

Long Hedge Example for Asset Purchase

- Initial Futures Price (F_1): 88.0
- Futures Price at Purchase (F_2): 89.1
- Spot Price at Purchase (S_2): 90.0
- Basis at Purchase (b_2): 0.9 (calculated as $S_2 - F_2$)

Hedging Outcome:

- The asset's cost is 90.0.
- The gain from futures is 1.1 ($F_2 - F_1$).
- The net amount paid effectively becomes 88.9, considering the gain from futures and the basis at purchase.

Short Hedge Example for Asset Sale

- Initial Futures Price (F_1): 0.98
- Futures Price at Sale (F_2): 0.925
- Spot Price at Sale (S_2): 0.92
- Basis at Sale (b_2): -0.005 (calculated as $S_2 - F_2$)

Hedging Outcome:

- The asset's sale price is 0.92.
- The gain from futures is 0.055 ($F_1 - F_2$).
- The net amount received effectively becomes 0.975, factoring in the gain from futures and the basis at sale.

2.8 Cross Hedging

Cross hedging involves using futures contracts of a commodity or asset that is different from, but correlated to, the asset being hedged. It is particularly useful when no futures market exists for the specific asset in question. The effectiveness of a cross hedge depends on the correlation between the price movements of the hedged asset and the futures contract used for hedging.

The **optimal hedge ratio** (proportion of exposure to be hedged), denoted as h^*, is calculated using the formula:

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

- ρ is the coefficient of correlation between ΔS and ΔF, indicating how closely the futures prices move in relation to the spot prices of the hedged asset.
- σ_S is the standard deviation of ΔS, representing the change in the spot price of the asset being hedged over the hedging period.
- σ_F is the standard deviation of ΔF, representing the change in the futures price over the hedging period.

To determine the **optimal number of futures contracts** (N^*) required for the hedge, **without adjusting for daily settlement**, is:

$$N^* = h^* \times \frac{Q_A}{Q_F}$$

- Q_A represents the size of the position being hedged, measured in units.
- Q_F denotes the size of one futures contract, also in units.

Example: Hedging Jet Fuel with Heating Oil Futures

Consider an airline that anticipates purchasing 2 million gallons of jet fuel in one month and decides to hedge the price risk using heating oil futures, based on their historical price movements and correlation:

- Historical data provides $\sigma_F = 0.0313$, $\sigma_S = 0.0263$, and $\rho = 0.928$.
- Applying the formula for h^*:

$$h^* = 0.928 \times \frac{0.0263}{0.0313} = 0.78$$

- Given that one heating oil futures contract covers 42,000 gallons, the optimal number of contracts to hedge the airline's exposure is calculated as:

$$0.78 \times \frac{2,000,000}{42,000} \approx 37$$

Thus, to optimally hedge against the price risk of jet fuel, the airline should purchase approximately 37 heating oil futures contracts.

2.8.1 Daily Settlement and Tailing Adjustment in Hedging

The daily settlement process and the concept of tailing adjustments are crucial for refining the hedging strategy, particularly in the context of futures markets. These adjustments are necessary to account for the nuances of daily price changes and the mechanics of futures contracts.

The optimal hedge ratio, **adjusted for daily settlement**, is given by:

$$\hat{h} = \hat{\rho}\frac{\hat{\sigma_S}}{\hat{\sigma_F}}$$

- $\hat{\rho}$ represents the correlation between percentage daily changes in the spot and futures prices. This measures how closely the futures prices move in relation to the spot prices on a day-to-day basis.
- $\hat{\sigma_S}$ is the standard deviation of the percentage daily changes in the spot price, quantifying the daily price volatility of the spot asset.
- $\hat{\sigma_F}$ is the standard deviation of the percentage daily changes in the futures price, indicating the daily volatility in the futures market.

The **optimal number of futures contracts**, incorporating tailing adjustments for daily settlement, can be calculated as follows:

$$N^* = \hat{h} \times \frac{V_A}{V_F}$$

- V_A represents the total value of the position being hedged, calculated as the spot price multiplied by the quantity of the asset (Q_A).
- V_F denotes the value of one futures contract, determined by the futures price multiplied by the quantity specified in one futures contract (Q_F).

This adjustment, often referred to as "tailing the hedge," modifies the hedge ratio to account for the impact of daily settlements on the futures position. It ensures that the hedge remains effective in offsetting the price risk of the underlying asset over the hedging period.

Example: Hedging with Tailing Adjustments

A transportation company, "TransCo," needs to purchase 500,000 gallons of diesel fuel in one month. However, there are no futures contracts available for diesel. TransCo decides to use heating oil futures as a hedge because heating oil prices are highly correlated with diesel prices. Each heating oil futures contract covers 42,000 gallons.

- Current diesel price (spot price): \$3.00 per gallon
- Current heating oil futures price for next month's delivery: \$2.90 per gallon

- Correlation between daily percentage changes in diesel and heating oil prices ($\hat{\rho}$): 0.9
- Standard deviation of daily percentage changes in diesel prices ($\hat{\sigma_S}$): 1.5%
- Standard deviation of daily percentage changes in heating oil futures prices ($\hat{\sigma_F}$): 1.8%

2.8.1.1 Solution

- Calculate the optimal hedge ratio

$$\hat{h} = \hat{\rho}\frac{\hat{\sigma_S}}{\hat{\sigma_F}} = 0.9 \times \frac{0.015}{0.018} = 0.75$$

- Determine the V_A and V_F

$$V_A = \$3.00 \times 500,000 = \$1,500,000$$
$$V_F = \$2.90 \times 42,000 = \$121,800$$

- Calculate the optimal number of futures contracts

$$N^* = \hat{h} \times \frac{V_A}{V_F} = 0.75 \times \frac{1,500,000}{121,800} \approx 9.23$$

Given that partial contracts cannot be purchased, TransCo would need to round to the nearest whole number. In practice, the company might choose to hedge with 9 contracts to avoid over-hedging, understanding that this leaves a slight unhedged exposure.

By purchasing 9 heating oil futures contracts, TransCo effectively hedges against the risk of rising diesel prices. The chosen hedge ratio of 0.75, derived from the correlation and volatility (standard deviation) differences between diesel and heating oil prices, optimizes the hedge despite the imperfect match between the two commodities. This example illustrates how cross-hedging can be applied when direct hedging instruments are not available, using tailing adjustments to fine-tune the hedge for daily settlement impacts.

2.9 Stock Index Futures

Stock index futures are a powerful tool for hedging the risk in a portfolio. By taking positions in these futures, investors can manage exposure to market fluctuations without the need to alter the composition of their portfolio.

2.9.1 Hedging Portfolio

The formula to determine the number of contracts to short for hedging purposes is:

$$\beta \times \frac{V_A}{V_F}$$

- V_A is the value of the portfolio being hedged.
- β measures the portfolio's sensitivity to market movements. A beta of 1 indicates that the portfolio moves in line with the market.
- V_F is the value of one futures contract, calculated as the futures price multiplied by a specified multiplier (e.g., $250 times the index for S&P 500 futures).

Example: Stock Index Futures Hedging

Consider a portfolio with the following characteristics:

- S&P 500 futures price: 1,000 points
- Portfolio value: $5 million
- Portfolio beta: 1.5
- Value of one futures contract: $250,000 ($250 times the index)

Necessary position in S&P 500 futures to hedge the portfolio:

$$1.5 \times \frac{5,000,000}{250,000} = 30$$

Thus, to hedge the portfolio against market movements, it would be necessary to short 30 S&P 500 futures contracts.

2.9.2 Adjusting Portfolio Beta

To modify the beta of a portfolio, the formula is:

$$(\beta^* - \beta) \times \frac{V_A}{V_F}$$

- β^* is the desired new beta level.
- A negative outcome suggests shorting futures to decrease beta, while a positive value indicates going long on futures to increase beta.

Example: Adjusting Portfolio Beta

A portfolio manager seeks to adjust the beta of a $10 million portfolio. The current beta of the portfolio is 1.2, but due to a bearish market outlook, the manager wishes to decrease the portfolio's beta to 0.8 to reduce exposure to market volatility. The futures contract used for hedging is based on an index that is currently priced at 3,000 points, with each contract representing $250 times the index value.

- Current portfolio value (V_A): $10,000,000
- Current portfolio beta (β): 1.2
- Desired portfolio beta (β^*): 0.8
- Index futures price: 3,000 points
- Value of one futures contract (V_F): $250 $\times$3,000 = $750,000

Calculate the number of futures contracts needed to adjust the portfolio's beta to 0.8.

$$N^* = -0.4 \times \frac{10,000,000}{750,000} = -0.4 \times 13.33 \approx -5.33$$

Since we cannot trade a fraction of a contract, we'll need to round to the nearest whole number. In this case, rounding to -5 suggests that the portfolio manager should short 5 futures contracts to achieve the desired beta adjustment.

2.9.3 Additional Notes About Hedging

2.9.3.1 Why Hedge Equity Returns?

Hedging with stock index futures is advantageous for several reasons:

- **Market Timing:** Allows investors to effectively "exit" the market without selling assets, ideal for avoiding transaction costs and capital gains taxes.
- **Risk Management:** Enables precise control over the portfolio's exposure to market risk.
- **Performance:** Ensures returns are based on the specific selection and performance of portfolio assets, beyond the general market movements.

Tip

Imagine your portfolio's stocks have an average beta of 1.0, aligning their performance with the market. Yet, you're confident these stocks will surpass market returns in any scenario. By hedging, you secure returns at the risk-free rate plus any outperformance of your stocks over the market. This strategy minimizes market volatility's impact, ensuring your portfolio's gains are primarily due to your stock selection skills.

2.9.3.2 Stack and Roll Strategy

This approach involves rolling futures contracts forward to manage future exposures:

- Start with futures contracts hedging up to a certain time horizon.
- Before these contracts mature, close them out and replace them with new contracts for the next period.
- This method allows continuous hedging over time, adjusting as market conditions and portfolio needs change.

However, it's crucial to be mindful of the liquidity risks and potential for realized losses.

- **Realized vs. Unrealized Gains/Losses:** Hedging can lead to scenarios where losses on the hedge are realized while corresponding gains in the underlying assets are unrealized, potentially causing liquidity issues.

> Warning
>
> The Metallgesellschaft (MG) case in the early 1990s serves as a cautionary example of how rolling forward hedges can lead to significant cash flow problems. MG sold long-term fixed-price contracts for heating oil and gasoline, hedging its exposure with short-dated long futures that were regularly rolled over. When oil prices dropped, MG faced substantial margin calls, creating severe short-term liquidity issues. Despite the hedging strategy's long-term rationale, the immediate financial strain alarmed the company's management and bankers. Ultimately, MG had to close out its hedge positions and cancel the fixed-price contracts with customers, resulting in a staggering loss of $1.33 billion.

2.10 Practice Questions and Problems

2.10.1 Fundamentals of Futures Trading

1. What are the most important aspects of the design of a new futures contract?
2. The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.
3. What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

4. "Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

2.10.2 Open Interest and Trading Volume

1. Distinguish between the terms open interest and trading volume.
2. "When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.
3. Why does the open interest usually decline during the month preceding the delivery month? On a particular day, there were 2,000 trades in a particular futures contract. This means that there were 2000 buyers (going long) and 2000 sellers (going short). Of the 2,000 buyers, 1,400 were closing out positions and 600 were entering into new positions. Of the 2,000 sellers, 1,200 were closing out positions and 800 were entering into new positions. What is the impact of the day's trading on open interest?

2.10.3 Margin Mechanics in Futures Trading

1. Explain how margin accounts protect investors against the possibility of default.
2. Suppose that you enter into a short futures contract to sell July silver for $17.20 per ounce. The size of the contract is 5,000 ounces. The initial margin is $4,000, and the maintenance margin is $3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?
3. A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is $6,000 per contract, and the maintenance margin is $4,500 per contract. What price change would lead to a margin call? Under what circumstances could $2,000 be withdrawn from the margin account?
4. A company enters into a short futures contract to sell 5,000 bushels of wheat for 750 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2,000. What price change would lead to a margin call? Under what circumstances could $1,500 be withdrawn from the margin account?

2.10.4 Basics of Hedging with Futures

1. Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?
2. Explain what is meant by basis risk when futures contracts are used for hedging.
3. Explain what is meant by a perfect hedge. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.
4. Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.

5. "For an asset where futures prices for contracts on the asset are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

2.10.5 Advanced Hedging Scenarios and Strategies

1. Sixty futures contracts are used to hedge an exposure to the price of silver. Each futures contract is on 5,000 ounces of silver. At the time the hedge is closed out, the basis is \$0.20 per ounce. What is the effect of the basis on the hedger's financial position if (a) the trader is hedging the purchase of silver and (b) the trader is hedging the sale of silver?
2. In the corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in a) June, b) July, and c) January.
3. Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a three-month contract? What does it mean?
4. The standard deviation of monthly changes in the spot price of live cattle is 1.2 (in cents per pound). The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

2.10.6 Practical Concerns and Risk Management

1. Give three reasons why the treasurer of a company might not hedge the company's exposure to a particular risk.
2. A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather." Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?
3. Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.
4. A futures contract is used for hedging. Explain why the daily settlement of the contract can give rise to cash flow problems.

2.10.7 Hedging with Stock and Commodity Futures

1. A company has a $20 million portfolio with a beta of 1.2. It would like to use futures contracts on a stock index to hedge its risk. The index futures is currently standing at 1080, and each contract is for delivery of $250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6? What should the company do if it wants to increase the beta of the portfolio to 1.5?
2. On July 1, an investor holds 50,000 shares of a certain stock. The market price is $30 per share. He decides to use the September Mini S&P 500 futures contract. The index is currently 1,500 and one contract is for delivery of $50 times the index. The beta of the stock is 1.3. What strategy should the investor follow? Under what circumstances will it be profitable?
3. It is now June. A company knows that it will sell 5,000 barrels of crude oil in September. It uses the October CME Group futures contract to hedge the price it will receive. Each contract is on 1,000 barrels of "light sweet crude". What position should it take? What price risks is it still exposed to after taking the position?

3 Determination of Forward and Futures Prices

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 5 - Determination of Forward and Futures Prices
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 2 - Basics of Derivative Pricing and Valuation
  - Chapter 3 - Pricing and Valuation of Forward Commitments

Learning Outcomes:

- Understand the principles of pricing and valuation for forward and futures contracts.
- Apply pricing models to specific assets, including commodities and financial instruments.
- Analyze the relationship between futures prices and expected spot prices.
- Understand the structure and valuation of Forward Rate Agreements (FRAs).

3.1 Pricing and Valuation of Forward/Futures Contracts

- **Pricing** of forward/futures contracts entails determining the fair market price or rate at the **initiation** of the contract. This process ensures that the contract's initial value is neutral, aligning the interests of both parties without giving an undue advantage to either side.
- **Valuation**, on the other hand, is the process of assessing the contract's current value **after its initiation**. The value fluctuates based on market conditions, reflecting the gain or loss to the contract holder.

Example: Forward Price vs. Value

- An investor enters a long position in a December Gold futures contract on 2 November, agreeing to a price of \$1,250/oz. Initially, the contract's value is set to \$0, indicating a fair agreement based on current market expectations.
- Post-initiation, the contract's value is subject to change due to market dynamics. For instance, if on 3 November, the market price for December Gold futures falls to \$1,225/oz, the value of the investor's long position declines, rendering a **negative value** to their investment.

3.2 Pricing and Valuation Methodology

- **Risk Aversion:** Investors require compensation for taking on additional risk, reflecting the principle that higher risk should be rewarded with higher potential returns.
- **Risk-Neutral Pricing:** Under this approach, it's assumed that investors are indifferent to risk. The pricing of derivatives through arbitrage opportunities ensures that the portfolio, combining the derivative and its underlying asset, yields the risk-free rate of return.
- **Arbitrage-Free Pricing:** This methodology prices derivatives based on the assumption that the market operates under risk-neutral conditions and is free of arbitrage opportunities, ensuring no risk-free profits can be made from market inefficiencies.

The framework for pricing and valuation operates on the principle that market prices adjust to preclude arbitrage profits. The **Law of One Price** asserts that identical cash flows must have the same price, irrespective of future outcomes.

Arbitrageurs follow two cardinal rules:

1. Do not use your own money.
2. Do not take any price risk.

3.2.1 Short Selling

Short selling entails the sale of securities not owned by the seller, typically facilitated by borrowing these securities from a broker. The seller aims to buy back the securities at a lower price to profit from the difference. Obligations include covering dividends and possibly a borrowing fee.

Example: Short Selling

Assuming you short sell 100 shares at \$100 each and close out the short position three months later at \$90, while a \$3 per share dividend is issued during this period:

- Your profit would be the difference in share prices minus the dividends paid, highlighting the risks and rewards of short selling.

$$100 \times (100 - 90 - 3) = \$700$$

- Conversely, buying 100 shares would result in a loss if the share price decreases.

$$100 \times (90 - 100 + 3) = \$ - 700$$

3.2.2 Consumption vs. Investment Assets

- **Investment Assets:** These are assets like gold, stocks, and bonds, held primarily for their potential to appreciate in value over time.
- **Consumption Assets:** Assets such as copper or oil are classified as consumption assets, held mainly for their utility in production or consumption rather than for investment.

3.3 Forward Price for an Investment Asset

3.3.1 Assumptions and Notation

3.3.1.1 Assumptions

1. Market participants incur no transaction costs when trading.
2. All net trading profits are taxed at the same rate for market participants.
3. Market participants can borrow and lend money at the identical risk-free interest rate.
4. Arbitrage opportunities are exploited immediately by market participants.

3.3.1.2 Notation

- S_0: Current spot price of the asset.
- F_0: Today's futures or forward price of the asset.
- T: Time to the delivery date of the contract (in years).

- r: Annualized risk-free interest rate over the period T.

3.3.2 Arbitrage Example

Consider an asset that provides no income, with a current price of \$40, subject to an annual interest rate of 5%, and associated with a forward contract that has a maturity of 3 months.

3.3.2.1 Case 1: Forward Price = \$43

- **Action now:** Borrow \$40 at 5% annual interest for 3 months. Purchase one unit of the asset and enter into a forward contract to sell it in 3 months at \$43.
- **Action in 3 months:** Sell the asset at the forward price of \$43. Repay the loan, which amounts to \$40.50 with interest.
- **Profit realized:** \$2.50.

3.3.2.2 Case 2: Forward Price = \$39

- **Action now:** Short sell one unit of the asset for \$40. Invest the proceeds at 5% annual interest for 3 months. Enter into a forward contract to buy the asset in 3 months at \$39.
- **Action in 3 months:** Buy back the asset at the forward price of \$39. Close the short position and retrieve \$40.50 from the investment.
- **Profit realized:** \$1.50.

3.3.3 The Forward Price Formula

The forward price F_0 for an investment asset that does not provide income and is deliverable in T years is given by the following formula, where r is the annualized risk-free interest rate for the period T.

$$F_0 = S_0 e^{rT}$$

In the given example, with $S_0 = 40$, $T = 0.25$ (3 months), and $r = 0.05$ (5%), the forward price is calculated as:

$$F_0 = 40e^{0.05\times0.25} = 40.50$$

This calculation demonstrates that the no-arbitrage forward price aligns with the theoretical value, preventing arbitrage opportunities under these conditions.

3.3.4 Adjustments for Known Income or Yield

When the asset provides a known income or yield, the forward price adjusts to account for this. For an asset generating a known income I (expressed as the present value of the income) over the contract's life, the formula modifies to:

$$F_0 = (S_0 - I)e^{rT}$$

For assets providing a yield q (expressed as a continuous compounding rate), the forward price formula adjusts to:

$$F_0 = S_0 e^{(r-q)T}$$

3.4 Valuing a Forward Contract

Initially, a forward contract's value is zero, except for potential impacts from the bid-offer spread. This neutrality in value reflects the agreement's fairness based on current market conditions.

As time progresses, the value of the forward contract can become positive or negative, reflecting changes in market conditions and the underlying asset's price relative to the contract's terms.

Let K represent the agreed-upon delivery price in the contract, and F_0 denote the forward price for a contract negotiated today for future delivery.

The valuation of forward contracts can be understood by comparing contracts with different delivery prices. This comparison reveals:

- The value of being in a long position in a forward contract, where one has agreed to buy the asset, is $(F_0 - K)e^{-rT}$. This formula represents the present value of the profit from the contract if the forward price F_0 set today is higher than the contract's delivery price K.
- Conversely, the value of a short forward contract, where one has agreed to sell the asset, is $(K - F_0)e^{-rT}$. This expresses the present value of the profit from the contract if the delivery price K exceeds today's forward price F_0.

These valuations underscore the importance of the risk-free rate r and the time to delivery T in determining the present value of future contract profits or losses.

3.5 Forward vs. Futures Prices

While forward and futures contracts are similar in their function of agreeing to buy or sell an asset at a future date, their pricing can diverge under certain conditions, despite the common assumption that they are equal when the maturity and asset prices coincide. One notable exception is Eurodollar futures, which often exhibit pricing anomalies due to their specific market characteristics.

The theoretical distinction in pricing between forwards and futures arises primarily from the volatility and uncertainty of interest rates:

- When there is a strong positive correlation between interest rates and the asset's price, futures prices tend to be slightly higher than forward prices. This is because the daily settlement of futures can lead to a net gain in environments where rising interest rates accompany rising asset prices, due to the reinvestment of gains at higher rates.
- Conversely, a strong negative correlation between interest rates and the asset's price suggests that forward prices may exceed futures prices. In such scenarios, the absence of daily settlement in forward contracts avoids the potential loss from having to reinvest at lower interest rates.

3.6 Forward Price for Specific Assets

3.6.1 Stock Index

A stock index can be regarded as an investment asset that effectively pays a dividend yield, analogous to the income generated by holding the underlying stocks.

The relationship between the futures price (F_0) and the spot price (S_0) of the index is captured by the formula:

$$F_0 = S_0 e^{(r-q)T}$$

Here, q represents the average dividend yield of the portfolio reflected by the index over the contract's life.

This formula assumes the index acts as a tradable investment asset, with changes in the index mirroring changes in the value of a corresponding tradable portfolio. For instance, the Nikkei 225, when viewed purely as a numerical value without considering its underlying assets, does not qualify as an investment asset in this context.

Index Arbitrage

Index arbitrage exploits discrepancies between the futures price and the spot price adjusted for the dividend yield and risk-free rate. When F_0 exceeds $S_0 e^{(r-q)T}$, arbitrageurs purchase the underlying stocks of the index while selling futures. Conversely, if $F_0 < S_0 e^{(r-q)T}$, they buy futures and short sell the index's underlying stocks.
This strategy requires executing trades in futures and various stocks simultaneously, often facilitated by computer algorithms to manage the complexity and timing. However, real-world frictions sometimes prevent perfect execution, leading to temporary deviations from the theoretical no-arbitrage relationship.

3.6.2 Exchange Rates

A foreign currency functions similarly to a security that yields interest, where the yield is the foreign country's risk-free interest rate.

Given r_f as the foreign risk-free interest rate, the forward exchange rate formula is:

$$F_0 = S_0 e^{(r-r_f)T}$$

Exchange Rate Arbitrage

Consider two strategies for converting 1,000 units of a foreign currency into dollars by time T, where S_0 represents the current spot exchange rate, F_0 the forward exchange rate, and r and r_f the domestic and foreign risk-free interest rates, respectively.

- **Strategy 1:** Convert the 1,000 units of foreign currency into dollars at the future time T by initially investing these units at the foreign risk-free rate r_f. Then, agree to a forward contract to sell the resulting amount for dollars at time T.
- **Strategy 2:** Immediately exchange the 1,000 units of foreign currency for dollars at the current spot rate S_0, then invest this dollar amount at the domestic risk-free rate r until time T.

The principle of no arbitrage implies that both strategies must yield the same outcome in dollar terms to prevent free profit opportunities. This condition leads to the following equation:

$$1,000 \times e^{r_f \times T} \times F_0 = 1,000 \times S_0 \times e^{r \times T}$$

Simplifying this equation provides the formula for the forward exchange rate:

$$F_0 = S_0 e^{(r-r_f)T}$$

3.6.3 Commodities

For **consumption assets** like commodities, storage costs represent a form of negative income, impacting the forward price:

$$F_0 \leq S_0 e^{(r+u)T}$$

Here, u stands for the storage cost per unit time as a percentage of the asset's value. Alternatively, incorporating the present value of storage costs U:

$$F_0 \leq (S_0 + U)e^{rT}$$

These formulas accommodate the costs associated with holding and storing physical commodities, from agricultural products to metals, affecting their forward pricing.

3.6.4 The Cost of Carry

The cost of carry (c) combines storage costs, interest expenses, and any income earned. For investment assets, the forward price formula simplifies to:

$$F_0 = S_0 e^{cT}$$

In contrast, for consumption assets, the formula accounts for potential lower bounds due to additional costs, hence:

$$F_0 \leq S_0 e^{cT}$$

The concept of a **convenience yield** (y) is introduced for consumption assets, acknowledging the benefits from holding the physical asset as opposed to a derivative position. It's defined such that:

$$F_0 = S_0 e^{(c-y)T}$$

3.7 Futures Prices and Expected Spot Prices

When analyzing futures contracts, we consider the expected return required by investors, denoted as k. This expected return plays a critical role in determining the futures price F_0 relative to the expected spot price at maturity $E(S_T)$.

An investment strategy involving futures contracts can be described as follows:

- By investing an amount $F_0 e^{-rT}$ at the risk-free rate r and entering a long futures contract, an investor can secure a cash inflow of S_T at maturity.
- The present value (PV) of this investment strategy is calculated as $-F_0 e^{-rT} + E(S_T)e^{-kT}$, where E denotes the expected value.

In efficient markets, investments are typically priced to yield a zero net present value (NPV). This principle leads to the fundamental pricing relation for futures contracts:

$$F_0 = E(S_T)e^{(r-k)T}$$

The relationship between futures prices and expected future spot prices varies with the underlying asset's systematic risk:

- **No Systematic Risk**: When the asset carries no systematic risk, the expected return from the asset k equals the risk-free rate r, leading to $F_0 = E(S_T)$.
- **Positive Systematic Risk**: For assets with positive systematic risk, such as stock indices, $k > r$ implies that futures prices will be lower than the expected spot prices $F_0 < E(S_T)$.
- **Negative Systematic Risk**: Conversely, assets like gold, which may exhibit negative systematic risk during certain periods, have $k < r$, resulting in futures prices exceeding expected spot prices $F_0 > E(S_T)$.

Contango and Backwardation

- CME Institute
- **Backwardation**: This market condition occurs when futures prices are below the expected future spot prices, often observed in markets expecting lower future prices.
- **Contango**: In contrast, a contango market condition is characterized by futures prices that exceed expected future spot prices, typically occurring in markets where future prices are anticipated to rise.

3.8 Forward Rate Agreement (FRA)

A Forward Rate Agreement (FRA) is a financial contract between two parties to exchange interest payments for a predetermined amount of principal at a future date. The agreement allows one party to pay a fixed interest rate and receive a floating interest rate, or vice versa. The key characteristics of FRAs make them valuable tools for financial risk management and speculation.

LIBOR (London Interbank Offered Rate) has been the predominant reference rate for FRAs. However, with the phase-out of LIBOR, alternative reference rates such as SOFR (Secured Overnight Financing Rate) and SONIA (Sterling Overnight Index Average) are gaining prominence.

i Example: FRA

Consider an FRA where Party A and Party B agree to exchange a fixed rate of 3% for three-month SOFR on a principal of \$100 million in two years. The interest rates are compounded quarterly.

- **Position of Parties:** Party A will pay the floating SOFR rate and receive a fixed 3%. Party B takes the opposite position.
- **Outcome:** If the three-month SOFR rate is 3.5% in two years, Party A receives a net payment from Party B calculated as follows:

$$\$100,000,000 \times (0.035 - 0.030) \times 0.25 = \$125,000$$

This amount would ideally be paid at 2.25 years. However, considering the advance determination of SOFR, the actual payment is adjusted to its present value and made at the 2-year mark (discounted for three months at 3.5%).

3.8.1 Advantages of Using FRAs

- **Risk Management:** FRAs allow businesses and financial institutions to hedge against interest rate fluctuations, securing borrowing costs or investment returns in advance.
- **Speculation:** Traders can speculate on the direction of future interest rates, entering FRAs to profit from their predictions.
- **Customization:** FRAs can be tailored to specific needs, such as the amount, duration, and the starting date of the agreement, providing flexibility.

3.8.2 Valuation and Pricing

- **Initial Value:** At inception, the fixed rate of an FRA is set equal to the prevailing forward rate for the contract period, making the initial value of the contract zero.
- **Value Over Time:** The value of an FRA changes as market interest rates fluctuate. If market rates rise above the FRA rate, the buyer (fixed rate payer) benefits, and vice versa.
- **Settlement:** The difference between the agreed fixed rate and the reference floating rate at the settlement date is paid by the losing party to the winning party, adjusted for the contract's term.

3.9 Practice Questions and Problems

3.9.1 Basics of Forward and Futures Contracts

1. Explain what happens when an investor shorts a certain share.
2. What is the difference between the forward price and the value of a forward contract?
3. What is meant by (a) an investment asset and (b) a consumption asset. Why is the distinction between investment and consumption assets important in the determination of forward and futures prices?
4. Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.
5. Explain carefully the meaning of the terms convenience yield and cost of carry. What is the relationship between futures price, spot price, convenience yield, and cost of carry?
6. What is the cost of carry for (a) a non-dividend-paying stock, (b) a stock index, (c) a commodity with storage costs, and (d) a foreign currency?

3.9.2 Calculation of Forward and Futures Prices

1. Suppose that you enter into a three-month forward contract on a non-dividend-paying stock when the stock price is \$108 and the risk-free interest rate (with continuous compounding) is 4% per annum. What is the forward price?
2. A four-months long forward contract on a non-dividend-paying stock is entered into when the stock price is \$150 and the risk-free rate of interest is 5.7% per annum with continuous compounding.
   - a) What are the forward price and the initial value of the forward contract?
   - b) Two months later, the price of the stock is \$168 and the risk-free interest rate is still 5.7%. What are the forward price and the value of the forward contract?

3. The risk-free rate of interest is 4.1% per annum with continuous compounding, and the dividend yield on a stock index is 6.2% per annum. The current value of the index is 2445. What is the one-month futures price?
4. A stock index currently stands at 725. The risk-free interest rate is 7.6% per annum (with continuous compounding) and the dividend yield on the index is 1.8% per annum. What should the futures price for a three-month contract be?
5. An index is 550. The three-month risk-free rate is 4.60% per annum and the dividend yield over the next three months is 5.80% per annum. The six-month risk-free rate is 5.34% per annum and the dividend yield over the next six months is 4.93% per annum. Estimate the futures price of the index for three-month and six-month contracts. All interest rates and dividend yields are continuously compounded.
6. The spot price of silver is $11 per ounce. The storage costs are $0.25 per ounce per year payable quarterly in advance. Assuming that interest rates are 1.80% per annum for all maturities, calculate the futures price of silver for delivery in nine months.
7. The spot price of oil is $39 per barrel and the cost of storing a barrel of oil for one year is $1.2, payable at the end of the year. The risk-free interest rate is 8.60% per annum, continuously compounded. What is an upper bound for the one-year futures price of oil?

3.9.3 Arbitrage and Risk Management

1. Suppose that the risk-free interest rate is 3.00% per annum with continuous compounding and that the dividend yield on a stock index is 0.60% per annum. The index is standing at 3646, and the futures price for a contract deliverable in ten months is 3701. What arbitrage opportunities does this create?
2. The eight-month interest rates in Switzerland and the United States are, respectively, 3.60% and 5.40% per annum with continuous compounding. The spot price of the Swiss franc is $0.9. The futures price for a contract deliverable in two months is also $0.9. What arbitrage opportunities does this create?
3. When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the daily settlement process does leave the company exposed to some risk. Explain the nature of this risk. In particular, consider whether the company is better off using a futures contract or a forward contract when

   - a) The value of the foreign currency falls rapidly during the life of the contract
   - b) The value of the foreign currency rises rapidly during the life of the contract
   - c) The value of the foreign currency first rises and then falls back to its initial value
   - d) The value of the foreign currency first falls and then rises back to its initial value
   - Assume that the forward price equals the futures price.

4 Swaps

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

5 Options

ℹ References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
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  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

6 Options Trading Strategies and Hedging

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

7 Reading Week

Tip

Take some rest and revise the first six topics!

8 Option Pricing - Binomial Trees

ℹ References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

9 Option Pricing - Black-Scholes Model

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

10 The Greeks and Volatility

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

11 Options on Stock Indices, Currencies, and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

12 Structured Products I

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

13 Structured Products II

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

A Time Value of Money

References

This material was originaly published HERE by Department of Mathematics, Penn State University Park.

Learning Outcomes:

- Understand the concept of the time value of money (TVM).
- Calculate the present value (PV) of both single and multiple future cash flows using appropriate discount rates.
- Calculate the future value (FV) of both single and ongoing investments using given interest rates.
- Apply TVM formulas to various financial scenarios, including loans, savings, and investments, to make informed decisions.
- Understand the effects of compounding frequency on FV and PV calculations.
- Critically analyze TVM problems, taking into account the impact of rate, time, and cash flows on financial decisions.

A.1 Notation and Terminology

A.1.1 Basic Notation and Terminology

- $P =$ Principal (i.e., value of initial deposit)
- $A =$ Accumulated amount (i.e., sum of the principal and interest)
- $r =$ Nominal interest rate
- $m =$ Number of conversion periods per year, (a conversion period is the interval of time between successive interest payments)

Annually	Semiannually	Quarterly	Monthly	Weekly	Daily
$m = 1$	$m = 2$	$m = 4$	$m = 12$	$m = 52$	$m = 365$

- \$t = \$ Term of investment (in years)

A.1.2 Simple Interest

Interest is always computed based on the original principal.

Interest Earned	Accumulated Amount
$I = Prt$	$A = P(1 + rt)$

A.1.3 Discrete Compound Interest

Interest payments are added to the principal at the end of each conversion period and therefore earn interest during future conversion periods.

Accumulated Amount	Present Value Formula
$A = P\left(1 + \frac{r}{m}\right)^{mt}$	$P = A\left(1 + \frac{r}{m}\right)^{-mt}$

A.1.4 Continuous Compound Interest

Continuous compounding of interest is equivalent to a discrete compounding of interest where m, the number of conversion periods per year, goes to infinity.

Accumulated Amount	Present Value Formula
$A = Pe^{rt}$	$P = Ae^{-rt}$

A.1.5 Effective Rate of Interest

The effective interest rate, r_{eff}, is the simple interest rate that produces the same accumulated amount in 1 year as the nominal rate, r, compounded m times a year.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

A.2 Future Value Examples

A.2.1 Example 1

Suppose \$1,000 is deposited into an account with an interest rate of 16% compounded annually. How much money is in the account after 3 years?

Step 1: Since interest is compounded annually, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 1$, and $t = 3$.

$$\begin{aligned} A &= 1000\left(1 + \frac{0.16}{1}\right)^{1\cdot 3} \\ &= 1000\left(1 + 0.16\right)^{3} \\ &= 1000\left(1.16\right)^{3} \approx \$1,560.90 \end{aligned}$$

Therefore, after 3 years of accumulating interest, the original investment of \$1,000 is worth \$1,560.90.

A.2.2 Example 2

Suppose \$1,000 is deposited into an account with an interest rate of 16% compounded quarterly. How much money is in the account after 3 years?

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 4$, and $t = 3$.

$$
\begin{aligned}
A &= 1000\left(1+\frac{0.16}{4}\right)^{4\cdot 3} \\
&= 1000\left(1+0.04\right)^{12} \\
&= 1000\left(1.04\right)^{12} \approx \$1,601.03
\end{aligned}
$$

Therefore, after 3 years of accumulating interest, the original investment of \$1,000 is worth \$1,601.03.

> **Observation**
>
> Compare the accumulated amounts in the above two examples. Both examples have the same principal, interest rate, and term. But since interest is compounded more frequently in Example 2 (4 times a year) than in Example 1 (1 time a year), the accumulated amount is higher in Example 2.

A.2.3 Example 3

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded quarterly.

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$
A = P\left(1+\frac{r}{m}\right)^{mt}
$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), $m = 4$, and $t = 5$.

$$
\begin{aligned}
6000 &= 3000\left(1+\frac{r}{4}\right)^{4\cdot 5} \\
&= 3000\left(1+\frac{r}{4}\right)^{20}
\end{aligned}
$$

Step 3: Solve for the interest rate, r.

Method 1

Divide both sides by 3000

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(2) &= \ln\left[\left(1 + \frac{r}{4}\right)^{20}\right] \\ &= 20\ln\left(1 + \frac{r}{4}\right) && \text{since } \ln(m^n) = n\ln(m) \end{aligned}$$

Divide both sides by 20.

$$\ln(2)/20 = \ln\left(1 + \frac{r}{4}\right)$$

Take the exponential of both sides.

$$\begin{aligned} e^{\ln(2)/20} &= e^{\ln(1+\frac{r}{4})} \\ &= 1 + \frac{r}{4} && \text{since } e^{\ln(x)} = x \end{aligned}$$

Subtract 1 from both sides.

$$e^{\ln(2)/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(e^{\ln(2)/20} - 1) \approx 0.1411.$$

Method 2

Here is an alternate method for solving for the interest rate r. We start with the following equation.

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Instead of taking the natural logarithm of both sides as we did before, now take the 20th root of both sides (i.e., raise both sides to the power of 1/20).

$$2^{1/20} = 1 + \frac{r}{4}$$

Subtract 1 from both sides.

$$2^{1/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(2^{1/20} - 1) \approx 0.1411$$

Note that this value of r is numerically equal in both methods since

$$\begin{aligned} e^{\ln(2)/20} &= e^{\ln(2^{1/20})} && \text{since } n\ln(m) = \ln(m^n) \\ &= 2^{1/20} && \text{since } e^{\ln(x)} = x \end{aligned}$$

Therefore, an interest rate of approximately 14.11% compounded quarterly is required for an investment of \$3,000 to double in value in 5 years.

A.2.4 Example 4

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded continuously.

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), and $t = 5$.

$$6000 = 3000e^{5r}$$

Step 3: Solve for the interest rate, r.

Divide both sides by 3000.

$$2 = e^{5r}$$

Take the natural logarithm of both sides.

$$
\begin{aligned}
\ln(2) &= \ln(e^{5r}) \\
&= 5r && \text{since } \ln(e^x) = x
\end{aligned}
$$

Divide both sides by 5.

$$r = \ln(2)/5 \approx 0.1386$$

Therefore, an interest rate of approximately 13.86% compounded continuously is required for an investment of $3,000 to double in value in 5 years.

> ⚠ Observation
>
> Compare the last two examples. Since continuous compounding of interest earns interest faster than discrete compounding, a lower interest rate is needed for an investment to double in value over a fixed term if interest is compounded continuously. In our examples, an interest rate of 13.86% was needed for the investment with continuous compound interest to double in value in 5 years, while an interest rate to 14.11% was needed for the investment with quarterly compound interest.

A.2.5 Example 5

How long will it take for $5,000 to grow to $8,000 if the investment earns interest at 6% per year compounded monthly?

Step 1: Since interest is compounded monthly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, $m = 12$, and $r = 0.06$.

$$
\begin{aligned}
8000 &= 5000\left(1 + \frac{0.06}{12}\right)^{12\cdot t} \\
&= 5000\left(1 + 0.005\right)^{12t}
\end{aligned}
$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = 1.005^{12t}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(8/5) &= \ln(1.005^{12t}) \\ &= 12t\ln(1.005) && \text{since } \ln(m^n) = n\ln(m) \end{aligned}$$

Divide both sides by $12\ln(1.005)$.

$$t = \frac{\ln(8/5)}{12\ln(1.005)} \approx 7.85$$

Therefore, it will take approximately 7.85 years for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded monthly.

A.2.6 Example 6

How long will it take for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded continuously?

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, and $r = 0.06$.

$$8000 = 5000e^{0.06t}$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = e^{0.06t}$$

Take the natural logarithm of both sides.

$$
\begin{aligned}
\ln(8/5) &= \ln(e^{0.06t}) \\
&= 0.06t && \text{since } \ln(e^x) = x
\end{aligned}
$$

Divide both sides by 0.06.

$$t = \frac{\ln(8/5)}{0.06} \approx 7.83$$

Therefore, it will take approximately 7.83 years for $5,000 to grow to $8,000 if the investment earns interest at 6% per year compounded monthly.

> ⚠ Observation
>
> Compare the last two examples. Both examples have the same principal, accumulated amount, and interest rate. But since continuous compounding of interest earns interest faster than discrete compounding, it should take less time for the investment to grow to $8,000 if interest is compounded continuously.

A.2.7 Example 7

Find the effective interest rate corresponding to a nominal interest rate of 10% compounded semiannually.

Step 1: Recall the formula for effective interest rate, r_{eff}.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

Step 2: Plug in the given values: $r = 0.1$ and $m = 2$.

$$
\begin{aligned}
r_{\text{eff}} &= \left(1 + \frac{0.1}{2}\right)^2 - 1 \\
&= 1.05^2 - 1 \\
&= 0.1025
\end{aligned}
$$

Therefore, an investment earning interest compounded semiannually at 10% earns the same amount of interest after 1 year as an investment earning simple interest at 10.25%.

A.2.8 Example 8

Suppose you have \$12,000 in the bank earning interest at a rate of 12% compounded quarterly. Your cousin calls you and needs \$12,000 to buy a new car. You are willing him to loan him the money, but you'd hate to lose out on the interest you would gather by simply leaving your money alone. If you charge your cousin an interest rate compounded continuously, what rate should you charge in order to earn the same amount of interest you otherwise would have?

Step 1: Assume your cousin is prepared to pay you back after t years.

We'll use t as the term in each of the following calculations. Eventually, we'll see that the interest rate you charge does not depend on the specific value of t.

Step 2: Compute the accumulated amount of the \$12,000 after t years assuming you leave your money in the bank.

$$\begin{aligned} A &= P\left(1+\frac{r}{m}\right)^{m\cdot t} \\ &= 12000\left(1+\frac{0.12}{4}\right)^{4t} \\ &= 12000\left(1.03\right)^{4t} \end{aligned}$$

Step 3: Compute the accumulated amount of the \$12,000 after t years assuming you let your cousin borrow the money.

This would be the amount that your cousin repays you after t years.

$$\begin{aligned} A &= Pe^{rt} \\ &= 12000e^{rt} \end{aligned}$$

Step 4: Equate the two accumulated amounts and solve for r.

$$12000\left(1.03\right)^{4t} = 12000e^{rt}$$

Divide both sides by 12000.

$$1.03^{4t} = e^{rt}$$

Take the natural logarithm of both sides.

$$\ln(1.03^{4t}) = \ln(e^{rt})$$

Simplify using properties of logarithms ($\ln(m^n) = n\ln(m)$ and $\ln(e^x) = x$).

$$4t\ln(1.03) = rt$$

And finally, divide both sides by t. Here is where we see that the time it would take your cousin to repay you does not affect the interest rate you would charge.

$$r = 4\ln(1.03) \approx 0.1182$$

Therefore, charging your cousin 11.82% interest compounded continuously earns the same amount of interest as leaving your money in the bank earning interest at a rate of 12% compounded quarterly.

A.3 Present Value Examples

A.3.1 Example 1

How much money should be deposited in a bank paying a yearly interest rate of 6% compounded monthly so that after 3 years, the accumulated amount will be $20,000?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded monthly, we'll use the present value formula for discrete compounding of interest.

$$P = A\left(1 + \frac{r}{m}\right)^{-mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$P = 20000\left(1 + \frac{0.06}{12}\right)^{-(12)(3)}$$

$$= 20000(1.005)^{-36} \approx \$16,712.90$$

Therefore, $16,712.90 invested at 6% interest compounded monthly will be worth $20,000 in 3 years.

A.3.2 Example 2

Use the accumulated amount for discrete compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for discrete compounding of interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$\begin{aligned} 20000 &= P\left(1 + \frac{0.06}{12}\right)^{(12)(3)} \\ &= P(1.005)^{36} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{20000}{1.005^{36}} \approx \$16,712.90$$

A.3.3 Example 3

Parents wish to establish a trust fund for their child's education. If they need \$170,000 in 7 years, how much should they set aside now if the money is invested at 20% compounded continuously?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded continuously, we'll use the present value formula for continuous compounding of interest.

$$P = Ae^{-rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$P = 170,000e^{-(0.2)(7)}$$

$$= 170,000e^{-1.4} \approx \$41,921.48$$

Therefore, \$41,921.48 invested at 20% interest compounded continuously will be worth \$170,000 in 7 years.

A.3.4 Example 4

Use the accumulated amount for continuous compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for continuous compounding of interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$\begin{aligned} 170000 &= Pe^{(0.2)(7)} \\ &= Pe^{1.4} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{170000}{e^{1.4}} \approx \$41,921.48$$

A.4 Try It Yourself

A.4.1 Exercise 1

If \$6,00 is invested at 7% compounded continuously, what will be the accumulated amount after 6 years?

Show answer

$A = 6000e^{0.42}$

A.4.2 Exercise 2

If \$7,000 is invested at 16% compounded quarterly, what will be the accumulated amount after 3 years?

Show answer

$A = 7000(1.04)^{12}$

A.4.3 Exercise 3

Find the interest rate r needed for an investment of \$2,000 to grow to \$8,000 in 7 years if compounded continuously.

Show answer

$r = \ln(4)/7$

A.4.4 Exercise 4

Find the interest rate r needed for an investment of \$7,000 to grow to \$12,000 in 21 years if compounded monthly.

Show answer

$r = 12\left[(12/7)^{1/252} - 1\right]$

A.4.5 Exercise 5

Find the time it would take for an investment of \$1,000 to grow to \$100,000 if interest is compounded quarterly at an annual rate of 8%.

Show answer

$t = \frac{\ln(100)}{4\ln(1.02)}$

A.4.6 Exercise 6

Find the time it would take for an investment of \$2,500 to grow to \$6,000 if interest is compounded continuously at an annual rate of 24%.

Show answer

$t = \frac{25}{6} \ln(12/5)$

A.4.7 Exercise 7

Calculate the effective rate of interest corresponding to a nominal interest rate of 52% compounded weekly.

Show answer

$r_{eff} = 1.01^{52} - 1$

A.4.8 Exercise 8

Your grandma would like to establish a trust fund for your education. How much should she set aside now if she wants \$50,000 in 9 years and interest is compounded monthly at an annual rate of 12%?

Show answer

$P = 50000(1.01)^{-108}$

A.4.9 Exercise 9

You are preparing to run for president and want to have \$100,000 in 6 years to start your campaign. How much money do you need now if interest is compounded continuously at an annual rate of 15%?

Show answer

$P = 100000e^{-0.9}$

A.4.10 Exercise 10

You have \$50,000 in the bank earning 7% interest compounded quarterly. However, your cousin needs a \$50,000 investment to start up his new financial consulting business. In order to get the same total return as leaving your money in the bank, what interest rate r should you request from your cousin if interest is compounded continuously?

Show answer

$r = 4\ln(1 + 0.07/4)$

B Interest Rates

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 4 - Interest Rates

Learning Outcomes:

- Understand the different types of interest rates.
- Define the risk-free rate and its significance in financial derivatives.
- Explain the concept of continuous compounding and its importance in pricing financial derivatives.

B.1 Types of Rates

B.1.1 Treasury Rate

- Rate on instrument issued by a government in its own currency.

B.1.2 The U.S. Fed Funds Rate

- Unsecured interbank overnight rate of interest.
- Allows banks to adjust the cash (i.e., reserves) on deposit with the Federal Reserve at the end of each day.
- The effective fed funds rate is the average rate on brokered transactions.
- The central bank may intervene with its own transactions to raise or lower the rate.
- Similar arrangements in other countries.

B.1.3 Repo Rate

- Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them for X and buy them bank in the future (usually the next day) for a slightly higher price, Y.
- The financial institution obtains a loan.
- The rate of interest is calculated from the difference between X and Y and is known as the repo rate.

B.1.4 LIBOR (ICE LIBOR)

- Detailed information about LIBOR: https://www.theice.com/iba/libor
- LIBOR is the rate of interest at which a AA bank can borrow money on an **unsecured** basis from another bank.
- Based on **submissions** from a panel of contributor banks (16 for each of USD and GBP).
- It is calculated daily for 5 currencies and 7 maturities.
- There have been some suggestions that banks manipulated LIBOR during certain periods.
- Why would they do this?

B.2 Alternative Reference Rates

Country/Currency/CODE	IBOR Rate	New Reference Rate
USA/Dollars/USD	USD ICE LIBOR	SOFR
UK/Pounds Sterling/GBP	GBP ICE LIBOR	SONIA
Switzerland/Swiss Francs/CHF	CHF ICE LIBOR	SARON
Japan/Yen/JPY	JPY ICE LIBOR, Tibor	TONAR
EU/Euro/EUR	Euribor	ESTER

B.2.1 SOFR (Secured Overnight Financing Rate)

- CME Group Education
- Administered by Federal Reserve Bank of New York (link)
- Transaction-based, calculated from overnight US Treasury repurchase (repo) activity.
- SOFR is a broad measure of the cost of borrowing USD cash overnight, collateralized by U.S. Treasury securities.
- SOFR is a good representation of general funding conditions in the overnight Treasury repo market.

- As such, it will reflect an economic cost of lending and borrowing relevant to the wide array of market participants active in the market.

B.2.2 SONIA (Sterling Overnight Index Average)

- CME Group Education
- Administered by Bank of England (link)
- Unsecured transaction-based index, wholesale based (beyond Interbank)
- It has been endorsed by the Sterling Risk-Free Reference Rate Working Group (Working Group) as the preferred risk-free reference rate for Sterling Overnight Indexed Swaps (OIS).
- In January 2018, the Working Group added banks, dealers, investment managers, non-financial corporates, infrastructure providers, trade associations and professional services firms.
- In April 2018, the BOE introduced a series of reforms of the SONIA benchmark.

B.2.3 €STR (or ESTER, Euro Short-Term Rate)

- Administered by European Central Bank (link)
- It is based on the unsecured market segment.
- The ECB developed an unsecured rate, because it is intended to complement the EONIA.
- Furthermore, a secured rate would be affected by the type of the collaterals.
- The money market statistical reporting covers the 50 largest banks in the euro area in terms of balance sheet size.
- While the EONIA (link) reflects the interbank market, the €STR extends the scope to money market funds, insurance companies and other financial corporations because banks developed significant money market activity with those entities.

B.3 OIS Rate

- An **overnight indexed swap** is swap where a fixed rate for a period (e.g. 3 months) is exchanged for the geometric average of overnight rates (or overnight rate compounded over the term of the swap).
- The underlying floating rate is typically the rate for overnight lending between banks, either non-secured or secured (SOFR, SONIA, €STR).
- For maturities up to one year there is a single exchange (swap term is not overnight).
- For maturities beyond one year there are periodic exchanges, e.g. every quarter.
- The OIS rate is a continually refreshed overnight rate.
- The fixed rate of OIS is typically an interest rate considered less risky than the corresponding interbank rate (LIBOR) because there is limited counterparty risk.

B.3.1 The Risk-Free Rate

- The Treasury rate is considered to be artificially low because:
  - Banks are not required to keep capital for Treasury instruments
  - Treasury instruments are given favorable tax treatment in the US
- OIS rates are now used as a proxy for risk-free rates in derivatives valuation.

B.4 Time Value of Money

B.4.1 Compounding Frequency

- When we compound m times per year at rate r an amount P grows to $P(1 + r/m)^m$ in one year.
- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
- Effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	Value of $100 at end of year ($)
Annually m = 1	110.00
Semiannually m = 2	110.25
Quarterly m = 4	110.38
Monthly m = 12	110.47
Weekly m = 52	110.51
Daily m = 365	110.52

B.4.2 Continuous Compounding

- **Rates used in option pricing are nearly always expressed with continuous compounding.**
- In the limit as we compound more and more frequently we obtain continuously compounded interest rates.
- Notation:
  - r: continuously compounded annual interest rate
  - T: time to maturity in years
  - e: Euler's number (mathematical constant)

$$\text{Future value} = P \times e^{rT}$$

$$\text{Present value} = P \times e^{-rT}$$

- USD 100 grows to $100 \times e^{rT}$ when invested at a continuously compounded rate r for time T.
- USD 100 received at time T discounts to $100 \times e^{-rT}$ at time zero when the continuously compounded discount rate is r.

B.4.3 Conversion Formulas

- r_c: continuously compounded rate
- r_m: same rate with compounding m times per year

$$r_c = m \ln(1 + \frac{r_m}{m})$$

$$r_m = m(e^{r_c/m} - 1)$$

Examples:

- 10% with semiannual compounding is equivalent to $2 \ln(1.05) = 9.758\%$ with continuous compounding.
- 8% with continuous compounding is equivalent to $4(e^{0.08/4} - 1) = 8.08\%$ with quarterly compounding.