Financial Derivatives

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Preamble

The course Financial Derivatives delves into the intricacies of derivative instruments, focusing on both fixed and conditional contracts, including forwards, futures, swaps, and options. It explores the utilization of derivatives within financial markets, particularly for hedging against various risks. Students will gain a comprehensive understanding of the essence and valuation of financial derivatives, along with detailed insights into the exchange trading mechanisms for futures and options.

Upon completing this course, the student will be able to:

- define financial derivatives and understand their specific characteristics
- explain the differences between fixed future contracts and options
- price financial derivatives
- suggest their use

References:

- HULL, John. Options, futures, and other derivatives. Global edition. Harlow: Pearson, 2018. ISBN 9781292212890.
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. xix, 597. ISBN 9781119381815.
- BLÜMKE, Andreas. How to invest in structured products : a guide for investors and investment advisors. Chichester: Wiley, 2009. xvi, 374. ISBN 9780470746790.

1 Introduction to Financial Derivatives

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Define and explain the concept and significance of derivatives in financial markets.
- Describe the structure and function of derivative markets, including exchange-traded and OTC markets.
- Differentiate between unconditional (forward commitments) and conditional (contingent claims) derivatives.
- Identify and briefly describe the main types of derivatives: forwards, futures, swaps, and options.
- Understand the applications of derivatives in hedging, speculation, and arbitrage.
- Recognize common criticisms and potential misuses of derivatives, and understand key regulatory safeguards.

1.1 What Are Derivatives?

- A **derivative** is a financial contract whose value is derived from the price of an **underlying asset**.
- The **underlying asset** can be a stock, currency, interest rate, commodity, bond, index, or even non-financial elements like weather conditions or insurance claims.
- The market price of the underlying is called the **cash price** or **spot price**.

1.1.1 The Significance of Derivatives

Derivatives play a critical role in modern financial markets by offering several key benefits:

- **Risk Management:** Allow parties to hedge and transfer various types of financial risks.
- **Investment Strategies:** Enable the construction of complex strategies and returns beyond simple stock or bond investments.
- **Market Expectations:** Reflect future price expectations and offer valuable market information.
- **Cost Efficiency:** Reduce transaction costs compared to trading the underlying asset.
- **Capital Efficiency:** Require lower initial capital outlay due to leverage.
- **Short Selling:** Simplify taking short positions compared to shorting the underlying directly.
- **Liquidity and Market Efficiency:** Improve liquidity and enhance the functioning of the underlying asset markets.

1.2 Derivative Markets

The global derivatives market is one of the largest financial markets. Derivatives are traded in two main venues:

- **organized exchanges** and
- **over-the-counter (OTC)** markets.

Feature	Exchange-Traded Derivatives	OTC Derivatives
Contract Standardization	Standardized terms (size, expiry, asset)	Fully customizable
Clearing Mechanism	Guaranteed by a clearing house	No central clearing (except post-2008 for some)
Counterparty Risk	Low (due to clearing house)	Higher credit risk
Transparency	High (regulated, prices public)	Lower (private negotiations)
Flexibility	Limited	High
Capital and Margin Requirements	Margin required	Negotiated, often lower pre-2008

1.2.1 OTC Market Evolution

- **Prior to 2008:** Largely unregulated, dominated by banks as market makers; transactions governed by master agreements, some cleared through central counterparties (CCPs).
- **Since 2008:** Regulatory reforms introduced:
  - Standardized OTC transactions must be cleared through CCPs.
  - All trades must be reported to a central repository.
  - Aim: Reduce systemic risk and enhance transparency.

The Lehman Bankruptcy

- **Event:** Lehman Brothers filed for bankruptcy on September 15, 2008—the largest in U.S. history.
- **Role in OTC Market:** Heavily involved in high-risk derivatives.
- **Cause:** Inability to refinance short-term debt.

At the time of its bankruptcy, Lehman Brothers had an extensive network of transactions, with hundreds of thousands outstanding across approximately 8,000 counterparties. The process of unwinding these transactions has posed significant challenges for both the Lehman liquidators and the involved counterparties, illustrating the complex and interconnected nature of modern financial markets.

1.3 Unconditional vs. Conditional Derivatives

Both unconditional (Forward Commitments) and conditional derivatives (Contingent Claims) are essential financial instruments that derive their value from the performance of an underlying asset, playing pivotal roles in global financial markets for hedging, speculation, and arbitrage.

Unconditional Derivatives (Forward Commitments)

Unconditional derivatives create a binding **obligation** to buy or sell an asset at a predetermined future date and price. They include:

- Forwards
- Futures
- Swaps

Conditional Derivatives (Contingent Claims)

Conditional derivatives offer the holder **the right, but not the obligation**, to buy or sell an asset under specified conditions. The primary form is:

- Options

1.4 Types of Derivatives

1.4.1 Forward Contract

Definition

A **forward contract** is a customized, over-the-counter derivative agreement between two parties, where the buyer agrees to purchase, and the seller agrees to sell, an underlying asset at a predetermined future date and price established at the contract's inception (forward price).

- **Long Position**: The party committing to purchase the asset.
- **Short Position**: The party committing to sell the asset.

1.4.1.1 Key Points

- **Popularity in Foreign Exchange**: Forward contracts are frequently used for hedging in the foreign exchange markets.
- **OTC Markets**: Typically involves at least one financial institution, allowing for customization.
- **Market Dynamics and Pricing**: Forward prices reflect the market's consensus on future price movements, adjusted for the time value of money, influenced by factors such as the underlying asset's current price, interest rates, and expected volatility.

1.4.2 Futures Contract

Definition

A **futures contract** is a standardized derivative, similar to a forward contract but traded on futures exchanges, like the CME Group or Intercontinental Exchange, facilitating the buying and selling of underlying assets at future dates.

Forwards	Futures
Customized terms, traded over-the-counter.	Standardized terms, traded on regulated exchanges.
Counterparty risk, with less regulatory oversight.	Mitigated counterparty risk through clearinghouses.
Settlement occurs at contract maturity.	Daily mark-to-market settlement.

1.4.3 Swap Contract

Definition

A **swap** is an OTC derivative where two parties exchange cash flow series over time. It can be viewed as a series of forward contracts. Swaps address multi-period risks and are commonly used to manage interest rate, currency, or commodity exposure.

- **Key Usage**: Interest rate swaps exchange fixed for floating interest rate payments to manage interest rate risk, while currency swaps exchange cash flows in different currencies to hedge currency risk.

1.4.4 Options

Definition

Options are versatile financial derivatives allowing the holder to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specific timeframe. The buyer of the option pays a premium to the seller (writer) for this right, without the obligation to execute the transaction.

1.4.4.1 Types of Options

- **Call Options**: Grant the holder the right to **purchase** the underlying asset at the strike price. Investors buy calls when they anticipate the underlying asset's price will increase.
- **Put Options**: Provide the holder the right to **sell** the underlying asset at the strike price. Puts are purchased when an investor expects the underlying asset's price to decline.

1.4.4.2 Exercise Styles

- **American Options**: Can be exercised at any point up to and including the expiration date, offering maximum flexibility to the holder.
- **European Options**: Can only be exercised on the expiration date itself, limiting the timing of execution to this single moment.

1.5 Applications of Derivatives

1.5.1 Hedging

Hedging aims to **reduce risk** by protecting against adverse price movements. It is widely used by businesses, investors, and financial institutions to manage exposure.

- **Forward Contracts**: Lock in a future price, eliminating uncertainty.
- **Options**: Provide insurance-like protection while allowing upside potential; the cost is the premium paid.

> **i** Examples
>
> 1. **Currency Hedge:** A U.S. company expects to pay GBP 10 million in 3 months. It uses a forward contract to secure a fixed exchange rate, avoiding currency risk.
> 2. **Stock Hedge:** An investor holds 1,000 Microsoft shares at \$28 each. To protect against price drops, they buy put options with a \$27.50 strike price, limiting losses while keeping upside potential. A two-month put option costs \$1 per share.

1.5.2 Arbitrage

Arbitrage exploits **price differences** across markets to earn **risk-free profits**. It relies on the **Law of One Price**: identical assets should have the same price.

- **Buy low in one market, sell high in another**.
- Arbitrage aligns prices and improves market efficiency.

> **i** Example
>
> A stock trades at GBP 100 in London and USD 150 in New York, with the exchange rate at 1.5300 GBP/USD.
>
> - Buy in New York for USD 150.
> - Sell in London for GBP 100.

- Convert GBP 100 to USD 153 for a **risk-free profit of USD 3**.

1.5.3 Speculation

Speculators seek **profit from price movements**, accepting higher risk for potential reward.

- **Futures**: High risk and reward; obligates buying/selling at a future date.
- **Options**: Lower risk; limited loss (premium), but potential for large gains.

Examples

1. **Buying Shares:** An investor with $2,000 buys stock directly, gaining full exposure to price movements.
2. **Buying Call Options:** The investor buys call options, controlling more stock with less money; loss is limited to the premium.
3. **Futures:** The investor takes a futures position betting on a price increase; potential for large gains but also large losses.

1.6 Criticisms and Misuses of Derivatives

While derivatives are vital for risk management and price discovery, they have been criticized for their potential misuse and the risks they can pose to the financial system. Key concerns include:

1. **Excessive Speculation:** Derivatives can encourage speculative trading akin to gambling, leading to large losses when markets move unexpectedly.
2. **Systemic Risk:** Large-scale or complex derivative positions can amplify financial instability. Failures can spread through interconnected markets, as seen in the 2008 crisis.
3. **Complexity:** Many derivatives are difficult to understand and value, even for experienced investors, increasing the risk of mispricing and misuse.
4. **Blurring of Roles:** Market participants may shift between hedging, speculation, and arbitrage, leading to excessive risk-taking under the guise of risk management.
5. **Need for Regulation:** Without adequate oversight, derivatives can be misused, increasing market instability. Strong controls ensure derivatives serve their intended purpose.

1.6.1 Risk Mitigation and Regulatory Measures

- **Central Clearing Counterparties (CCPs):** Reduce counterparty risk and increase transparency by acting as intermediaries.

- **Margin Requirements:** Ensure traders hold sufficient capital to cover potential losses.
- **Post-2008 Reforms:** Regulations like the **Dodd-Frank Act (U.S.)** and **EMIR (EU)** enhance market transparency, reduce systemic risk, and promote responsible use of derivatives.

1.7 Practice Questions and Problems

1.7.1 Time Value of Money

1. A bank quotes an interest rate of 7% per annum with quarterly compounding. How much you will earn from $100 investment after (a) 1 year and (b) 3 years? What is the equivalent rate with (a) continuous compounding and (b) annual compounding? Verify your results.

1.7.2 Theoretical Foundations of Derivatives

1. Explain carefully the difference between hedging, speculation, and arbitrage.
2. What is the difference between the over-the-counter market and the exchange-traded market?
3. "Options and futures are zero-sum games." What do you think is meant by this statement?
4. What is the difference between a long forward position and a short forward position?
5. What is the difference between entering into a long forward contract when the forward price is $50 and taking a long position in a call option with a strike price of $50?
6. Explain carefully the difference between selling a call option and buying a put option.
7. When first issued, a stock provides funds for a company. Is the same true of an exchange-traded stock option? Discuss.

1.7.3 Practical Applications of Forwards and Futures

1. A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?
2. An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.5000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.4900 and (b) 1.5200?

1.7.4 Practical Applications of Options

1. A trader buys a call option with a strike price of \$30 for \$3. Does the trader ever exercise the option and lose money on the trade. Explain.
2. Suppose that you write a put contract with a strike price of \$40 and an expiration date in three months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?
3. Suppose you own 5,000 shares that are worth \$25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?
4. You would like to speculate on a rise in the price of a certain stock. The current stock price is \$29, and a three-month call with a strike of \$30 costs \$2.90. You have \$5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

1.7.5 Hedging Strategies

1. Explain why a futures contract can be used for either speculation or hedging.
2. A US company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract and (b) an option.
3. The CME Group offers a futures contract on long-term Treasury bonds. Characterize the investors likely to use this contract.

2 Forwards and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 2 - Mechanics of futures markets
  - Chapter 3 - Hedging Strategies Using Futures
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Understand forwards and futures, including their characteristics, payoff structures, and key differences.
- Gain knowledge of exchange and over-the-counter (OTC) markets, focusing on their functionalities and distinctions.
- Learn the basics of hedging with futures, managing basis risk, and applying cross hedging techniques.
- Explore the role of stock index futures in portfolio risk management and speculation.

2.1 Forwards and Futures Characteristics

Definition

Forwards and futures are agreements where two parties **commit to** exchange an asset at a set price on a future date.

Forward/Futures Price:

- The agreed-upon price is fixed when the contract starts.
- Different expiration dates can lead to different prices based on market expectations.

Positions:

- **Long Position:** The buyer promises to purchase the asset.
  - Benefits if the asset's price goes up.
- **Short Position:** The seller promises to sell the asset.
  - Benefits if the asset's price goes down.

Risk Exposure:

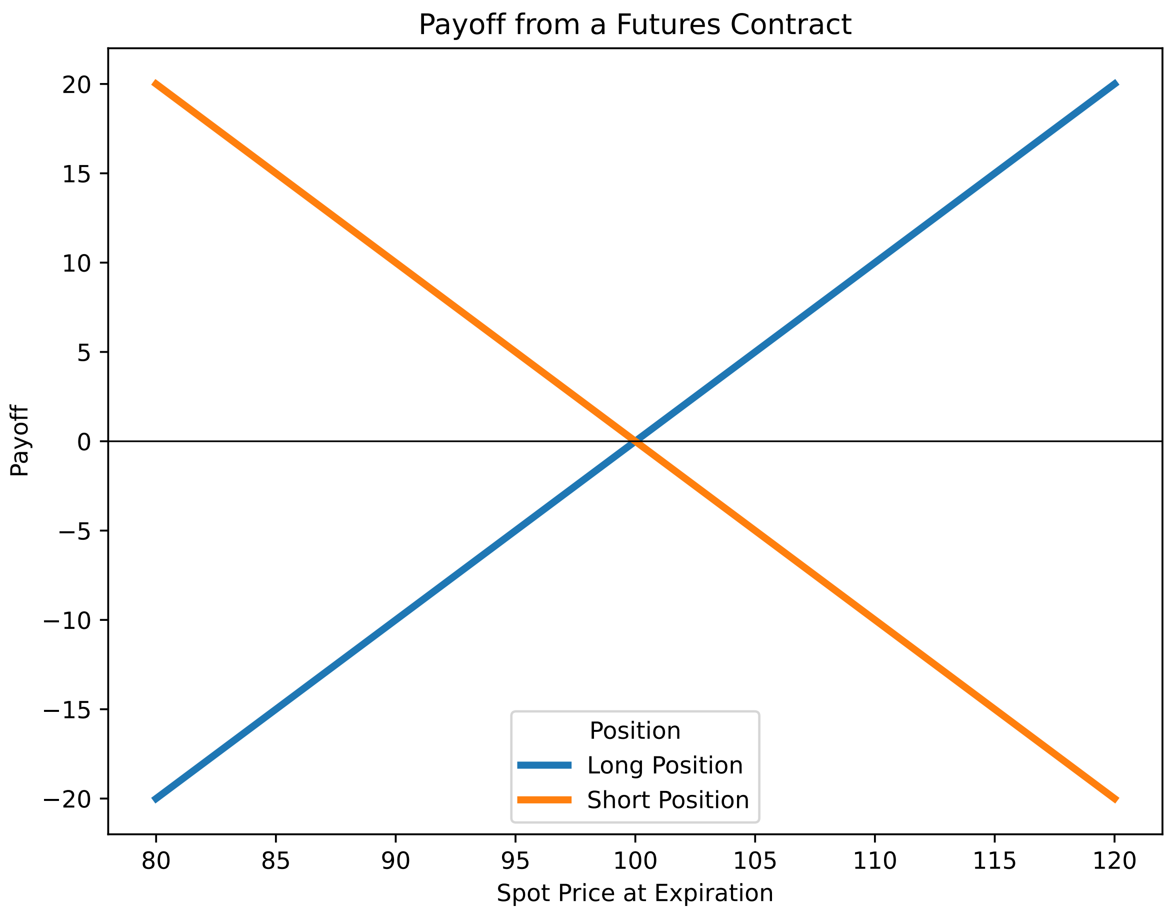
- **Long Position:** Loss can be large if the asset's price drops significantly.
- **Short Position:** Loss can be unlimited if the asset's price rises dramatically.

Contract Details:

- **Asset Specification:** What asset is being traded and any quality standards.
- **Delivery Terms:** Where and when the asset will be delivered.

Settlement Methods:

- **Futures Contracts:** Settled daily (marking to market).
- **Forward Contracts:** Settled at the end of the contract term by comparing the agreed price with the final market price.



2.2 Payoff From a Forward Contract

- **Start of Contract:** Time $= 0$, with an initial spot price S_0.
- **End of Contract:** Time $= T$, with a final spot price S_T.
- **Forward Price:** The fixed price agreed upon, F_0.

Long Position (Buyer):

- Profit if S_T is higher than F_0; loss if it is lower.

$$\text{Payoff}_{\text{Long}} = S_T - F_0$$

Short Position (Seller):

- Profit if S_T is lower than F_0; loss if it is higher.

$$\text{Payoff}_{\text{Short}} = -(S_T - F_0)$$

Example: Gold Purchase Agreement

Barbara Nix agreed to buy 1 kilogram of gold at a price of $38,000 per kilo from Metals Inc. in 90 days. After 90 days, the spot price of gold reached $38,500 per kilo. What is the payoff for each party?

- **For the Long Party (Barbara Nix):** The gain is $500, calculated as

$$S_T - F_0 = \$38,500 - \$38,000 = \$500$$

- **For the Short Party (Metals Inc.):** The loss is $500

$$-(S_T - F_0) = -(\$38,500 - \$38,000) = -\$500$$

2.3 Exchange Markets (Futures)

Exchange markets provide a regulated, **standardized** environment for trading futures contracts. They enable participants to hedge risks or speculate on market movements.

- **Regulation:**
  The Commodity Futures Trading Commission (CFTC) in the U.S. oversees these markets to protect participants and ensure fair trading.
- **Contract Specifications:**
  Futures contracts are standardized, clearly defining the asset, quality, delivery location, and date. For example, see CME Group's Gold Futures.
- **Settlement:**
  Most futures contracts are closed out before maturity, avoiding physical delivery through offsetting trades.
  - **Physical Delivery:** When required, contracts specify the deliverable asset, delivery location, and timeframe. Typically, the short (seller) sets the delivery details.
  - **Cash Settlement:** Some contracts, such as stock index futures, settle in cash rather than through the physical delivery of assets.

2.3.1 Market Quotes and Trading Activity

- **Settlement Price:** The closing price used for daily settlements.
- **Open Interest:** The total number of outstanding contracts (either long or short), reflecting market liquidity.
- **Trading Volume:** The total number of contracts traded during a day, regardless of changes in open interest.

> Example of Open Interest and Trading Volume
>
> Imagine a wheat futures market that starts the day with **1,000 open contracts**. During the day, the following transactions occur:
>
> 1. **Transaction 1:** A new buyer and a new seller agree on a contract.
>   - **Open Interest:** Increases by 1 (total: 1,001).
>   - **Trading Volume:** +1 contract.
> 2. **Transaction 2:** An existing long position is sold to a new participant.
>   - **Open Interest:** Remains at 1,001 (one party is simply replaced).
>   - **Trading Volume:** +1 contract.
> 3. **Transaction 3:** Two new participants (a buyer and a seller) enter with 5 new contracts.
>   - **Open Interest:** Increases by 5 (total: 1,006).
>   - **Trading Volume:** +5 contracts.
> 4. **Transaction 4:** An existing buyer and seller close their positions with an offsetting contract.
>   - **Open Interest:** Decreases by 1 (total: 1,005).
>   - **Trading Volume:** +1 contract.
>
> Final Totals:
>
> - **Open Interest:** 1,005 contracts (active positions at the end of the day).
> - **Trading Volume:** 8 contracts traded throughout the day.
>
> This example shows that **open interest** reflects the number of active contracts, while **trading volume** is the total number of contracts exchanged during the day.

2.3.2 Daily Settlement and Margins

- **Mark-to-Market:**
  The clearinghouse adjusts each trader's margin account daily to reflect gains or losses based on the settlement price.
- **Margin Requirements:**
  - **Initial Margin:** The deposit required to open a position.
  - **Maintenance Margin:** The minimum balance that must be maintained. Falling below this triggers a **margin call** (additional funds must be deposited to meet the initial margin requirement).

Example: Margin and Margin Call

Alex enters a crude oil futures contract valued at $60,000.

- **Initial Margin (5%):** $3,000
- **Maintenance Margin (3%):** $1,800

If the contract value drops to $58,000, Alex loses $2,000, reducing his margin account to $1,000—below the maintenance margin. He then receives a margin call and must deposit an additional $2,000 to restore his account to $3,000.
If Alex fails to meet the margin call, his position may be closed out by the broker to limit further losses.

Some Useful Links

- Margin: Know What's Needed
- The Benefits of Futures Margins
- Understanding Futures Margin

2.4 OTC Markets (Forwards)

Over-the-Counter (OTC) markets allow forward contracts and other derivatives to be traded directly between two parties without an exchange. This setup offers **flexibility and customization** but traditionally lacks the transparency and regulation found in exchange-traded markets.

Pre-Crisis Era:

- OTC markets operated with minimal oversight, contributing to systemic risk before the 2007–2008 financial crisis.

Post-Crisis Reforms:

- In response to the crisis, regulations such as the Dodd-Frank Act (U.S.) and EMIR (EU) were introduced.

- These reforms require reporting OTC transactions and often mandate central clearing to enhance transparency and reduce risk.

2.4.1 Collateral Requirements

To manage counterparty risk in OTC markets, collateral is required:

- **Initial Margin:**
  - An upfront deposit to cover potential losses immediately after a default.
- **Variation Margin:**
  - Additional funds required to adjust for daily price changes, ensuring the collateral always matches the current exposure.

2.4.2 Clearing Mechanisms

Bilateral Clearing:

- Each transaction is settled directly between the two parties.

- This method allows customization but lacks centralized risk management.

Central Clearing:

- A Central Counterparty (CCP) steps in between the buyer and seller, reducing counterparty risk by standardizing margin requirements and daily mark-to-market adjustments.
- Although OTC contracts are customized, central clearing helps bring some uniformity to risk management and settlement processes.

2.5 Hedging with Futures and Managing Basis Risk

Hedging with futures is a strategy used to lock in prices for assets you plan to buy or sell in the future, helping to manage price volatility in commodities, currencies, interest rates, and more.

- **Long Futures Hedge:**
  Used when you expect to purchase an asset. Locking in a price now protects against future price increases.
- **Short Futures Hedge:**
  Used when you plan to sell an asset. Locking in a price now protects against future price decreases.

Advantages of hedging:

- **Business Focus:** Reduces exposure to market risks, allowing companies to concentrate on their core activities.
- **Cost Predictability:** Helps with budgeting by stabilizing costs and revenues.
- **Risk Management:** Mitigates risks from volatile market factors.

Disadvantages of hedging:

- **Shareholder Autonomy:** Some argue that shareholders can manage risks through personal diversification.
- **Competitive Dynamics:** Hedging may add extra risk if competitors do not hedge.
- **Complexity:** Hedges can be difficult to implement and explain, especially if outcomes differ from expectations.

2.5.1 Understanding Basis Risk

Definition

Basis risk is the uncertainty that the difference between the current (spot) price and the futures price will not behave as expected when the hedge is closed.

This risk arises from:

1. **Asset Mismatch:** The asset being hedged may not exactly match the underlying asset of the futures contract.
2. **Timing Uncertainty:** Variations in the timing of the actual purchase or sale can affect the hedge.
3. **Early Contract Closure:** Exiting a futures contract before its delivery date can alter the expected price difference.

Mitigation Strategies:

- **Delivery Month Selection:** Choose a contract with a delivery month as close as possible to, but after, your expected transaction date.
- **Cross Hedging:** If a direct futures contract isn't available, use a contract with a price that closely correlates with the asset's price.

2.5.2 Hedging Examples

Long Hedge Example for Asset Purchase

- Initial Futures Price (F_1): 88.0
- Futures Price at Purchase (F_2): 89.1
- Spot Price at Purchase (S_2): 90.0
- Basis at Purchase (b_2): $S_2 - F_2 = 0.9$

Outcome:

- Asset cost: \$90.0
- Gain from futures: $F_2 - F_1 = 1.1$
- **Net cost:** $\$90.0 - \$1.1 = \$88.9$

Short Hedge Example for Asset Sale

- Initial Futures Price (F_1): 0.98
- Futures Price at Sale (F_2): 0.925
- Spot Price at Sale (S_2): 0.92
- Basis at Sale (b_2): $S_2 - F_2 = -0.005$

Outcome:

- Asset sale price: \$0.92
- Gain from futures: $F_1 - F_2 = 0.055$

- **Net sale price:** $\$0.92 + \$0.055 = \$0.975$

2.6 Cross Hedging and Tailing Adjustments

Definition

Cross hedging is used when no futures market exists for the asset you want to hedge. Instead, you use a futures contract on a correlated asset. The effectiveness of a cross hedge depends on how closely the prices of the two assets move together.

2.6.1 Optimal Hedge Ratio and Number of Contracts

The **optimal hedge ratio** (h^*) without daily settlement adjustments is given by:

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

- ρ is the correlation coefficient between changes in the spot price (ΔS) and the futures price (ΔF).
- σ_S is the standard deviation of the changes in the spot price.
- σ_F is the standard deviation of the changes in the futures price.

Once h^* is determined, the **optimal number of futures contracts** (N^*) needed is calculated as:

$$N^* = h^* \times \frac{Q_A}{Q_F}$$

- Q_A is the size of the position being hedged (in units).
- Q_F is the size of one futures contract (in units).

Example: Hedging Jet Fuel with Heating Oil Futures

An airline plans to purchase 2 million gallons of jet fuel in one month and decides to hedge using heating oil futures.

- $\sigma_F = 0.0313$, $\sigma_S = 0.0263$, and $\rho = 0.928$
- Compute the hedge ratio:

$$h^* = 0.928 \times \frac{0.0263}{0.0313} \approx 0.78$$

- One heating oil futures contract covers 42,000 gallons.

- Optimal number of contracts:

$$N^* = 0.78 \times \frac{2,000,000}{42,000} \approx 37$$

  Thus, the airline should use about 37 heating oil futures contracts.

2.6.2 Daily Settlement and Tailing Adjustments

Futures contracts are settled daily, which can affect the hedge's performance. To account for these daily changes, we use an **adjusted optimal hedge ratio** ($\hat{h}$):

$$\hat{h} = \hat{\rho}\frac{\hat{\sigma_S}}{\hat{\sigma_F}}$$

- $\hat{\rho}$ is the correlation between the daily percentage changes in spot and futures prices.
- $\hat{\sigma_S}$ and $\hat{\sigma_F}$ are the standard deviations of these daily percentage changes.

Using $\hat{h}$, the adjusted **optimal number of futures contracts** is:

$$N^* = \hat{h} \times \frac{V_A}{V_F}$$

- V_A is the total value of the position being hedged (spot price $\times$ quantity).
- V_F is the value of one futures contract (futures price $\times$ contract size).

i Example: Hedging with Tailing Adjustments

A transportation company, TransCo, needs to purchase 500,000 gallons of diesel in one month. With no diesel futures available, it hedges using heating oil futures (each covering 42,000 gallons).

- Diesel spot price: \$3.00 per gallon

- Heating oil futures price: \$2.90 per gallon

- Correlation of daily changes, $\hat{\rho}$: 0.9

- Daily volatility: $\widehat{\sigma_S} = 1.5\%$ for diesel, $\widehat{\sigma_F} = 1.8\%$ for heating oil

Step 1: Calculate the adjusted hedge ratio

$$\hat{h} = 0.9 \times \frac{0.015}{0.018} = 0.75$$

Step 2: Determine the values

$$V_A = \$3.00 \times 500,000 = \$1,500,000$$

$$V_F = \$2.90 \times 42,000 = \$121,800$$

Step 3: Calculate the optimal number of contracts

$$N^* = 0.75 \times \frac{1,500,000}{121,800} \approx 9.23$$

Since partial contracts aren't possible, TransCo would hedge with 9 contracts to avoid over-hedging.

2.7 Stock Index Futures

Stock index futures are powerful tools for managing portfolio risk. They allow investors to hedge against market fluctuations or adjust their portfolio's sensitivity to market movements without altering its actual composition.

To determine the number of futures contracts needed, use:

$$N^* = (\beta^* - \beta) \times \frac{V_A}{V_F}$$

- V_A: Portfolio value.
- β: Current portfolio beta.
- β^*: Target beta (if fully hedging, $\beta^* = 0$).
- V_F: Value of one futures contract (e.g., futures price multiplied by the contract multiplier).
- A negative N^* means shorting futures; a positive value means going long.

This approach allows investors to either fully hedge their portfolio ($\beta^* = 0$) or fine-tune their market exposure based on strategic objectives.

Example: Hedging a Portfolio

A portfolio valued at \$5 million has a beta of 1.5. Using S&P 500 futures, where each contract is worth \$250,000, to fully hedge ($\beta^* = 0$):

$$N^* = (0 - 1.5) \times \frac{5,000,000}{250,000} = -30$$

Shorting 30 futures contracts would hedge the portfolio against market fluctuations.

Example: Adjusting Portfolio Beta

A portfolio manager wants to lower a \$10 million portfolio's beta from 1.2 to 0.8. Using an index futures contract priced at 3,000 points with a multiplier of \$250 (thus, $V_F = 250 \times 3,000 = \$750,000$):

$$N^* = (0.8 - 1.2) \times \frac{10,000,000}{750,000} = -0.4 \times 13.33 \approx -5.33$$

Rounding to -5 indicates shorting 5 contracts to reduce the portfolio beta.

2.7.1 Additional Hedging Considerations

2.7.1.1 Why Hedge with Stock Index Futures?

- **Market Timing:** Allows an exit from market exposure without selling underlying assets, avoiding transaction costs and capital gains taxes.
- **Risk Management:** Offers precise control over market risk.
- **Performance Focus:** Helps ensure returns are driven by the performance of selected assets rather than broad market movements.

Tip

Imagine your portfolio's stocks have an average beta of 1.0, aligning their performance with the market. Yet, you're confident these stocks will surpass market returns in any scenario. By hedging, you secure returns at the risk-free rate plus any outperformance of your stocks over the market. This strategy minimizes market volatility's impact, ensuring your portfolio's gains are primarily due to your stock selection skills.

2.7.1.2 Stack and Roll Strategy

This strategy involves rolling futures contracts forward to maintain continuous hedging:

- **Process:**
  - Initiate futures contracts for a specific time horizon.
  - Before maturity, close out the contracts and replace them with new ones.
- **Risks:**
  - Liquidity issues and the possibility of realizing losses on the hedge while gains in the underlying assets remain unrealized.

> ⚠ Warning
>
> The Metallgesellschaft (MG) case in the early 1990s is a reminder that rolling hedges can lead to severe liquidity issues. MG's strategy of hedging long-term exposure with short-dated futures resulted in massive margin calls and a $1.33 billion loss when oil prices dropped, forcing the closure of hedge positions.

2.8 Practice Questions and Problems

2.8.1 Fundamentals of Futures Trading

1. What are the most important aspects of the design of a new futures contract?
2. The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.
3. What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?
4. "Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

2.8.2 Open Interest and Trading Volume

1. Distinguish between the terms open interest and trading volume.
2. "When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.
3. Why does the open interest usually decline during the month preceding the delivery month?

4. On a particular day, there were 2,000 trades in a particular futures contract. This means that there were 2000 buyers (going long) and 2000 sellers (going short). Of the 2,000 buyers, 1,400 were closing out positions and 600 were entering into new positions. Of the 2,000 sellers, 1,200 were closing out positions and 800 were entering into new positions. What is the impact of the day's trading on open interest?

2.8.3 Margin Mechanics in Futures Trading

1. Explain how margin accounts protect investors against the possibility of default.
2. Suppose that you enter into a short futures contract to sell July silver for $17.20 per ounce. The size of the contract is 5,000 ounces. The initial margin is $4,000, and the maintenance margin is $3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

> **i** Solution
>
> $S = \$17.40$

3. A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is $6,000 per contract, and the maintenance margin is $4,500 per contract. What price change would lead to a margin call? Under what circumstances could $2,000 be withdrawn from the margin account?

> **i** Solution
>
> $S = 166.67$ cents

4. A company enters into a short futures contract to sell 5,000 bushels of wheat for 750 cents per bushel. The initial margin is $3,000 and the maintenance margin is $2,000. What price change would lead to a margin call? Under what circumstances could $1,500 be withdrawn from the margin account?

> **i** Solution
>
> Margin call: $S = 770$ cents; Withdraw $1500: $S = 720$ cents

2.8.4 Basics of Hedging with Futures

1. Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?
2. Explain what is meant by basis risk when futures contracts are used for hedging.

3. Explain what is meant by a perfect hedge. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.
4. Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.
5. "For an asset where futures prices for contracts on the asset are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

2.8.5 Advanced Hedging Scenarios and Strategies

1. Sixty futures contracts are used to hedge an exposure to the price of silver. Each futures contract is on 5,000 ounces of silver. At the time the hedge is closed out, the basis is \$0.20 per ounce. What is the effect of the basis on the hedger's financial position if (a) the trader is hedging the purchase of silver and (b) the trader is hedging the sale of silver?
2. In the corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in a) June, b) July, and c) January.
3. Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a three-month contract? What does it mean?

> **Solution**
>
> $h = 64.2\%$

4. The standard deviation of monthly changes in the spot price of live cattle is 1.2 (in cents per pound). The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

> **Solution**
>
> Long 3 contracts.

2.8.6 Practical Concerns and Risk Management

1. Give three reasons why the treasurer of a company might not hedge the company's exposure to a particular risk.
2. A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather." Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?
3. Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.
4. A futures contract is used for hedging. Explain why the daily settlement of the contract can give rise to cash flow problems.

2.8.7 Hedging with Stock and Commodity Futures

1. A company has a $20 million portfolio with a beta of 1.2. It would like to use futures contracts on a stock index to hedge its risk. The index futures is currently standing at 1080, and each contract is for delivery of $250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6? What should the company do if it wants to increase the beta of the portfolio to 1.5?

> **Solution**
>
> - short 89 contracts
> - short 44 contracts to reduce beta
> - long 22 contracts to increase beta

2. On July 1, an investor holds 50,000 shares of a certain stock. The market price is $30 per share. He decides to use the September Mini S&P 500 futures contract. The index is currently 1,500 and one contract is for delivery of $50 times the index. The beta of the stock is 1.3. What strategy should the investor follow? Under what circumstances will it be profitable?

> **Solution**
>
> Short 26 contracts.

3. It is now June. A company knows that it will sell 5,000 barrels of crude oil in September. It uses the October CME Group futures contract to hedge the price it will receive. Each

contract is on 1,000 barrels of "light sweet crude". What position should it take? What price risks is it still exposed to after taking the position?

3 Determination of Forward and Futures Prices

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 5 - Determination of Forward and Futures Prices
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 2 - Basics of Derivative Pricing and Valuation
  - Chapter 3 - Pricing and Valuation of Forward Commitments

Learning Outcomes:

- Understand the principles of pricing and valuation for forward and futures contracts.
- Apply pricing models to specific assets, including commodities and financial instruments.
- Analyze the relationship between futures prices and expected spot prices.
- Understand the structure and valuation of Forward Rate Agreements (FRAs).

3.1 Pricing and Valuation of Forward/Futures Contracts

- **Pricing** determines the fair market rate at contract **initiation**, ensuring neutrality for both parties.
- **Valuation** assesses the contract's value **after initiation**, fluctuating based on market conditions.

Example: Forward Price vs. Value

- On November 2, an investor takes a **long position** in a December Gold futures contract at **$1,250/oz**. At initiation, the contract's value is **$0** (fair market agreement).

- On November 3, if the market price drops to **$1,225/oz**, the contract's value turns **negative**, reflecting a loss for the investor.

- **Risk Aversion:** Investors require compensation for taking on additional risk, reflecting the principle that higher risk should be rewarded with higher potential returns.
- **Risk-Neutral Pricing:** Under this approach, it's assumed that investors are indifferent to risk. The pricing of derivatives through arbitrage opportunities ensures that the portfolio, combining the derivative and its underlying asset, yields the risk-free rate of return.
- **Arbitrage-Free Pricing:** This methodology prices derivatives based on the assumption that the market operates under risk-neutral conditions and is free of arbitrage opportunities, ensuring no risk-free profits can be made from market inefficiencies.
- **Law of One Price:** asserts that identical cash flows must have the same price, irrespective of future outcomes.
- **Arbitrage Principles:**
  1. Do not use your own money.
  2. Do not take any price risk.
- **Short Selling:** Short selling involves selling borrowed securities, aiming to repurchase them at a lower price. Sellers must cover dividends and borrowing fees.

Example: Short Selling

Short 100 shares at **$100 each** and close the position at **$90**, while a **$3/share** dividend is issued:

- **Profit Calculation:**

$$100 \times (100 - 90 - 3) = 700$$

- If instead you buy 100 shares, a price drop results in a loss:

$$100 \times (90 - 100 + 3) = -700$$

- **Investment Assets:** Held for potential value appreciation over time (e.g., gold, stocks, bonds).
- **Consumption Assets:** Used in production or consumption (e.g., oil, copper).

3.2 Forward Price for an Investment Asset

Assumptions:

1. No transaction costs in trading.

2. All net trading profits are taxed equally.

3. Borrowing and lending occur at the same **risk-free interest rate**.

4. Arbitrage opportunities are **immediately** exploited.

Notation:

- S_0: Current spot price of the asset.
- F_0: Today's futures or forward price of the asset.
- T: Time to contract maturity (in years).
- r: Annualized risk-free interest rate over T.

3.2.1 Arbitrage Example

Consider an asset that provides with **no income**, priced at **$40**, with a **5% annual interest rate** and a **3-month forward contract**.

3.2.1.1 Case 1: Forward Price = $43

- **Now**: Borrow $40, buy the asset, and enter a forward contract to sell at $43.

- **In 3 months**: Sell at $43, repay the loan ($40.50 with interest).

- **Profit**: $2.50.

3.2.1.2 Case 2: Forward Price = $39

- **Now**: Short sell the asset for $40, invest the proceeds at 5% interest, and enter a forward contract to buy at $39.

- **In 3 months**: Buy back at $39, close the short position, and receive $40.50 from the investment.

- **Profit**: $1.50.

3.2.2 Forward Price Formula

For an **investment asset** with **no income**, the no-arbitrage forward price is:

$$F_0 = S_0 e^{rT}$$

> **i** Example: Eliminate arbitrage
>
> Using $S_0 = 40$, $T = 0.25$ (3 months), and $r = 0.05$ (5%):
>
> $$F_0 = 40e^{0.05\times0.25} = 40.50$$
>
> This aligns with the **theoretical fair price**, ensuring no arbitrage.

- If the asset generates a **known income** I (expressed as the present value of the income), the formula adjusts to:

$$F_0 = (S_0 - I)e^{rT}$$

- If the asset provides a **yield** q (expressed as a continuously compounded annualized rate):

$$F_0 = S_0 e^{(r-q)T}$$

3.3 Valuing a Forward Contract

At initiation, a forward contract has **zero value**, ensuring fairness for both parties (excluding bid-offer spreads). Over time, its value can become **positive or negative** as market conditions change.

Let:

- K = **Agreed delivery price** in the contract.
- F_0 = **Forward price** for a contract negotiated today.

The value of a forward contract depends on the difference between the agreed **delivery price** and the current **forward price**, discounted at the risk-free rate:

- **Long position (buying the asset):**

$$V_{\text{long}} = (F_0 - K)e^{-rT}$$

- **Short position (selling the asset):**

$$V_{\text{short}} = -(F_0 - K)e^{-rT}$$

These formulas show how the **risk-free rate** r and **time to maturity** T impact the present value of expected gains or losses.

3.4 Forward vs. Futures Prices

Although **forward** and **futures** contracts both involve agreeing to buy/sell an asset at a future date, their prices may diverge due to **interest rate volatility**.

- If interest rates and asset prices are **positively correlated** $\rightarrow$ **Futures price** $>$ **Forward price**
  - Daily settlement allows gains to be reinvested at **higher rates**.
- If interest rates and asset prices are **negatively correlated** $\rightarrow$ **Futures price** $<$ **Forward price**
  - No daily settlement avoids reinvesting at **lower rates**.

One notable example is **Eurodollar futures**, where pricing anomalies arise due to unique market characteristics.

3.5 Forward Price for Specific Assets

3.5.1 Stock Index

A **stock index** is an **investment asset** that effectively pays a **dividend yield**, similar to the income generated by holding its underlying stocks.

The relationship between the **futures price** (F_0) and the **spot price** (S_0) is:

$$F_0 = S_0 e^{(r-q)T}$$

where, q represents the average dividend yield of the portfolio reflected by the index over the contract's life.

Warning

This formula assumes the index represents a **tradable investment asset**. However, a purely numerical index (e.g., **Nikkei 225** without its underlying assets) does not qualify as a true investment asset.

Index Arbitrage

Arbitrageurs exploit price discrepancies between **futures** and **spot prices** adjusted for dividend yield and the risk-free rate:

- If $F_0 > S_0 e^{(r-q)T} \rightarrow$ Buy the underlying stocks, sell futures.
- If $F_0 < S_0 e^{(r-q)T} \rightarrow$ Buy futures, short-sell the index's underlying stocks.

High-frequency trading algorithms often execute these strategies. However, **real-world frictions** (e.g., execution delays, transaction costs) may prevent perfect arbitrage.

3.5.2 Exchange Rates

A **foreign currency** behaves like a security yielding **interest**, where the yield is the **foreign risk-free rate** (r_f). The forward exchange rate is given by:

$$F_0 = S_0 e^{(r-r_f)T}$$

Exchange Rate Arbitrage

Consider two equivalent strategies for converting **1,000 units** of a foreign currency into dollars by time T:

1. **Invest in the foreign currency** at rate r_f, then convert at the forward rate F_0.
2. **Convert immediately** at spot rate S_0, then invest in dollars at rate r.

No-arbitrage implies both must yield the same result, leading to:

$$1,000 \times e^{r_f T} \times F_0 = 1,000 \times S_0 \times e^{rT}$$

Solving for F_0 confirms the forward rate formula:

$$F_0 = S_0 e^{(r-r_f)T}$$

3.5.3 Commodities

For **consumption assets** (e.g., oil, wheat, copper), **storage costs** act as **negative income**, affecting forward pricing:

$$F_0 \leq S_0 e^{(r+u)T}$$

where:

- u = **Storage cost per unit time** as a percentage of the asset's value.

Alternatively, if U represents the **present value** of storage costs:

$$F_0 \leq (S_0 + U)e^{rT}$$

These formulas accommodate the costs associated with holding and storing physical commodities, from agricultural products to metals, affecting their forward pricing.

3.5.4 The Cost of Carry

The **cost of carry** (c) represents the total cost of holding an asset, including:

- Storage costs
- Financing costs
- Income earned (e.g., dividends, yields)

For **investment assets**, the forward price follows:

$$F_0 = S_0 e^{cT}$$

For **consumption assets**, storage and financing costs may create **lower price bounds**:

$$F_0 \leq S_0 e^{cT}$$

A **convenience yield** (y) represents the **benefit of physically holding** a consumption asset rather than a derivative. The forward price adjusts to:

$$F_0 = S_0 e^{(c-y)T}$$

where y reflects **scarcity, supply-chain security, and non-financial benefits** of physical possession.

3.6 Futures Prices and Expected Spot Prices

- **Contango** ($F_t > S_t$): Futures prices are higher than the current or expected future spot price.
  - Occurs when carrying costs (storage, interest) exceed convenience yield.
  - Common in gold, silver, natural gas, and agricultural commodities.
  - Contango is more common across most futures markets due to carrying costs.
- **Backwardation** ($F_t < S_t$): Futures prices are lower than the expected future spot price.
  - Happens when convenience yield is high (scarcity or strong demand for immediate delivery).
  - Common in crude oil, perishable goods, and equity index futures.
- **Futures prices are poor predictors of future spot prices.** Empirical evidence shows biases due to risk premia and market inefficiencies. Speculative demand, liquidity constraints, and irrational expectations can further distort pricing.
- While futures prices provide useful signals, they should not be relied upon as perfect forecasts of future spot prices.

> Tip
>
> - What is Contango and Backwardation - CME Institute

3.7 Forward Rate Agreement (FRA)

An **FRA** is a financial contract where parties exchange a **fixed interest rate** for a **floating rate** (e.g., LIBOR, SOFR, SONIA) at a future date, based on a specified **notional principal**.

- No principal exchange occurs—only the difference in interest payments is settled.
- The **initial value** of an FRA is **zero** since the fixed rate equals the forward rate at inception.
- As the forward rate changes over time, the FRA's value fluctuates.

Example: FRA

Party A and Party B agree to exchange a fixed rate of 3% for the three-month SOFR on a $100 million notional in two years (compounded quarterly).

- **Party A** pays floating SOFR and receives a fixed 3%.
- **Party B** takes the opposite position.

If in two years the SOFR rate is 3.5%, Party A receives a payment from Party B:

$$100,000,000 \times (0.035 - 0.030) \times 0.25 = 125,000$$

This amount is discounted for three months at **3.5%**, since payments are made at the two-year mark.

- A **zero rate** is the interest rate for an investment lasting n years, with interest and principal paid at maturity (no intermediate payments).
- A **forward rate** represents the future interest rate implied by current zero rates for future periods. It reflects the cost of borrowing/lending at a future time.

For a forward rate between T_1 and T_2, given zero rate R_1 and R_2:

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

Example: Forward Rate Calculation

Year	Zero Rate (% per annum)	Forward Rate for nth Year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

For **year 4**, given $T_1 = 3$, $T_2 = 4$, $R_1 = 4.6\%$, and $R_2 = 5.0\%$:

$$R_F = \frac{0.05 \times 4 - 0.046 \times 3}{4 - 3} = 6.2\%$$

This forward rate (**6.2%**) is higher than the **zero rate (5.0%)**, reflecting an **upward-sloping yield curve**.

The **value of an FRA** is determined by:

- The difference between the **agreed fixed rate** (R_K) and the **current forward rate** (R_F).
- The **contract period** (t).
- Discounting the difference to **present value**.

3.8 Practice Questions and Problems

1. Explain what happens when an investor shorts a certain share.
2. What is the difference between the forward price and the value of a forward contract?
3. Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.
4. Explain carefully the meaning of the terms convenience yield and cost of carry. What is the relationship between futures price, spot price, convenience yield, and cost of carry?
5. What is the cost of carry for (a) a non-dividend-paying stock, (b) a stock index, (c) a commodity with storage costs, and (d) a foreign currency?
6. Suppose that you enter into a three-month forward contract on a non-dividend-paying stock when the stock price is $108 and the risk-free interest rate (with continuous compounding) is 4% per annum. What is the forward price?

Solution

$F_0 = 109.085$

7. A four-months long forward contract on a non-dividend-paying stock is entered into when the stock price is $150 and the risk-free rate of interest is 5.7% per annum with continuous compounding.

    - a) What are the forward price and the initial value of the forward contract?

- b) Two months later, the price of the stock is \$168 and the risk-free interest rate is still 5.7%. What are the forward price and the value of the forward contract?

> **Solution**
>
> Forward price $F_0 = 152.88$; initial value 0; final value 169.6

8. The risk-free rate of interest is 4.1% per annum with continuous compounding, and the dividend yield on a stock index is 6.2% per annum. The current value of the index is 2445. What is the one-month futures price?

> **Solution**
>
> $F_0 = 2440.725$

9. A stock index currently stands at 725. The risk-free interest rate is 7.6% per annum (with continuous compounding) and the dividend yield on the index is 1.8% per annum. What should the futures price for a three-month contract be?

> **Solution**
>
> $F_0 = 735.589$

10. An index is 550. The three-month risk-free rate is 4.60% per annum and the dividend yield over the next three months is 5.80% per annum. The six-month risk-free rate is 5.34% per annum and the dividend yield over the next six months is 4.93% per annum. Estimate the futures price of the index for three-month and six-month contracts. All interest rates and dividend yields are continuously compounded.

> **Solution**
>
> 3-month $F_0 = 548.352$; 6-month $F_0 = 551.129$

11. The spot price of silver is \$11 per ounce. The storage costs are \$0.25 per ounce payable quarterly in advance. Assuming that interest rates are 1.80% per annum for all maturities, calculate the futures price of silver for delivery in nine months.

> **Solution**
>
> $F_0 = 11.91$

12. The spot price of oil is \$39 per barrel and the cost of storing a barrel of oil for one year is \$1.2, payable at the end of the year. The risk-free interest rate is 8.60% per annum, continuously compounded. What is an upper bound for the one-year futures price of oil?

> **Solution**
>
> $F_0 = 43.70$

13. Suppose that the risk-free interest rate is 3.00% per annum with continuous compounding and that the dividend yield on a stock index is 0.60% per annum. The index is standing at 3646, and the futures price for a contract deliverable in ten months is 3701. What arbitrage opportunities does this create?

> **Solution**
>
> $F_0 = 3719.654$

14. The eight-month interest rates in Switzerland and the United States are, respectively, 3.60% and 5.40% per annum with continuous compounding. The spot price of the Swiss franc is \$0.9. The futures price for a contract deliverable in two months is also \$0.9. What arbitrage opportunities does this create?

> **Solution**
>
> $F_0 = 0.911$

15. When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the daily settlement process does leave the company exposed to some risk. Explain the nature of this risk. Assume that the forward price equals the futures price. In particular, consider whether the company is better off using a futures contract or a forward contract when

    - a) The value of the foreign currency falls rapidly during the life of the contract
    - b) The value of the foreign currency rises rapidly during the life of the contract
    - c) The value of the foreign currency first rises and then falls back to its initial value
    - d) The value of the foreign currency first falls and then rises back to its initial value

4 Swaps

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 7 - Swaps
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 2 - Basics of Derivative Pricing and Valuation
  - Chapter 3 - Pricing and Valuation of Forward Commitments

Learning Outcomes:

- Understand the fundamental principles and nature of swaps, including their purpose and functionality in financial markets.
- Analyze the mechanics and applications of interest rate swaps, identifying the roles of comparative advantage and valuation methods.
- Explore currency swaps, including their types, uses, and the valuation process for fixed-for-fixed currency swaps.
- Examine additional swap arrangements outside of interest rate and currency swaps, recognizing their unique features and applications.

4.1 Nature of Swaps

Definiton

A swap is an over-the-counter (OTC) contract where two parties agree to exchange cash flows at specified intervals based on predetermined terms. Due to its bilateral nature and customization, it carries default risk.

- **Comparison with Forward Contracts**: While forward contracts involve a single cash flow exchange at a future date, swaps facilitate multiple exchanges over time, making them more flexible and useful for various financial strategies.

- **Conceptual View**: A swap can be seen as a series of bundled forward contracts. This perspective highlights its role in hedging and financing, where cash flow timing and valuation are crucial.
- **Initial Market Value**: At inception, a swap has a market value of zero, reflecting a balanced structure where some forward components may hold positive value while others hold negative value. Over time, market fluctuations and underlying asset performance alter the swap's value.

4.2 Interest Rate Swaps

This section explores a **Plain Vanilla Interest Rate Swap**, using a transaction between **Apple and Citigroup** as an example. The swap spans three years and demonstrates the core mechanics of interest rate swaps.

- Apple agrees to pay Citigroup a **fixed 3% annual interest rate**, with quarterly payments based on a **notional principal of $100 million**.
- In return, Citigroup pays Apple interest at the **three-month LIBOR rate**, calculated on the same principal.
- Apple is the fixed-rate payer, while Citigroup is the floating-rate payer.

The table below illustrates potential cash flow outcomes, maintaining the **$100 million notional principal**. The net cash flow for Apple varies due to LIBOR fluctuations, sometimes resulting in payments and other times in receipts.

Date	SOFR Rate (%)	Floating Cash Flow Received ($'000s)	Fixed Cash Flow Paid ($'000s)	Net Cash Flow ($'000s)
June 8, 2022	2.20	550	750	-200
Sept. 8, 2022	2.60	650	750	-100
Dec. 8, 2022	2.80	700	750	-50
Mar. 8, 2023	3.10	775	750	+25
June 8, 2023	3.30	825	750	+75
Sept. 8, 2023	3.40	850	750	+100

Date	SOFR Rate (%)	Floating Cash Flow Received ($'000s)	Fixed Cash Flow Paid ($'000s)	Net Cash Flow ($'000s)
Dec. 8, 2023	3.60	900	750	+150
Mar. 8, 2024	3.80	950	750	+200

4.2.1 Typical Applications of Interest Rate Swaps

Interest rate swaps are widely used for managing interest rate risk, allowing firms to **adjust their exposure** by switching between fixed and floating rates based on financial strategy and market outlook.

- Apple Converts a Floating-Rate Liability to Fixed:
- Intel Converts a Fixed-Rate Liability to Floating:
- Apple Converts a Fixed-Rate Asset to Floating:
- Intel Converts a Floating-Rate Asset to Fixed:

4.3 The Comparative-Advantage Argument

The **comparative-advantage argument** explains why companies engage in interest rate swaps to **optimize borrowing costs**. It highlights how firms can **leverage differences in borrowing rates** to achieve more favorable financing conditions.

4.3.1 Case Study: AAACorp and BBBCorp

Consider two hypothetical companies:

- **AAACorp** prefers **floating-rate borrowing**.
- **BBBCorp** prefers **fixed-rate borrowing**.

Their respective borrowing costs are:

Entity	Fixed Rate	Floating Rate
AAACorp	4.0%	Floating - 0.1%

Entity	Fixed Rate	Floating Rate
BBBCorp	5.2%	Floating + 0.6%

- **Direct Swap Mechanism:** AAACorp borrows at its preferred floating rate, while BBBCorp borrows at its preferred fixed rate. They swap obligations, effectively allowing each company to access the preferred rate indirectly.
- **Indirect Swap via a Financial Institution:** A financial institution can act as an intermediary, structuring the swap between AAACorp and BBBCorp. This setup reduces negotiation complexities and often enhances efficiency.

4.3.2 Criticism of the Comparative-Advantage Argument

While this argument supports the rationale for interest rate swaps, it has limitations:

Mismatch in Rate Terms:

- Fixed rates (e.g., 4.0% for AAACorp and 5.2% for BBBCorp) apply to **5-year loans**.
- Floating rates (e.g., Floating - 0.1% for AAACorp and Floating + 0.6% for BBBCorp) are based on **6-month terms**.
- This disparity complicates direct comparisons of borrowing advantages.

Future Rate Uncertainty:

- BBBCorp's **future borrowing costs** depend on the spread over the floating rate at each reset.
- If interest rates **fluctuate significantly**, BBBCorp may face unexpected financial exposure.

4.4 Valuation of Interest Rate Swaps

Interest rate swaps are used to manage interest rate exposure, and their value fluctuates as market conditions change. At inception, a swap's value is **approximately zero**, reflecting an initial balance between fixed and floating cash flows. Over time, valuation techniques—particularly those using **Forward Rate Agreements (FRAs)**—help determine its market value.

4.4.1 Valuation Framework

The swap valuation process consists of three key steps:

1. **Calculate Floating Forward Rates:** Use market data to estimate future floating interest rates, which determine floating-rate payments.
2. **Determine Swap Cash Flows:** Apply calculated forward rates to estimate future cash flows for both fixed and floating legs of the swap.
3. **Discount Cash Flows to Present Value:** Discount future cash flows using Overnight Indexed Swap (OIS) rates, adjusting for time value and credit risk.

Example: Swap Valuation

4.4.2 Swap Agreement Overview

- **Structure**: A swap where one party pays a fixed annual rate of 3% and receives SOFR semi-annually on a $100 million principal.
- **Remaining Term**: 1.2 years, with cash flow exchanges at 0.2, 0.7, and 1.2 years.
- **Risk-Free Rates** (OIS zero rates for different maturities):
  - 3 months: 2.8%
  - 9 months: 3.2%
  - 15 months: 3.4%
- **Historical SOFR Rate**: 2.9% (already fixed for the upcoming 3-month exchange).
- **Projected Forward SOFR Rates**:

Time (years)	Continuous Compounding	Semi-Annual Compounding
0.2	2.50%	2.516%
0.7	3.36%	3.388%
1.2	3.68%	3.714%

4.4.3 Cash Flow Valuation

The table below details the swap's fixed and floating cash flows, their net impact, and their **present value** after discounting.

Time (years)	Fixed Cash Flow ($M)	Floating Cash Flow ($M)	Net Cash Flow ($M)	Discount Factor	Present Value ($M)
0.2	-1.500	+1.258	-0.242	0.9944	-0.241
0.7	-1.500	+1.694	+0.194	0.9778	+0.190
1.2	-1.500	+1.857	+0.357	0.9600	+0.343
Total					**+0.292**

- **Fixed Cash Flow** (at 0.7 years):

$$0.5 \times 0.03 \times 100 = -1.5 \text{ million}$$

- **Floating Cash Flow** (based on 3.388% semi-annual rate for 0.7 years):

$$0.5 \times 0.03388 \times 100 = 1.694 \text{ million}$$

- **Present Value of Net Cash Flow** (discounting at OIS rate for 0.7 years):

$$0.194 \times e^{-0.032 \times 0.7} = 0.190 \text{ million}$$

- **Final Swap Value**:
  The sum of all present values across time points, resulting in a swap valuation of **$0.292 million**.

- This valuation assumes **simplifications** (e.g., ignoring holiday calendars and day count conventions), which may affect precise cash flows.

4.5 Currency Swaps

Currency swaps are financial instruments that enable **the exchange of principal and interest payments in different currencies** between two parties. These contracts help firms manage currency exposure, reduce borrowing costs, and optimize investments across global markets.

In a **fixed-for-fixed currency swap**, parties exchange fixed interest rate payments in one currency for fixed payments in another currency, along with the principal exchange at the start and end of the contract.

Currency swaps are widely used for:

1. **Liability Management:** Firms **convert liabilities** from one currency to another to match their currency exposure and reduce exchange rate risk.
2. **Investment Optimization:** Investors **swap investment returns** between currencies to access favorable interest rates or hedge against currency fluctuations.

Case Study: British Petroleum and Barclays Swap Agreement

4.5.0.1 Agreement Overview

- **Parties Involved**: British Petroleum and Barclays.
- **Term**: 5 years.
- **Structure**:
  - British Petroleum pays a fixed 3% interest rate in USD.
  - British Petroleum receives a fixed 4% interest rate in GBP.
- **Principal Exchange**:
  - USD 15 million exchanged for GBP 10 million at inception.
  - Principal amounts are re-exchanged at the end of the swap.
- **Interest Payments**: Made annually, based on the fixed rates in each currency.

4.5.0.2 Yearly Cash Flow Summary

The table below illustrates the cash flows for British Petroleum exchanged over the swap's duration:

Date	USD Cash Flow (millions)	GBP Cash Flow (millions)
Feb 1, 2022	+15.00	-10.00
Feb 1, 2023	-0.45	+0.40
Feb 1, 2024	-0.45	+0.40
Feb 1, 2025	-0.45	+0.40
Feb 1, 2026	-0.45	+0.40
Feb 1, 2027	-15.45	+10.40

4.5.1 Comparative Advantage in Currency Swaps

A **comparative advantage** in currency swaps arises when companies face different borrowing costs in various currencies due to **taxation, credit ratings, or market conditions**.

- **Efficiency**: Currency swaps enable companies to **lower borrowing costs** and manage currency risk.
- **Strategic Use**: They help **convert liabilities and investments** across different currencies.
- **Comparative Advantage**: Firms leverage **differential borrowing costs** to gain financial benefits.

Case Study: General Electric & Qantas Airways

- **Scenario**:
  - General Electric (GE) wants to borrow in Australian dollars (AUD).
  - Qantas Airways wants to borrow in U.S. dollars (USD).
- **Borrowing Costs (after tax adjustments)**:

	USD	AUD
General Electric	5.0%	7.6%
Qantas Airways	7.0%	8.0%

Since **General Electric has a lower borrowing cost in USD** and **Qantas Airways has a lower borrowing cost in AUD**, they can **swap their borrowing obligations** to access cheaper rates than they would get individually.

4.6 Valuation of Fixed-for-Fixed Currency Swaps

Fixed-for-fixed currency swaps involve **exchanging fixed interest payments in two different currencies** over a set period, alongside an initial and final principal exchange. These swaps can be viewed as **a series of forward foreign exchange contracts**, where future exchange rates determine the valuation of expected cash flows.

The core assumption in valuing these swaps is that **forward exchange rates reflect the market's expectations of future currency values**. The valuation process, therefore, involves discounting cash flows in each currency and converting them using forward exchange rates.

- **Forward Exchange Rates Matter**: The swap's valuation depends on market expectations of future exchange rates.
- **Discounting Reflects Present Value**: Future cash flows must be discounted using the domestic interest rate to obtain today's value.
- **Principal Exchange is Crucial**: The final principal repayment significantly impacts the swap's overall valuation.

Example: Valuation of a Currency Swap

Consider a **U.S. dollar (USD) - Japanese yen (JPY) swap** with the following terms:

- **Fixed Interest Rates**:
  - USD: 2.5% per annum (continuous compounding).
  - JPY: 1.5% per annum (continuous compounding).
- **Swap Payments**:
  - Receives 3% interest in JPY.
  - Pays 4% interest in USD.
- **Principal Exchange**:
  - USD 10 million exchanged for 1,200 million JPY at inception and at maturity.
- **Remaining Swap Duration**: 3 years.
- **Current Spot Exchange Rate**: 110 JPY per USD.

The table below outlines the **expected cash flows** in both currencies, their conversion using forward exchange rates, and their **discounted present values**:

Time (years)	Dollar Cash Flow (million)	Yen Cash Flow (million)	Forward Exchange Rate	Dollar Value of Yen Cash Flow (million)	Net Cash Flow (million)	Present Value (million)
1	-0.4	+36	0.009182	0.3306	-0.0694	-0.0677

2	-0.4	+36	0.009275	0.3339	-0.0661	-0.0629
3	-10.4	+1236	0.009368	11.5786	+1.1786	+1.0934
Total						**+0.9629**

- **USD Payment:**

$$0.04 \times 10 = -0.4 \text{ million USD per year}$$

- **JPY Receipt:**

$$1,200 \times 0.03 = +36 \text{ million JPY per year}$$

- **Principal Exchange at Maturity (Year 3):**
  - Pays: USD 10 million
  - Receives: 1,200 million JPY

4.6.0.1 Forward Exchange Rate Calculations

The valuation **relies on forward exchange rates**, derived from the interest rate differential between USD and JPY:

$$F_t = S_0 \times e^{(r-r_f)\times t}$$

- $S_0 = 0.009091$ (initial spot rate, 1/110 JPY per USD).
- $r = 2.5\%$ (USD interest rate).
- $r_f = 1.5\%$ (JPY interest rate).
- t = time in years.

4.6.0.2 Forward Rates for Each Year

- **Year 1**:

$$0.009091 \times e^{(0.025-0.015)\times 1} = 0.009182$$

- **Year 2**:

$$0.009091 \times e^{(0.025-0.015)\times 2} = 0.009275$$

- **Year 3**:

$$0.009091 \times e^{(0.025-0.015)\times 3} = 0.009368$$

4.6.0.3 Valuation Process

1. **Convert Future JPY Cash Flows to USD Using Forward Rates**
   - Example (Year 1):
   $$36 \times 0.009182 = 0.3306 \text{ million USD}$$
   - **Net Cash Flow** (Year 1):
   $$0.3306 - 0.4 = -0.0694 \text{ million USD}$$
2. **Discount Net Cash Flows Using USD Interest Rate (2.5%)**
   - Example (Year 1):
   $$-0.0694 \times e^{-0.025 \times 1} = -0.0677 \text{ million USD}$$
3. **Repeat for Years 2 and 3**
   - The same process applies for each year's cash flows, using the respective forward rates and discount factors.
4. **Sum Present Values to Obtain Swap Valuation**
   - The total present value of net cash flows is $0.9629 million USD, representing the swap's market value.

4.7 Other Currency Swaps

4.7.1 Fixed-for-Floating Currency Swaps

A fixed-for-floating currency swap combines elements of **a fixed-for-fixed currency swap and a fixed-for-floating interest rate swap**. In this arrangement, one party pays a fixed interest rate in one currency while receiving a floating interest rate in another currency.

Example

Consider a swap agreement where:

- A party pays a floating interest rate on GBP 7 million.
- The same party receives a fixed 3% interest rate on USD 10 million, with semiannual payments over 10 years.

This swap can be broken down into two components:

1. **Currency Swap Component**:
    - **Receives** 3% fixed interest on USD 10 million.
    - **Pays** 4% fixed interest on GBP 7 million.
2. **Interest Rate Swap Component**:
    - **Receives** 4% fixed interest.
    - **Pays** floating sterling interest on GBP 7 million.

4.7.2 Floating-for-Floating Currency Swaps

A floating-for-floating currency swap involves exchanging **floating interest payments in two different currencies**. It effectively combines a fixed-for-fixed currency swap with two floating interest rate swaps.

Example

A party exchanges:

- Sterling floating interest payments on GBP 7 million.
- For dollar floating interest payments on USD 10 million.

This structure consists of:

1. **Base Swap**:
    - **Pays** 3% fixed interest on USD 10 million.
    - **Receives** 4% fixed interest on GBP 7 million.
2. **Interest Rate Swap #1**:
    - **Pays** 4% fixed interest.
    - **Receives** floating sterling interest on GBP 7 million.
3. **Interest Rate Swap #2**:
    - **Pays** 3% fixed interest.

- **Receives** floating USD interest on USD 10 million.

4.8 Other Types of Swaps

Beyond currency and interest rate swaps, financial markets offer specialized swaps tailored for **risk management, investment strategies, and arbitrage opportunities**.

- **Amortizing/Step-Up Swap**: The notional principal gradually increases or decreases to match an underlying asset or liability.
- **Compounding Swap**: Interest payments are reinvested, compounding over the swap's duration.
- **Constant Maturity Swap (CMS)**: The interest rate resets periodically based on the rate of a constant maturity instrument (e.g., a 10-year Treasury note).
- **LIBOR-in-Arrears Swap**: The interest rate is determined at the end of the payment period instead of the beginning, increasing uncertainty.
- **Accrual Swap**: Interest accrues only when a benchmark rate (or index) meets specific conditions.
- **Equity Swap**: Exchanges equity returns (e.g., stock performance) for either fixed or floating interest payments.
- **Cross-Currency Interest Rate Swap**: A variation of currency swaps where one or both legs involve floating rates in different currencies.
- **Diff Swap**: An interest rate differential swap, where payments depend on the difference between two reference rates.
- **Commodity Swap**: One party pays a fixed price for a commodity, while the other pays the floating market price over time.
- **Variance Swap**: A contract based on future volatility, where payments depend on the variance of an underlying asset's price.

4.9 Practice Questions and Problems

1. A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?
2. Explain the difference between the credit risk and the market risk in a financial contract.

3. Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

4. Why is the expected loss from a default on a swap less than the expected loss from the default on a loan to the same counterparty with the same principal?

5. A corporate treasurer tells you that he has just negotiated a five-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at six-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

6. Companies A and B have been offered the following rates per annum on a $20 million five-year loan:

	Fixed Rate	Floating Rate
Company A	5.0%	Floating + 0.1%
Company B	6.4%	Floating + 0.6%

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

7. Company X wishes to borrow U.S. dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes:

	Yen	Dollars
Company X	5.0%	9.6%
Company Y	6.5%	10.0%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

8. Companies X and Y have been offered the following rates per annum on a $5 million 10-year investment:

	Fixed Rate	Floating Rate
Company X	8.0%	Floating
Company Y	8.8%	Floating

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

9. A $100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, six-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid-offer rate being exchanged for six-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding (5.063% with semi-annual compounding). The six-month LIBOR rate was 4.6% per annum two months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

10. A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on 20 million GBP for interest at 6% on 30 million USD once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding (the continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum). The current exchange rate (dollars per pound sterling) is 1.5500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

5 Options

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 10. Mechanics of Options Markets
  - Chapter 11. Properties of Stock Options
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Comprehend the fundamentals and definitions of options, including the mechanics behind option payoffs and profits.
- Identify and understand other option characteristics, adjustments, and the relationship with related assets.
- Grasp the principles underlying option pricing dynamics, along with the factors influencing option values.
- Analyze the upper and lower bounds for option prices and the concept of put-call parity in option trading strategies.
- Evaluate the specific considerations for American options, including the impact of early exercise decisions and dividends on option valuation.

5.1 Understanding Options

Definition

Options are financial derivatives that give the buyer the **right, but not the obligation**, to buy (call option) or sell (put option) an asset at a predetermined price (strike price) on or before a specific date. The seller (writer) receives a premium in exchange for granting this right.

Types of options:

- **Call Option**: Gives the holder the right **to buy an asset** at the strike price before expiration. Buyers expect the asset's price to rise.
- **Put Option**: Gives the holder the right **to sell an asset** at the strike price before expiration. Buyers expect the asset's price to fall.

Execution Styles:

- **American Option**: Can be exercised **at any time before expiration**, offering more flexibility.
- **European Option**: Can only be exercised **on the expiration date**, making it more predictable but less flexible.

Options can be written on various assets, including:

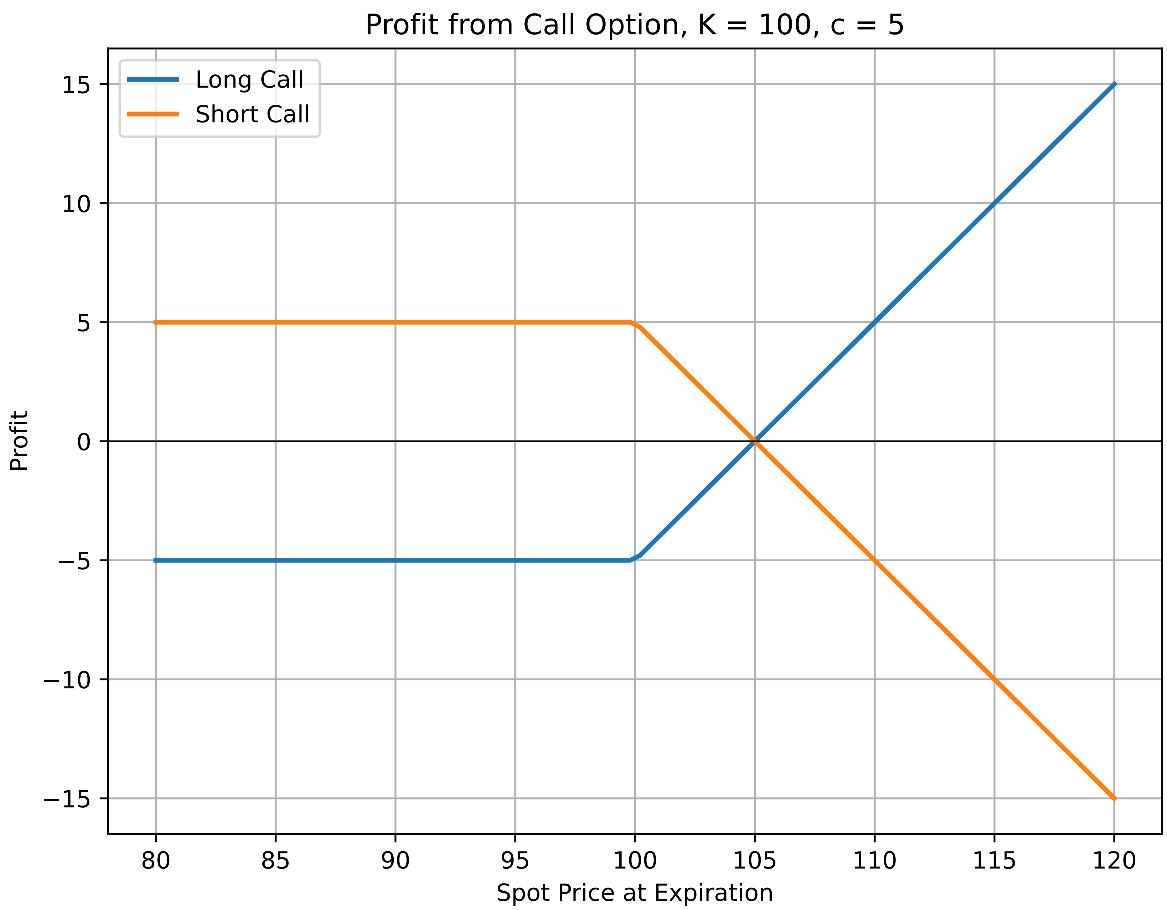
- Stocks
- ETFs and ETPs
- Foreign currency
- Stock indices
- Futures

Margin account:

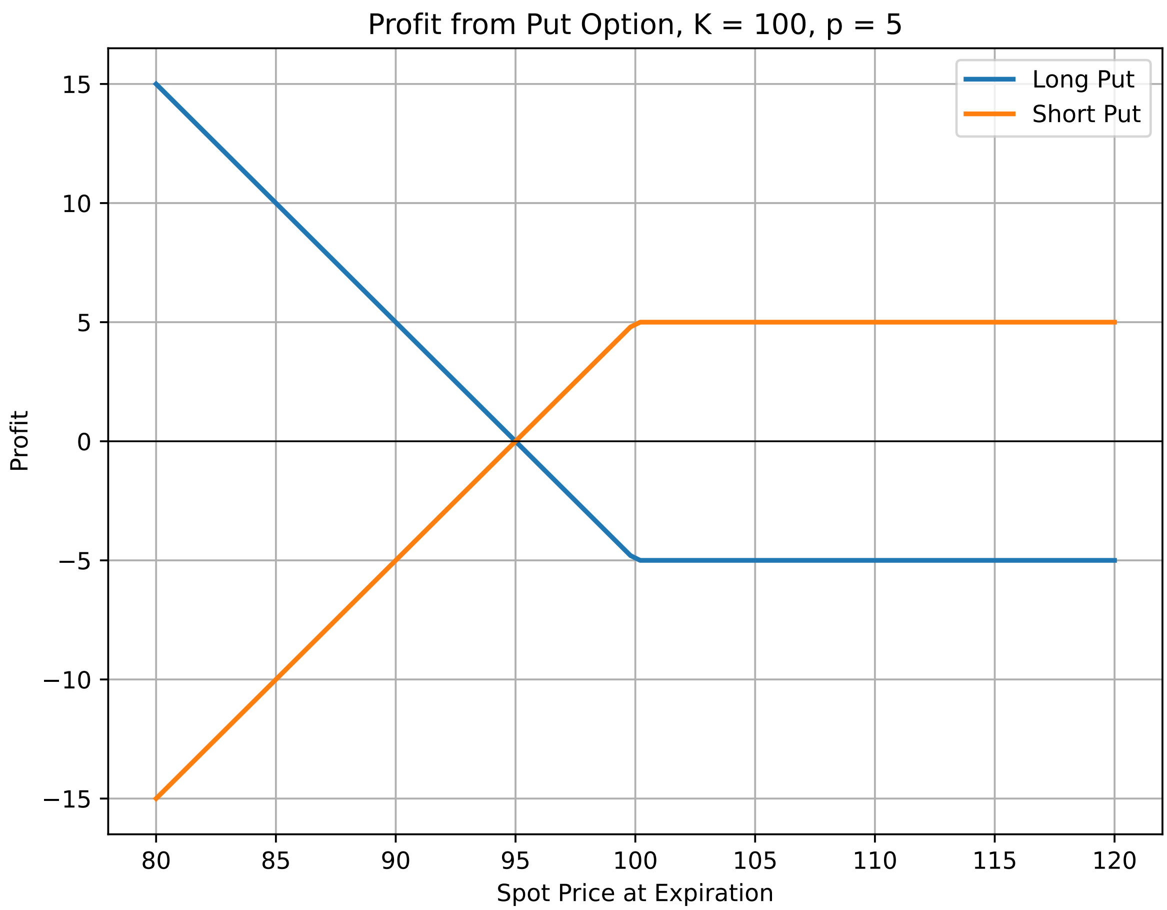
- Buying options **does not require margin** as the buyer's risk is limited to the premium paid.
- Selling options **requires margin** due to the obligation to fulfill the contract.

Option Value Components:

- **Intrinsic Value**: The profit from immediate exercise.
- **Time Value**: The extra value reflecting potential future price changes.



Profit from Call Option, K = 100, c = 5
Long Call
Short Call
Profit
Spot Price at Expiration
15
10
5
0
−5
−10
−15
80
85
90
95
100
105
110
115
120



5.2 Option Payoffs and Profits

5.2.1 Understanding Option Payoffs

The option buyer's payoff depends on the relationship between the price of the underlying asset at expiration (S_T) and strike price of the option (K).

Moneyness determines whether exercising the option would be profitable:

- **In the Money (ITM)**: Exercising the option yields a positive payoff.
- **At the Money (ATM)**: The option's strike price equals the asset price ($S_T = K$). There is no intrinsic value.
- **Out of the Money (OTM)**: Exercising the option results in zero payoff.

Moneyness is crucial for pricing and trading decisions, as ITM options have intrinsic value, while ATM and OTM options are purely speculative.

- **Call Option Payoff**:
$$c_T = \max(0, S_T - K)$$
- **Put Option Payoff**:
$$p_T = \max(0, K - S_T)$$

> **i** Example: Option Payoff
>
> Given an underlying asset price at expiration S_T of \$28 and a strike price K of \$25, the payoffs for call and put options are calculated as follows:
>
> - **Call Buyer Payoff**: $c_T = \max(0, \$28 - \$25) = \$3$
> - **Put Buyer Payoff**: $p_T = \max(0, \$25 - \$28) = \$0$
>
> The call is **in the money** with a payoff of \$3, while the put is **out of the money** with no payoff.

5.2.2 Calculating Option Profit

To determine profit, the option premium (c_0 for calls, p_0 for puts) must be considered:

- **Call Option Profit**:
$$\Pi_{call} = \max(0, S_T - K) - c_0$$
- **Put Option Profit**:
$$\Pi_{put} = \max(0, K - S_T) - p_0$$

> **i** Example: Option Profit
>
> Considering CBX stock options with a strike price $K = \$30$, where the call and put premiums are \$1 and \$2 respectively, and the stock price at expiration S_T is \$27.50:
>
> - **Call Option Profit**: $\Pi_{call} = \max(0, \$27.5 - \$30) - \$1 = -\$1$
> - **Put Option Profit**: $\Pi_{put} = \max(0, \$30 - \$27.5) - \$2 = \$0.5$
>
> The call buyer loses \$1, while the put buyer profits \$0.5.

5.3 Dividend Adjustments and Related Assets

5.3.1 Adjustments for Dividends and Stock Splits

- **Cash Dividends**: Typically do not affect options, as they are factored into pricing.

- **Stock Splits & Stock Dividends**: Adjustments ensure option value remains unchanged.

For an n**-for-**m **stock split**:

- New Strike Price:
$$K' = \frac{m}{n} \times K$$
- New Number of Options:
$$N' = \frac{n}{m} \times N$$

Example: Option Adjustments

5.3.1.1 2-for-1 Stock Split

A call option for 100 shares at \$20 per share:

- **New Strike Price**: $\frac{1}{2} \times 20 = 10$
- **New Number of Options**: $\frac{2}{1} \times 100 = 200$

5.3.1.2 5% Stock Dividend

A call option for 100 shares at \$20 per share:

- **New Strike Price**: $\frac{1}{1.05} \times 20 = 19.05$
- **New Number of Options**: $\frac{1.05}{1} \times 100 = 105$

5.3.2 Related Assets

5.3.2.1 Warrants

- Issued by corporations, granting the right to buy shares at a fixed price.
- Exercising warrants leads to new stock issuance.

5.3.2.2 Employee Stock Options

- Compensation-based options, often issued at-the-money.
- Exercising creates new company shares.

5.3.2.3 Convertible Bonds

- Bonds that can be converted into equity at predetermined terms.
- Often callable, allowing issuers to force conversion.

5.4 Understanding the Dynamics of Option Pricing

The value of stock options depends on several key factors, each influencing option prices under the **ceteris paribus** (all else equal) assumption. The table below summarizes the impact of these variables on European and American options:

Variable	European Call	European Put	American Call	American Put
Current Stock Price (S_0)	+	−	+	−
Strike Price (K)	−	+	−	+
Time to Expiration (T)	?	?	+	+
Volatility (σ)	+	+	+	+
Risk-free Rate (r)	+	−	+	−
Amount of Future Dividends (D)	−	+	−	+

- c, p: Prices of European call and put options.
- C, P: Prices of American call and put options.
- S_0: Current stock price.
- S_T: Stock price at expiration.
- K: Strike price.
- T: Time to expiration.
- σ: Stock price volatility.
- D: Present value of future dividends.
- r: Risk-free interest rate (continuous compounding).

5.4.1 American vs. European Options

American options, which can be exercised anytime before expiration, are **always worth at least as much** as their European counterparts:

$$C \geq c, \quad P \geq p$$

This is due to the added flexibility of early exercise, making American options inherently more valuable or, at minimum, equal in price to European options.

5.5 Upper and Lower Bounds for Option Prices

5.5.0.1 Upper Bound - Call Option

The price of a call option, whether American or European, cannot exceed the current stock price:

$$c \leq S_0, \quad C \leq S_0$$

This prevents arbitrage, as no rational investor would pay more for a call than the stock itself.

5.5.0.2 Upper Bound - Put Option

- **American put options**: Capped at the strike price since the holder can sell at K:

$$P \leq K$$

- **European put options**: Limited by the present value of K, as they can only be exercised at expiration:

$$p \leq Ke^{-rT}$$

5.5.0.3 Lower Bound - European Call Option

The minimum price of a European call on a non-dividend paying stock is:

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$

Example: European Call Lower Bound

Given: $S_0 = 20$, $K = 18$, $T = 1$, $r = 10\%$, $D = 0$

$$\text{Lower bound} = 20 - 18e^{-0.1} = 3.71$$

If the call price is \$3, an arbitrageur could **short the stock, buy the call, and invest at 10%**.

5.5.0.4 Lower Bound - European Put Option

For puts on non-dividend paying stock, the lower bound is:

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

Example: European Put Lower Bound

Given: $S_0 = 37$, $K = 40$, $T = 0.5$, $r = 5\%$, $D = 0$

$$40e^{-0.05\times0.5} - 37 = 2.01$$

If the put price is \$1, an arbitrageur could **borrow \$38 for six months to buy both the put and stock**.

5.5.1 American Options and Early Exercise

5.5.1.1 American Call Options (Non-Dividend Paying Stocks)

It is never optimal to exercise early because:

- **Preserving capital**: Delaying exercise defers payment of K.
- **Insurance benefit**: Holding the call provides downside protection.
- **Maximizing value**: Selling the option retains its time value.

Thus, their price bounds are the same as European calls:

$$C \geq \max(S_0 - Ke^{-rT}, 0)$$

5.5.2 American Put Options (Non-Dividend Paying Stocks)

Early exercise may be optimal, especially when deep in the money:

- **Immediate value realization**: $K - S_0$ can be collected now.
- **Time value of money**: Receiving proceeds sooner is preferable.
- **Stock price limitations**: Since $S_0 \geq 0$, the put's value is capped.

Thus, the lower bound for American puts is adjusted:

$$P \geq \max(K - S_0, 0)$$

5.6 Put-Call Parity

Put-call parity establishes a fundamental relationship between European call and put options with the same strike price and expiration. It ensures pricing consistency in the absence of arbitrage opportunities.

5.6.1 Put-Call Parity for Non-Dividend Paying Stocks

Consider two equivalent portfolios:

- **Portfolio A**: A European call option (c) and a zero-coupon bond paying K at expiration.
- **Portfolio B**: A European put option (p) and the underlying stock (S_0).

At expiration (T), both portfolios yield the same payoff:

		$S_T > K$ (Above Strike)	$S_T < K$ (Below Strike)
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio B	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

Since both portfolios must have the same present value, we get the **put-call parity equation**:

$$c + Ke^{-rT} = p + S_0$$

Arbitrage Opportunities via Put-Call Parity

Consider an example where $S_0 = \$31$, $r = 10\%$, the call option price $c = \$3$, and the strike price $K = \$30$. The table outlines potential arbitrage strategies based on discrepancies in put option pricing, illustrating the mechanism for securing risk-free profits by leveraging the put-call parity principle.

Three-month put price = \$2.25	Three-month put price = \$1
Action now:	*Action now:*
Buy call for \$3	Borrow \$29 for 3 months
Short put to realize \$2.25	Short call to realize \$3
Short the stock to realize \$31	Buy put for \$1
Invest \$30.25 for 3 months	Buy the stock for \$31
...	
Action in 3 months if $S_T > 30$:	*Action in 3 months if* $S_T > 30$:
Receive \$31.02 from investment	Call exercised: sell stock for \$30
Exercise call to buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27
...	

Action in 3 months if $S_T < 30$:
Receive \$31.02 from investment
Put exercised: buy stock for \$30
Net profit = \$1.02

Action in 3 months if $S_T < 30$:
Exercise put to sell stock for \$30
Use \$29.73 to repay loan
Net profit = \$0.27

5.6.2 Extension to American Options

Put-call parity strictly applies to **European** options. However, for **American** options, which allow early exercise, the pricing relationship is bounded by:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

This range accounts for the added flexibility of early exercise, particularly for American put options, which may be exercised early if deep in the money.

5.7 Effect of Dividends on Options

Dividends impact option valuation and exercise strategies, particularly for **American call options**. The present value of expected dividends over the option's life is denoted as D. Understanding the impact of dividends is crucial for accurate option pricing and strategic decision-making.

- **Dividends reduce call option value** due to lost dividend income.
- **Dividends increase put option value** as they lower the stock price.
- For **American call options**, early exercise is typically suboptimal. However, if a dividend is expected, it may be beneficial to **exercise just before the ex-dividend date** to capture the dividend payout. Apart from this scenario, early exercise remains suboptimal.

5.7.1 Adjusted Lower Bound Valuations

Dividends modify the lower bounds of option prices:

- **Call Options:** The lower bound decreases since early exercise forfeits dividends:

$$c \geq S_0 - D - Ke^{-rT}$$

- **Put Options:** The lower bound increases, as dividends reduce the stock price:

$$p \geq D + Ke^{-rT} - S_0$$

5.7.2 Adjusted Put-Call Parity with Dividends

Dividends also modify the **put-call parity** relationship:

- **European Options with Dividends**:

$$c + D + Ke^{-rT} = p + S_0$$

- **American Options with Dividends**:

$$S_0 - D - K < C - P < S_0 - Ke^{-rT}$$

5.8 Practice Questions and Problems

5.8.1 Option Profitability and Exercise Conditions

1. Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

2. An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

3. An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

4. Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

5.8.2 Margin Requirements, Market Choices, and Contract Adjustments

5. Explain why margin accounts are required when clients write options but not when they buy options.

6. A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the NASDAQ OMX and (b) the over-the-counter market for trading?

7. The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

8. Consider an exchange-traded call option contract to buy 500 shares with a strike price of \$40 and maturity in four months. Explain how the terms of the option contract change when there is

    1. A 10% stock dividend
    2. A 10% cash dividend
    3. A 4-for-1 stock split

5.8.3 Option Pricing Bounds

9. Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

10. Explain why an American option is always worth at least as much as its intrinsic value.

11. What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?

12. What is a lower bound for the price of a one-month European put option on a non-dividend paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

5.8.4 Early Exercise and Put-Call Parity

13. Give at least two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal.

14. The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put. Explain this statement.

15. The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of \$20?

16. List and explain the six factors affecting stock option prices.

6 Options Trading Strategies and Hedging

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 12 - Trading Strategies Involving Options
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 5 - Derivatives Strategies

Additonal sources on option trading strategies:

- tastylive
- options playbook
- investopedia

Learning Outcomes:

- Understand the structure and purpose of Principal Protected Notes, including how they safeguard the principal amount while offering potential investment gains.
- Analyze strategies that combine positions in the underlying asset with options to manage risk and enhance potential returns, focusing on protective puts and covered calls.
- Explore various option spread strategies, such as bull spreads, bear spreads, calendar spreads, and butterfly spreads, to understand their risk/reward profiles and market outlook implications.
- Examine option combination strategies like straddles, strangles, strips, and straps, highlighting their use in volatile markets to capitalize on significant price movements in either direction.

```
# Load python libraries
import plotly.graph_objects as go
import numpy as np
```

6.1 Principal Protected Note

Principal Protected Notes (PPNs) enable investors to engage in potentially high-reward investments without the fear of losing their principal amount. This dual-feature mechanism is facilitated through a blend of a zero-coupon bond and a derivative instrument, typically a call option.

> **i** Example: PPN
>
> Consider an investment in a PPN valued at \$1,000, structured as follows:
>
> 1. **Zero-Coupon Bond Component**: A 3-year zero-coupon bond with a face value of \$1,000 ensures the principal protection. Given a continuous compounding interest rate of 6%, the present value of this bond is calculated as:
>
> $$PV = \$1,000 \times e^{-0.06\times 3} = \$835.27$$
>
> This calculation confirms that an initial investment of \$835.27 will mature to \$1,000 over 3 years, effectively protecting the principal.
>
> 2. **Call Option Component**: The remaining funds, amounting to \$164.73 (\$1,000 - \$835.27), are utilized to purchase a 3-year at-the-money call option on a stock portfolio. This option provides the upside potential.

The feasibility of structuring a PPN profitably hinges on several market conditions:

- **Dividend Levels**: Higher dividends reduce the attractiveness of the call option component.
- **Interest Rates**: The zero-coupon bond's cost is inversely related to interest rates; higher rates make principal protection cheaper.
- **Portfolio Volatility**: Higher volatility increases the price of the call option, but enhancing the investment's upside potential.

To cater to diverse investor preferences, PPNs can be customized with various features:

- **Strike Price Adjustments**: Options may be set out of the money to increase potential returns.
- **Return Caps**: Limits may be imposed on the maximum return to reduce the cost of the call option.
- **Structural Innovations**: Features like knock-outs, averaging mechanisms, etc., can be incorporated to tailor risk and return profiles.

6.2 Combining Underlying and Option

Combining positions in the underlying stock with options can create tailored risk/reward profiles that suit various investment strategies and market views. Two popular strategies that illustrate this principle are the protective put and the covered call.

6.2.1 Protective Put

A protective put strategy involves buying a stock (or holding a currently owned stock) and buying a put option for the same stock. This strategy is used to protect against a decline in the stock's price while allowing for participation in any upside. It guarantees a minimum selling price (the strike price of the put) for the stock until the put's expiration, effectively setting a floor on the potential losses without capping the potential gains.

```
# Parameters for the protective put strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
S0 = 100  # Initial stock price
K = 100  # Strike price of the put option
premium_put = 5  # Premium paid for the put option
initial_investment = S0 + premium_put  # Total initial investment including the put premium

# Profit calculations for protective put
stock_profit = spot_prices - S0  # Profit from the stock alone
put_option_profit = np.maximum(K - spot_prices, 0) - premium_put  # Profit from the long put
protective_put_profit = stock_profit + put_option_profit  # Total profit from the protective

# Create the figure
fig = go.Figure()

# Add traces for the individual components' profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=stock_profit,
        mode="lines",
        name="Stock Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
```

```
        x=spot_prices,
        y=put_option_profit,
        mode="lines",
        name="Put Option Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the protective put net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=protective_put_profit,
        mode="lines",
        name="Protective Put",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference, adjusted for the initial investment
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Protective Put Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

```
Unable to display output for mime type(s): text/html
```

6.2.2 Covered Call

A covered call strategy involves owning a stock and selling a call option on that stock. This strategy is aimed at generating additional income from the option premium, which can enhance the overall returns on the stock, especially in flat or slightly bullish markets. It limits the upside potential to the strike price of the sold call but provides premium income that offers some protection against a decline in the stock's price.

```
# Parameters for the covered call strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
S0 = 100  # Initial stock price
K = 100  # Strike price of the call option
premium_call = 5  # Premium received for the call option

# Profit calculations for covered call
stock_profit = spot_prices - S0  # Profit from the stock alone
call_option_profit = np.where(spot_prices > K, K - spot_prices, 0) + premium_call  # Profit f
covered_call_profit = stock_profit + call_option_profit  # Total profit from the covered call

# Create the figure
fig = go.Figure()

# Add traces for the individual components' profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=stock_profit,
        mode="lines",
        name="Stock Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_option_profit,
        mode="lines",
        name="Call Option Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
```

```python
# Add trace for the covered call net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=covered_call_profit,
        mode="lines",
        name="Covered Call",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Covered Call Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3 Option Spreads

Option spread strategies involve holding two or more options of the same type simultaneously to capitalize on movements in the underlying asset's price. These strategies can be tailored to reflect various market outlooks, from bullish to bearish or even neutral.

6.3.1 Bull Spread Strategy

A bull spread is a strategy used when an investor expects a moderate increase in the price of the underlying asset. It can be constructed using either calls or puts.

6.3.1.1 Using Calls (Call Bull Spread)

- **Position**: Buy a call option with a lower strike price and sell another call option with a higher strike price. Both options share the same expiration date.
- **Profit Potential**: Maximum profit is limited to the difference between the two strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Point**: Lower strike price + net premium paid.

```
# Parameters for the bull spread strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the long call option
K2 = 105  # Strike price of the short call option
premium_long = 5  # Premium paid for the long call
premium_short = 2  # Premium received for the short call

# Profit calculations
long_call_profit = np.maximum(spot_prices - K1, 0) - premium_long
short_call_profit = -(np.maximum(spot_prices - K2, 0) - premium_short)
bull_spread_profit = long_call_profit + short_call_profit  # Net profit of the bull spread

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call_profit,
        mode="lines",
        name="Long Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call",
        line=dict(width=3, dash='dot'),
```

```
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bull spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bull_spread_profit,
        mode="lines",
        name="Bull Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bull Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.1.2 Using Puts (Put Bull Spread)

- **Position**: Buy a put option with a higher strike price and sell another put option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the difference between the strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Point**: Higher strike price - net premium paid.

```
# Parameters for the bull spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the long put option (lower strike)
K2 = 105  # Strike price of the short put option (higher strike)
premium_long = 2  # Premium paid for the long put
premium_short = 5  # Premium received for the short put

# Profit calculations
long_put_profit = np.maximum(K1 - spot_prices, 0) - premium_long
short_put_profit = -(np.maximum(K2 - spot_prices, 0) - premium_short)
bull_spread_profit = long_put_profit + short_put_profit  # Net profit of the bull spread usi

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put_profit,
        mode="lines",
        name="Long Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bull spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bull_spread_profit,
```

```
        mode="lines",
        name="Bull Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bull Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.2 Bear Spread Strategy

A bear spread is used when an investor expects a moderate decline in the price of the underlying asset. Similar to the bull spread, it can be executed using calls or puts.

6.3.2.1 Using Calls (Call Bear Spread)

- **Position**: Buy a call option with a higher strike price and sell another call option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the net premium received for the spread.
- **Risk**: Limited to the difference between the strike prices minus the net premium received.
- **Break-even Point**: Lower strike price + net premium received.

```
# Parameters for the bear spread strategy using call options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 90  # Strike price of the short call option (lower strike)
K2 = 105  # Strike price of the long call option (higher strike)
```

```
premium_short = 5  # Premium received for the short call
premium_long = 2  # Premium paid for the long call

# Profit calculations for bear spread using call options
short_call_profit = -(np.maximum(spot_prices - K1, 0) - premium_short)
long_call_profit = np.maximum(spot_prices - K2, 0) - premium_long
bear_spread_profit = short_call_profit + long_call_profit  # Net profit of the bear spread

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call_profit,
        mode="lines",
        name="Long Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bear spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bear_spread_profit,
        mode="lines",
        name="Bear Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
```

```
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bear Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.2.2 Using Puts (Put Bear Spread)

- **Position**: Buy a put option with a higher strike price and sell another put option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the net premium received for the spread.
- **Risk**: Limited to the difference between the strike prices minus the net premium received.
- **Break-even Point**: Higher strike price - net premium received.

```
# Parameters for the bear spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 105  # Strike price of the long put option (higher strike)
K2 = 95  # Strike price of the short put option (lower strike)
premium_long = 5  # Premium paid for the long put
premium_short = 2  # Premium received for the short put

# Profit calculations for bear spread using put options
long_put_profit = np.maximum(K1 - spot_prices, 0) - premium_long
short_put_profit = -(np.maximum(K2 - spot_prices, 0) - premium_short)
bear_spread_profit = long_put_profit + short_put_profit  # Net profit of the bear spread

# Create the figure
fig = go.Figure()
```

```python
# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put_profit,
        mode="lines",
        name="Long Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bear spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bear_spread_profit,
        mode="lines",
        name="Bear Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bear Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
```

```
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.3 Butterfly Spread Strategy

The butterfly spread is a neutral option strategy that is used when an investor expects little to no movement in the underlying asset's price. It can be constructed using either calls or puts and involves both buying and selling options at three different strike prices.

6.3.3.1 Using Calls (Call Butterfly Spread)

- **Position**:
  - Buy one call option at a lower strike price (A).
  - Sell two call options at a middle strike price (B).
  - Buy one call option at a higher strike price (C).
  - All options have the same expiration date, and the strike prices are equidistant.
- **Profit Potential**: Maximum profit is achieved if the underlying asset's price is equal to the middle strike price at expiration. The profit is the difference between the middle and lower (or higher) strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid to establish the spread.
- **Break-even Points**: There are two break-even points:
  - Lower break-even: Lower strike price (A) + net premium paid.
  - Upper break-even: Higher strike price (C) - net premium paid.

```
# Parameters for the butterfly spread strategy using call options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the first long call option (lower strike)
K2 = 100  # Strike price of the short call options (middle strike)
K3 = 105  # Strike price of the second long call option (higher strike)
premium_long1 = 8  # Premium paid for the first long call
premium_short = 5  # Premium received for each short call (twice for the middle strike)
premium_long2 = 4  # Premium paid for the second long call
```

```
# Profit calculations for butterfly spread using call options
long_call1_profit = np.maximum(spot_prices - K1, 0) - premium_long1
short_call_profit = 2 * (-(np.maximum(spot_prices - K2, 0) - premium_short))
long_call2_profit = np.maximum(spot_prices - K3, 0) - premium_long2
butterfly_spread_profit = long_call1_profit + short_call_profit + long_call2_profit  # Net pr

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call1_profit,
        mode="lines",
        name="Long Call 1",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call (2x)",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call2_profit,
        mode="lines",
        name="Long Call 2",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the butterfly spread net profit
```

```
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=butterfly_spread_profit,
        mode="lines",
        name="Butterfly Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Butterfly Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.3.2 Using Puts (Put Butterfly Spread)

- **Position**:
  - Buy one put option at a higher strike price (A).
  - Sell two put options at a middle strike price (B).
  - Buy one put option at a lower strike price (C).
  - All options have the same expiration date, and the strike prices are equidistant.
- **Profit Potential**: Maximum profit is similar to the call butterfly and is achieved if the underlying's price equals the middle strike price at expiration.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Points**: Identical in concept to the call butterfly, adjusted for the put options' strike prices.

```
# Parameters for the butterfly spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 105  # Strike price of the first long put option (higher strike)
K2 = 100  # Strike price of the short put options (middle strike)
K3 = 95  # Strike price of the second long put option (lower strike)
premium_long1 = 8  # Premium paid for the first long put
premium_short = 5  # Premium received for each short put (twice for the middle strike)
premium_long2 = 4  # Premium paid for the second long put

# Profit calculations for butterfly spread using put options
long_put1_profit = np.maximum(K1 - spot_prices, 0) - premium_long1
short_put_profit = 2 * (-(np.maximum(K2 - spot_prices, 0) - premium_short))
long_put2_profit = np.maximum(K3 - spot_prices, 0) - premium_long2
butterfly_spread_profit = long_put1_profit + short_put_profit + long_put2_profit  # Net profi

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put1_profit,
        mode="lines",
        name="Long Put 1",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put (2x)",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
```

```
        y=long_put2_profit,
        mode="lines",
        name="Long Put 2",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the butterfly spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=butterfly_spread_profit,
        mode="lines",
        name="Butterfly Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Butterfly Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.4 Key Characteristics of Option Spreads

- **Market Outlook**: Bull spreads are used when expecting a moderate price increase, while bear spreads are for a moderate price decrease. Butterfly spread is best suited for markets expected to be stable or within a specific range.

- **Risk and Reward**: Bull and bear spreads strategies offer limited risk and reward, making them attractive for traders with a specific market view and risk tolerance. Butterfly spreado ffers a high reward-to-risk ratio if the market remains stable but with limited profit potential outside the narrow price range.
- **Cost Efficiency**: Spreads can be cost-effective ways to gain market exposure compared to outright option purchases.
- **Flexibility**: Investors can adjust the width of the spread and the strike prices to manage the risk-reward ratio according to their market outlook and risk appetite.

> Calendar Spreads
>
> Calendar spreads, also known as time or horizontal spreads, involve options of the same underlying asset, strike price, but **different expiration dates**. They capitalize on the difference in time decay rates (theta) between options. Typically, a calendar spread is constructed by selling a short-term option and buying a longer-term option of the same type (call or put).
> For more information check, e.g., tastylive

6.4 Option Combinations

Option combinations involve two or more options of different types. It allow traders and investors to express complex market views and hedge positions in ways that single option positions cannot. This section covers four popular option combination strategies: straddles, strangles, strips, and straps.

6.4.1 Straddles

A straddle involves buying or selling both a call and a put option with the same strike price and expiration date.

6.4.1.1 Long Straddle

A long straddle is created by buying both a call and a put option. It is a bet on volatility without taking a directional stance. Traders expect the underlying asset to move significantly, but they are unsure in which direction. The maximum loss is limited to the total premium paid for both options, while the profit potential is unlimited if the underlying moves significantly in either direction.

```python
# Parameters for the long straddle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K = 100  # Strike price for both the call and put options
premium_call = 5  # Premium paid for the call option
premium_put = 5 # Premium paid for the put option

# Profit calculations for long straddle
call_profit = np.maximum(spot_prices - K, 0) - premium_call
put_profit = np.maximum(K - spot_prices, 0) - premium_put
long_straddle_profit = call_profit + put_profit  # Net profit of the long straddle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the long straddle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_straddle_profit,
        mode="lines",
```

```
        name="Long Straddle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Long Straddle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.4.1.2 Short Straddle

Conversely, a short straddle involves selling a call and a put option. This strategy bets on low market volatility, with the trader expecting the underlying asset to remain stable. The maximum profit is limited to the premiums received, while the risk is theoretically unlimited, as the underlying asset can move significantly in either direction.

```
# Parameters for the short straddle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K = 100  # Strike price for both the call and put options
premium_call = 5  # Premium received for the call option
premium_put = 5  # Premium received for the put option

# Profit calculations for short straddle
call_profit = np.minimum(K - spot_prices, 0) + premium_call
put_profit = np.minimum(spot_prices - K, 0) + premium_put
short_straddle_profit = call_profit + put_profit  # Net profit of the short straddle

# Create the figure
```

```
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the short straddle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_straddle_profit,
        mode="lines",
        name="Short Straddle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Short Straddle Strategy",
```

```
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.4.2 Strangles

A strangle is similar to a straddle but involves options with different strike prices. Typically, the call option has a higher strike price than the put.

6.4.2.1 Long Strangle

A long strangle is less expensive than a long straddle due to the options being out of the money. It requires a larger move in the underlying asset's price to be profitable but still benefits from significant volatility without a clear directional bias. The maximum loss is limited to the total premium paid.

```
# Parameters for the long strangle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K_call = 105  # Strike price of the call option (higher strike)
K_put = 95  # Strike price of the put option (lower strike)
premium_call = 2  # Premium paid for the call option
premium_put = 2  # Premium paid for the put option

# Profit calculations for long strangle
call_profit = np.maximum(spot_prices - K_call, 0) - premium_call
put_profit = np.maximum(K_put - spot_prices, 0) - premium_put
long_strangle_profit = call_profit + put_profit  # Net profit of the long strangle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
```

```
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the long strangle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_strangle_profit,
        mode="lines",
        name="Long Strangle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Long Strangle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)
```

```
# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.4.2.2 Short Strangle

Selling a strangle involves selling an OTM call and an OTM put. This strategy profits from low volatility and time decay. The risk is significant if the underlying moves dramatically, as one of the sold options could become deeply in the money.

```
# Parameters for the short strangle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K_call = 105  # Strike price of the call option (higher strike)
K_put = 95  # Strike price of the put option (lower strike)
premium_call = 2  # Premium received for the call option
premium_put = 2  # Premium received for the put option

# Profit calculations for short strangle
call_profit = np.minimum(K_call - spot_prices, 0) + premium_call
put_profit = np.minimum(spot_prices - K_put, 0) + premium_put
short_strangle_profit = call_profit + put_profit  # Net profit of the short strangle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
```

```
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the short strangle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_strangle_profit,
        mode="lines",
        name="Short Strangle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Short Strangle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.4.3 Strips and Straps

Strips and straps are modifications of the straddle strategy, offering a directional bias while still betting on volatility.

- **Strip**: A strip consists of buying one call option and two put options with the same strike price and expiration. This strategy bets on volatility but with a bearish outlook, as the additional put provides extra profit if the underlying asset's price falls.
- **Strap**: A strap involves buying two call options and one put option with the same strike price and expiration. It is similar to a strip but with a bullish outlook, benefiting from an upward move in the underlying asset's price.

6.4.4 Key Characteristics of Option Combinations

- **Volatility Sensitivity**: All these strategies are sensitive to changes in implied volatility. Long positions in straddles, strangles, strips, and straps benefit from an increase in volatility, while short positions benefit from a decrease.
- **Directional Bias**: Straddles and strangles are primarily non-directional strategies that profit from significant price movements in either direction. Strips and straps introduce a directional bias, offering asymmetric payoffs based on the underlying asset's movement.
- **Risk and Reward**: The risk-reward profile varies significantly between these strategies. Long positions have limited risk and potentially unlimited reward, while short positions offer limited profit potential with significant risk.
- **Breakeven Points**: These strategies have two breakeven points, one on each side of the underlying asset's price at entry. The underlying must move beyond these points for long strategies to profit.

6.5 Practice Questions and Problems

1. What is meant by a protective put? What position in call options is equivalent to a protective put?
2. Explain two ways in which a bear spread can be created.
3. When is it appropriate for an investor to purchase a butterfly spread?
4. What trading strategy creates a reverse calendar spread?
5. What is the difference between a strangle and a straddle?
6. A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?
7. Explain how an aggressive bear spread can be created using put options.
8. Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Draw and explain profit/loss.

9. An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

7 Reading Week

> Tip
>
> Take some rest and revise the first six topics!

8 Option Pricing - Binomial Trees

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 13 - Binomial Trees
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 4 - Valuation of Contingent Claims
- Cox, J. C., S. A. Ross, and M. Rubinstein. "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (October 1979): 229–64.

Learning Outcomes:

- Understand the basic principles of the one-step binomial model for option pricing, including how to set up a riskless portfolio and calculate the option's value.
- Grasp the concept of risk-neutral valuation and its application in simplifying the pricing of derivatives by focusing on discounting expected payoffs at the risk-free rate.
- Learn how to extend the binomial model to two steps, including calculating the option value at each node and understanding the implications for option pricing.
- Comprehend the methodology for choosing the up (u) and down (d) factors in binomial models, based on the asset's volatility and the length of the time step, as per the Cox, Ross, and Rubinstein (CRR) model.
- Gain insights into how the binomial model can be adapted for different types of assets, including nondividend paying stocks, stock indices, currencies, and futures contracts.

8.1 A One-Step Binomial Model

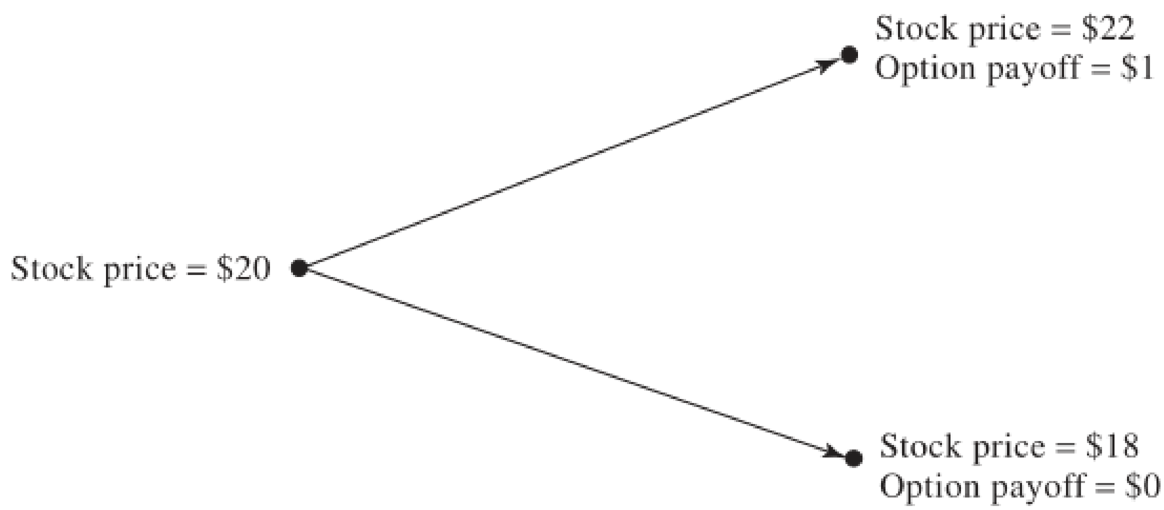
This section provides an overview of a fundamental financial derivative pricing method using a simplified binomial model. The model offers an intuitive approach to option pricing, grounded in the concept of constructing a riskless portfolio and applying arbitrage-free pricing principles.

This generalized model provides a foundational approach for valuing derivatives using discrete-time, binomial frameworks. It not only clarifies the mechanics of risk-neutral valuation but also lays the groundwork for understanding more complex derivatives and pricing methods in continuous-time models.

8.1.1 A Simple Binomial Model for a Call Option

Option Characteristics:

- 3-month call option on a stock.
- The option's strike price is \$21.
- Currently, the stock is priced at \$20.
- In three months, the stock's price can either rise to \$22 or fall to \$18.
- Risk-free rate of interest is 4%.



8.1.1.1 Option Pricing

1. **Constructing a Riskless Portfolio:**

- The portfolio consists of being long Δ shares of the stock and short 1 call option.
- To keep the portfolio risk-free over the option's life, we find Δ such that the portfolio's value is independent of the stock's final price:
  - At stock price \$22: Portfolio value = $\$22\Delta - 1$ (assuming the option is exercised).
  - At stock price \$18: Portfolio value = $\$18\Delta - 0$ (the option expires worthless).
  - Setting these values equal to each other yields $\Delta = 0.25$.

- Thus, a riskless portfolio entails being long 0.25 shares of the stock and short 1 call option.

2. **Valuing the Riskless Portfolio:**

- With a risk-free rate of 4%, we calculate the portfolio's present value.
- Future value of the portfolio in 3 months: $\$22 \times 0.25 - 1 = \$18 \times 0.25 = \$4.50$.
- Discounting this back to the present yields: $4.50 \times e^{-0.04 \times 0.25} = \$4.455$.

3. **Determining the Call Option's Value:**

- The portfolio, consisting of long 0.25 shares and short 1 option, is valued today at \$4.455.
- The current value of 0.25 shares is $0.25 \times \$20 = \$5$.
- Subtracting the portfolio's value from the shares' value gives the option's price: $\$5 - \$4.455 = \$0.545$.

8.1.2 Generalized Framework for Binomial Option Pricing

This section extends the simple binomial model to a more general framework, illustrating how option prices can be systematically derived under varying market conditions. This generalization leverages the fundamental concepts of arbitrage-free pricing and the construction of a synthetic, risk-free portfolio.

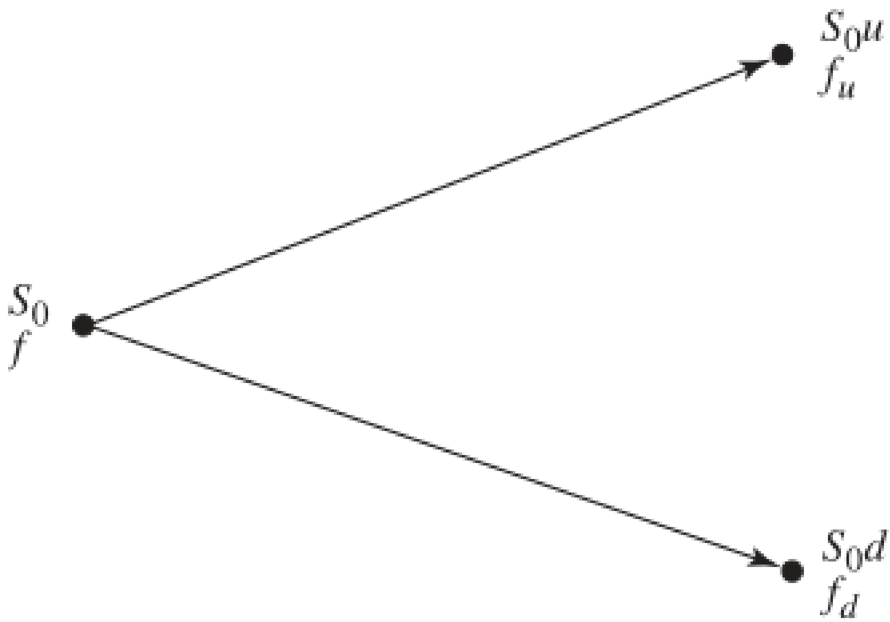
8.1.2.1 Core Principles

- The ending value of a portfolio that contains Δ shares of stock and one short call option can vary based on the stock's final price movement:
  - Following an upward movement, the value is $S_0 u\Delta - f_u$.
  - Following a downward movement, the value is $S_0 d\Delta - f_d$.
- **Creation of a Riskless Portfolio:** To eliminate risk, the values from both upward and downward movements must be equalized, leading to the determination of Δ:

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

  This equation highlights that Δ effectively measures the sensitivity of the option's price to changes in the underlying stock's price, akin to the "delta" in continuous-time models.
- **No-Arbitrage Condition:** The portfolio's risk-free nature implies it should yield returns at the risk-free rate, reinforcing the no-arbitrage principle in financial markets.

8.1.2.2 Analytical Representation



- At time T, the portfolio's value is $S_0u\Delta - f_u$.
- Discounted to present value, this becomes $(S_0u\Delta - f_u)e^{-rT}$.
- The initial cost, or present value, of establishing this portfolio is $S_0\Delta - f$.

By equating the initial investment to the present value of the future payoff, we derive the option's pricing formula:

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

Substituting Δ yields the generalized option pricing equation:

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

where p represents the risk-neutral probability of an upward price movement:

$$p = \frac{e^{rT} - d}{u - d}$$

8.1.2.3 Interpretation of p as a Probability

- The parameter p and its complement $(1-p)$ can be interpreted as the probabilities of the stock's upward and downward price movements, respectively.
- This interpretation leads to a significant insight: the value of an option, or more broadly any derivative, in this model is equivalent to its expected payoff in a "risk-neutral" world, discounted at the risk-free rate. This reflects the fundamental principle that, in an arbitrage-free market, derivative pricing must align with the expected payoff under risk neutrality.

8.1.3 Key Formulas in Binomial Option Pricing

The binomial model provides a straightforward method for determining the price of an option using a discrete-time framework. The central formulas include:

- **Option Price Calculation:** This formula represents the present value of the expected payoff of the option, adjusted for risk neutrality.

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

- **Risk-neutral Probability:** p calculates the risk-neutral probability of the stock price moving up, crucial for determining the expected option payoff.

$$p = \frac{e^{rT} - d}{u - d}$$

8.1.3.1 Variable Definitions:

- f: The current price/premium of the option.
- p: The risk-neutral probability of the stock's price moving up (upward movement).
- $(1-p)$: The risk-neutral probability of the stock's price moving down (downward movement).
- f_u: The payoff from the option in the event of an up movement in the stock price.
- f_d: The payoff from the option in the event of an down movement in the stock price.
- u: The factor by which the stock price increases in the event of an upward movement. This multiplicative factor represents the potential growth of the stock's price in the model's "up" state.
- d: The factor by which the stock price decreases in the event of a downward movement. This multiplicative factor indicates the potential decline of the stock's price in the model's "down" state.
- T: The time to maturity (or expiration) of the option, expressed in years.

- r: The annualized, continuously compounded risk-free interest rate.

Example: Binomial Option Pricing

Consider an option with parameters: $u = 1.1$, $d = 0.9$, $r = 0.04$, $T = 0.25$ (three months), $f_u = 1$ (payoff if the stock price increases), and $f_d = 0$ (payoff if the stock price decreases).

- **Calculating p:**

$$p = \frac{e^{0.04 \times 0.25} - 0.9}{1.1 - 0.9} = 0.5503$$

This result indicates a 55.03% risk-neutral probability of the stock price increasing.

- **Determining f (Option Price):**

$$f = e^{-0.04 \times 0.25}(0.5503 \times 1 + 0.4497 \times 0) = 0.545$$

The calculated option price is \$0.545, reflecting the present value of the option's expected payoff, discounted at the risk-free rate.

8.2 Risk-Neutral Valuation Framework

Risk-neutral valuation is a pivotal concept in financial mathematics, offering a streamlined and theoretically robust method for pricing derivatives. This approach is grounded in the notion that the expected return on the underlying asset can be assumed to be the risk-free rate when valuing derivatives. Here, we elaborate on this principle and its practical implications, using a binomial model as the illustrative framework.

8.2.1 Core Concept

- **Expected Stock Price Dynamics:** In a binomial model where the probability of upward and downward movements are denoted by p and $1 - p$ respectively, the expected stock price at time T, discounted back to the present, is mathematically equivalent to the stock's initial price compounded at the risk-free rate, $S_0 e^{rT}$. This equivalence underscores the principle that, under risk-neutral valuation, the stock is assumed to grow at the risk-free rate over time.
- **Binomial Trees and Derivative Pricing:** The use of binomial trees in financial modeling demonstrates a general principle: to value a derivative, it is sufficient to assume

that the underlying asset yields a return equal to the risk-free rate, and similarly, the discount rate applied to future payoffs is the risk-free rate. This methodology, known as risk-neutral valuation, simplifies the derivative pricing process by abstracting away from the actual expected return of the underlying asset.

Example Revisited

Consider an option with parameters: $u = 1.1$, $d = 0.9$, $r = 0.04$, $T = 0.25$ (three months), $f_u = 1$ (payoff if the stock price increases), and $f_d = 0$ (payoff if the stock price decreases).
The calculation proceeds as follows:

$$22p + 18(1-p) = 20e^{0.04\times 0.25} \Rightarrow p = 0.5503$$

- This implies a 55.03% probability of the option being worth \$1 (if the stock price increases) and a 44.97% probability of it being worth \$0 (if the stock price decreases) at the end of 3 months. The expected value of the option, calculated under the risk-neutral probability, is therefore:

$$0.5503 \times 1 + 0.4497 \times 0 = 0.5503$$

- Discounting this expected value at the risk-free rate gives the present value of the option:

$$0.5503e^{-0.04\times 0.25} = 0.545$$

- Remarkably, this value aligns with the outcome obtained through no-arbitrage arguments, affirming the congruence between no-arbitrage principles and risk-neutral valuation in deriving the same pricing result for the derivative.

8.2.2 The Irrelevance of the Stock's Expected Real-World Return

- A key insight from risk-neutral valuation is the irrelevance of the real-world probabilities of the underlying asset's future price movements when valuing options.
- Since **the current stock price embeds these real-world probabilities**, derivative pricing can proceed without needing to account for them again.
- This perspective aligns with a broader financial theory that the expected return of the underlying asset, as anticipated in the real world, does not affect the valuation of derivatives based on that asset.

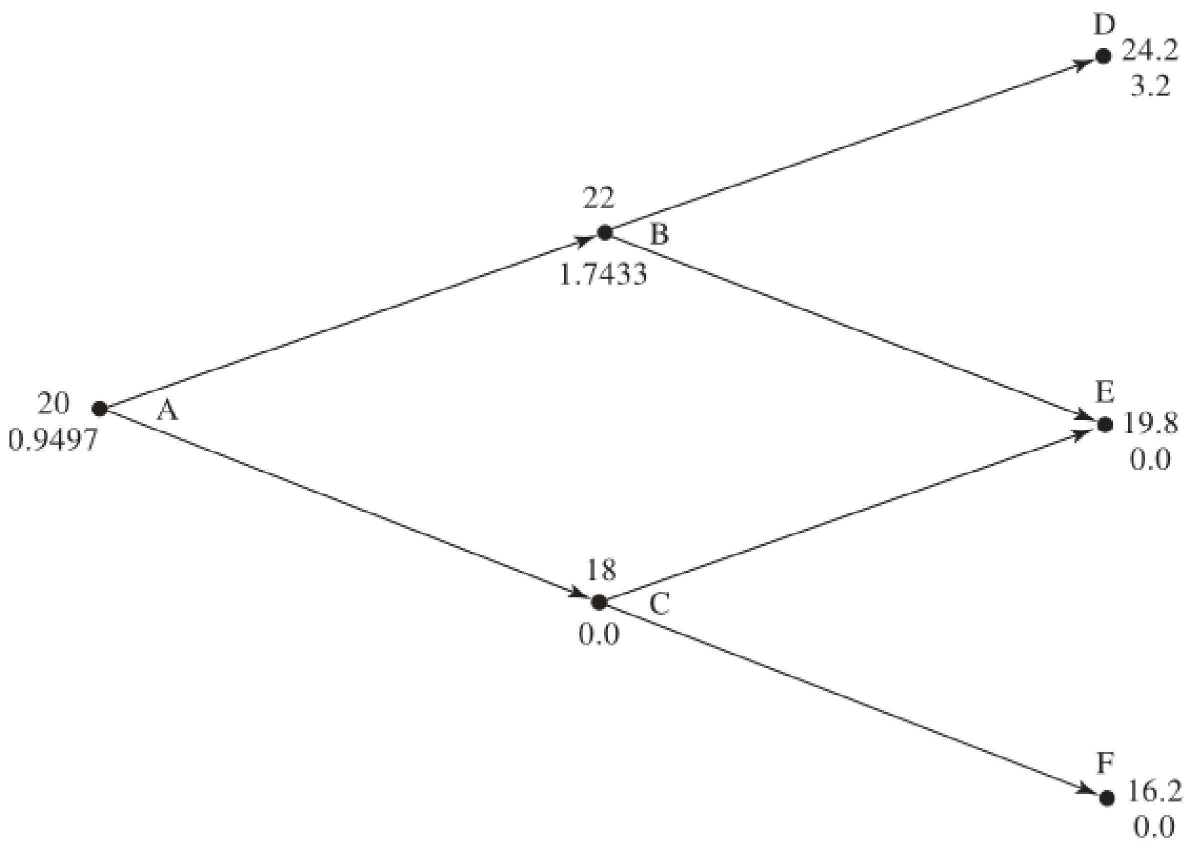
- The valuation hinges instead on the construct of a risk-neutral world where all assets are assumed to grow at the risk-free rate.

8.3 Two-Step Binomial Trees

Two-step binomial trees extend the single-period model to allow for a more detailed examination of option pricing over multiple periods. This method can value both call and put options by simulating possible paths the underlying asset's price might take and then discounting expected payoffs back to the present value.

8.3.1 Valuing a Call Option

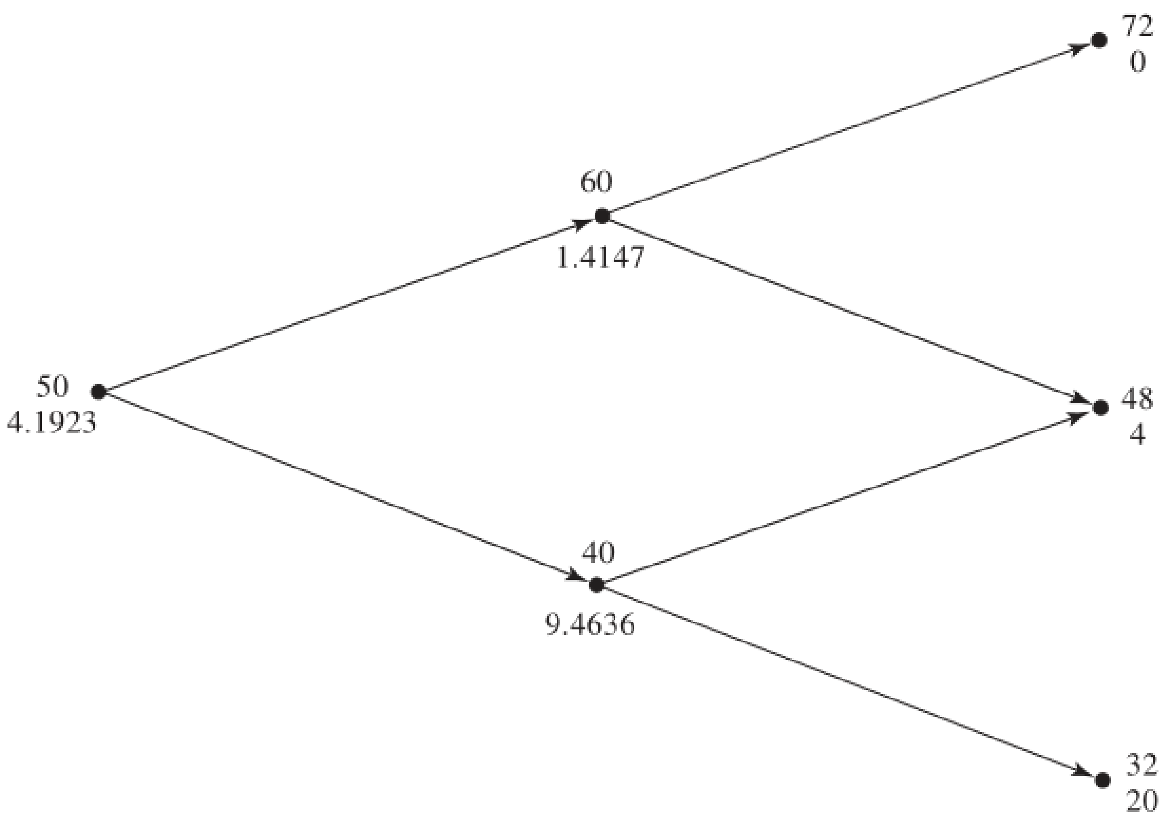
Consider a call option with a strike price (K) of \$21, where each time step represents 3 months. The annual risk-free rate (r) is 4%, with up (u) and down (d) factors of 1.1 and 0.9, respectively, and the risk-neutral probability (p) is 0.5503.



- At the final nodes (D, E, F), the option value simply reflects the option payoff at expiration.
- At nodes B and C, the option's value is calculated as the present value of its expected payoff:
  - Value at node $B = e^{-0.04\times0.25}(0.5503 \times 3.2 + 0.4497 \times 0) = \$1.7433$
  - Value at node $C = 0 + 0 = \$0$
- Moving back to the initial node (A), we compute the present value of the expected option payoff from nodes B and C:
  - Value at node $A = e^{-0.04\times0.25}(0.5503 \times 1.7433 + 0.4497 \times 0) = \$0.9497$

8.3.2 Valuing a Put Option

For a put option with a strike price (K) of \$52 and a time step of 1 year, the parameters are: $r = 5\%$, $u = 1.2$, $d = 0.8$, and $p = 0.6282$.

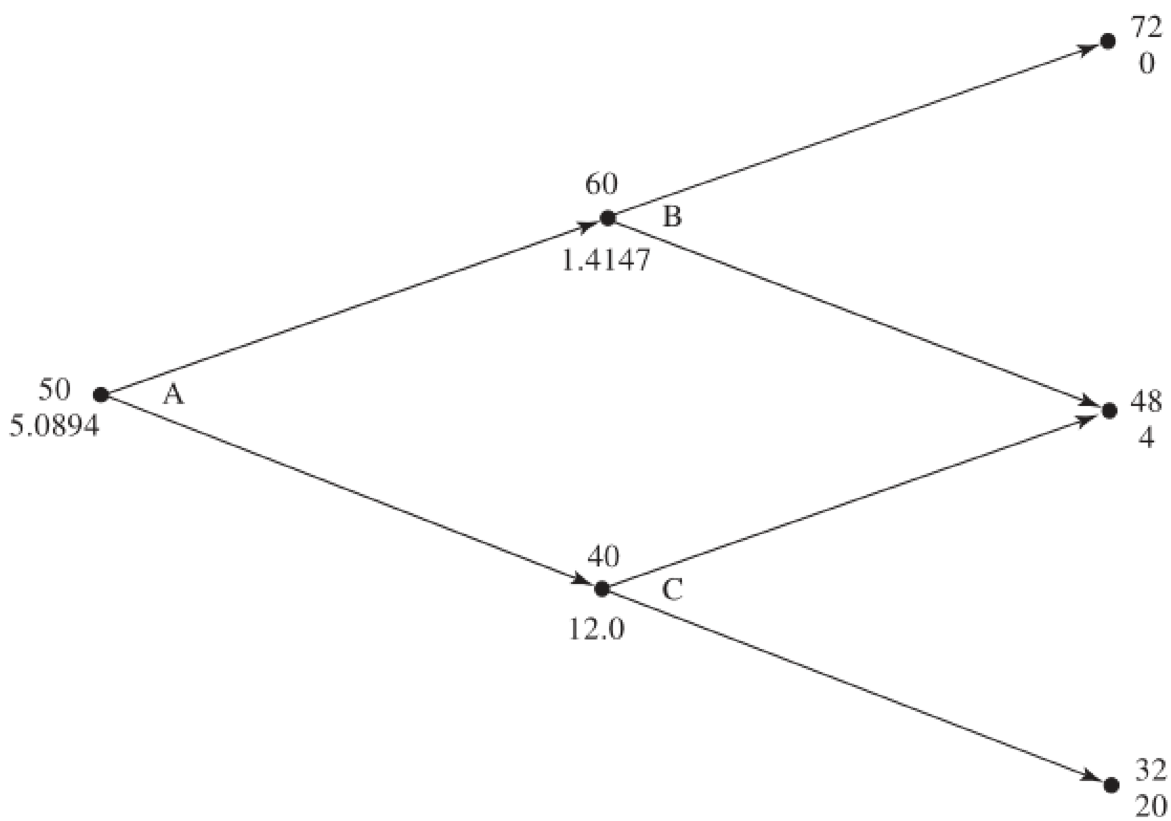


- Following the same approach and determining option values at each node from right to left, we obtain:
  - Value at node $C = e^{-0.05\times 1}(0.6282 \times 4 + 0.3718 \times 20) = \$9.4636$
  - Value at node $B = e^{-0.05\times 1}(0.6282 \times 0 + 0.3718 \times 4) = \$1.4147$
  - Value at node $A = e^{-0.05\times 1}(0.6282 \times 1.4147 + 0.3718 \times 9.4636) = \$4.1923$

8.3.3 What Happens When the Put Option is American?

American options allow for early exercise, affecting their valuation. Specifically, for the previous example with a put option:

- The exercising early increases the option's value at node C to \$12.
- The recalculated value at node A, incorporating the potential for early exercise, is:
  - Value at node $A = e^{-0.05\times 1}(0.6282 \times 1.4147 + 0.3718 \times 12.0000) = \$5.0894$



- The introduction of the American feature, hence, increases the put option's value from \$4.1923 to \$5.0894, reflecting the added value of early exercise flexibility.
- The same approach applies to call options.

8.4 Choosing u and d for Binomial Models

In binomial option pricing models, accurately modeling the underlying asset's price dynamics is crucial. One effective approach to align these models with the asset's volatility—a key measure of its price fluctuations—is to set the up (u) and down (d) factors based on volatility. This method ensures that the model captures the essence of the asset's risk and return characteristics over the specified time steps.

8.4.1 Cox, Ross, and Rubinstein (CRR) Methodology

Cox, Ross, and Rubinstein's seminal work in 1979 introduced a practical and widely adopted method for setting u and d, which directly ties these parameters to the asset's volatility (σ) and the length of the model's time step (Δt):

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

This approach ensures that the binomial model reflects the underlying asset's volatility, allowing for a more accurate and realistic simulation of its price movements. The choice of u as an exponentiation of volatility times the square root of the time step captures the log-normal distribution of stock prices over time, consistent with empirical observations.

Implications of Girsanov's Theorem

Girsanov's Theorem provides a critical theoretical foundation for applying real-world volatility measurements within a risk-neutral pricing framework. The theorem suggests that while the expected return on an asset may differ between the real world and the risk-neutral world, the asset's volatility (σ) remains consistent across both.

- **Volatility Consistency:** Girsanov's Theorem assures us that volatility—a fundamental input in modeling asset price dynamics—is invariant whether we're considering a real-world or risk-neutral probability measure. This invariance allows us to observe and measure volatility in the real world and then directly apply these measurements to construct a binomial tree for option pricing in the risk-neutral world.

- **Risk-Neutral Valuation:** Despite the potential differences in expected returns between the real and risk-neutral worlds, the theorem supports the use of risk-neutral valuation. By ignoring risk preferences and market expectations, risk-neutral valuation simplifies the pricing of derivatives by focusing solely on the discounting of expected payoffs at the risk-free rate, underpinned by the consistent application of real-world volatility measures.

8.5 The Binomial Tree Model Summary

8.5.1 Core Formulas

8.5.1.1 Price Movement Factors

The magnitude of up (u) and down (d) movements in the asset's price is modeled to reflect the asset's volatility (σ) and the length of the time step (Δt):

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

These factors ensure that the model captures the log-normal distribution of asset price changes, consistent with empirical observations.

8.5.1.2 Probability of an Up Move

The risk-neutral probability (p) of an upward price movement is derived as:

$$p = \frac{a-d}{u-d}$$
$$a = e^{r\Delta t}$$

Here, a represents the asset's expected growth factor over a single time step under the risk-free rate (r), adjusting for any dividends or interest rate differentials.

8.5.1.3 Option Value Calculation

The value of an option in a one-step binomial model is determined by:

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

where f_u and f_d are the option's payoffs in the event of an up or down move, respectively, and T is the total time to maturity.

8.5.2 Application to Various Assets

While the basic binomial tree structure remains constant, adjustments to the calculation of p accommodate different types of underlying assets, reflecting their unique characteristics.

- **Nondividend Paying Stocks:**
  - $a = e^{r\Delta t}$
- **Stock Indices:** For indices that pay dividends, the dividend yield (q) is subtracted from the risk-free rate:
  - $a = e^{(r-q)\Delta t}$
- **Currencies:** For options on currencies, the foreign risk-free rate (r_f) is subtracted from the domestic rate:
  - $a = e^{(r-r_f)\Delta t}$
- **Futures Contracts:** For futures, the growth factor (a) simplifies to 1 since futures prices already embody the cost-of-carry, negating the need for discounting or growth adjustments:
  - $a = 1$

8.6 Practice Questions and Problems

1. A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

> **Solution**
>
> Option price = 1.69

2. A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a six-month European put option with a strike price of \$50?

> **Solution**
>
> Option price = 1.16

3. Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

4. A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

> **Solution**
>
> Option price = 9.6104

5. For the situation considered in Problem 4, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put-call parity.

> **Solution**
>
> Option price = 1.9203

6. Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

> **Solution**
>
> u = 1.0352, d = 0.966, p = 0.4553

7. A stock index is currently 1,500. Its volatilityis 18%. The risk-free rate is 4% per annum (continuously compounded) for all maturities and the dividend yield on the index is 2.5%. Calculate values for u, d, and p when a six-month time step is used. What is the value a 12-month American put option with a strike price of 1,480 given by a two-step binomial tree.

Solution

Option price $= 78.41$

9 Option Pricing - Black-Scholes Model

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 15 - the Black-Scholes-Merton Model
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 4 - Valuation of Contingent Claims
- Black, F., & Scholes, M. (1973). "The pricing of options and corporate liabilities". *Journal of political economy*, 81(3), 637-654.
- Merton, Robert (1973). "Theory of Rational Option Pricing". *Bell Journal of Economics and Management Science*. 4 (1): 141–183.

Learning Outcomes:

- Understand the foundational principles of the Black-Scholes-Merton model, including its formulation and applications in financial markets.
- Interpret key parameters within the Black-Scholes-Merton model, explore its properties, assumptions and limitations in real-world scenarios.
- Grasp the concept of risk-neutral valuation and its significance in pricing financial derivatives and managing financial risk.
- Analyze the effect of dividends on option pricing within the Black-Scholes-Merton framework and understand its practical implications for investors and traders.
- Discuss the role of volatility in financial markets, specifically how it influences option pricing and investor strategies.

9.1 Black-Scholes-Merton Model

The Black-Scholes model, developed by Fischer Black and Myron Scholes in 1973, is a groundbreaking mathematical framework for valuing European options. It provides a closed-form solution for the price of a European call or put option based on stock volatility, the risk-free rate, the strike price, and the time to expiration. The model assumes a lognormal distribution of stock prices, continuous trading, and no dividends, among other factors.

The Binomial Option Pricing Model, introduced by Cox, Ross, and Rubinstein in 1979, offers a more flexible approach to option valuation, accommodating American options and dividends. It constructs a discrete-time (binomial) lattice for asset price movements, allowing for an iterative calculation of option prices. This model can be viewed as a simplified, discrete approximation of the continuous processes underlying the Black-Scholes model. As the number of steps in the binomial model increases, its results converge to those of the Black-Scholes formula, illustrating their intrinsic relationship. The binomial model's adaptability to different types of options and its intuitive framework make it a vital complement to the Black-Scholes model in financial theory.

9.1.1 Model Assumptions

The Black-Scholes-Merton model is predicated on several key assumptions to facilitate the derivation of a closed-form solution for pricing European options. These assumptions include:

1. **European-style Options:** The model is applicable to European options, which can only be exercised at expiration.
2. **No Dividends:** It is assumed that the underlying asset does not pay dividends during the life of the option.
3. **No Arbitrage:** The market is efficient, prohibiting the possibility of riskless arbitrage profits.
4. **Short Selling:** Investors are allowed to short sell the underlying asset without restrictions.
5. **No Market Frictions:** The market operates without transaction costs, taxes, or regulatory constraints, implying perfect market conditions.
6. **Constant Risk-Free Rate:** The risk-free interest rate, underpinning the time value of money, is constant over the option's life and the same for all maturities.
7. **Known and Constant Volatility:** The volatility of the underlying asset's returns, a measure of its price fluctuations, is known and remains constant over time.
8. **Geometric Brownian Motion:** The price of the underlying asset follows a geometric Brownian motion, characterizing price changes as continuous and random without sudden jumps.
9. **Continuous Trading and Liquidity:** Trading is possible at any moment during market hours, and the asset is perfectly liquid, allowing for immediate buy and sell transactions.

9.1.2 The Black-Scholes-Merton Formulas for Options Pricing

The model provides explicit formulas to compute the price of call and put options, defined as follows:

- c = price of a European call option
- p = price of a European put option
- $N(x)$ = the standard normal cumulative distribution function, representing the probability that a normally distributed variable with a mean of zero and a standard deviation of one is less than x.
- r = annualized continuously compounded risk-free rate
- σ = annualized constant volatility of the stock returns
- S = current stock price
- K = option strike price
- T = time to expiration (in years)

The core equations are:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

9.1.3 The $N(x)$ Function

The function $N(x)$ is pivotal in calculating the probabilities essential for the Black-Scholes-Merton formula. It quantifies the likelihood that a random draw from a standard normal distribution falls below a specific value x. For practical computations, one may refer to statistical tables or the NORM.DIST function in Excel.

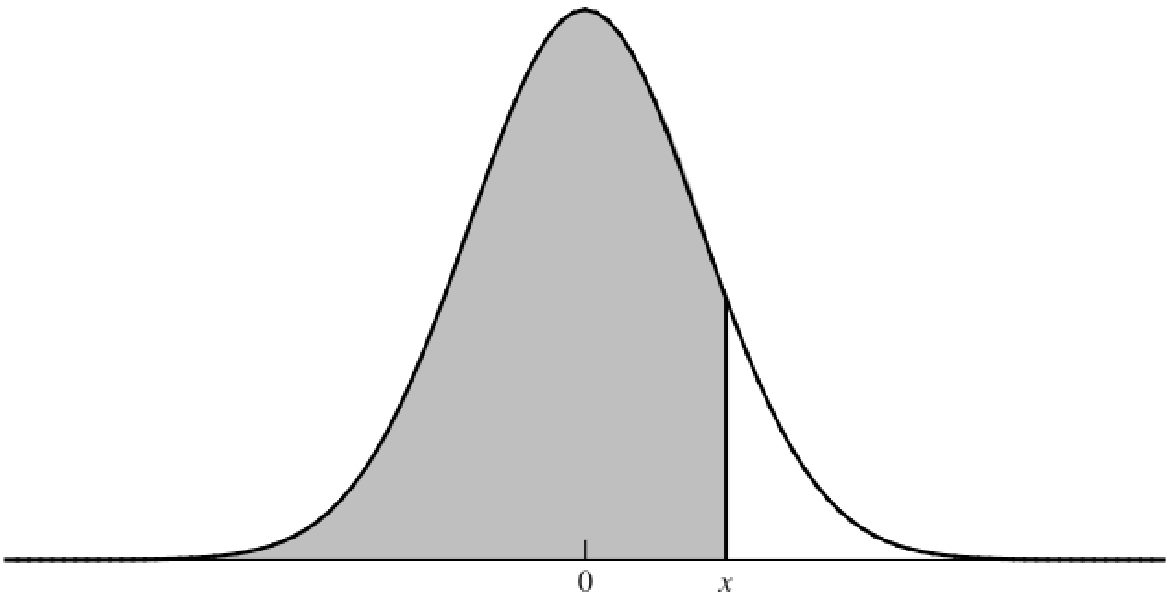


Figure 9.1: image

Example: Black-Scholes-Merton Model

Consider an option with the following parameters:

- $S_0 = 42$ (Current stock price)
- $K = 40$ (Strike price)
- $r = 10\%$ (Risk-free rate)
- $\sigma = 20\%$ (Volatility)
- $T = 0.5$ (Time to expiration in years)

Using these parameters, we calculate:

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = 0.7693 - 0.2\sqrt{0.5} = 0.6278$$

Employing the NORM.DIST function in Excel for d_1 and d_2 yields:

- $N(0.7693) = 0.7791$; $N(0.6278) = 0.7349$;
- $N(-0.7693) = 0.2209$; $N(-0.6278) = 0.2651$.

The discounted strike price is calculated as:

$$Ke^{-rT} = 40e^{-0.1\times 0.5} = 38.049$$

Finally, the call (c) and put (p)

$$c = 42 \times 0.7791 - 38.049 \times 0.7349 = 4.76$$

$$p = 38.049 \times 0.2651 - 42 \times 0.2209 = 0.81$$

9.2 Interpretation of $N(d_1)$ and $N(d_2)$

Understanding the terms $N(d_1)$ and $N(d_2)$ within the Black-Scholes formula is fundamental for grasping the model's insights into option pricing dynamics. These terms provide a probabilistic interpretation that enriches the model's application and theoretical comprehension.

9.2.1 Interpretation #1

- **$N(d_2)$:** This term represents the probability that the option will be exercised at maturity. It quantifies the likelihood that the stock price at maturity will exceed the strike price, making it beneficial for the holder to exercise the option.
- **$S_0 e^{rT} N(d_1)$:** This expression estimates the expected stock price at maturity in a risk-neutral world, adjusting for instances when the stock price falls below the strike price, which are considered as zero value. This adjustment mirrors the option's payoff structure, where only in-the-money scenarios contribute to the expected payoff.
- **Expected Payoff and Present Value:** The expected payoff of holding a call option can be expressed as $S_0 e^{rT} N(d_1) - KN(d_2)$. To derive the option's current value, we discount this expected payoff to the present, yielding the Black-Scholes call option pricing formula: $c = S_0 N(d_1) - Ke^{-rT} N(d_2)$.

9.2.2 Interpretation #2

Another lens to view the call option pricing formula is:

$$c = e^{-rT} N(d_2) \left(S_0 e^{rT} N(d_1)/N(d_2) - K \right)$$

- **e^{-rT}:** This factor discounts the expected payoff to its present value, reflecting the time value of money.
- **$N(d_2)$:** Represents the probability of the option being exercised, acting as a weight for the expected payoff.
- **$S_0 e^{rT} N(d_1)/N(d_2)$:** Expected stock price in a risk-neutral world if option is exercised.
- **K:** The strike price, indicating the cost to exercise the option.

9.3 Properties of the Black-Scholes Formula

The behavior of the call (c) and put (p) option prices relative to changes in the underlying stock price (S_0) is critical for understanding the model's implications:

- **As S_0 increases significantly:**
  - The call option price (c) converges to $S_0 - Ke^{-rT}$, because both d_1 and d_2 grow large, making $N(d_1)$ and $N(d_2)$ approach 1. This reflects the option's intrinsic value, as the benefit from exercise becomes almost certain.
  - The put option price (p) approaches zero, as $N(-d_1)$ and $N(-d_2)$ approach zero, indicating a negligible chance of the option being in-the-money.
- **As S_0 decreases significantly:**
  - The call option price (c) trends towards zero, mirroring the diminishing likelihood of the option being exercised.
  - The put option price (p) converges to $Ke^{-rT} - S_0$, reflecting the increasing intrinsic value of the option as the likelihood of exercise grows.

9.4 Risk-Neutral Valuation

The concept of Risk-Neutral Valuation is a cornerstone in the field of financial derivatives, offering a powerful framework for option pricing, particularly within the Black-Scholes-Merton model. This approach simplifies valuation by assuming that all investors are indifferent to risk, thereby pricing assets solely based on the risk-free rate rather than the expected return under actual market conditions.

In the derivation of the Black-Scholes-Merton differential equation, an intriguing observation is that the expected return of the underlying asset, denoted by μ, does not influence the equation. This omission signifies that the equation—and by extension, the option's price—is not affected by investors' risk preferences. The remarkable outcome is that the solution to this differential equation remains consistent whether we consider a risk-neutral world or the real, risk-averse world. This observation leads to the principle of risk-neutral valuation, which posits that under certain conditions, the market can be modeled as if all investors were risk-neutral.

This methodological approach allows the valuation of options and other derivatives in a theoretically consistent manner, abstracting away from the complexities introduced by varying risk preferences among investors. The elegance and simplicity of risk-neutral valuation underpin its widespread application in financial economics and the pricing of derivatives.

9.4.1 Key Points of Risk-Neutral Valuation

- **Indifference to Risk:** In a risk-neutral world, investors expect to earn the risk-free rate on all investments, regardless of the risk involved. This assumption is purely theoretical but facilitates the practical application of option pricing models.
- **Exclusion of μ:** The fact that the differential equation does not incorporate the expected return μ indicates the irrelevance of personal risk preferences in pricing derivatives through this model.
- **Universal Pricing Principle:** The risk-neutral valuation framework suggests that the price of a derivative is determined by the expected payoff under risk neutrality, discounted at the risk-free rate.

9.4.2 Application of Risk-Neutral Valuation

Applying risk-neutral valuation to price options involves a straightforward three-step process:

1. **Assumption of Risk-Free Returns:** Initially, it is assumed that the expected return on the stock (or the underlying asset) is equal to the risk-free rate. This simplification aligns the future stock price dynamics with a risk-neutral perspective, where all assets are expected to grow at this rate.
2. **Calculation of Expected Payoff:** The next step involves computing the expected payoff of the option at expiration. This calculation is performed under the assumption that the stock prices follow a risk-neutral probability distribution, reflecting a world where the expected rate of return on all assets is the risk-free rate.
3. **Discounting at the Risk-Free Rate:** Finally, the expected payoff is discounted back to the present using the risk-free rate. This discounting reflects the principle that future cash flows must be adjusted to present value at a rate that reflects their time value of money, devoid of risk premiums.

> Example: Risk-Neutral Valuation of a Call Option
>
> Consider a call option with a strike price of K, and let's assume that under the risk-neutral world, the expected stock price at expiration is $E[S_T]$. The expected payoff from holding this call option can be represented as $E[max(S_T - K, 0)]$, where S_T is the stock price at expiration. By discounting this expected payoff at the risk-free rate r, we can calculate the present value of the call option.

9.5 The Effect of Dividends

The inclusion of dividends in the valuation of options introduces additional complexity, as dividends can significantly affect option prices. The adjustment for dividends in the Black-Scholes model and considerations for American call options are pivotal for accurate pricing.

9.5.1 Valuing European Options on Dividend-Paying Stocks

To accommodate dividends in the Black-Scholes model, **the stock price is adjusted by subtracting the present value of dividends expected to be paid during the life of the option**. This adjustment reflects the reduction in stock price that typically occurs on the ex-dividend date, thereby affecting the option's intrinsic value. It's crucial to include only those dividends that are expected to be paid before the option expires, as these are the ones that will impact the option holder.

For a dividend-paying stock, the adjusted stock price S_{adj} in the Black-Scholes formula becomes:

$$S_{adj} = S_0 - PV(\text{dividends})$$

where $PV(\text{dividends})$ is the present value of dividends paid during the option's life. This adjustment aims to capture the expected drop in the stock price due to dividend payouts, ensuring the option valuation accurately reflects this anticipated decrease.

9.5.2 American Calls on Dividend-Paying Stocks

The traditional wisdom that American call options should not be exercised early due to their time value does not always hold in the presence of dividends. The possibility of early exercise becomes relevant when the stock pays dividends.

9.5.2.1 Criteria for Early Exercise

For an American call option on a dividend-paying stock, early exercise might be optimal just before an ex-dividend date if the dividend amount exceeds the loss of time value from early exercise, specifically if the dividend is greater than:

$$K[1 - e^{-r(t_{i+1} - t_i)}]$$

Here, K is the strike price, r is the risk-free rate, and t_i to t_{i+1} represents the interval between dividend payments. This condition evaluates whether the immediate gain from capturing the dividend outweighs the potential benefits of holding the option for its time value.

9.5.3 Black's Approximation for American Call Options with Dividends

Black's Approximation offers a pragmatic method for approximating the price of American call options on dividend-paying stocks. This approximation involves comparing two European call option prices:

1. **First European Price:** This is calculated for a European call option with the same expiration date as the American option, incorporating the adjusted stock price for dividends.
2. **Second European Price:** This is calculated for a European call option expiring just before the final ex-dividend date within the American option's life.

The value of the American call option is approximated as the maximum value derived from these two European call options. This method acknowledges the critical influence of dividends on option pricing and provides a practical solution for incorporating this factor into American option valuation.

Black's Approximation simplifies the valuation process by sidestepping the complex optimization problem of determining the exact optimal early exercise strategy in the presence of dividends. It offers a balance between accuracy and computational efficiency, making it a valuable tool for practitioners in the financial markets.

9.6 Volatility in Financial Markets

Volatility is a pivotal concept in finance, encapsulating the degree of variation in the price of a financial instrument over time. It is central to the valuation of options, where it measures the extent of uncertainty or risk associated with the price change of the underlying asset.

- **Definition:** Volatility, denoted as σ, quantifies the expected variability in the returns of a stock or any financial asset. It reflects the degree of uncertainty or risk about the magnitude of changes in an asset's value.
- **Range:** Typically, stocks exhibit annual volatilities in the range of 15% to 60%. This range indicates the expected yearly change in the stock price, highlighting the variability in asset prices across different markets and conditions.
- **Impact on Options:** The value of an option is exceptionally sensitive to changes in volatility. Since options are derivatives based on the underlying asset, increased volatility translates to higher option premiums, all else being equal.

- **Observability:** Unlike direct observables like stock prices, volatility is not directly observable and must be estimated. Historical volatility is derived from past market prices, while implied volatility is inferred from current market prices of options.

9.6.1 Historical Volatility

- **Calculation:** Historical volatility is the standard deviation of the continuously compounded returns of an asset over a specified period, typically annualized. This measure provides a backward-looking estimate of volatility based on actual market data.
- **Daily Volatility Example:** For a stock priced at \$30 with an annual volatility of 25%, the standard deviation of the stock's price change over one day (Δt) can be approximated as $25\% \times \sqrt{\frac{1}{252}} = 1.57\%$, assuming 252 trading days in a year. This calculation aids in understanding daily price variability.
- **Annualization:** By annualizing the standard deviation of daily returns (multiplied by $\sqrt{252}$), investors can compare volatility across assets with different time frames.

9.6.2 Nature of Volatility

- **Market Hours:** Volatility tends to be higher during trading hours compared to when the market is closed, reflecting the immediate impact of news and events on asset prices.
- **Trading Days:** For options valuation, time is measured in trading days, not calendar days, acknowledging the fact that price changes are primarily driven by market activity. This convention ensures that the time to maturity for options reflects actual market exposure.

9.6.3 Implied Volatility

- **Definition:** Implied volatility is the volatility implied by the market price of an option, based on the Black-Scholes-Merton model. It represents the market's expectation of future volatility over the life of the option.
- **VIX Index:** The VIX S&P 500 Volatility Index is a widely recognized measure of market expectation of near-term volatility conveyed by S&P 500 stock index option prices. The VIX S&P 500 Volatility Index.
- **Application:** Implied volatility is crucial in options trading, serving as a standard measure that reflects the market's view on future volatility. It is instrumental in pricing, trading strategies, and risk management.
- **Volatility Smile and Surface:** The volatility smile and surface graphically represent implied volatility across different strike prices and maturities. Deviations from a flat surface—expected under the BSM model assumptions—reveal market anomalies, such as varying risk preferences and the impact of market events on different options.

- **Market Anomalies:** The observed curvature in the volatility smile and the dynamic topology of the volatility surface challenge the BSM model's assumptions, demonstrating real-market frictions and behaviors not accounted for by the model. These phenomena offer valuable insights into the complexity of market dynamics and the factors influencing option pricing beyond theoretical models.

9.6.3.1 Role of Implied Volatility in Option Trading

- **Market's Perception of Value:** Implied volatility reflects the market's anticipation of future price fluctuations of the underlying asset. Higher implied volatility indicates greater expected price movement and, consequently, a higher option premium.
- **Pricing Medium:** In options markets, implied volatility is often used as a standard metric to quote prices. A call option priced at \$10, corresponding to an implied volatility of 25%, illustrates how traders communicate and gauge option value through volatility levels.
- **Valuation of Options:** Implied volatility aids in comparing the relative value of options by standardizing price differences attributable to intrinsic value and time decay. Options with lower implied volatility are considered cheaper relative to those with higher implied volatility, offering insights into potential over- or under-valuation.
- **Revaluation Over Time:** As market conditions and perceptions change, so does implied volatility. Tracking its evolution allows traders and portfolio managers to reassess the value of their positions, adapting strategies accordingly.
- **Market Sentiment Indicator:** The level of implied volatility reflects market participants' expectations about future volatility, serving as a barometer for market sentiment.
- **Communication Tool:** For regulators, banks, and compliance officers, implied volatility serves as a universal language to discuss and assess the risk and valuation of options portfolios.

9.7 Practice Questions and Problems

1. Name some assumptions of the Black-Scholes-Merton model. Compare with the real world and explain potential issues.

2. Explain the principle of risk-neutral valuation.

3. Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. (Tip: Use Excel function NORMDIST to calculate $N(x)$)

Solution

Put option price = 2.37

4. What difference does it make to your calculations in the previous Problem if a dividend of \$1.50 is expected in two months? (Tip: Use Excel function NORMDIST to calculate $N(x)$)

Solution

Put option price = 3.03

5. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

Solution

Call option price = 5.06

6. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

Solution

Put option price = 6.40

7. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

   a. What is the price of the option if it is a European call?
   b. What is the price of the option if it is an American call?
   c. What is the price of the option if it is a European put?
   d. Verify that put–call parity holds.

Solution

European call = 2.52 European put = 1.05

8. What is implied volatility? How can it be calculated?

9. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

> **Solution**
>
> 1-day std. $= 1.9\%$

10. What is the actual implied volatility of the S&P 500? Try to interpret the value.

11. A call option on a non-dividend-paying stock has a market price of \$2.50. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% perannum. What is the implied volatility?

> **Solution**
>
> Implied volatility $= 39.6\%$

10 Options on Indices, Currencies, and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 17 - Options on Stock Indices and Currencies
  - Chapter 18 - Futures Options and Black's Model
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 4 - Valuation of Contingent Claims

Learning Outcomes:

- Master the use of index options for hedging and portfolio protection; understand the benefits compared to direct stock transactions.
- Apply models like Black-Scholes for valuing index options.
- Utilize currency options for hedging foreign exchange risks; grasp valuation techniques using both spot and forward prices.
- Evaluate European futures options using Black's model; distinguish between options on futures and physical commodities.

10.1 Options on Stock Indices

Options on stock indices allow investors to speculate on the direction of an entire market index or hedge their portfolio market risk without directly transacting the component stocks. Some of the most widely traded stock index options in the United States are:

- **S&P 100 Index (OEX and XEO):** These options provide a broad market exposure to the top 100 stocks in the S&P index. OEX options are American-style, allowing exercise at any point before the expiration date, whereas XEO options are European-style, exercisable only at expiration.

- **S&P 500 Index (SPX):** A benchmark index for U.S. equities, these European-style options reflect the performance of 500 large-cap stocks and are a key tool for institutional investors for hedging and exposure adjustment.
- **Dow Jones Index (DJX):** Priced at 1/100th of the Dow Jones Industrial Average, these options are a cost-effective way to gain exposure to the movements of the Dow Jones, consisting of 30 significant stocks.
- **Nasdaq 100 Index (NDX):** These European-style options represent the 100 largest non-financial stocks from the Nasdaq, making them crucial for investors focused on technology and innovation sectors.

All mentioned index options are exchange-traded, have their contract size set at 100 times the index value, and are settled in cash, which simplifies the transaction and eliminates the need for physical delivery of the underlying securities.

10.1.1 Using Index Options for Portfolio Insurance

Portfolio insurance is a strategy used to protect against market declines through the use of put options. The strategy involves purchasing puts to cover potential losses in a portfolio's value due to adverse market movements.

- **Initial Setup:**
  - Let S_0 represent the current index value.
  - Let K be the strike price of the put option.
  - The portfolio's beta (β) indicates its sensitivity to market movements relative to the index.
- **Insurance Strategy:**
  - For a portfolio with $\beta = 1.0$, purchase one put option for every $\$100 \times S_0$ of portfolio value to hedge against downturns.
  - If $\beta \neq 1.0$, adjust the number of put options accordingly, purchasing β options per $\$100 \times S_0$.
  - In both cases, K is chosen to give the appropriate insurance level.

Example: Portfolio Insurance, $\beta = 1.0$

- **Portfolio Details:**
  - Portfolio $\beta = 1.0$
  - Portfolio Value = \$500,000
  - Current Index Value = 1,000
- **Objective:** Insure the portfolio so its value does not fall below \$450,000 in the

next three months.

- **Implementation Strategy:**
  - The manager opts to buy put options as a form of insurance. Since each contract covers 100 times the index value, the manager needs one put option for every \$100,000 of the portfolio value (i.e., $500,000/100 \times 1,000 = 5$ put options).
  - These options have a strike price (K) of 900 (10% decrease).

- **Mechanism of Insurance:**
  - If the index drops to 880 in three months, the portfolio's expected value would decrease proportionately to about \$440,000.
  - The put options' payoff can be calculated as follows:

$$\text{Payoff} = 5 \times [(900 - 880) \times 100] = \$10,000$$

  - This payoff compensates for the decrease in the portfolio's value, effectively bringing it up to the insured value of \$450,000.

Example: Portfolio Insurance, $\beta \neq 1.0$

Consider a portfolio with a β of 2.0. This high beta implies that the portfolio's returns are expected to be twice as volatile as the underlying index's returns. Currently, the portfolio is valued at \$500,000, while the index stands at 1,000. With a risk-free rate of 12% per annum and a dividend yield of 4% on both the portfolio and the index.

- **Objective:** Insure the portfolio so its value does not fall below \$450,000 in the next three months.
- **Required Number of Contracts:** $(500,000/100,000) \times 2 = 10$ put option contracts are needed.

10.1.1.1 Determining the Appropriate Strike Price Using CAPM Model

- **Index Return Scenario:** Assume the index rises to 1,040 in three months, reflecting a 4% return.
- **Total Return Including Dividends:** 4% (index return) + 1% (dividends) = 5%.
- **Excess Return Over Risk-Free Rate:** 5% - 3% (equivalent quarterly risk-free rate) = 2%.

- **Excess Return for Portfolio:** $2\% \times 2 = 4\%$ (due to beta of 2.0).
- **Net Portfolio Return Calculation:** Excess return (4%) + risk-free rate over three months (3%) - dividend yield (1%) = 6%.
- **Projected Portfolio Value:** $\$500,000 \times 1.06 = \$530,000$.
- **Result:** Similar calculations can be carried out for other values of the index at the end of the three months. Appropriate strike price for the 10 put option contracts that are purchased is 960 (or 955 when we include dividends).

Value of Index in Three Months	Value of Portfolio in Three Months ($)
1,080	570,000
1,040	530,000
1,000	490,000
960	450,000
920	410,000
880	370,000

10.2 Valuation of Stock Index Options

The valuation of European stock index options hinges on the understanding of asset price dynamics under different conditions. The probability distribution of the asset price at maturity T remains consistent in two distinct scenarios: 1. When the asset originates at a price S_0 with a yield of q. 2. When the asset starts at a discounted price of $S_0 e^{-qT}$, assuming it yields no income.

By utilizing this equivalence, we simplify the valuation process for European options. Instead of accounting for dividends directly, we **adjust the initial stock price** to $S_0 e^{-qT}$ and **proceed with the valuation as if the stock pays no dividends**. This approach effectively streamlines the calculation by isolating the impact of yields and focusing on pure price movements.

10.2.1 Extending Lower Bounds and Put-Call Parity

The intrinsic value forms the lower bound for option prices, ensuring that the option price does not fall below its immediate exercise value. These bounds are critical in preventing arbitrage opportunities.

- **Lower Bound for Call Options:**

$$c \geq \max(S_0 e^{-qT} - Ke^{-rT}, 0)$$

- **Lower Bound for Put Options:**

$$p \geq \max(Ke^{-rT} - S_0 e^{-qT}, 0)$$

Put-call parity establishes a risk-neutral relationship between the prices of European put and call options with identical strike prices and expirations.

- **Put-Call Parity**

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

10.2.2 Extending Black-Scholes Model

The Black-Scholes formula for pricing European options underlies much of modern financial derivatives trading. This extedned version considers the adjusted stock price that accounts for divid yield (q). The formulas for calls and puts are as follows:

- **Call Option Price:**

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

- **Put Option Price:**

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

10.2.3 Alternative Formulas Using Forward Prices

The valuation of European options can also be efficiently conducted using forward prices, which simplifies the treatment of dividends and risk-free rates. This approach directly incorporates the expected future price of the asset, eliminating the need to adjust the current price for dividends and rates separately.

- **Call Option Price:**

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

- **Put Option Price:**

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

Where F_0 is the forward price of the asset, representing the expected price at time T, adjusted for the risk-free rate r and the dividend yield q. The terms d_1 and d_2 in these formulas are modified to incorporate the forward price:

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- The forward price $F_0 = S_0 e^{(r-q)T}$ effectively encapsulates the expected growth of the asset price due to the net effect of interest rates and dividend yields.
- This methodology is particularly useful for valuing options on assets that pay dividends, as it directly uses the futures or forward index price, which corresponds to a contract maturing at the same time as the option.

10.2.4 Implied Forward Prices and Dividend Yields

Forward prices and dividend yields can also be derived from market prices of European calls and puts, enabling a deeper understanding of market expectations and assisting in strategic investment decisions.

- **Implied Forward Price:**

  $$F_0 = K + (c - p)e^{rT}$$

  This formula calculates the implied forward price from the current prices of European calls and puts with the same strike price and time to maturity. It reflects the market's expectation of the asset price at the option's expiry.

- **Implied Dividend Yield:**

  $$q = -\frac{1}{T}\ln\frac{c - p + Ke^{-rT}}{S_0}$$

  This expression estimates the average dividend yield expected during the life of the option, derived from the observed prices of calls and puts.

- These calculations are essential for estimating term structures of forward prices and dividend yields, particularly relevant in over-the-counter (OTC) European options.
- For American options, which can be exercised at any point up to and including the date of expiration, understanding the term structure of dividend yields is crucial due to their potential impact on early exercise decisions.

10.3 Currency Options

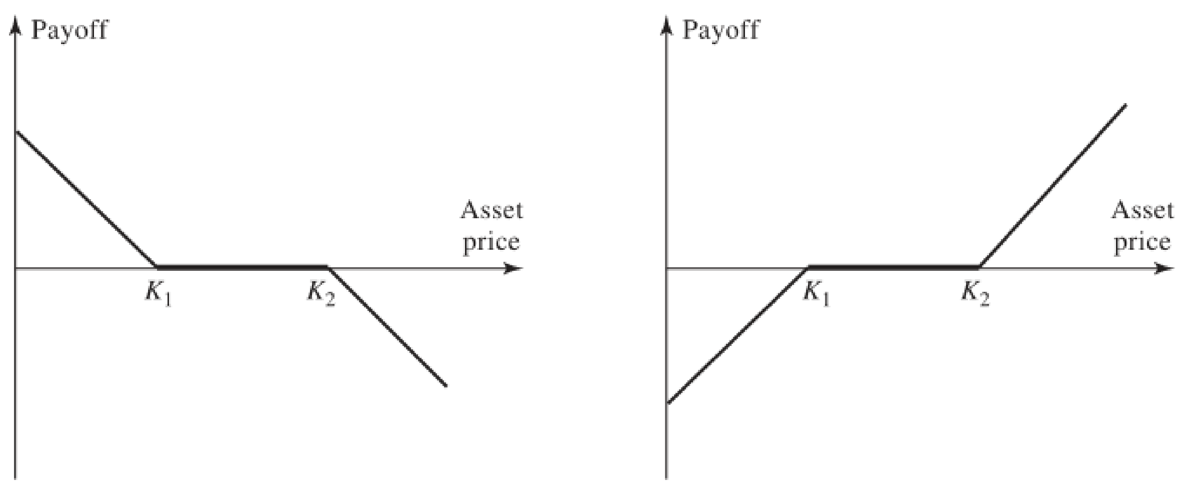
Currency options are financial instruments that provide an essential hedging mechanism against foreign exchange risk, utilized predominantly by corporations with international exposure. These options are traded on structured exchanges such as NASDAQ OMX and extensively in the over-the-counter (OTC) market, reflecting their importance in financial risk management.

Currency as an asset class behaves similarly to stocks that provide a yield, with the yield in this case being the foreign interest rate, r_f. This similarity allows the application of stock option pricing methods to currency options, considering r_f as analogous to the dividend yield in stock options.

10.3.1 Range Forward Contracts

Range forward contracts are bespoke financial agreements that help manage currency risk by setting bounds on the exchange rate fluctuations:

- When expecting to pay currency, one typically sells a put option with a lower strike price, K_1, and buys a call option with a higher strike price, K_2 ($K_2 > K_1$). This structure ensures that the exchange rate remains between K_1 and K_2.
- Conversely, when expecting to receive currency, one buys a put option at K_1 and sells a call option at K_2.
- The cost of the put typically offsets the premium received for the call, balancing the overall cost of the contract.



(a) (b)

These contracts are particularly advantageous for businesses that seek to mitigate the risk of adverse currency movements while retaining some potential for benefiting from favorable movements within a predetermined range.

10.3.2 Extending Black-Scholes Model

Using the Black-Scholes model adjusted for currencies, we derive the following formulas:

- **Call Option Price:**
$$c = S_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2)$$

- **Put Option Price:**
$$p = K e^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1)$$

Where:
$$d_1 = \frac{\ln(S_0/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

10.3.3 Alternative Formulas Using Forward Prices

The use of forward prices further simplifies the valuation by incorporating expectations about future exchange rates:

- **Call Option Price:**
$$c = e^{-rT}[F_0 N(d_1) - K N(d_2)]$$

- **Put Option Price:**
$$p = e^{-rT}[K N(-d_2) - F_0 N(-d_1)]$$

Where:
$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Where $F_0 = S_0 e^{(r - r_f)T}$ represents the forward exchange rate, adjusted for the interest rate differential between the two currencies. This model encapsulates the market's expectations of future exchange rates, reducing the need for separate adjustments for interest rates and foreign yields in the pricing formulas.

10.4 Options on Futures and Black's Model

Options on futures are derivatives that confer the right, but not the obligation, to enter into a futures contract at a predetermined price before a specified expiration date. These options are categorized by the maturity month of the underlying futures contract. Typically, these options are American-style, meaning they can be exercised at any time up to and including the expiration day, which usually aligns with or occurs a few days before the earliest delivery date of the underlying futures contract.

10.4.1 Mechanics of Futures Options

10.4.1.1 Call Options

When a call option on a futures contract is exercised, the holder assumes a **long position in the futures contract**. Additionally, the holder receives a cash amount equivalent to the difference between the futures price at the most recent settlement and the strike price of the option.

- **Example:**
  - A September call option on copper futures has a strike price of 320 cents per pound.
  - The option is exercised when the futures price is 331 cents, with the most recent settlement price at 330 cents.
  - Each contract represents 25,000 pounds of copper.
  - Upon exercising, the trader receives a long position in the September futures contract and a cash payout of $25,000 \times 10$ cents = \$2,500.
  - If desired, the position in the futures contract can be closed out immediately to get $25,000 \times (331 - 330)$ cents = \$250.
  - The total payoff from exercising the option is \$2,750.

10.4.1.2 Put Options

Conversely, when a put futures option is exercised, the holder gains a **short position in the futures contract**. Similar to the call option, there is a cash payout which in this case is the difference between the strike price and the futures price at the most recent settlement.

- **Example:**
  - A December put option on corn futures has a strike price of 600 cents per bushel.
  - The option is exercised when the futures price is 580 cents per bushel, with the most recent settlement price at 579 cents per bushel.
  - Each contract covers 5,000 bushels.

- The trader receives a short position in the December futures contract and a cash payout of $5,000 \times (600 - 579)$ cents = $\$1,050$.
- If desired, the position in the futures contract can be closed out immediately to get $5,000 \times (579 - 580)$ cents = $-\$50$.
- The total payoff from exercising the option is $1,000.

10.4.2 Payoffs of Futures Options

The payoff for futures options, if the position in the futures contract is closed out immediately after exercising the option, is straightforward:

- **Call Option Payoff:**
$$\text{Payoff} = F - K$$
Where F is the futures price at the time of exercise, and K is the strike price of the option.
- **Put Option Payoff:**
$$\text{Payoff} = K - F$$
Again, F represents the futures price at the time of exercise.

10.4.3 Potential Advantages of Futures Options over Spot Options

Futures options offer several distinct advantages over spot options under certain conditions:

1. **Equivalence with Spot Options:**
    - European futures options and European spot options are equivalent in value when the futures contract matures concurrently with the option's expiration. This equivalence arises because both types of options will reflect the same underlying economic exposures at expiration.
2. **Market Liquidity:**
    - Futures contracts often enjoy greater liquidity compared to many underlying assets, particularly in markets where the underlying assets are large or illiquid by nature (e.g., commodities, certain financial instruments). This increased liquidity generally makes it easier and potentially less costly to trade futures options.
3. **Settlement Characteristics:**
    - Upon exercising a futures option, the holder acquires a position in the futures contract rather than the immediate delivery of the underlying asset. This feature can be particularly advantageous in markets where physical delivery is less desirable or practical.

4. **Unified Trading Platforms:**
   - Futures options and the underlying futures contracts typically trade on the same exchange. This unification can simplify access, monitoring, and execution for traders and investors, enhancing market efficiency.
5. **Cost Efficiency:**
   - The transaction costs associated with trading futures options can be lower than those for trading spot options, largely due to standardized contract terms and centralized trading venues which foster more competitive pricing.

10.4.4 Put-Call Parity and Lower Bounds for European Futures Options

10.4.4.1 Put-Call Parity

Consider the following two portfolios:

1. European call plus Ke^{-rT} of cash.
2. European put plus long futures plus cash equal to F_0e^{-rT}.

Both portfolios must be worth the same at time T. Therfore, for European futures options, the put-call parity condition is expressed as:

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

Here, c and p represent the prices of the European call and put options, respectively, K is the strike price, F_0 is the current futures price, and r is the risk-free interest rate. This equation ensures that no arbitrage opportunities exist between buying a call and selling a put when adjusted for the present value of the strike price and the futures price.

10.4.4.2 Lower Bounds

The lower bounds for the prices of futures options reflect the minimum value these options must hold to prevent arbitrage:

- **Call Option Lower Bound:**
$$c \geq (F_0 - K)e^{-rT}$$
- **Put Option Lower Bound:**
$$p \geq (K - F_0)e^{-rT}$$

10.4.5 Black's Model for Valuing Futures Options

In the pricing of futures contracts, a pivotal assumption is that no initial investment is required, leading to a **zero expected return in a risk-neutral environment**. This implies that the expected growth rate of the futures price is also zero, essentially treating the futures price as equivalent to a stock paying a continuous dividend yield equal to the risk-free rate r. This approach simplifies the valuation by aligning the futures price dynamics with those of a dividend-paying stock.

Black's model extends the classic Black-Scholes framework to futures options, incorporating unique aspects of futures pricing:

- **Model Parameters:**
  - S_0, the current futures price, is denoted by F_0.
  - q, typically representing the dividend yield in stock options, is set to the domestic risk-free rate r in the context of futures. This setting neutralizes the expected growth of the futures price, maintaining the risk-neutral valuation framework.
- **Historical Context:**
  - The model was first proposed by Fischer Black in 1976 and is specifically adapted for European options on futures, providing a practical solution that sidesteps the complexities of estimating income on the underlying asset.

10.4.6 Pricing Formulas

The formulas for valuing European futures options using Black's model are derived as follows:

- **Call Option Price:**
$$c = e^{-rT}[F_0 N(d_1) - K N(d_2)]$$
- **Put Option Price:**
$$p = e^{-rT}[K N(-d_2) - F_0 N(-d_1)]$$

Where:
$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

10.4.7 Futures Option Price vs. Spot Option Price

Comparing futures options to spot options under different market conditions reveals nuanced valuation dynamics:

- **Normal Market Conditions (Contango):**
  - If futures prices are higher than spot prices, an American call option on futures is typically more valuable than a similar call option on the spot market. Conversely, an American put option on futures is less valuable than its spot counterpart.
- **Inverted Market Conditions (Backwardation):**
  - In situations where futures prices are lower than spot prices, the value relationships reverse: American calls on futures become less valuable, whereas puts on futures gain in relative value.

10.5 Practice Questions and Problems

10.5.1 Index Options

1. A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

> Solution
>
> Lower bound = 16.90

2. Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

> Solution
>
> Put option value = 3.84

3. Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

ℹ Solution

Call option value = 11.15

4. An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

ℹ Solution

Put option value = 40.6

10.5.2 Currency Options

1. A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

ℹ Solution

Lower bound European = 0.069 Lower bound American = 0.10

2. Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

ℹ Solution

Put option value = 0.0162

3. A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value (a) a European four-month call option with a strike price of 0.79 and (b) an American four-month call option with the same strike price.

Solution

- European Option Value: \$0.0235
- American Option Value: \$0.0250

10.5.3 Futures Options

1. Consider a two-month futures call option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Solution

Lower bound European = 6.88 Lower bound American = 7

2. Consider a four-month futures put option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Solution

Lower bound European = 2.90 Lower bound American = 3

3. Calculate the value of a five-month European futures put option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

Solution

Put option value = 1.50

4. A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

Solution

Call option value = 2.01

5. A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month European

call option on the futures with a strike price of 60. If the call were American, would it ever be worth exercising it early?

> **Solution**
>
> - European Option Value: 4.3155
> - American Option Value: 4.4026

6. Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

> **Solution**
>
> Arbitrage profit = \$1

10.5.4 Strategic Considerations

1. Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.
2. Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.
3. Explain how corporations can use range forward contracts to hedge their foreign exchange risk when they are due to receive a certain amount of a foreign currency in the future.
4. An index currently stands at 1,500. Six-month European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The risk-free rate is 5%. What is the implied dividend yield?

> **Solution**
>
> Implied dividend yield = 1.99%

5. Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

11 The Greek Letters

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 19 - The Greek Letters
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 4 - Valuation of Contingent Claims

Learning Outcomes:

- Identify and describe the Greek letters used in options trading.
- Explain the concept of Delta and its importance in hedging strategies.
- Understand Gamma and its impact on the curvature of the options price relative to the stock price.
- Comprehend Theta and how time decay affects the value of options.
- Explore Vega and its relationship with volatility in the pricing of options.
- Recognize Rho and its sensitivity to changes in interest rates.
- Develop strategies for managing Delta, Gamma, and Vega to optimize risk and return in options portfolios.

11.1 Case Study

- A bank sells a European call option for \$300,000, covering 100,000 shares of a stock that does not pay dividends.
- **Initial Stock Price** (S_0): \$49
- **Strike Price** (K): \$50
- **Risk-Free Rate** (r): 5%
- **Volatility** (σ): 20%
- **Time to Expiration** (T): 20 weeks
- **Expected Return** (μ): 13%
- **Black-Scholes-Merton Valuation**: \$240,000

- **Objective**: Determine how the bank can hedge its risk to secure a \$60,000 profit.

11.1.1 Naked Position

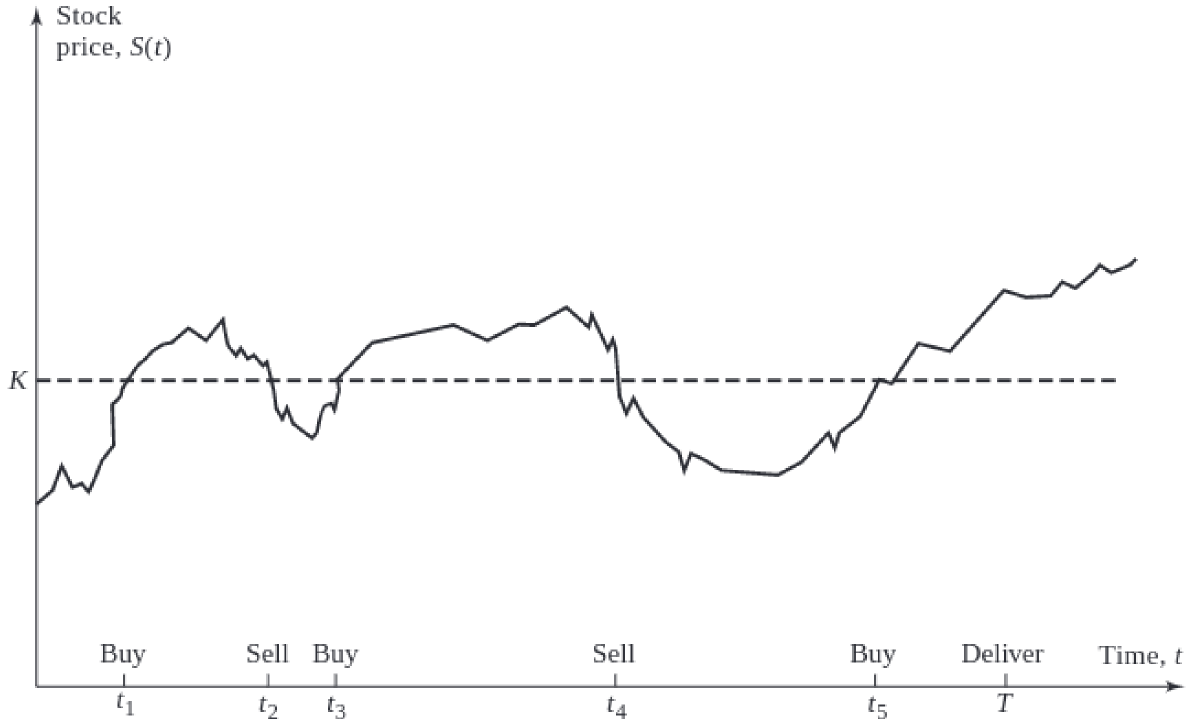
- **Description**: The bank takes no protective action against the position's exposure.
- **Risk Analysis**: This strategy leaves the position fully exposed to market fluctuations, with potential unlimited losses if the stock price rises significantly.

11.1.2 Covered Position

- **Description**: The bank purchases 100,000 shares immediately.
- **Risk Analysis**: Although this strategy can mitigate the risk if the stock's price increases above the strike price, it requires significant capital investment and is exposed to loss if the stock price declines.

11.1.3 Dynamic Hedging: Stop-Loss Strategy

- **Buy**: 100,000 shares are purchased when the stock price hits \$50.
- **Sell**: 100,000 shares are sold when the stock price drops below \$50.
- **Risk**: This approach attempts to limit losses by dynamically adjusting the position based on stock price movements. However, it may lead to high transaction costs and potential slippage; the actual transaction price may not always align with the trigger price due to market volatility.



The most efficient way to hedge would be to use **Greek letters.**

11.2 The Greek Letters

Greek letters are vital tools in options trading, providing a quantitative understanding of the sensitivity of an option's price to various factors. These metrics are crucial for effective risk management and strategic decision-making in trading portfolios.

- The Greeks are typically calculated using the Black-Scholes-Merton (BSM) model, a foundational tool in financial mathematics for valuing options.
- The Greek letters are the partial derivatives with respect to the model parameters that are liable to change.
- In practice, the volatility parameter in the BSM model is set to the implied volatility of the option, a modification often referred to as the "practitioner Black-Scholes" model.
- The discussion here pertains primarily to European options on non-dividend-paying stocks.

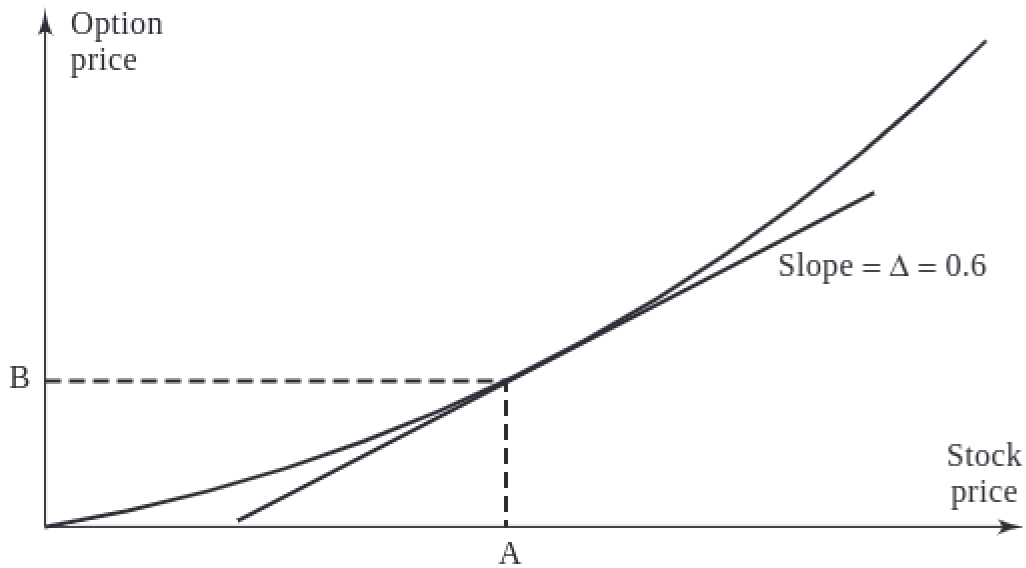
11.2.1 Description of Key Greeks

The Greeks measure the sensitivity of the option's price to one of the underlying parameters, **holding all other parameters constant**. Each Greek addresses a different risk factor:

- **Delta (Δ)**: Measures the rate of **change of the option's price** with respect to changes in the underlying asset's price. Specifically, delta is the first derivative of the option's price with respect to the stock's price. For instance, a delta of 0.6 suggests the option's price will move approximately $0.60 for every $1.00 movement in the underlying stock.
- **Gamma (Γ)**: Represents the rate of **change of delta** with respect to changes in the underlying asset's price. Gamma helps assess the stability of an option's delta, providing insight into how delta might change as the stock price varies. High gamma values indicate that delta is highly sensitive to changes in the stock price.
- **Theta (Θ)**: Measures the sensitivity of the option's price to **the passage of time**, known as "time decay." Theta indicates the expected rate of change in the option's price for a one-day decrease in its time to expiration. This is particularly important as the option approaches its expiry date.
- **Vega (ν)**: Quantifies the sensitivity of the option's price to **changes in the volatility** of the underlying asset. A vega of 1.5 suggests that the option's price is expected to change by $1.50 for every 1% change in the implied volatility of the underlying stock.
- **Rho (ρ)**: Measures the sensitivity of the option's price to **changes in the risk-free interest rate**. For instance, a rho of 0.05 indicates that the option's price will change by $0.05 for every 1% change in interest rates.

11.3 Understanding Delta (Δ)

Delta is a measure of an option's price sensitivity relative to changes in the price of the underlying asset. It's expressed as the amount an option's price is expected to move for a one-unit change in the price of the underlying asset.



The followint graph showing how delta varies with stock price for call and put options.

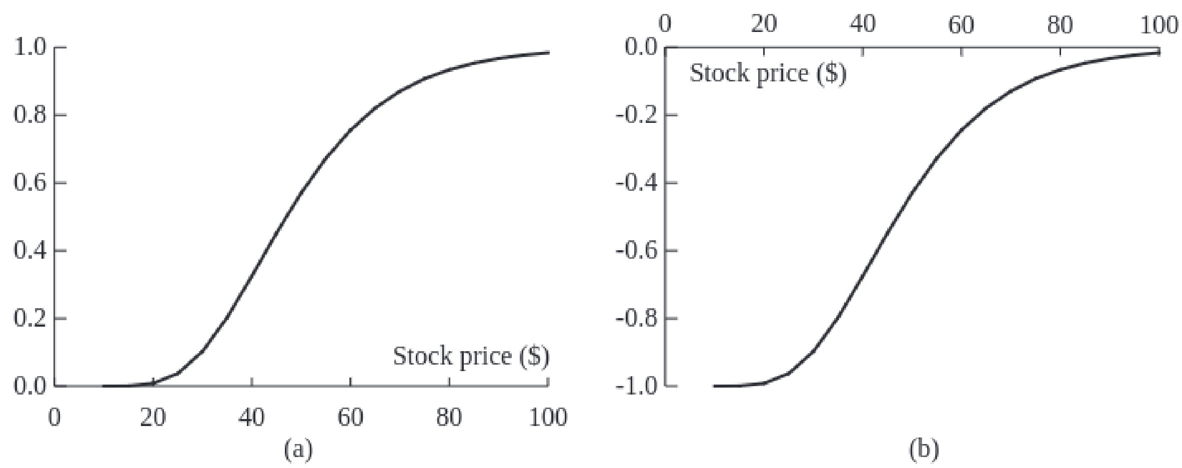


Figure 11.1: Variation of Delta With Stock Price (K=50)

- **Call Option (a)**: Delta increases as the stock price rises, peaking near 1 as the option goes deep in-the-money.
- **Put Option (b)**: Delta decreases (becomes more negative) as the stock price drops, indicating increased sensitivity as the option moves further in-the-money.

The next graph showing how delta changes as time to maturity decreases.

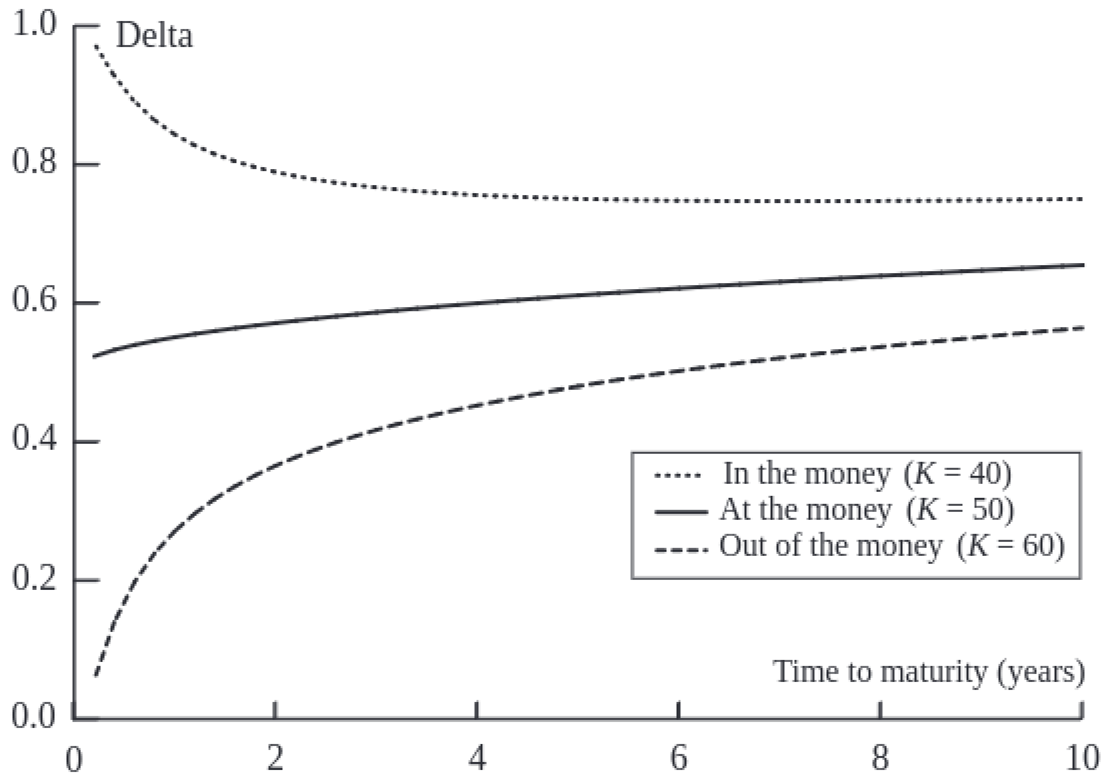


Figure 11.2: Variation of Delta with Time to Maturity

Delta's sensitivity to time decay depends on the moneyness of the option. For options that are exactly at-the-money, delta converges to 0.5. For out-of-the-money options, it approaches zero because such options are likely to expire worthless, whereas for in-the-money options, delta converges to 1.

11.3.1 Delta Hedging: Principles and Practices

Delta hedging is a strategy used to mitigate risk in option trading by setting up a position in the underlying asset. The position's size is determined by the option's delta, aiming to neutralize the effect of small price movements of the underlying asset.

To achieve delta neutrality, the required number of units to hedge, N_H, is calculated using the formula:

$$N_H = -\left(\frac{\text{Portfolio delta}}{\text{Delta of the hedging instrument}}\right)$$

A delta-neutral portfolio implies a total delta of zero, indicating no sensitivity to small price movements in the underlying stock. Note that the **delta of the stock itself is always 1**.

- **Rebalancing**: Delta values change as the stock price fluctuates and as time passes, necessitating periodic rebalancing of the hedge (buy high, sell low).
- **Delta for Calls and Puts**: The delta of a European call option on a non-dividend paying stock is represented by $N(d_1)$, whereas for a put, it's $N(d_1) - 1$.

Example: Delta Hedging I

Consider an investor who holds a short position in 1,000 call options, each with a delta of -0.6. Create a delta-neutral portfolio to shield against small price movements.

- **Current Delta**: $1,000 \times -0.6 = -600$
- **Hedging Action**: Purchase 600 shares of the stock (since each stock has a delta of 1).

This strategy ensures that any gain or loss on the options is countered by a corresponding loss or gain in the stock holdings.

Example: Delta Hedging II

Consider a portfolio comprising 1,000 shares of a non-dividend-paying stock. Each share contributes a delta of 1 to the portfolio, resulting in a total portfolio delta of 1,000.
To achieve delta neutrality using call options with a delta of 0.40, we calculate the required number of calls to sell:

$$N_H = -\left(\frac{1,000}{0.40}\right) = -2,500$$

This means selling 2,500 call options ensures that the positive delta from the stock holdings is exactly offset by the negative delta contribution from the short call positions.

Example: Delta Hedging III

Consider a scenario where you have a short position in 10,000 shares of a non-dividend-paying stock:

11.3.1.1 Using Call Options for Hedging:

- **Option Delta** (Δ_c): 0.668
- To neutralize the negative delta of -10,000 from the short stock position, you would need to purchase call options to increase delta:

$$N_H = -\left(\frac{-10,000}{0.668}\right) = 14,970$$

Purchasing 14,970 call options will add a positive delta, balancing out the negative delta from the short stock position.

11.3.1.2 Using Put Options for Hedging:

- **Option Delta** (Δ_p): -0.332
- To balance the negative delta, you can use put options which inherently have a negative delta:

$$N_H = -\left(\frac{-10,000}{-0.332}\right) = -30,120$$

Selling 30,120 put options would counteract the negative delta from the short stock position, achieving neutrality.

11.3.2 Hedging Cost Scenarios from Case Study

The following table presents a simulation of hedging dynamics when the option expires in-the-money, with total hedging costs amounting to $263,300:

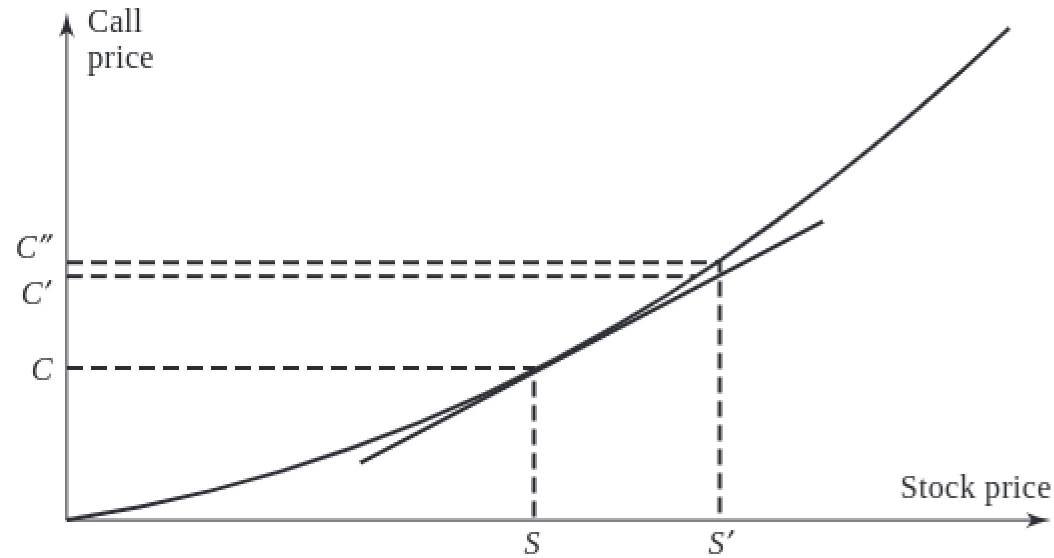
Week	Stock price	Delta	Shares purchased	Cost of shares purchased ($000)	Cumulative cost including interest ($000)	Interest cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

- The cost reflects the higher intrinsic value and sensitivity (delta) as the option gains worth.
- If the option expires out-of-the-money, the costs should be slightly lower, as reduced delta correlates with a decreased likelihood of the option finishing in-the-money.

11.4 Understanding Gamma (Γ)

Gamma (Γ) is the second derivative of the option's price with respect to the underlying asset's price. It measures the rate of change of delta (Δ) and is crucial for understanding the curvature or the convexity of the option's value in relation to the stock price. It's crucial for assessing the stability of an option's delta, thus influencing how often a delta-hedged portfolio needs rebalancing. Here are some key characteristics of gamma:

- **Zero for Stocks**: Gamma for a long or short position in a single share of stock is zero because a stock's delta does not change.
- **Symmetry for Calls and Puts**: Gamma is identical for both call and put options.
- **Non-negativity**: Gamma is always non-negative. It reaches its highest value when an option is at-the-money, highlighting increased sensitivity at this point.
- **Risk Measurement**: Gamma quantifies the non-linearity risk—risk that remains in a delta-neutral portfolio due to price movements of the underlying asset.



11.4.1 Gamma and Hedging Dynamics

Gamma plays a critical role in addressing potential errors in delta hedging, particularly when stock prices exhibit significant movement:

- **Small Changes**: For minor fluctuations in stock price, delta hedging generally performs well.
- **Larger Shifts**: For more substantial stock price movements, delta-plus-gamma hedging provides greater accuracy, accommodating the curvature in the relationship between an option's price and the stock price.
- **Gamma Risk**: This term refers to the risk arising from sudden, significant movements in stock prices—often referred to as jumps—which can leave a previously well-hedged position exposed. Gamma risk is particularly relevant in markets characterized by high volatility or discontinuous price movements.
- **Theta and Gamma Interaction**: Typically, when theta is large and negative, indicating substantial time decay benefits, gamma tends to be large and positive. This inverse relationship highlights the trade-offs between potential profitability from time decay and the risk from larger price movements.

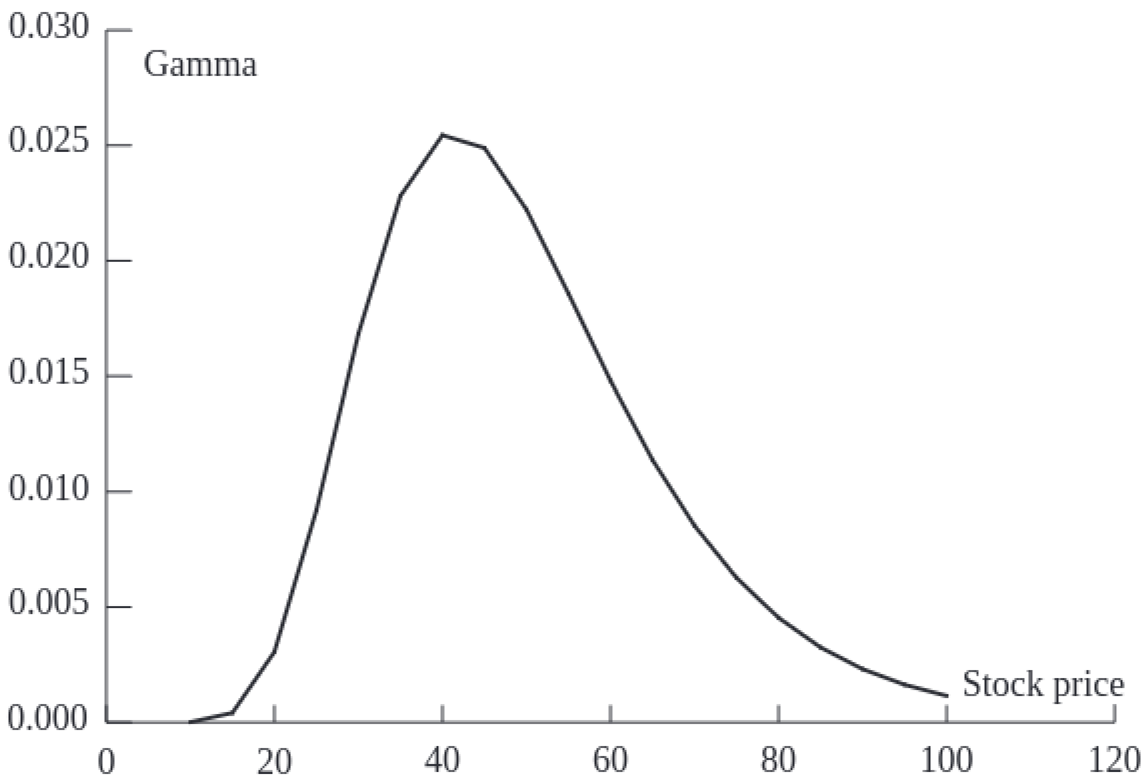


Figure 11.3: Gamma variation with stock price

This chart demonstrates how gamma peaks when an option is at-the-money and diminishes as the option moves deeper into or out of the money. This pattern underscores the heightened sensitivity and potential risk/reward near the at-the-money point.

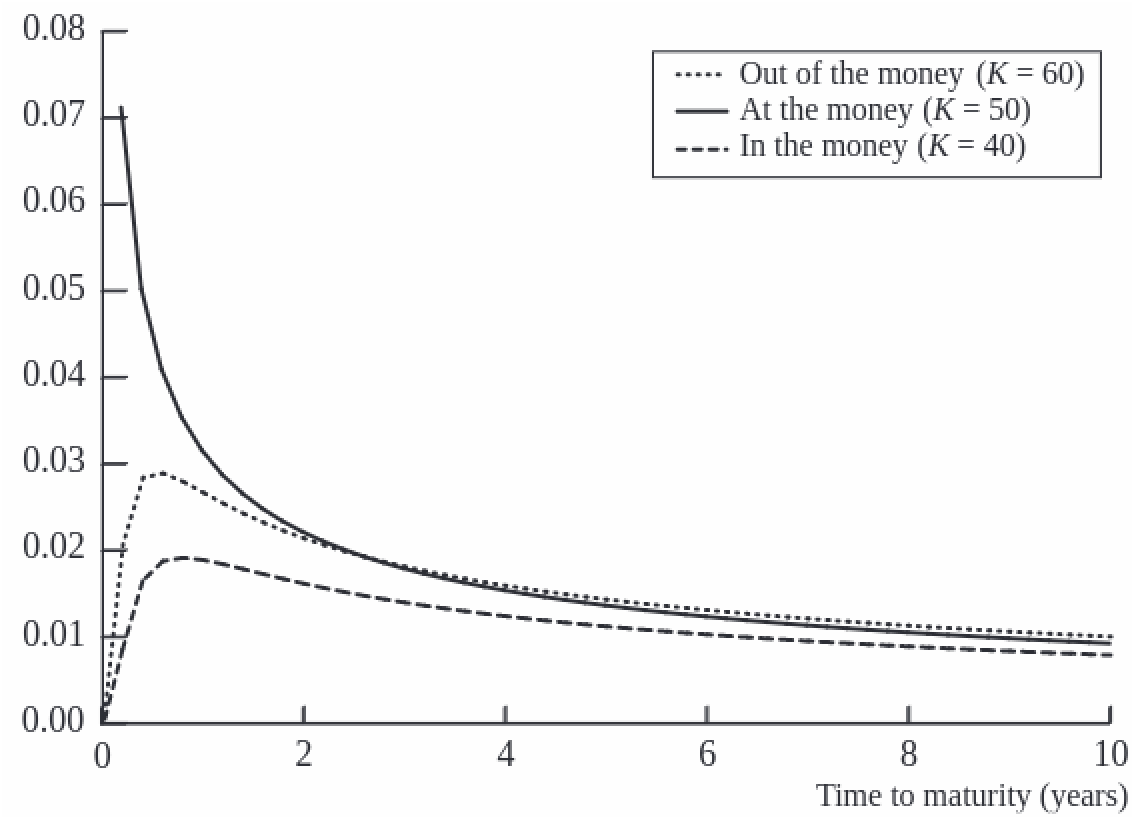


Figure 11.4: Gamma variation over time

Gamma increases as the option approaches expiration, particularly for at-the-money options. This increase reflects growing sensitivity to stock price movements as time to exercise decreases, emphasizing the importance of precise hedging strategies in the final days before an option's expiration.

11.5 Understanding Theta (Θ)

Theta (Θ) represents the sensitivity of the price of an option or a portfolio of derivatives to the passage of time, assuming all other factors remain constant. It is a critical component in options pricing, reflecting the temporal decay of an option's value.

- **Directionality**: Theta is typically negative for both calls and puts. This reflects the loss in time value as options approach their expiration date.
- **Non-applicability to Stocks**: Since stocks do not have an expiration date, they inherently have a theta of zero.

- **Impact on Options**: Each day closer to expiration decreases the option's time value, reducing its price if other conditions remain unchanged.

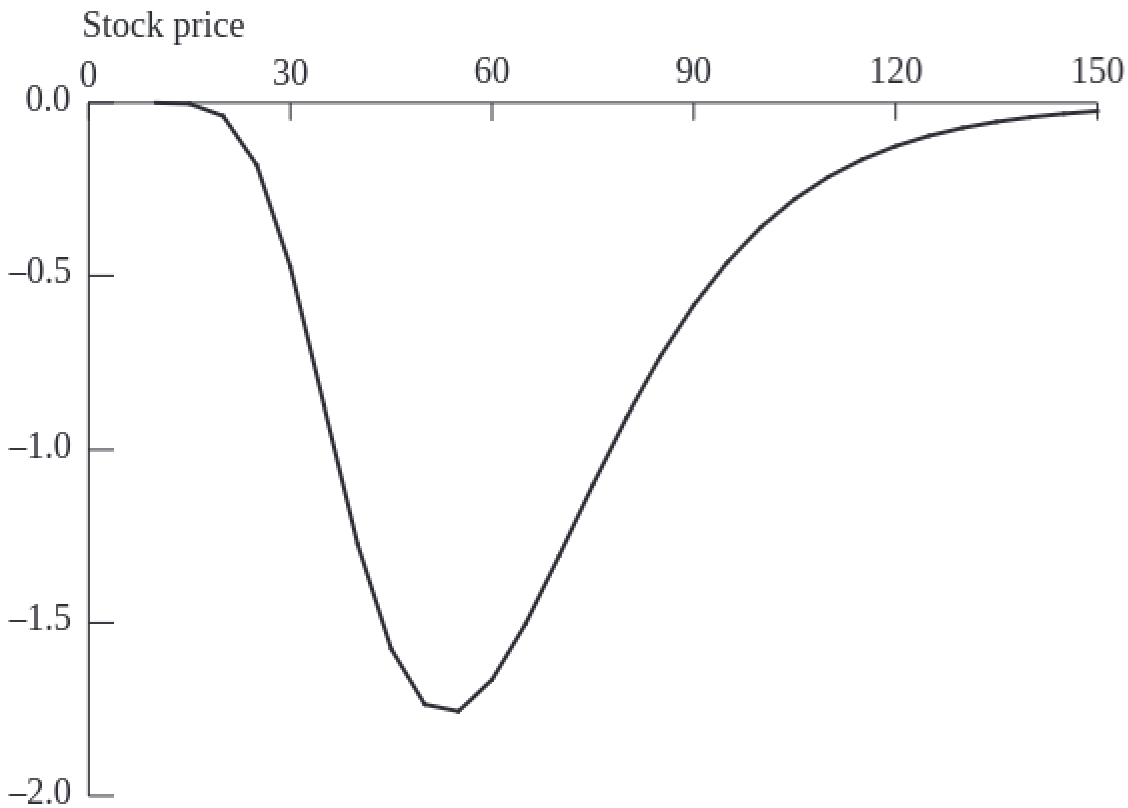


Figure 11.5: Theta variation with stock price

The graph illustrates how theta behaves relative to the stock price for a call option. Notably, theta becomes increasingly negative as the option moves around at-the-money.

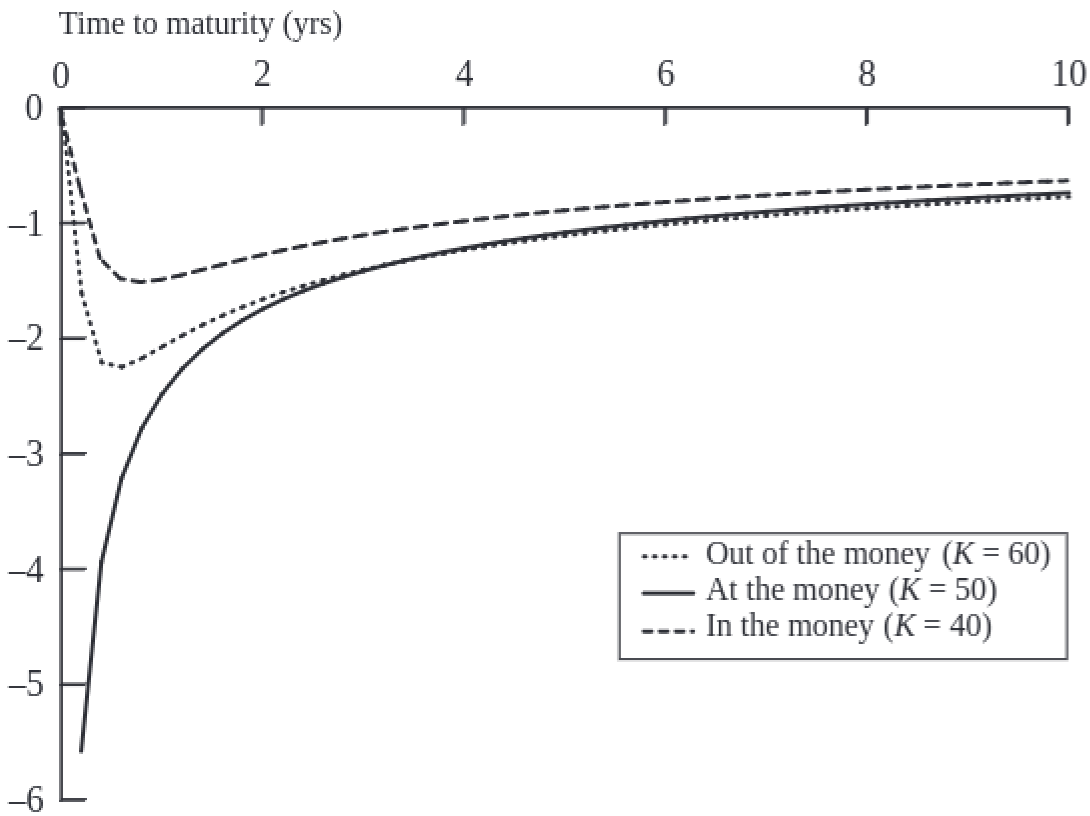


Figure 11.6: Theta variation over time

This visualization shows theta's behavior as the option nears its expiry. Theta typically becomes more negative as expiration approaches, indicating an acceleration in the rate of time value decay. This is most pronounced for at-the-money options, where the uncertainty about finishing in-the-money is highest.

- **Theta vs. Hedge Parameters**: Unlike delta, which can be hedged, theta represents a guaranteed decline in value over time and is not a hedgeable risk. However, its predictability and inevitability make it a significant factor in options strategy.
- **Descriptive Utility**: Despite its non-hedgeable nature, theta is considered a useful descriptive statistic for portfolios, particularly in strategies aimed at earning through time decay, such as selling options.
- In a delta-neutral setting, where the portfolio is insulated against small price movements in the underlying asset, theta can serve as an indirect measure of gamma exposure. This relationship arises because high gamma values, which indicate greater sensitivity to price changes, usually accompany high rates of time decay (theta). Thus, monitoring theta provides insights into potential gamma risks in the portfolio.

11.6 Understanding Vega (ν)

Vega represents the rate of change in the value of an option or a derivatives portfolio with respect to changes in volatility. This Greek is crucial in options trading, reflecting how the price of options reacts to fluctuations in the underlying asset's volatility.

Vega highlights the impact of volatility, an essential but unobservable market factor that significantly influences option pricing. Future volatility, being an estimation, adds a layer of complexity and risk in predicting option values.

- **Directionality**: Vega is always positive. An increase in the implied volatility of the underlying asset generally leads to an increase in the value of both call and put options.
- **Symmetry**: The vega of a call option is equal to the vega of a put option with the same terms.
- **Calculation**: For a portfolio, vega can be understood as a weighted average of the vegas of the individual positions, reflecting the aggregate sensitivity to a uniform shift in implied volatilities across the portfolio.
- **Volatility Sensitivity**: Of all the Black-Scholes-Merton (BSM) model variables (stock price, strike price, time to expiration, risk-free rate, and volatility), an option's price is most sensitive to changes in volatility. This sensitivity makes vega a key focus in risk management and trading strategies.

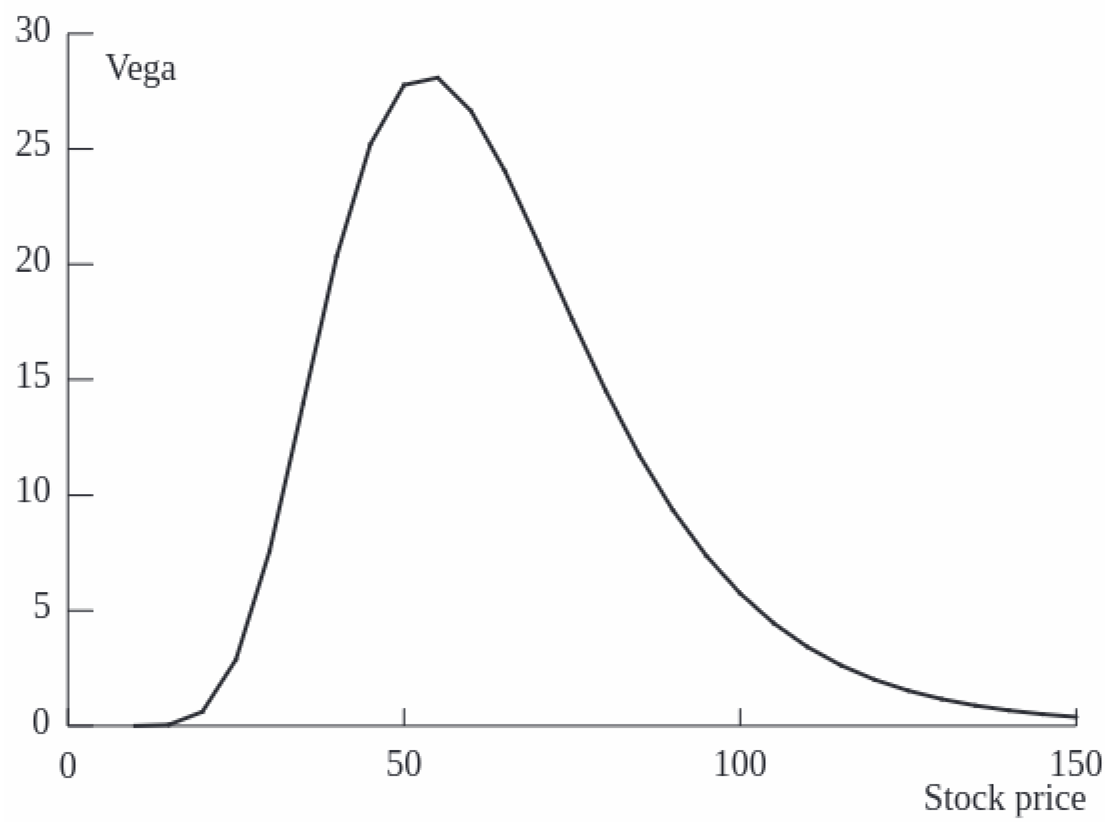


Figure 11.7: Vega variation with stock price

The chart illustrates how vega changes in relation to the stock price. Vega tends to be higher when an option is at-the-money and decreases as the option moves deeper into or out of the money. This pattern reflects the increased sensitivity to volatility changes when the option's strike price is near the current stock price, where the uncertainty about the outcome at expiration is greatest.

11.7 Understanding Rho (ρ)

Rho represents the sensitivity of an option or a derivatives portfolio's value to changes in the risk-free interest rate. It is one of the lesser-focused Greeks but plays a crucial role in environments where interest rate fluctuations are significant.

- **Directionality**: Rho is typically positive for call options and negative for put options. This reflects the different financial implications of options:

- **Call Options**: A positive Rho for call options indicates that their value increases as interest rates rise. This is because higher rates reduce the present value of the exercise price paid at expiration, making it cheaper in present terms to buy the stock at a future date.
- **Put Options**: Conversely, a negative Rho for put options means their value decreases as interest rates rise. Higher rates increase the present value of the proceeds received from selling the stock in the future, thus decreasing the attractiveness of holding a put.

The sensitivity to interest rates through Rho might seem minor compared to other Greeks like Delta or Vega, but it can become significant in certain financial environments:

- **High Interest Rate Volatility**: In periods of significant interest rate changes, Rho becomes a more critical factor for traders, especially those dealing with long-dated options where the cumulative impact of rate changes can be substantial.
- **Portfolio Management**: For portfolios that include a mix of long-dated options and bonds, understanding and managing Rho is vital to hedge interest rate risks effectively.

11.8 Strategic Management of Delta, Gamma, and Vega

Proper management of the Greek values—Delta, Gamma, and Vega—is essential for maintaining balanced options portfolios that align with specific risk management objectives. Here, we outline techniques to achieve neutrality in these Greeks through strategic positioning in options and the underlying asset.

- **Delta**: This Greek can be adjusted directly by taking positions in the underlying asset. For instance, buying or selling stock can increase or decrease the portfolio's delta.

- **Gamma and Vega**: Adjustments to gamma and vega generally require engaging in positions in options or other derivatives, as these Greeks measure sensitivity to changes in the underlying asset's price and volatility, respectively.

The general approach involves **first hedging vega and/or gamma** using options, and then as a **final step, using underlying assets** to reduce delta to zero. This sequence is important because hedging vega and gamma can alter delta, whereas hedging using underlying assets does not affect vega or gamma.

Example: Achieving Greek Neutrality

	Delta	Gamma	Vega
Portfolio	0	-5000	-8000

Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

1. **Achieving Delta and Gamma Neutrality**:
   - **Required Position**: Long 10,000 units of Option 1 and short 6,000 units of the underlying asset.
   - **Rationale**: This positioning uses the delta and gamma from Option 1 to offset the existing negative gamma in the portfolio while also balancing delta.
2. **Achieving Delta and Vega Neutrality**:
   - **Required Position**: Long 4,000 units of Option 1 and short 2,400 units of the underlying asset.
   - **Rationale**: This configuration uses Option 1's vega to counteract the portfolio's negative vega while adjusting the delta with a corresponding position in the underlying asset.
3. **Achieving Delta, Gamma, and Vega Neutrality Across Multiple Options**:
   - **System of Equations**:
     - For Gamma:
$$-5000 + 0.5w_1 + 0.8w_2 = 0$$
     - For Vega:
$$-8000 + 2.0w_1 + 1.2w_2 = 0$$
   - **Solution**: Long 400 units of Option 1 and 6,000 units of Option 2, with a short position of 3,240 units in the underlying asset to achieve complete neutrality in delta, gamma, and vega.

11.8.1 Hedging in Practice

- **Daily Delta Neutrality**: Traders typically adjust their portfolios to be delta-neutral at least once per day to manage risk effectively against price movements in the underlying asset.
- **Opportunistic Gamma and Vega Adjustments**: While daily adjustments for delta are common, adjustments for gamma and vega are made as opportunities arise, allowing traders to better manage the curvature and volatility risks associated with their positions.
- **Economies of Scale**: As portfolios increase in size, the relative cost of hedging per option decreases, providing cost efficiencies in larger portfolios.
- **Scenario Analysis**: This involves testing different market scenarios and their effects on the portfolio, considering various assumptions about asset prices and volatilities. This

analysis helps in understanding potential impacts and preparing for various market conditions.

11.9 Practice Questions and Problems

1. Explain how a stop-loss trading rule can be implemented for the writer of an out-of-the money call option. Why does it provide a relatively poor hedge?
2. What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?
3. What does it mean to assert that the theta of an option position is -0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?
4. What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?
5. A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of 80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?
6. A financial institution has the following portfolio of stock options:

Type	Position	Delta	Gamma	Vega
Call	-1,000	0.5	2.2	1.8
Call	-500	0.8	0.6	0.2
Put	-2,000	-0.4	1.3	0.7
Call	-500	0.7	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

1. What position would make the portfolio delta neutral?
2. What position in the traded option and in stocks would make the portfolio both gamma neutral and delta neutral?
3. What position in the traded option and in stocks would make the portfolio both vega neutral and delta neutral?

4. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Solution

1. Long 450 stocks.
2. Long 4,000 options and short 1,950 stocks.
3. Long 5,000 options and short 2,550 stocks.
4. Long 3,200 options 1, long 2,400 options 2, and short 1,710 stocks.

12 Structured Products I

References

- BLÜMKE, Andreas. How to invest in structured products: a guide for investors and investment advisors. Chichester: Wiley, 2009. xvi, 374. ISBN 9780470746790.

Learning Outcomes:

- Identify the key characteristics and benefits of structured products as investment vehicles.
- Describe the role and impact of the European Structured Investment Products Association (EUSIPA) in the structured products market.
- Differentiate between various types of structured products, including investment certificates and barrier options.
- Evaluate the mechanisms and behavior of capital protection and participation investment products through theoretical and case study approaches.

12.1 Introduction to Structured Products

12.1.1 Advantages of Investing in Structured Products

Structured products enjoy significant popularity across **European states**, primarily due to their capacity to balance risk and reward. These financial instruments offer several compelling advantages:

1. **Capital Protection and Market Participation**: Investors are drawn to structured products because they provide a mechanism to protect all or part of the invested capital while still allowing participation in the gains of the capital markets.

2. **Accessibility to Regional Markets**: These products facilitate access to regional markets and asset classes that may otherwise be inaccessible through direct investments.

3. **Diverse Return Profiles**: Structured products can be designed with a wide array of return profiles, adapting to various investor needs and market conditions. They remain effective across different market movements—providing potential returns in rising, sideways, or falling markets.

4. **Liquidity**: Market makers enhance the liquidity of structured products by continuously buying and selling them, which ensures a stable market presence and availability.
5. **Sophisticated Investment Strategies**: Investors can implement complex investment strategies typically available to advanced traders through the acquisition of a single structured product.
6. **Regulatory Oversight and Transparency**: These products are often listed on official, regulated markets, adding a layer of security and transparency for investors.
7. **Tax Advantages**: In some jurisdictions, structured products offer favorable tax conditions, enhancing their attractiveness as investment options.

12.1.2 European Structured Investment Products Association (EUSIPA)

- Official Website: EUSIPA
- Market Insights: EUSIPA Market Reports

Founded in 2009, EUSIPA represents a collective of national issuer associations from multiple European countries including Austria, France, Germany, Italy, Sweden, Belgium, the UK, Switzerland, and The Netherlands. As an international non-profit association governed under Belgian law, EUSIPA also maintains a presence in the EU transparency register.

EUSIPA aims to foster transparency and establish uniform market standards across Europe. It serves as a pivotal platform for its members to engage in meaningful dialogue with European policymakers, ensuring that the voices of issuers are heard in the regulatory landscape.

12.2 Definition and Nature of Structured Products (Investment Certificates)

> Definition by Andreas Blümke
>
> Structured products are financial assets, which consist of various elemental components, combined to generate a specific risk-return profile (not replicable with stocks and bonds) adapted to an investor's needs.

Structured products, often referred to as investment certificates, are essentially securitized derivatives. These are complex financial contracts encapsulated within a single security that trades on exchanges similar to stocks. These instruments are crafted and issued by financial institutions and are utilized by both retail and institutional investors. They can be traded on stock exchanges or dealt directly between parties in over-the-counter (OTC) transactions.

12.2.1 Credit Risk Associated with Structured Products

Structured products carry inherent credit risks as they **are issued in the form of bearer bonds**. This means the issuer's entire assets back the liability on these products. The quality and safety of structured products are intrinsically linked to the creditworthiness of the issuer. Like traditional bonds, these products are **exposed to issuer risk**, which implies that in the event of issuer bankruptcy, both bonds and structured products are treated as part of the bankruptcy estate.

To mitigate such risks, investors are advised to:

- Opt for products issued by financially robust institutions.
- Diversify their investments across various issuers.
- Continuously monitor the financial health of the issuers over time.

12.2.2 Categorization of Structured Products

Structured products do not follow a universal standard for categorization. However, common classifications are often derived from industry associations such as:

- **EUSIPA**: European Structured Investment Products Association provides a framework for categorizing structured products within Europe.
  - View the EUSIPA Derivatives Map for details.
- **SVSP**: Swiss Structured Products Association offers an alternative categorization scheme.
  - Explore the SVSP Derivative Map for comparison.

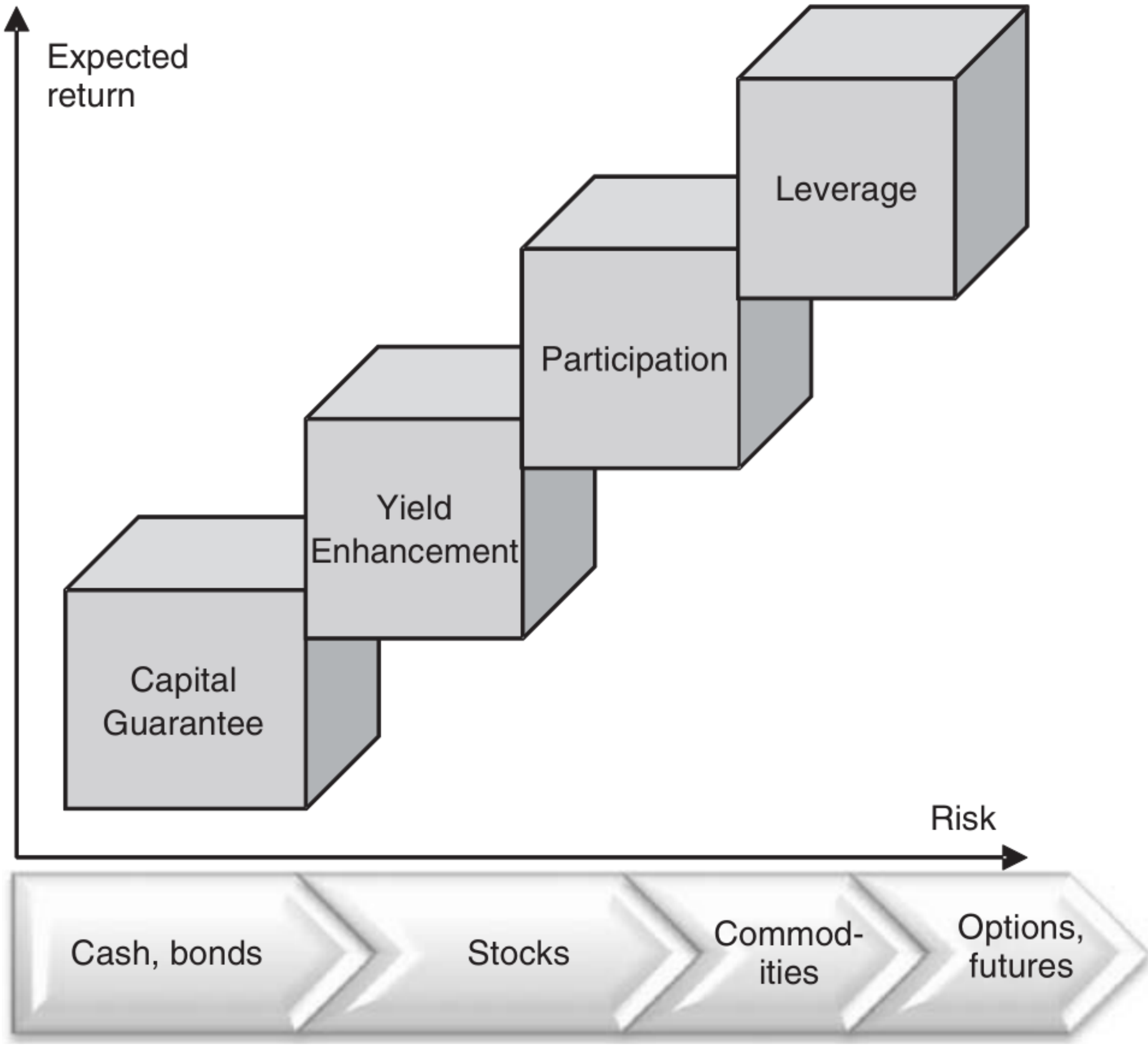


Figure 12.1: Classical structured products representation

12.2.3 Investors and Sellers of Structured Products

12.2.3.1 Investors: Private and Institutional Buyers

Structured products attract both private and institutional investors when traditional investment avenues do not meet their specific needs. These needs might include the desire for returns higher than the risk-free rate while still benefiting from capital protection. Structured products are particularly appealing in scenarios where conventional investments either do not provide sufficient returns or fail to address **specific financial goals and risk profiles**.

12.2.3.2 Sellers: Financial Institutions and Their Motivations

Sellers of structured products, typically financial institutions, are driven by profit. They utilize sophisticated mathematical models to determine a 'fair value' of the product at issuance. This fair value is then increased by a spread which covers various costs associated with the product over its lifetime. These costs include but are not limited to:

- Secondary market activities
- Listing fees
- Production of term-sheets
- Settlement processes

Among these, the most **significant cost factor is hedging**. Hedging expenses are challenging to predict in advance as they depend on market dynamics over the product's life. The profit from structured products, however, is only realized at their expiry, and having a large and diverse portfolio helps in more effectively hedging as some products can offset the risks of others.

12.2.4 Platforms to Find Structured Products

Structured products can be accessed and traded through various platforms, which serve as marketplaces for these financial instruments:

- **Deutsche Bank X-markets**: Offers a wide range of structured products for different investment strategies.
  - Website: Deutsche Bank X-markets
- **Börse Stuttgart**: Known for its user-friendly approach to trading structured products among other securities.
  - Website: Börse Stuttgart
- **Börse Frankfurt**: One of Europe's largest trading centers for securities, including derivatives and structured products.
  - Website: Börse Frankfurt

12.2.5 Additional References

For further research and detailed insights into the market for structured products, the following resources are invaluable:

- European Structured Investment Products Association (EUSIPA): Website
- German Derivatives Association: Website

- Swiss Structured Products Association (SVSP): Website
- UK Structured Products Association: Website
- Italian Association of Certificates and Investment Products (ACEPI): Website

12.3 Barrier Options

> Definition
>
> Barrier options are exotic call or put options that include a barrier condition placed above or below the strike that, when crossed, either transfroms the exotic option into a plain vanilla option (‘in’ barriers) or cancels it altogether (‘out’ barriers).

Barrier options are integrated into various structured financial products such as barrier reverse convertibles and bonus certificates. Recognized for their complexity, these options introduce a conditional component to the standard option mechanism, making the final payoff uncertain until the option’s maturity.

12.3.1 Types of Barrier Options

Barrier options can be classified into four main types based on the direction of the barrier and its effect:

- **Up & Out**: The option becomes void if the underlying asset’s price goes above the barrier.
- **Up & In**: The option comes into existence when the underlying asset’s price goes above the barrier.
- **Down & Out**: The option becomes void if the underlying asset’s price falls below the barrier.
- **Down & In**: The option comes into existence when the underlying asset’s price falls below the barrier.

Additionally, these options can feature a rebate, a predefined amount paid to the option holder if the barrier is breached before maturity.

12.3.2 Pricing Dynamics

Barrier options are generally more cost-effective than their plain vanilla counterparts due to the added condition of the barrier. The pricing dynamics vary significantly between 'in' and 'out' options:

- **For 'In' Options**: As the maturity increases, the price approaches that of a plain vanilla option, especially as the price of the underlying asset approaches the barrier.
- **For 'Out' Options**: Longer maturities reduce the price, potentially approaching zero as the asset's price nears the barrier.

12.3.3 Barrier Styles

Structured products typically incorporate barrier options with specific monitoring styles:

- **American-style Barriers**: These allow the barrier condition to be triggered at any point during the option's life, including intraday events, and are known for continuous monitoring.
- **European-style Barriers**: These restrict the barrier condition to only be checked at the maturity of the option.

Although most structured products utilize American-style barriers due to cost-effectiveness, the investor focus generally lies elsewhere rather than on the barrier type itself.

Example	American Barrier (Original)	European Barrier
Shark Note	Barrier 131.5%, Rebate 7.5%	Barrier 123%, Rebate 7.5%
Barrier Reverse Convertible	Coupon 10.4%, Barrier 75%	Coupon 8.6%, Barrier 75%
Bonus Certificate	Bonus 9%, Barrier 65%	Bonus 2.5%, Barrier 65%

- **Window-style Barriers**: Situated between American and European styles, window barrier options are designed to activate or deactivate only during specific periods within the product's life. For example, the barrier in some reverse convertibles might only be relevant during the final three months of a one-year term. Although not common, window barriers provide an opportunity for investors to avoid premature knockouts, with the value difference between American and window options typically being minimal.

12.4 Capital Protection Investment Products

Definition

Capital guaranteed products ensure the redemption of the initial capital invested at maturity, while also allowing participation to varying degrees in the performance of an underlying risky asset.

These products are distinguished by three primary features:

1. **Limited Loss Potential**: The potential loss is confined to the level of the capital guarantee, not accounting for the issuer's credit risk.
2. **Participation in Underlying Assets**: Investors gain exposure to the performance of selected assets.
3. **Minimal Guaranteed Income**: Typically, these products offer low or no guaranteed income, focusing instead on capital preservation and growth through asset performance.

It's crucial to consider the **opportunity costs**, such as foregone dividends or the risk-free rate. While attractive at first glance, the actual benefit depends significantly on the performance of the underlying asset at maturity:

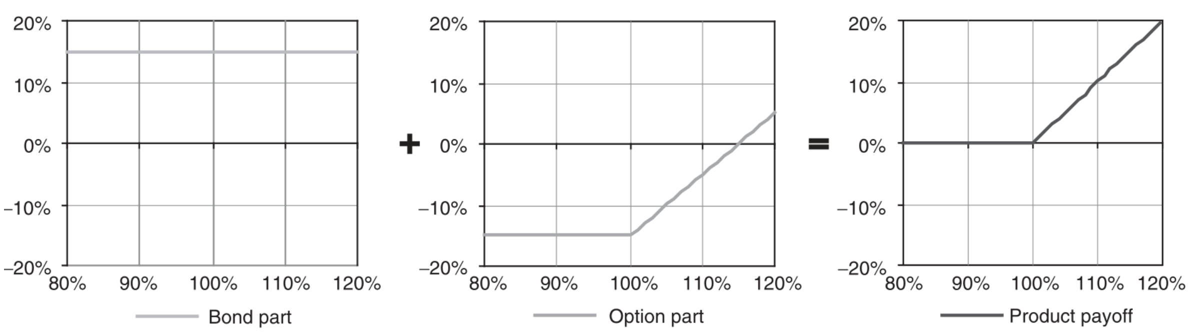
- If the asset performs well, the capital guarantee becomes redundant.
- If the asset performs poorly, it might have been better not to invest.

Capital guaranteed products serve as a prudent option for investors who prefer not to remain in cash but are also uncertain about future market directions.

12.4.1 Uncapped Capital Protection

Classical construction of capital guaranteed products involves:

1. Issuing a zero-coupon bond matching the maturity of the product to ensure capital return.
2. Purchasing a call option on the underlying risky asset to allow for profit participation.



12.4.1.1 Discount and Participation Formula

If we consider an interest rate of 4% with a 5-year maturity:

$$\text{Zero-bond Price} = \frac{100\%}{(1+4\%)^5} = 82.19\%$$

The available funds to purchase options (the discount) is:

$$\text{Discount} = 100\% - 82.19\% = 17.81\%$$

Participation rate is calculated as:

$$\text{Participation} = \frac{\text{Discount}}{\text{Option cost}}$$

12.4.1.2 Factors Influencing Product Viability

Two critical factors affect the desirability and effectiveness of capital guaranteed products:

- **Interest Rates**: Higher rates increase the discount, thereby enhancing the capacity to purchase options.
- **Volatility of the Underlying**: Lower volatility reduces option costs, improving participation rates.

Options for enhancing attractiveness include reducing the capital guarantee below 100%, adding caps or exotic features like knock-out barriers, and utilizing out-of-the-money options.

12.4.2 Capital Protection Products Modifications

12.4.2.1 Exchangeable Certificates

Figure 12.2: Exchangeable Certificates

- **Market Expectations:** Rising volatility, sharply rising or falling underlying.
- Minimum redemption at maturity equals the capital protection (e.g., 100% of nominal).
- Value may fall below capital protection during the product's life.
- Unlimited upside above the strike price, with possible coupon payments.

12.4.2.2 Capped Capital Protection

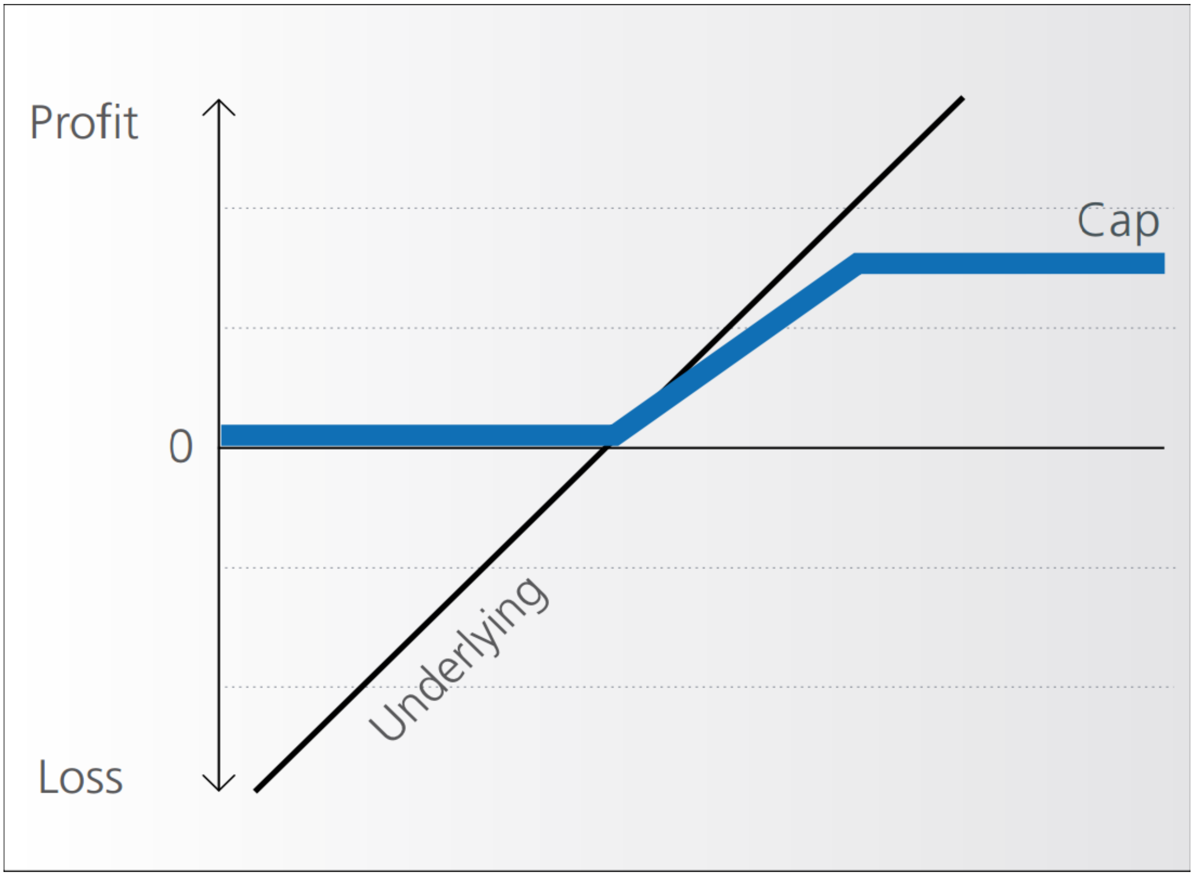


Figure 12.3: Capped Capital Protection

- **Market Expectations:** Rising underlying, potential for sharp falls.
- Guaranteed minimum redemption at expiry.
- Participation in positive performance up to a specified cap.
- Limited profit potential due to the cap.

12.4.2.3 Capital Protection with Knock-Out (Shark Note)

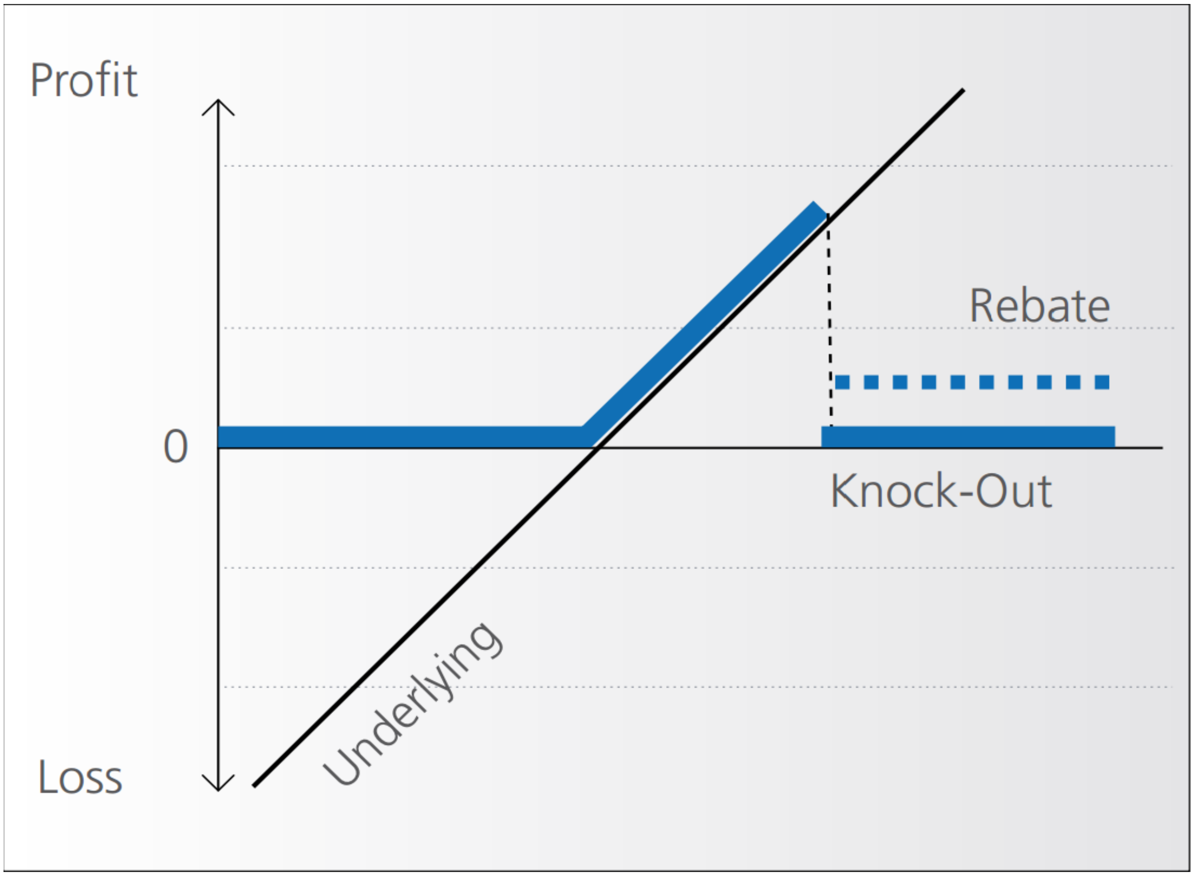


Figure 12.4: Shark Note - Knock-Out

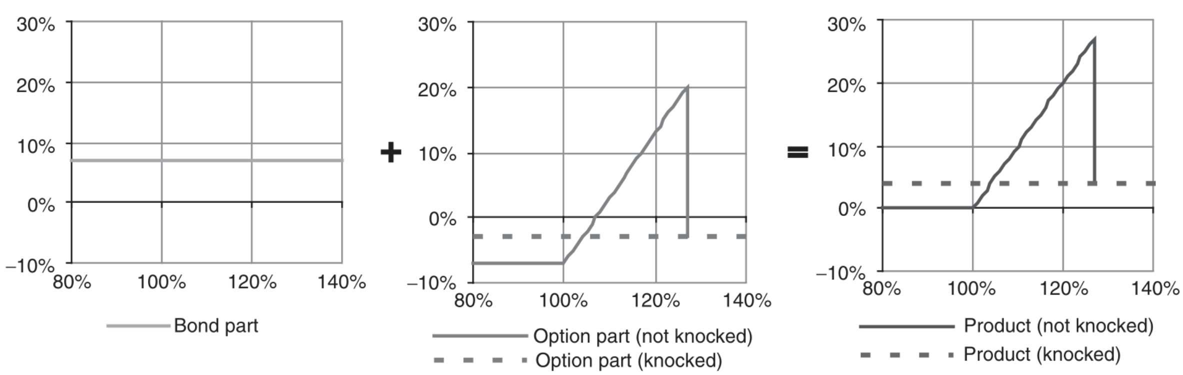


Figure 12.5: Shark Note Construction

- **Market Expectations:** Rising underlying, unlikely to breach a set barrier.
- Full capital protection with a short- to medium-term horizon.
- Uses an up-and-out call option; if the barrier is touched, a rebate may be paid.
- Participation in performance until a barrier is hit; if breached, participation ends and a rebate might be paid.

Redemption Scenarios:

- If the underlying is below 100% of its initial value at maturity: 100% capital guarantee.
- If above 100% without touching the barrier: 100% + participation.
- If the barrier is touched: 100% + rebate.

Strategic Use:

- Set a high barrier to minimize knock-out risk or a high rebate to enhance returns if knock-out occurs.
- Autocall feature allows early redemption if the barrier is breached, suitable for reinvestment without waiting for product maturity.

12.4.2.4 Capital Protection with Coupon

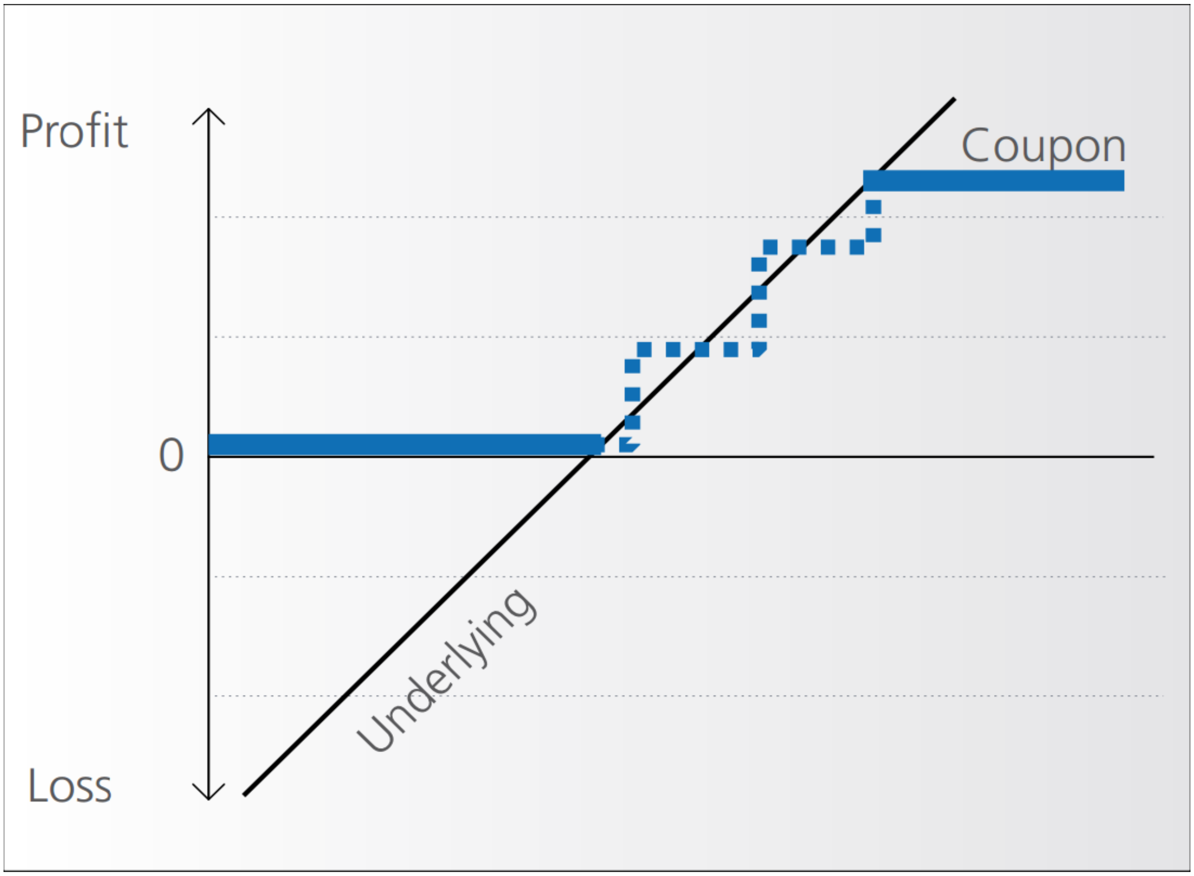


Figure 12.6: Capital Protection with Coupon

- **Market Expectations:** Rising underlying, potential sharp falls.
- Guaranteed capital protection at maturity.
- Periodic coupon payments linked to the performance of the underlying.
- Limited upside potential.

12.4.3 General Recommendations

- Match the product with your investment horizon; the capital guarantee is effective only at maturity.
- Avoid products with lower capital guarantee than 90% (e.g., 90% guarantee means that underlying must perform by more than 10% to be break even, not considering any opportunity cost).
- Ensure the participation rate is at least 80%.

- Verify the issuer's credit rating.
- Prefer shorter maturity periods for products like Shark Notes to reduce risk (no more than two or three years).

12.5 Behavior of Capital Guarantee Products - Case Study

Characteristic	Details
Underlying Risky Asset	Eurostoxx50 Index
Maturity	4 years
Implied Volatility	23%
Asset's Dividend Yield (p.a.)	4%
Interest Rate Level (4-year swap rate, p.a.)	4.5%
Capital Guarantee Level	100%
Initial Participation	100%

Key Variables:

- **Interest Rate:** A primary determinant of the price of the zero-coupon bond component of the product.
- **Volatility:** Crucial for the valuation of the call option embedded in the product. High volatility increases the potential upside, impacting the option's price more significantly when the asset is at-the-money.

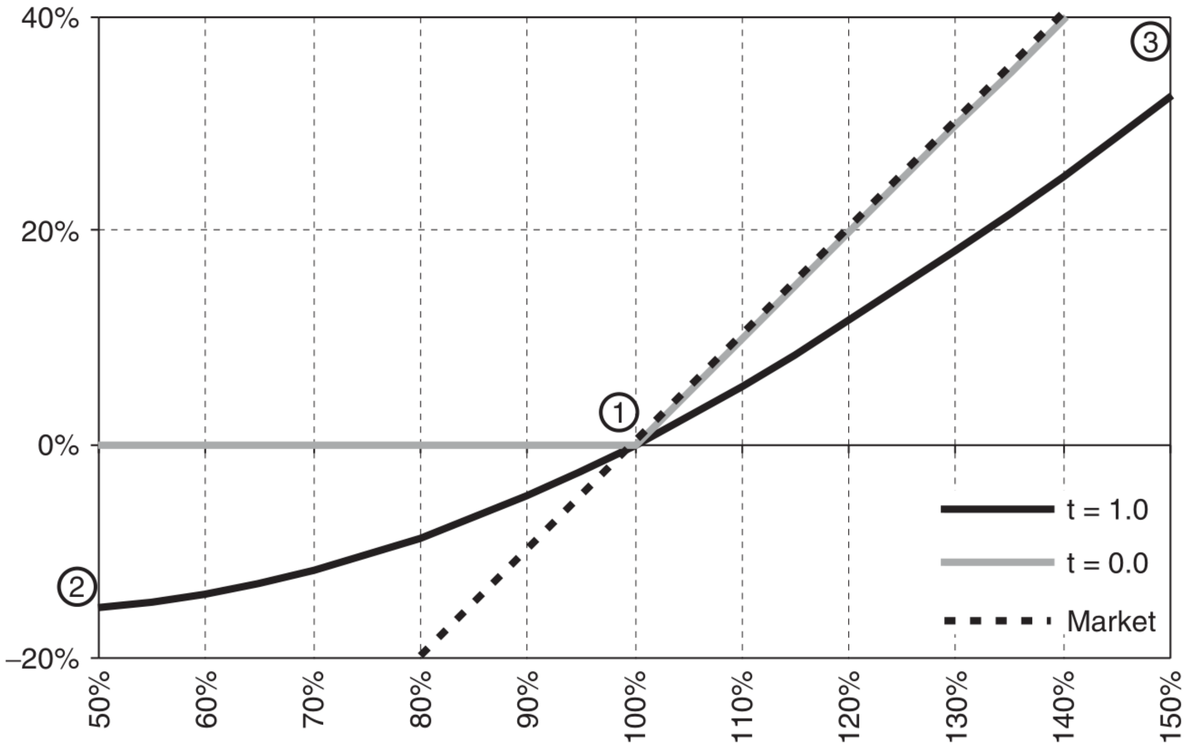


Figure 12.7: Spot price variations for classical capital guaranteed products

This graph demonstrates the relationship between the spot price of the underlying asset and the price of the capital guaranteed product. The slope of the line, particularly when it approaches a 45-degree angle, indicates a delta of 100%, where the product's price moves one-to-one with the underlying asset.

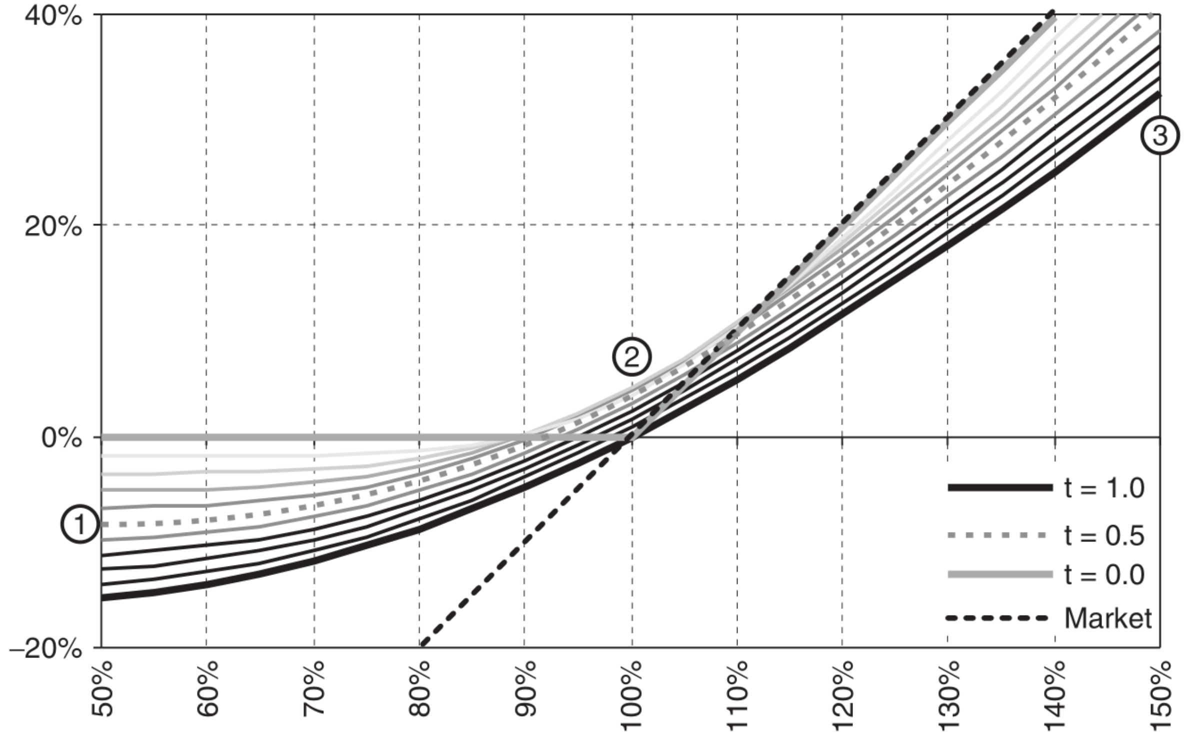


Figure 12.8: Price as a function of maturity

The slope of this graph illustrates how the delta of the product changes with time. Typically, the product's participation at inception ranges from 40%-60% of its maturity level, reflecting the initial risk profile and pricing.

Note: Capital guarantee is contingent on purchasing the product at par and is valid *only at maturity, not before. The guarantee level remains constant throughout the* product's life.

12.5.1 Impact of Volatility and Interest Rate Variations

- **Implied Volatility Fluctuations:**
  - An increase in implied volatility (e.g., from 23% to 33%) can enhance the value of the call option component, as shown in the corresponding graph.
  - A decrease (e.g., from 23% to 13%) typically lowers the call option's value due to reduced potential for high returns.

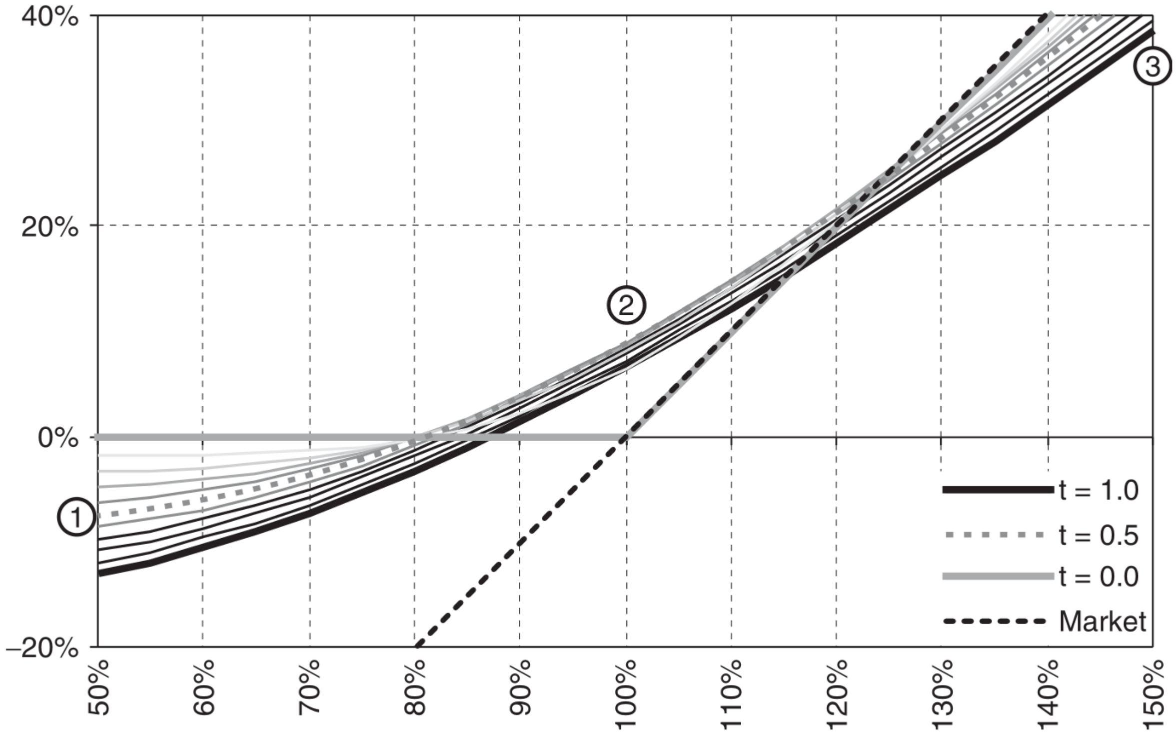


Figure 12.9: Implied volatility increase (from 23% to 33%)

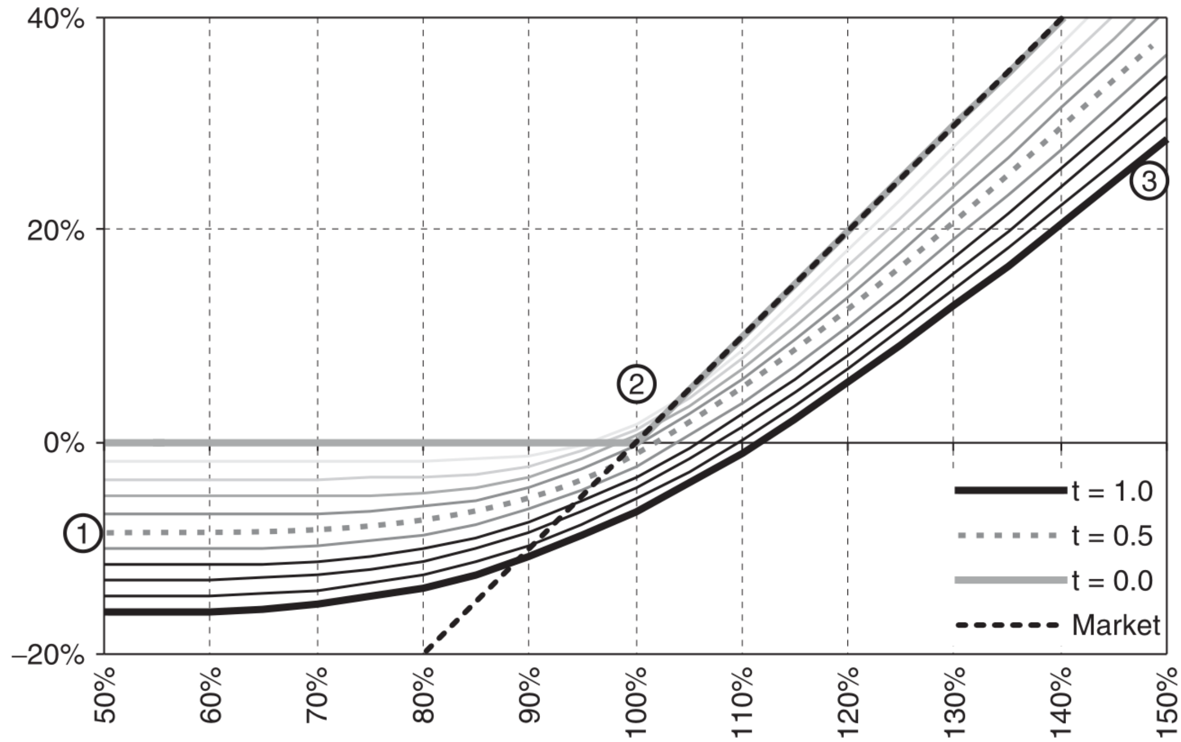


Figure 12.10: Implied volatility decrease (from 23% to 13%)

Volatility is more influential when the product is at-the-money and can mitigate some losses through increased option premiums during downturns in the asset's price.

- **Interest Rate Impact:**
  - Rising interest rates lead to lower prices for the zero-coupon bond component, influencing the overall valuation negatively, especially if the underlying asset's price falls simultaneously.
  - Conversely, falling interest rates increase the bond's value, cushioning any adverse effects from a drop in the underlying asset's price.

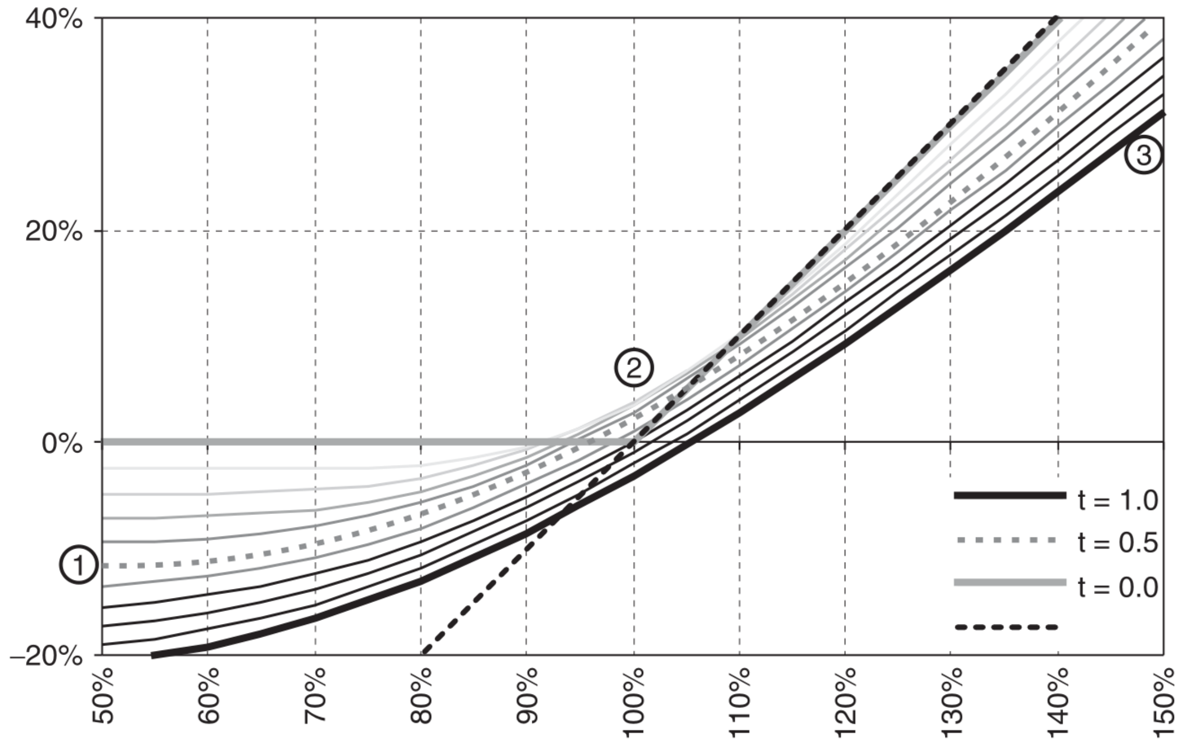


Figure 12.11: Interest rate increase (from 4.5% to 6.5%)

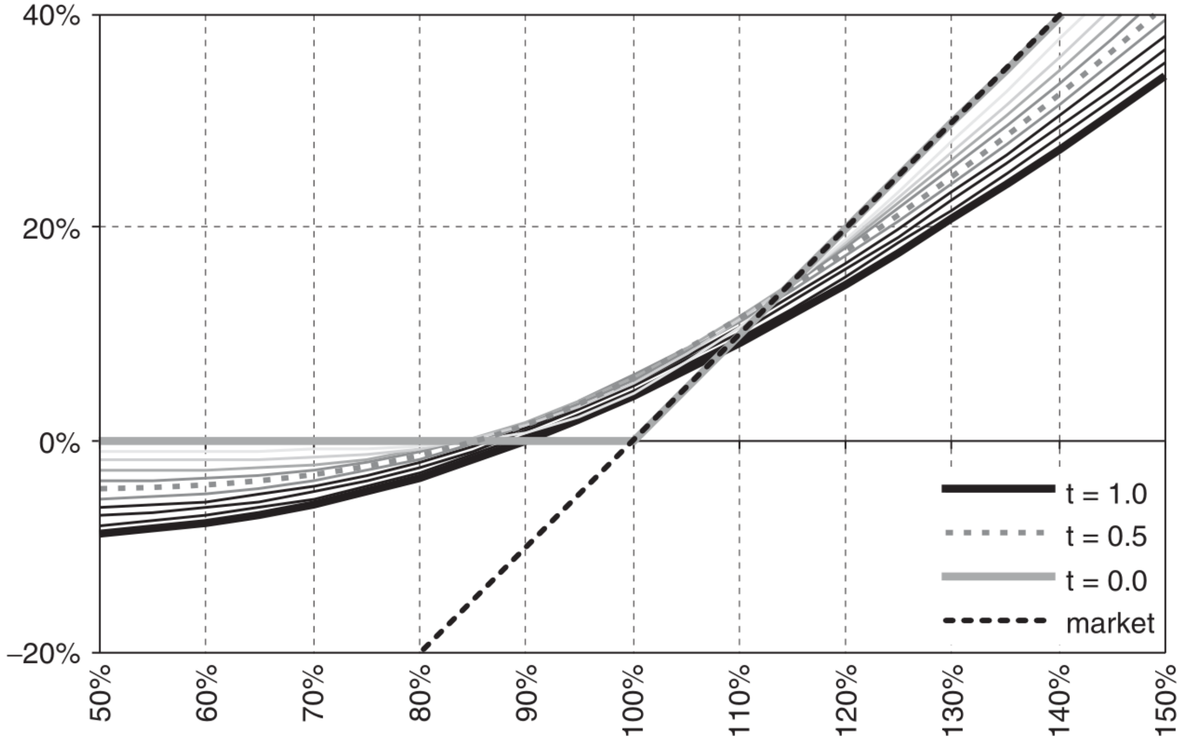


Figure 12.12: Interest rate decrease (from 4.5% to 2.5%)

12.5.2 Conclusive Insights

- Shifts in implied volatility influence the capital guaranteed product most when ***at-the-money***.
- An ***increase in volatility*** tends to ***increase the value*** of the product.
- A positive shift in ***interest rates lowers the value*** of the product most when the embedded call is ***out-of-the-money***.
- When the call is deep in-the-money, an interest rate shift has less impact.
- Everything else held equal, ***the passing of time*** (without movement on the spot) ***is positive for the value*** of the product over time, since the time value lost on the call is more than offset by the gain in the zero bond value.

12.6 Participation Investment Products

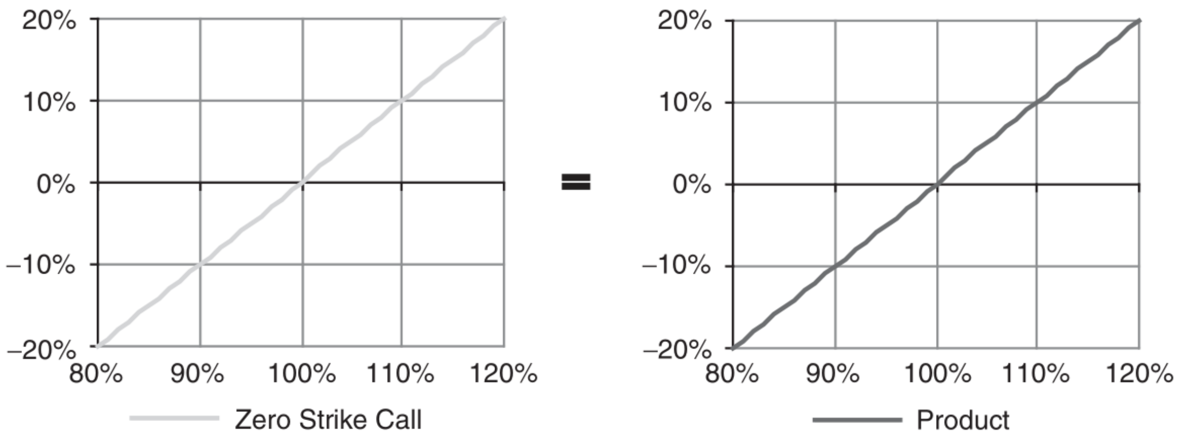
> **Definition**
>
> Participation products are investment vehicles that link returns directly to the performance of their underlying assets, sometimes featuring conditional downside protection or a leveraged upside.

Key Characteristics:

- **Risk Profile:** Generally higher risk compared to capital protection and yield enhancement products due to the absence of capital guarantees.
- **Underlying Assets:** Typically stocks or stock indices, but can also include commodities, real estate, and more exotic assets.
- **Liquidity and Efficiency:** Often very liquid, these products compete directly with ETFs in providing exposure to specific markets, themes, or regions.

12.6.1 Tracker Certificate

- **Function:** Mirrors the performance of one or more underlying assets. Commonly tracks excess returns (excluding dividends/yields).
- **Structure:** Comprised of a zero-strike call option, which values the asset minus any discounted dividends or yields.



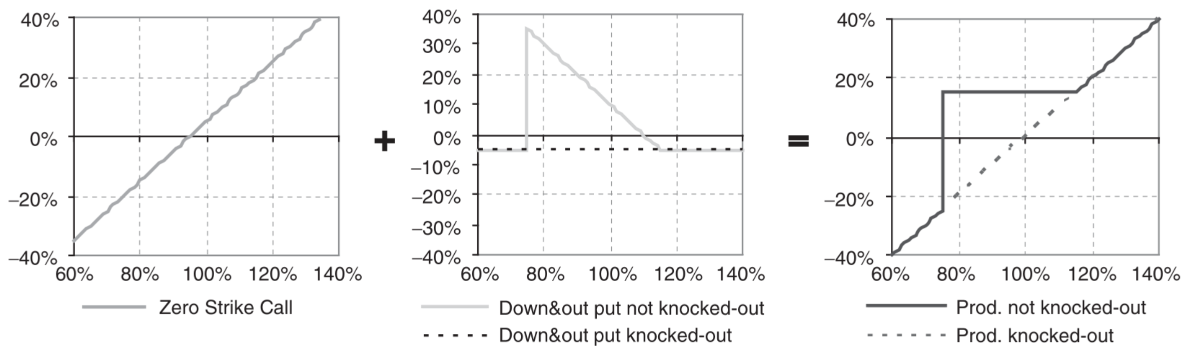
Advantages:

- **Access to Difficult Markets:** Allows investment in markets or assets that are otherwise inaccessible via traditional instruments, such as private equity or certain commodities.

- **Tax Efficiency:** May offer favorable tax treatments compared to direct investments in the underlying assets.

12.6.2 Bonus Certificate

- **Function:** Combines the features of a tracker certificate with conditional downside protection.
- **Structure:** Includes a zero-strike call option and a long down-and-out put option.

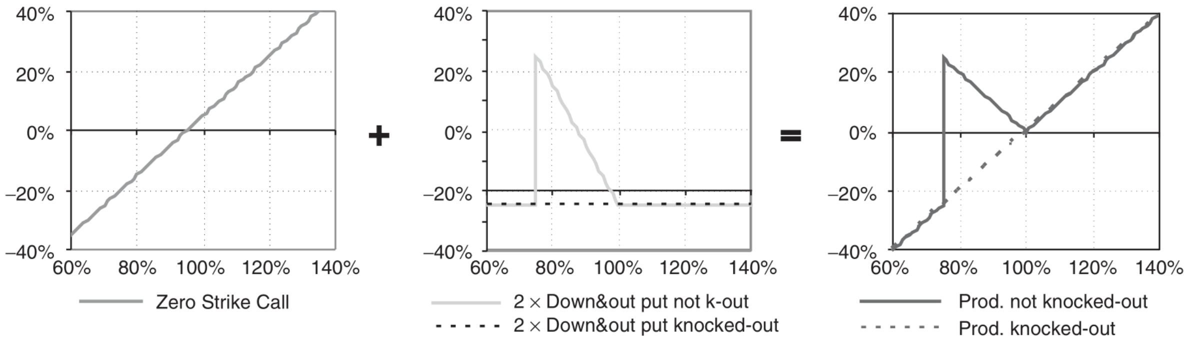


Key Parameters:

- **Participation Level:** Degree to which the investor gains from positive performance of the underlying.
- **Bonus Level:** Additional return offered if the underlying performs above a certain threshold without breaching a downside barrier.
- **Barrier Level:** The price level below which the downside protection is activated.
- **Maturity:** Typically short, recommended no longer than two to three years.

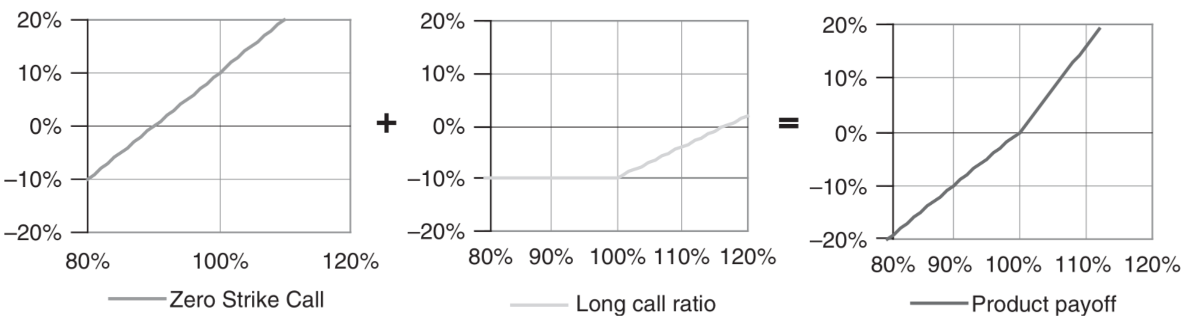
12.6.3 Twin-Win Certificate

- **Function:** Provides positive participation in both the upside and downside movements of the underlying asset.
- **Structure:** Consists of a long zero-strike call and double down-and-out put options.



12.6.4 Outperformance Certificate

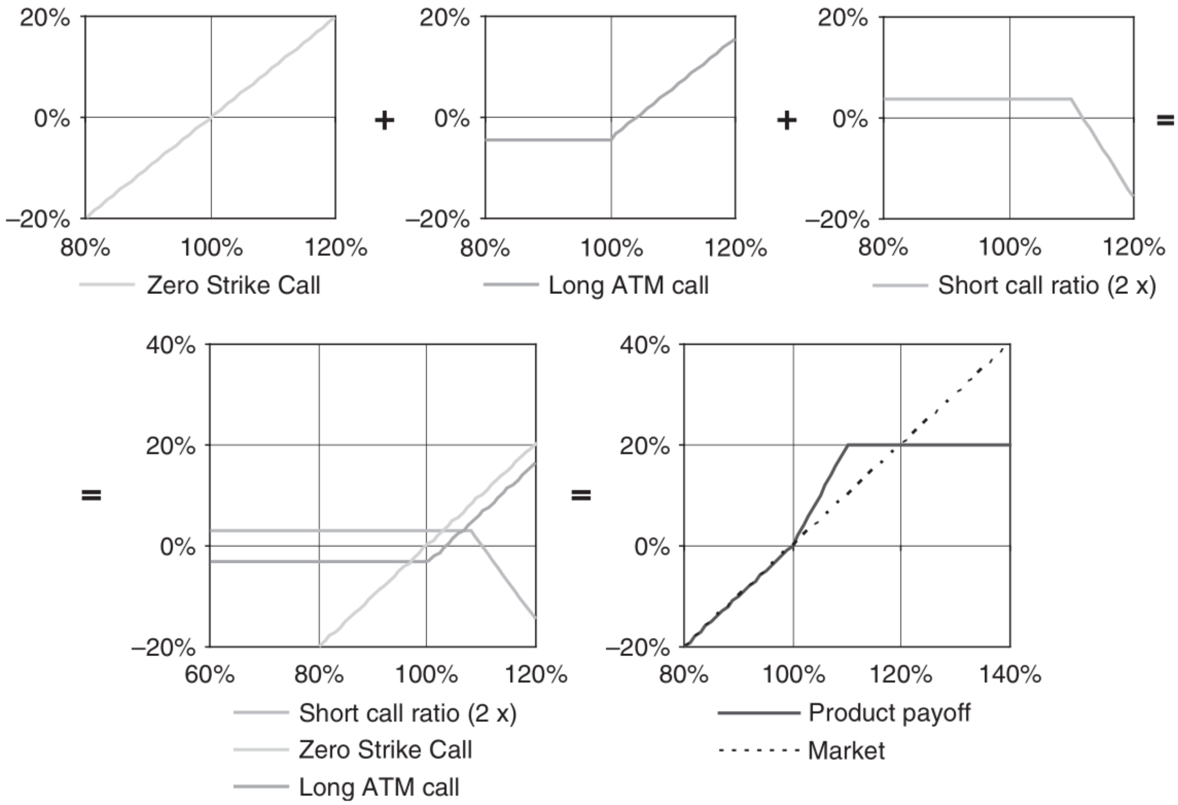
- **Function:** Designed for aggressive investment strategies, providing enhanced returns if the underlying outperforms expected dividends.
- **Structure:** Combines a zero-strike call with multiple long at-the-money calls, funded by dividends.



- **Volatility and Dividends:** Optimal conditions include low volatility (for cheaper options) and high dividend yields.

12.6.5 Capped Outperformance Certificate

- **Function:** Provides leveraged exposure to the underlying asset up to a capped level.
- **Structure:** A zero-strike call combined with a long at-the-money call and two short calls at higher strikes.



- **Scenario Planning:** Best utilized in a moderately bullish scenario with falling volatility.
- **Timing and Maturity:** Critical due to the short duration of the product, typically 3-9 months.

12.7 Behavior of Participation Products

Participation products such as bonus, turbo, airbag, and outperformance certificates are heavily influenced by factors like implied volatility, dividend yields, and interest rates. These factors shape the product's performance and its strategic suitability for different market conditions.

12.7.1 Bonus Certificate - Case Study

Factor (Increase)	Impact on Product's Price	Effect Level
Spot Price	Positive	Maximum
Implied Volatility	Variable	High
Implied Correlation	Positive	Medium-Low

Factor (Increase)	Impact on Product's Price	Effect Level
Interest Rates	Negative	Low
Dividends	Negative	Medium

- **Initial Sensitivity:** At issuance, the delta of a bonus certificate is approximately 1, meaning its price moves almost one-for-one with the underlying asset. However, this sensitivity decreases if the spot price approaches the barrier.
- **Price Stability:** Bonus certificates tend to underperform during market downturns due to the drop in market value, despite their protective features.

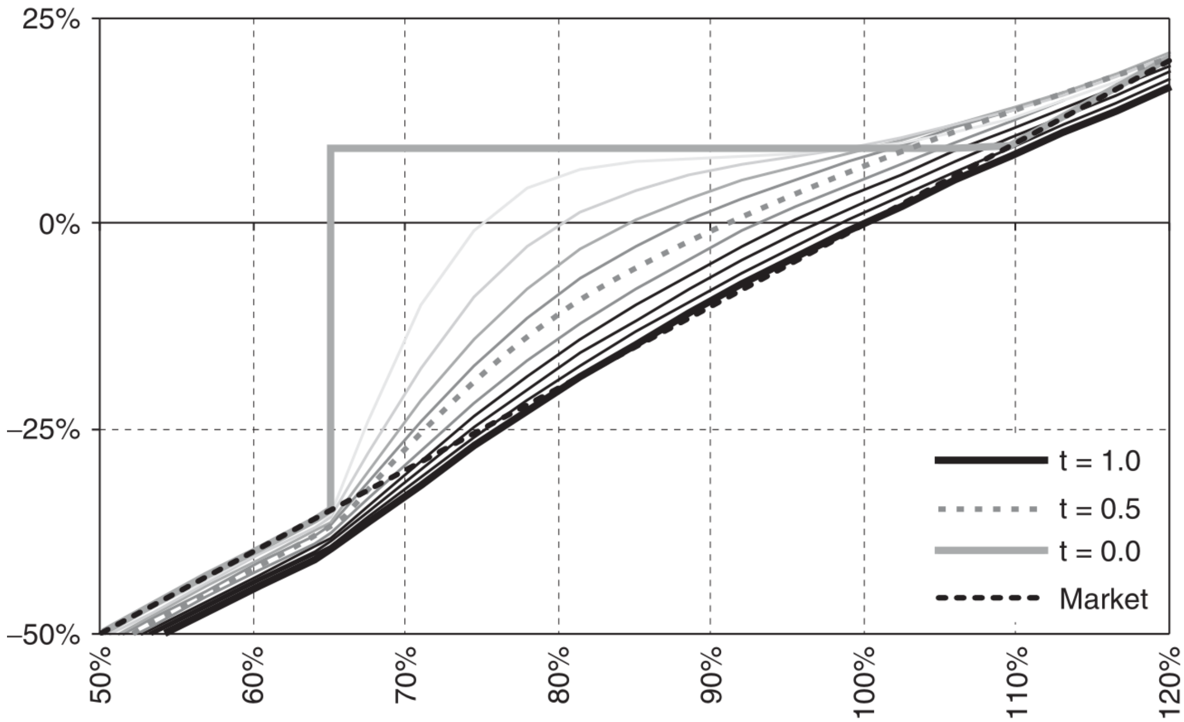


Figure 12.13: Price as a function of maturity

- **Maturity:** Shorter maturities are preferable to reduce exposure to prolonged market volatility.
- **Barrier Level:** The barrier should be set considering the worst-case market scenario to ensure effective downside protection.
- **Leverage and Sensitivity:** As the spot price nears the barrier, the certificate's delta becomes highly volatile, which can lead to significant price swings.

12.7.1.1 Impact of Implied Volatility

- **Initial Conditions:** High implied volatility at issuance allows for better pricing of protective options.
- **Post-Issuance Volatility:** A decrease in implied volatility post-issuance generally benefits the mark-to-market value of the certificate.

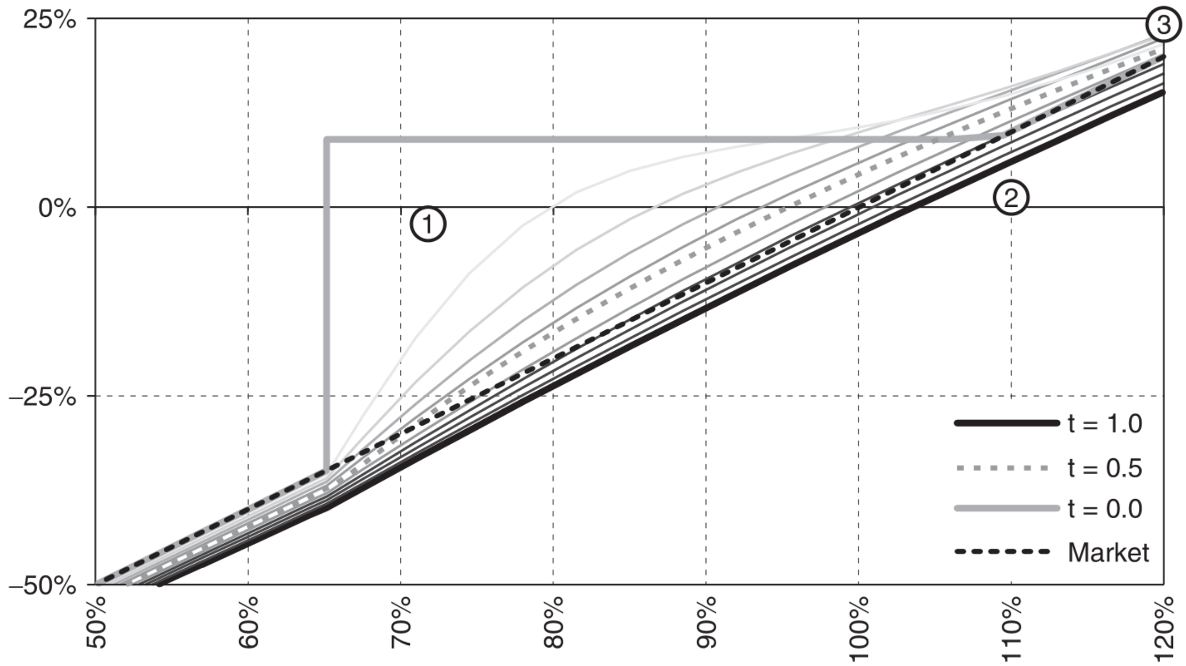


Figure 12.14: Increase in implied volatility (from 23% to 33%)

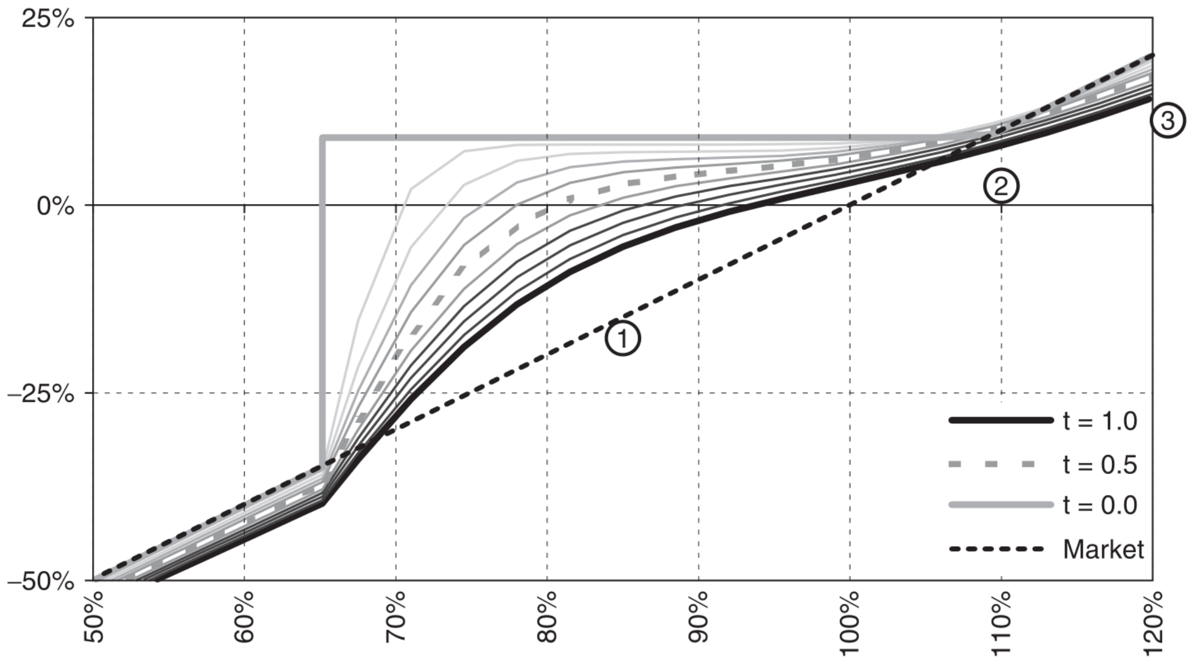


Figure 12.15: Decrease in implied volatility (from 23% to 13%)

12.8 Practice Questions and Problems

1. What is the value of the capital guarantee certificate at maturity if the value of the underlying asset ends at 120, the strike price of the certificate is 100, level of guarantee 100%, and participation 80%?

> **Solution**
>
> 116

2. What is the value of the capital guarantee certificate at maturity if the value of the underlying asset ends at 90, the strike price of the certificate is 100, level of guarantee 100%, and participation 80%?

> **Solution**
>
> 100

3. What is the value of the capital guarantee certificate at maturity if the value of the underlying asset ends at 63, the strike price of the certificate is 40, the cap is 50, level of guarantee 100%, and participation 130%?

Solution

53

4. What is the value of the capital guarantee certificate with knock-out at maturity if the value of the underlying asset ends at 50, the strike price of the certificate is 55, the knock-out barrier is 67, level of guarantee 100%, and participation 110%? The highest value of the underlying asset during maturity was 70.

Solution

55

5. What is the value of the capital guarantee certificate with knock-out at maturity if the value of the underlying asset ends at 63, the strike price of the certificate is 55, the knock-out barrier is 67, level of guarantee 100%, and participation 110%? The highest value of the underlying asset during maturity was 65.

Solution

63.8

6. What is the value of the outperformance certificate at maturity if the value of the underlying asset ends at 56, the strike price of the certificate is 40, and the participation is 170%?

Solution

67.2

7. What is the value of the outperformance certificate at maturity if the value of the underlying asset ends at 77, the strike price of the certificate is 40, the cap is 70, and the participation is 130%?

Solution

79

8. What is the value of the bonus certificate at maturity if the value of the underlying asset ends at 35, the strike price of the certificate is 40, the knock-out barrier is 30? The lowest value of the underlying asset during maturity was 35.

Solution

40

9. What is the value of the bonus certificate at maturity if the value of the underlying asset ends at 35, the strike price of the certificate is 40, the knock-out barrier is 30? The lowest value of the underlying asset during maturity was 25.

Solution

35

10. What is the value of the twin-win certificate at maturity if the value of the underlying asset ends at 44, the strike price of the certificate is 50, the knock-out barrier is 40? The lowest value of the underlying asset during maturity was 35.

Solution

44

11. What is the value of the twin-win certificate at maturity if the value of the underlying asset ends at 44, the strike price of the certificate is 50, the knock-out barrier is 40? The lowest value of the underlying asset during maturity was 41.

Solution

56

13 Structured Products II

References

- BLÜMKE, Andreas. How to invest in structured products: a guide for investors and investment advisors. Chichester: Wiley, 2009. xvi, 374. ISBN 9780470746790.

Learning Outcomes:

- Identify and explain the mechanisms and purposes of different yield enhancement structures such as reverse convertibles and discount certificates.
- Analyze strategic variations in yield enhancement products, including ways to enhance the coupon and general investment strategies.
- Evaluate the behavioral dynamics of yield enhancement products under varying market conditions.
- Utilize case studies, such as knock-in reverse convertibles, to understand practical applications and implications of product behaviors.
- Describe common special features in structured products, including autocall and callable options, and their impact on product performance and investor options.

13.1 Yield Enhancement Investment Products

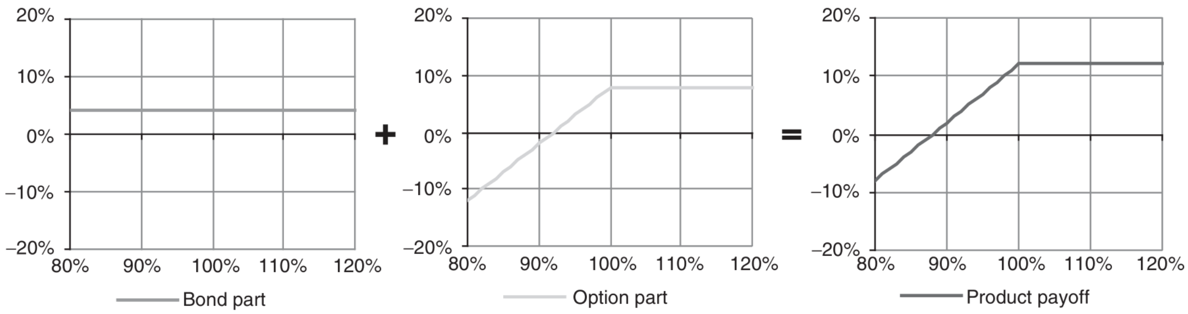
Definition

Yield enhancement products are financial constructs with capped upside potential and without capital guarantee, aiming to generate high returns relative to bond yields. The risk level may approach that of the underlying assets in unfavorable market scenarios.

- **Category Significance:** This is the largest category of structured products in terms of both the variety of structures available and the volume of investments.
- **Capped Upside:** Historically, all yield enhancement products feature a capped upside, contrasting with participation products that offer unlimited upside potential.
- **Popular Types:** The most prevalent forms are reverse convertibles and discount certificates, which share similar payout profiles but differ in construction.

13.1.1 Reverse Convertibles

- **Components:**
  1. **Zero-Coupon Bond:** Purchased to cover the product's lifetime, providing the base return.
  2. **Short Plain Vanilla Put Option:** Sold on the underlying risky asset to generate additional premium.

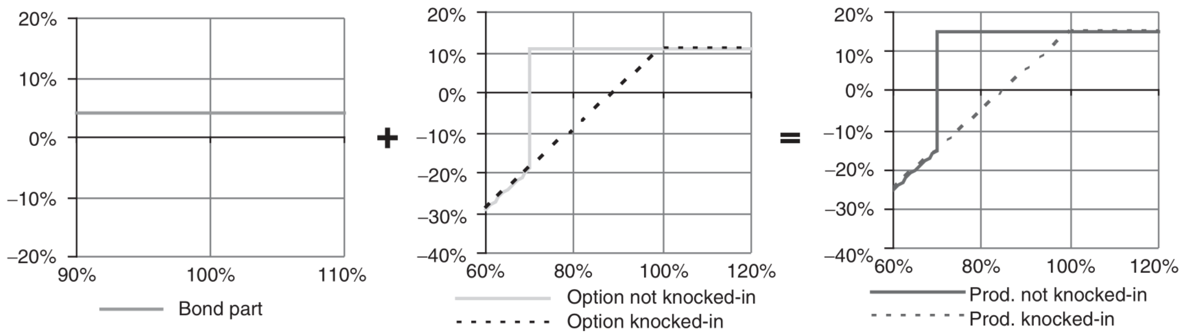


> **ℹ Example**
>
> The zero-coupon bond yields 4%, and the short put option provides a premium of 8%. The combined coupon is 12%, allowing the underlying asset to decline by up to 12% before the investor experiences a loss.

- **Risk Considerations:** The maximum potential loss is limited to the coupon received. However, the risk of loss increases if the underlying asset's price falls significantly beyond the cushion provided by the coupon.

13.1.2 Worst-of Barrier Reverse Convertible

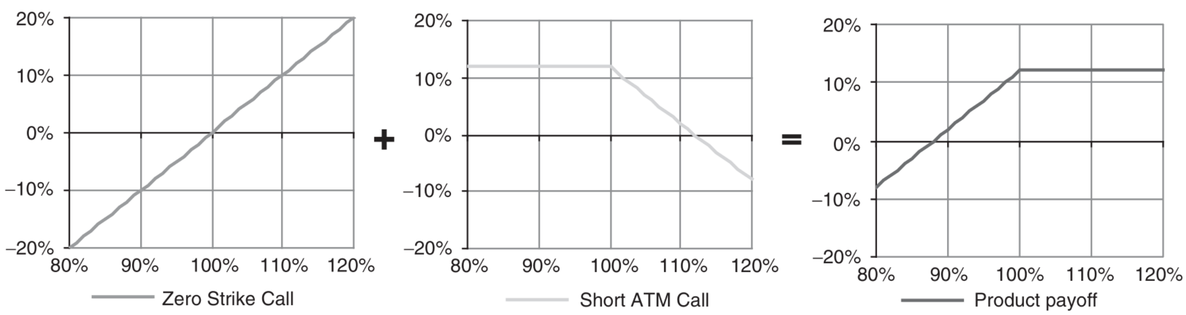
- **Enhanced Features:**
  1. **Multiple Underlyings:** The payoff depends on the performance of the worst-performing asset.
  2. **Knock-in Barrier:** Added to the short put option to provide conditional capital protection unless the barrier is breached.



- **Behavioral Dynamics:** The presence of a barrier can lead to a sharp price adjustment if the underlying reaches this level, activating the put option.

13.1.3 Discount Certificates

- **Structure:**
  1. **Long Zero-Strike Call Option:** Replicates the performance of the underlying asset.
  2. **Short Call Option:** Set at- or out-of-the-money to provide upfront income by reducing the purchase price.



- **Economic Rationale:** The premium from the short call lowers the effective purchase price of the zero-strike call, providing an immediate financial benefit, which contrasts with the deferred coupon payment structure of reverse convertibles.
- **Tax Efficiency:** In some jurisdictions, the structure of discount certificates can offer tax advantages over other investment products.

13.1.4 Strategic Variations and Recommendations

- **Barrier-Linked Coupons:** Linking the coupon payment to the breaching of one or more barriers can significantly increase the potential return but also adds to the risk.

- **Individualized Barriers:** In worst-of barrier products, setting specific barrier levels for each underlying can tailor the risk to more accurately reflect the volatility and performance expectations of each asset.
- **Barrier Perception:** Investors should not underestimate the risk associated with barriers, as these are high-risk products without capital guarantees.
- **Product Complexity:** Avoid investing in products based on too many underlying assets to manage risk more effectively. Preferably, limit the number of underlyings to two.
- **Barrier Style:** Opt for European-style barriers when market volatility is high and the medium-term market trajectory is uncertain, as these barriers typically offer a lower coupon but reduce the risk of early activation.

13.2 Behavior of Yield Enhancement Products

13.2.1 Knock-In Reverse Convertibles - Case Study

Knock-in reverse convertibles are complex financial instruments designed to enhance yields by incorporating a conditional protection feature, which activates under specific market conditions. Here's a detailed analysis of the behavior of such products, using a specific example for illustration.

Characteristic	Details
Underlying Risky Asset	Eurostoxx50 Index
Maturity	1 year
Implied Volatility	23%
Asset's Dividend Yield (p.a.)	4.0%
Interest Rate (4-year swap rate, p.a.)	4.5%
Barrier Level (% of spot)	75%
Coupon	10.4%

Factor (Increase)	Impact on Product's Price	Impact Level
Spot Price	Positive	Maximum
Implied Volatility	Negative	High
Implied Correlation	Positive	Medium-Low
Interest Rates	Negative	Low
Dividends	Negative	Low

13.2.2 Variations in Spot Price

- **At Issue Date:** The product exhibits bond-like characteristics when the spot price of the underlying asset increases, yet behaves akin to the asset itself with a decline in price, increasing the delta.
- **Near Maturity:** A significant gap develops; the delta can exceed 1, indicating increased sensitivity to spot price movements.
- **Delta Analysis:** The delta of approximately 34% suggests a 34% probability that the underlying's price decline will activate the knock-in feature (e.g., a 25% or greater drop in the Eurostoxx50).
- **Investor Perception:** While less risky than direct exposure to the underlying asset, the delta range (30%-40%) illustrates that these products are considerably more volatile than bonds.

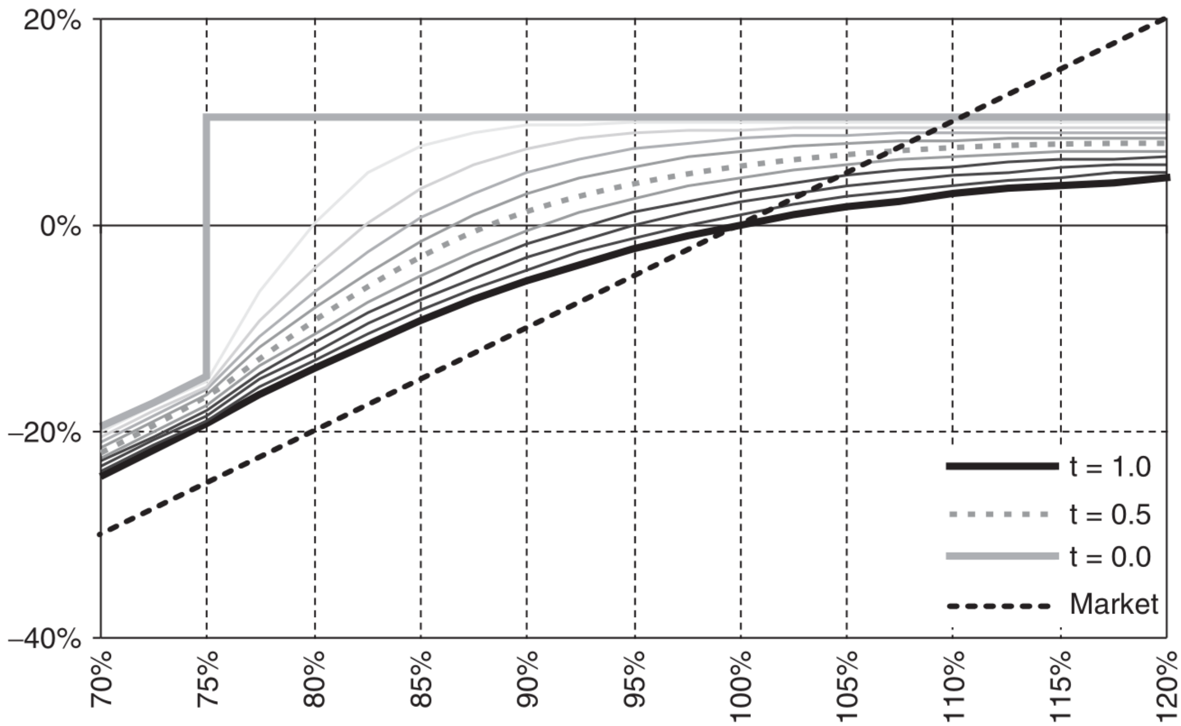


Figure 13.1: Reverse convertible as a function of spot and time to maturity

13.2.3 Impact of Implied Volatility Changes

- **Price Sensitivity:** Implied volatility significantly affects the product's price, particularly near the barrier. A lower volatility reduces price sensitivity to spot variations except near this critical threshold.

- **Investment Timing:** Higher initial implied volatility is advantageous as it allows for purchasing more protective options at a lower cost. Conversely, a drop in volatility after issuance benefits the mark-to-market valuation.
- **Product Lifespan:** Don't default to one-year products; sometimes shorter or longer maturities may align better with market conditions or portfolio strategies.

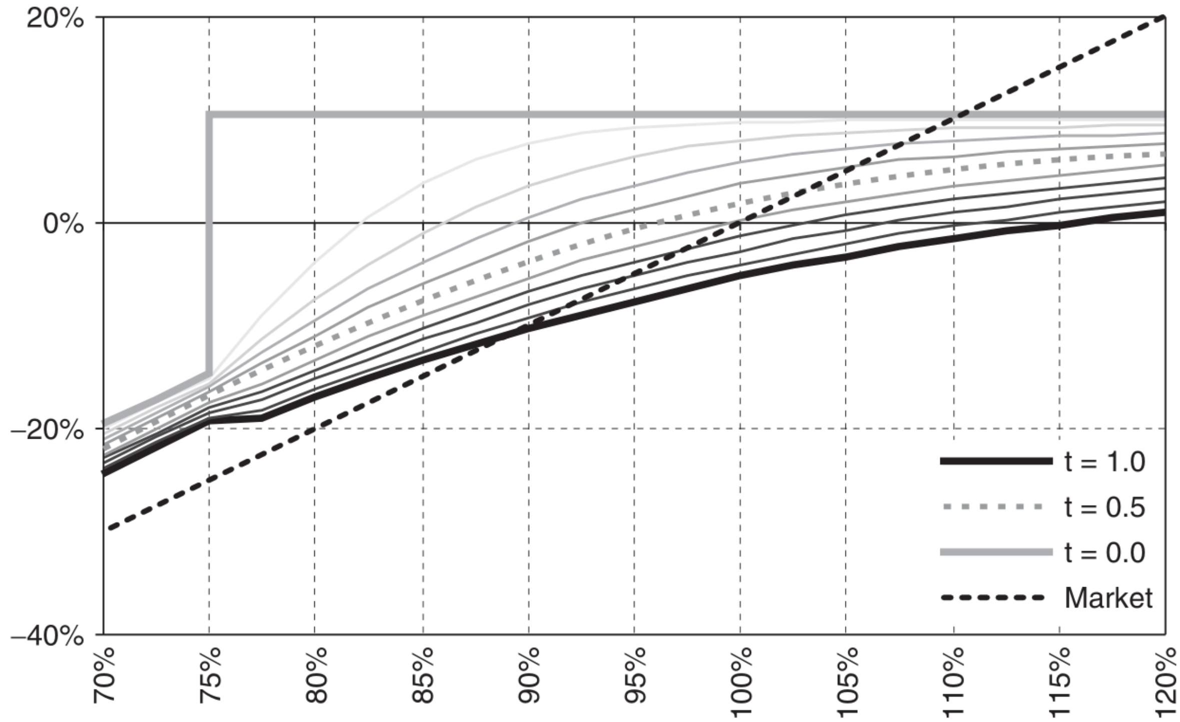


Figure 13.2: Increase in implied volatility (from 23% to 33%)

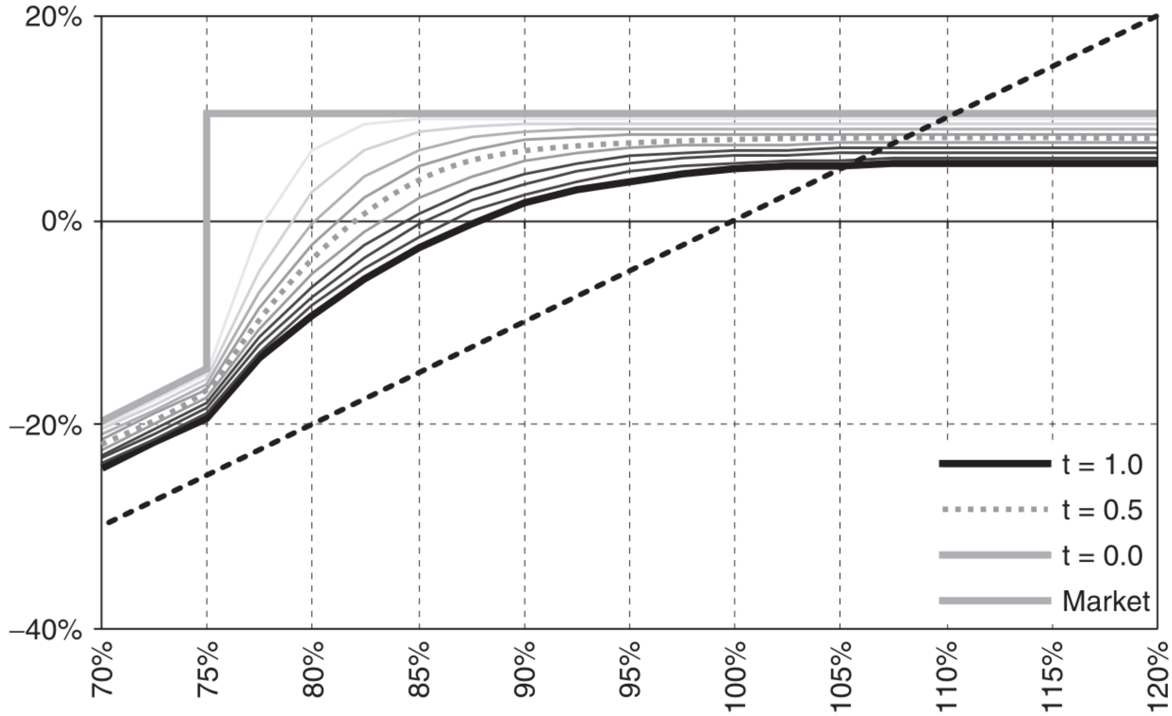


Figure 13.3: Decrease in implied volatility (from 23% to 13%)

13.2.4 Understanding Implied Correlation

Definition

Implied correlation reflects the expected linear relationship between two or more assets, as inferred from market prices.

- **Significance in Multi-Asset Products:** Essential for assessing the risk and potential return in products with multiple underlyings, such as 'worst-of' options, where a higher correlation can lead to enhanced coupon rates.
- **Asset Addition Risks:** Introducing additional underlyings for a modest increase in coupon (1%-2%) is generally not justified by the associated increase in risk, particularly for volatile stocks.
- **Risk Assessment:** Carefully evaluate the added value versus risk when including multiple assets. The incremental risk often outweighs the potential return enhancement, especially when the volatility of additional assets is high.

13.3 Leverage Products

Leverage products are financial instruments that provide a cost-effective way to gain exposure to various assets (shares, indices, commodities, currencies) with a smaller investment than purchasing the assets directly. These products amplify potential returns and increase the risk of loss, making them suitable for speculative purposes or for hedging against market movements.

13.3.1 Warrants

> Definition
>
> Warrants are financial instruments granting the holder the right, but not the obligation, to buy (Call Warrants) or sell (Put Warrants) an underlying asset at a predetermined price (strike price) on or before a specific date.

- **Characteristics:**
  - Warrants are similar to options but usually have longer maturities and cover a broader range of underlying assets.
  - Issued by financial institutions, warrants are bespoke products with terms set by the issuer.
  - Investors face the credit risk of the issuer since warrants are not backed by the assets themselves.
  - Warrants are highly liquid and can be used to leverage positions, hedge risks, or exploit arbitrage opportunities.

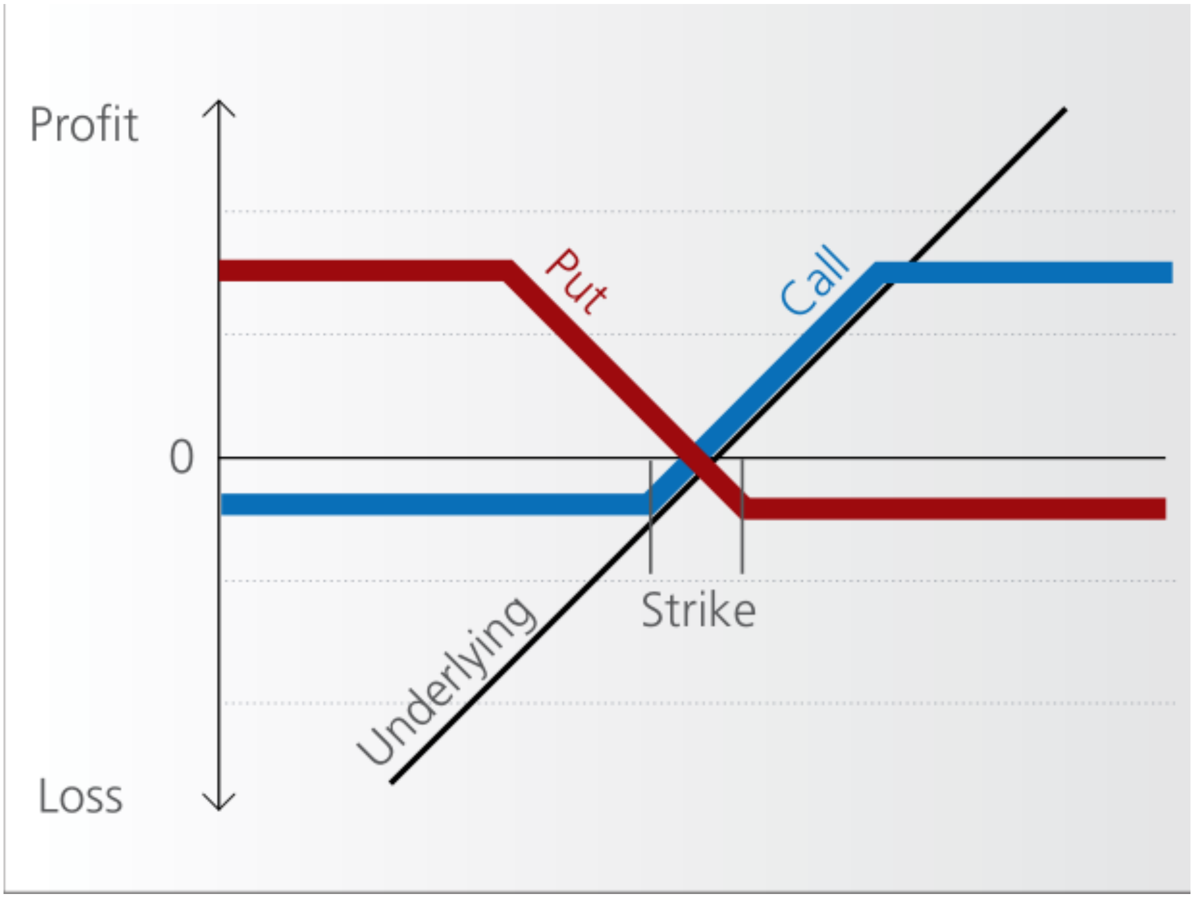
13.3.1.1 Types of Warrants

- **Traditional Warrants:** Issued by companies, often alongside bonds ("warrant-linked bonds") to raise capital at a lower interest cost. These warrants can be detached and sold separately.
- **Covered Warrants:** Issued by financial institutions and backed by assets the institution holds or can acquire. They can cover a wide range of assets beyond equities.
- **Naked Warrants:** Issued without backing assets; issuers hedge their exposure through other means.

13.3.1.2 Spread Warrants

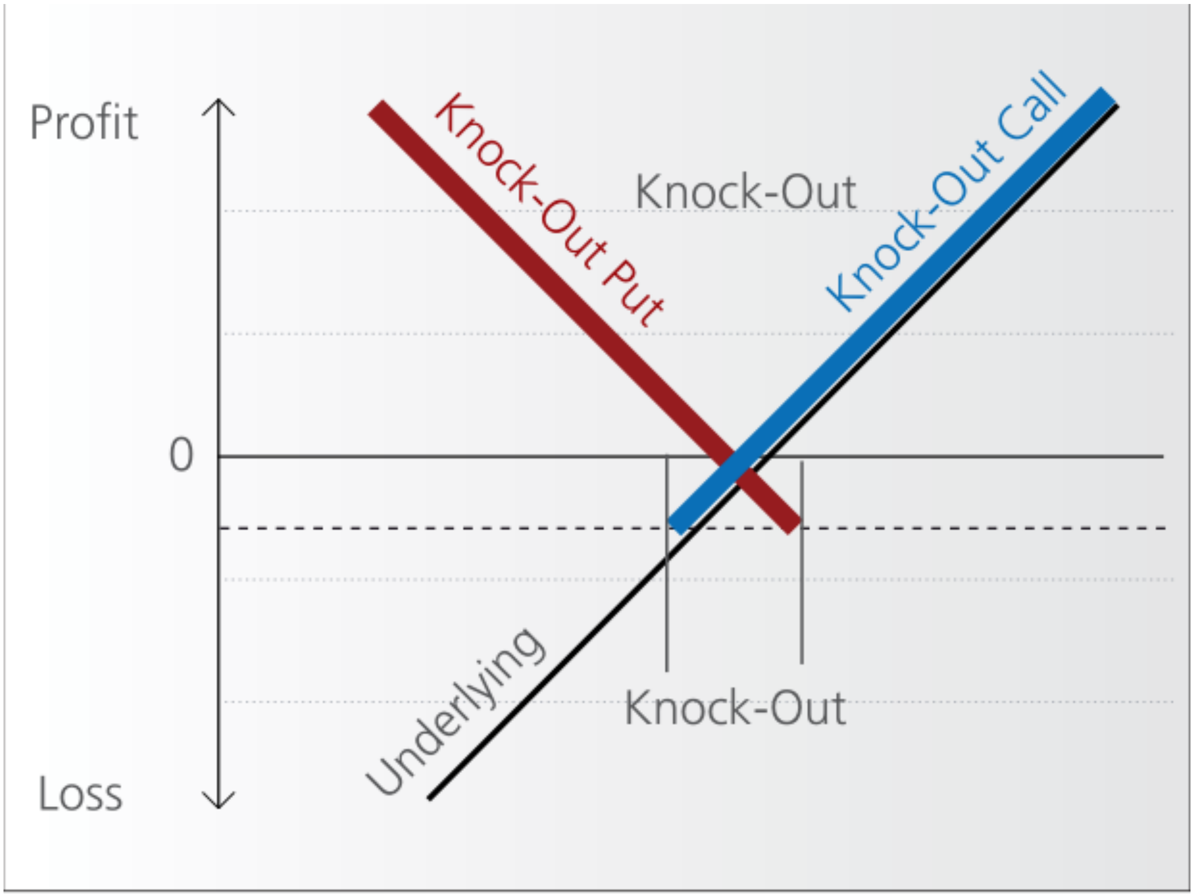
- **Purpose:** Designed to benefit from rising (bull warrants) or falling (bear warrants) markets.

- **Risks and Rewards:** While the initial investment is small and can yield significant leveraged gains, the maximum loss and gain are capped, and the value deteriorates as expiration approaches.



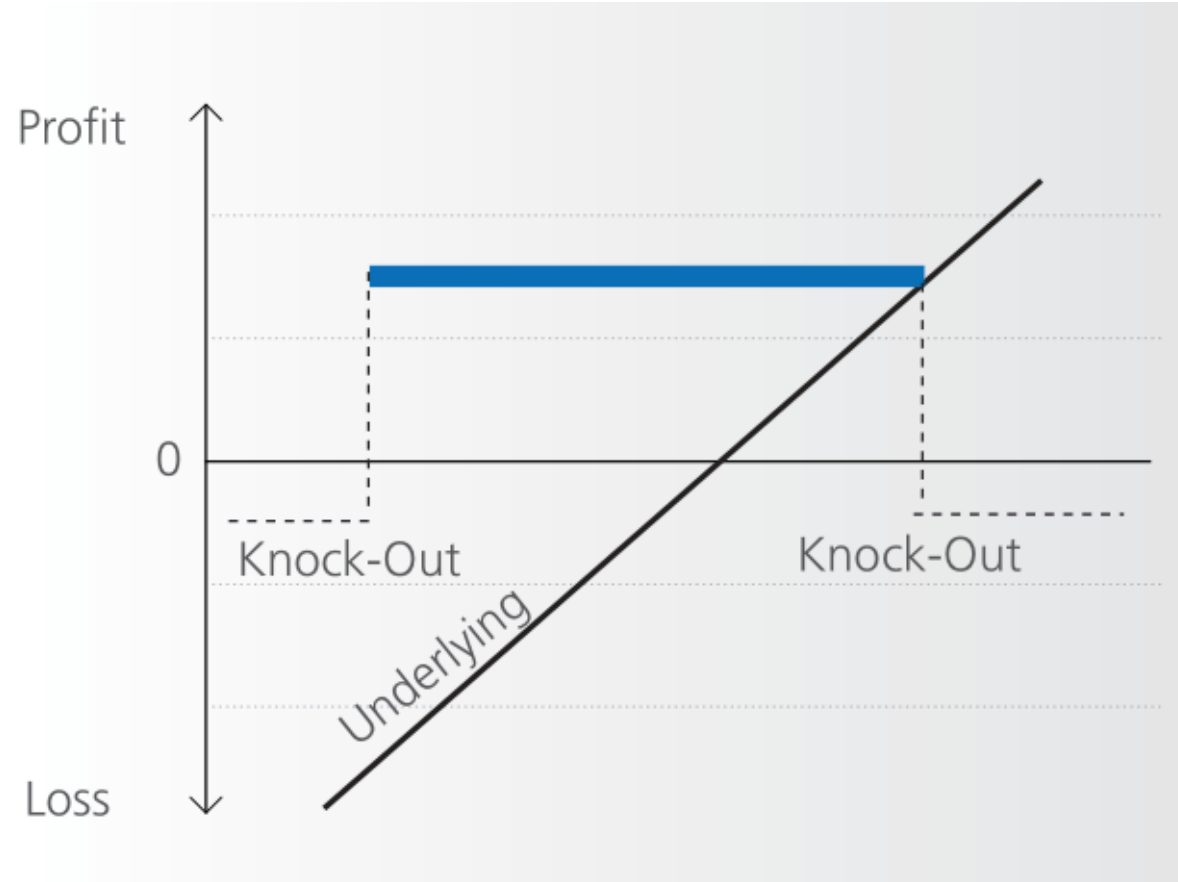
13.3.1.3 Knock-Out Warrants

- **Feature:** These products have an embedded barrier that, if reached, causes the warrant to expire worthless immediately.
- **Market Suitability:** Ideal for scenarios with clear directional market expectations and a significant buffer between the current price and the knock-out level.
- **Risk Profile:** Higher leverage compared to traditional warrants, with little to no time value, making them highly sensitive to market movements.



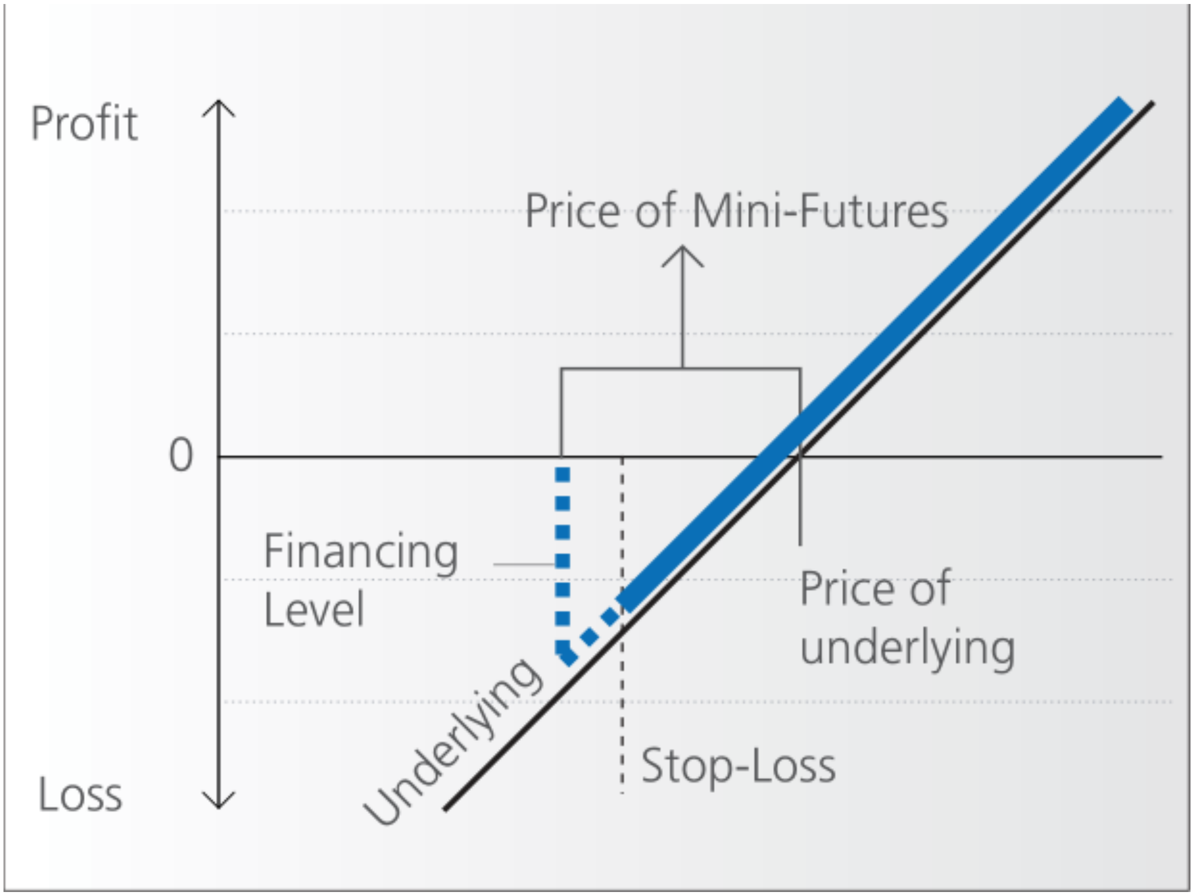
13.3.1.4 Double Knock-Out Warrants

- **Design:** These warrants have two barriers (upper and lower), offering a balanced approach to leveraging both upward and downward market movements.



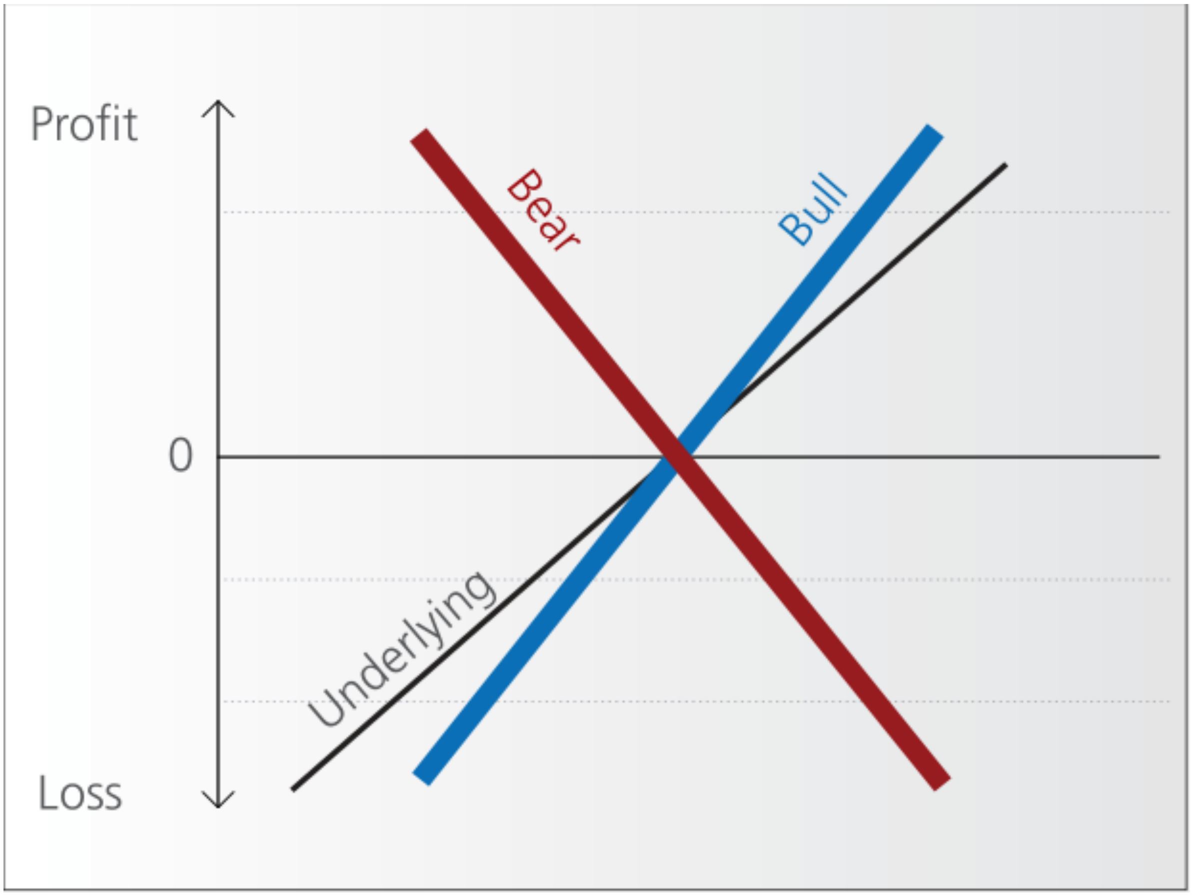
13.3.2 Mini-Futures

- **Comparison with Warrants:** Similar to knock-out products but open-ended, with no fixed expiration date unless the knock-out threshold is breached.
- **Investment Strategy:** Suitable for short-term speculative bets or hedging, with potential for residual value redemption after a stop-loss event.



13.3.3 Constant Leverage Certificates

- **Function:** These certificates provide fixed daily leverage on the price movements of an underlying asset, without any knock-out barriers.
- **Benefits:** They offer continuous exposure to price movements, unaffected by the underlying's volatility, and are not limited by time decay.



13.3.4 Additional Links

- https://www.investopedia.com/terms/w/warrant.asp
- https://en.wikipedia.org/wiki/Warrant_(finance)
- https://en.wikipedia.org/wiki/Covered_warrant
- https://www.six-structured-products.com/en/know-how/product-know-how/leverage-products-without-knockout/warrants
- https://www.six-structured-products.com/en/know-how/product-know-how/leverage-product-with-knock-out/knock-out-warrants

13.4 Common Special Features of Structured Products

Structured products often incorporate complex features to enhance their appeal to investors. These features, such as autocalls, callable options, and choices between physical or cash settlements, add flexibility and potentially higher returns under certain conditions, but also introduce specific risks and complexities.

13.4.1 Autocall and Callable Options

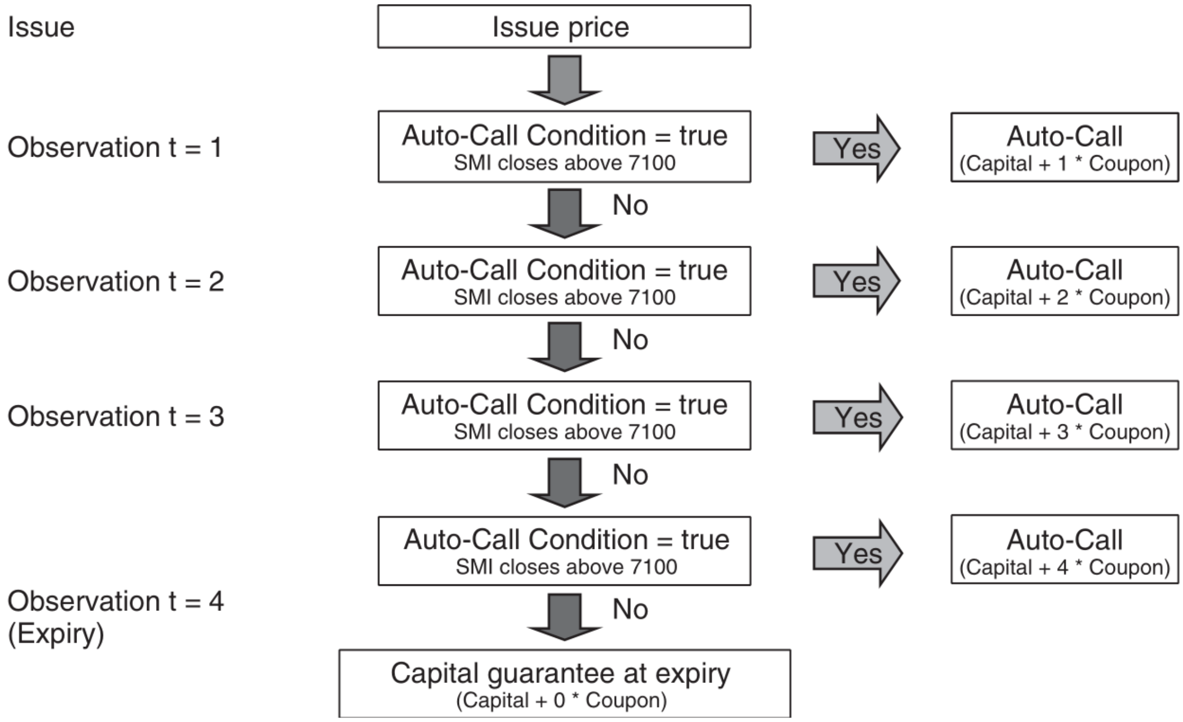
Definition

An autocall is a feature where the issuer is obligated to redeem the product when a predefined event occurs, which can be set to trigger at any time or on a specific date, at a predetermined price.

- **Callable Options:** Allow the issuer to decide whether to redeem the product on predetermined dates, providing more control over the product's lifecycle.

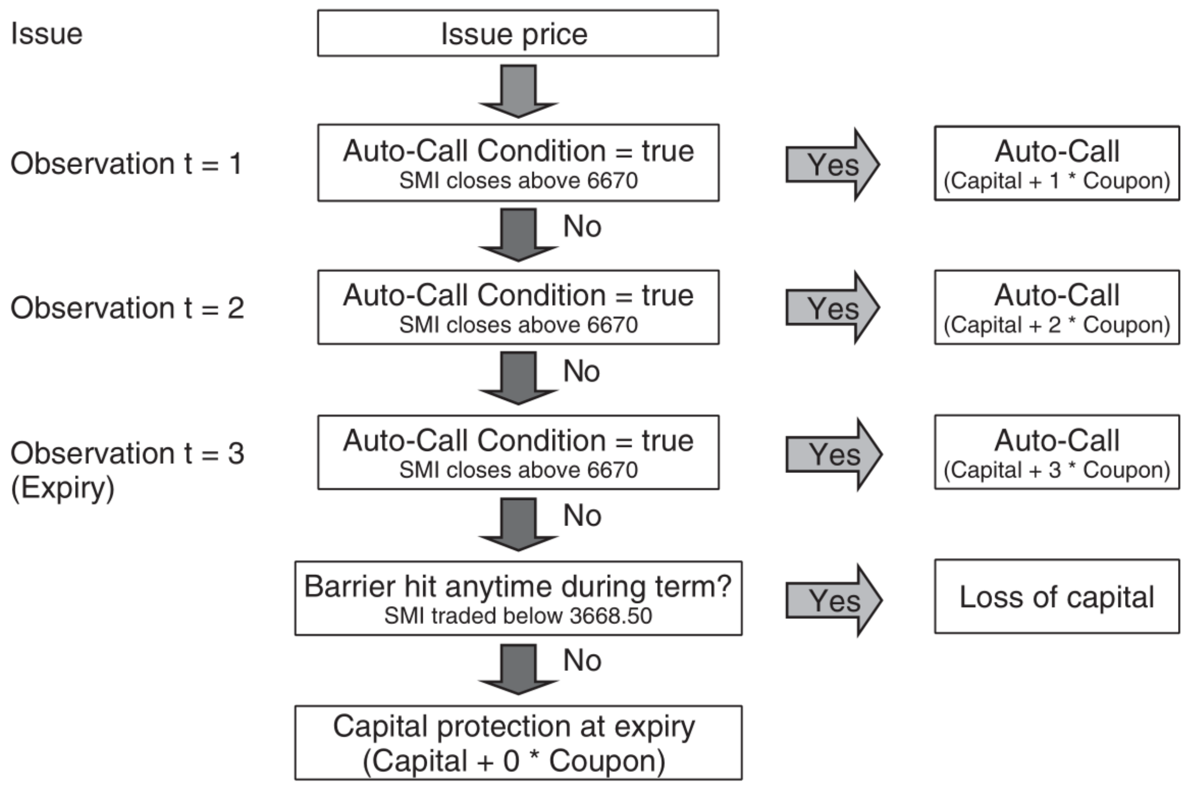
13.4.1.1 Autocallable Notes

- **Structure:** Typically includes full capital protection with the issuer redeeming the product if the underlying asset closes above a certain threshold (e.g., 100%) at predefined observation dates.
- **Returns:** Investors receive their capital plus a coupon, which is a percentage multiplied by the number of years the product has run.



13.4.1.2 Express Certificate

- **Features:** Combines an autocallable structure with conditional capital protection, usually involving a down-and-in barrier option.
- **Volatility Sensitivity:** High sensitivity to volatility due to the barrier; investors are effectively short volatility, losing out when it rises.



13.4.1.3 Autocallable Worst-Of Barrier Reverse Convertibles

- **Mechanism:** Functions like standard barrier reverse convertibles but includes an auto-call feature that triggers early redemption if all underlying assets exceed a certain level (e.g., 105% of initial spot).
- **Correlation Impact:** The correlation between the underlyings is crucial, as it affects the likelihood of all assets meeting the trigger condition simultaneously.

13.4.2 Settlement Types in Equity-Based Products

Structured products based on equities, such as reverse convertibles or bonus certificates, can offer different types of settlements:

1. **Physical Delivery:** Investors receive the actual shares at the redemption date, which might occur weeks after the expiry date of the embedded options.
2. **Cash Settlement:** Provides investors with a cash amount equivalent to the value of the shares at expiry.

- **Investor Considerations:** Cash settlements are often preferred as they provide immediate liquidity and avoid the market risk associated with holding the actual shares until the redemption date.
- **Market Preferences:** Retail products are typically settled physically, appealing to investors' behavioral tendencies to hold onto stocks rather than realizing losses.

13.4.3 Issue Minimum/Maximum Size and Liquidity Considerations

13.4.3.1 Minimum Size

- **Cost Implications:** The issuer will only launch a product if it can cover fixed and variable costs such as listing fees, back-office operations, and computational expenses.
- **Investor Strategy:** Prefer products from issuers with lower operational costs, as these savings can be passed on in the product's pricing.

13.4.3.2 Maximum Size

- **Risk Management:** Issuers limit the size of a product based on their capacity to hedge the associated risks effectively, particularly for less liquid underlyings.
- **Market Influence:** Large hedging operations can inadvertently affect the market price of the underlying, potentially impacting the performance of the product.

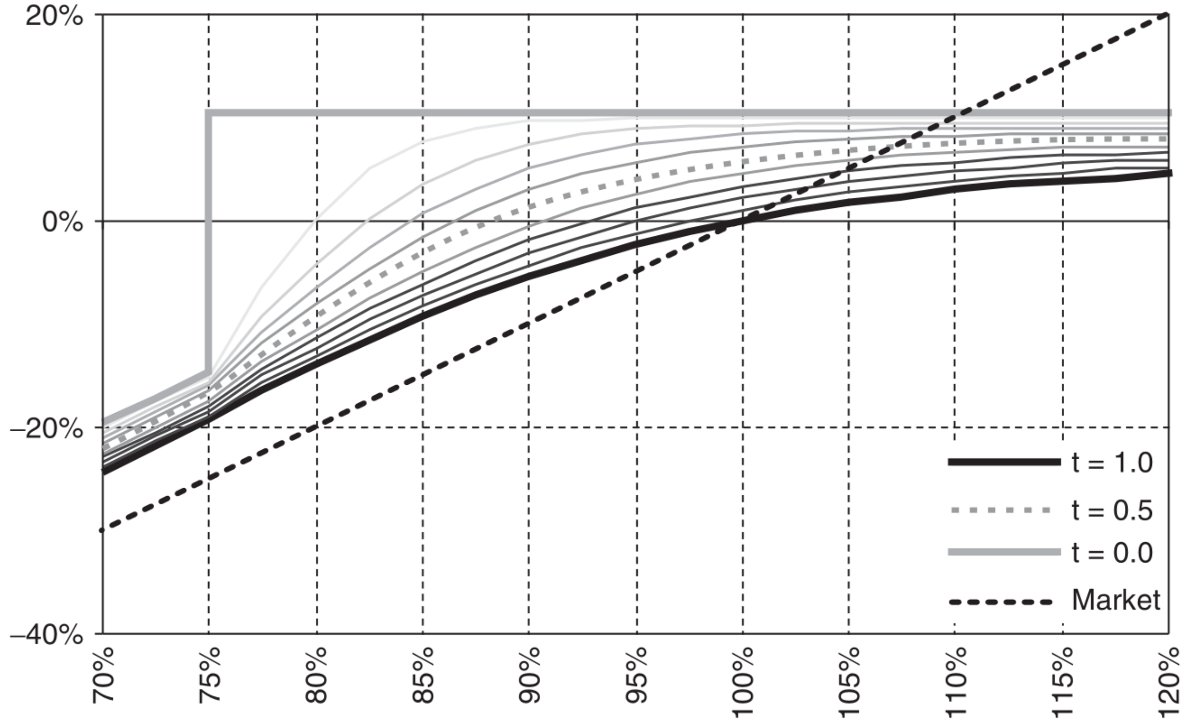


Figure 13.4: Reverse convertible as a function of spot price and time to maturity

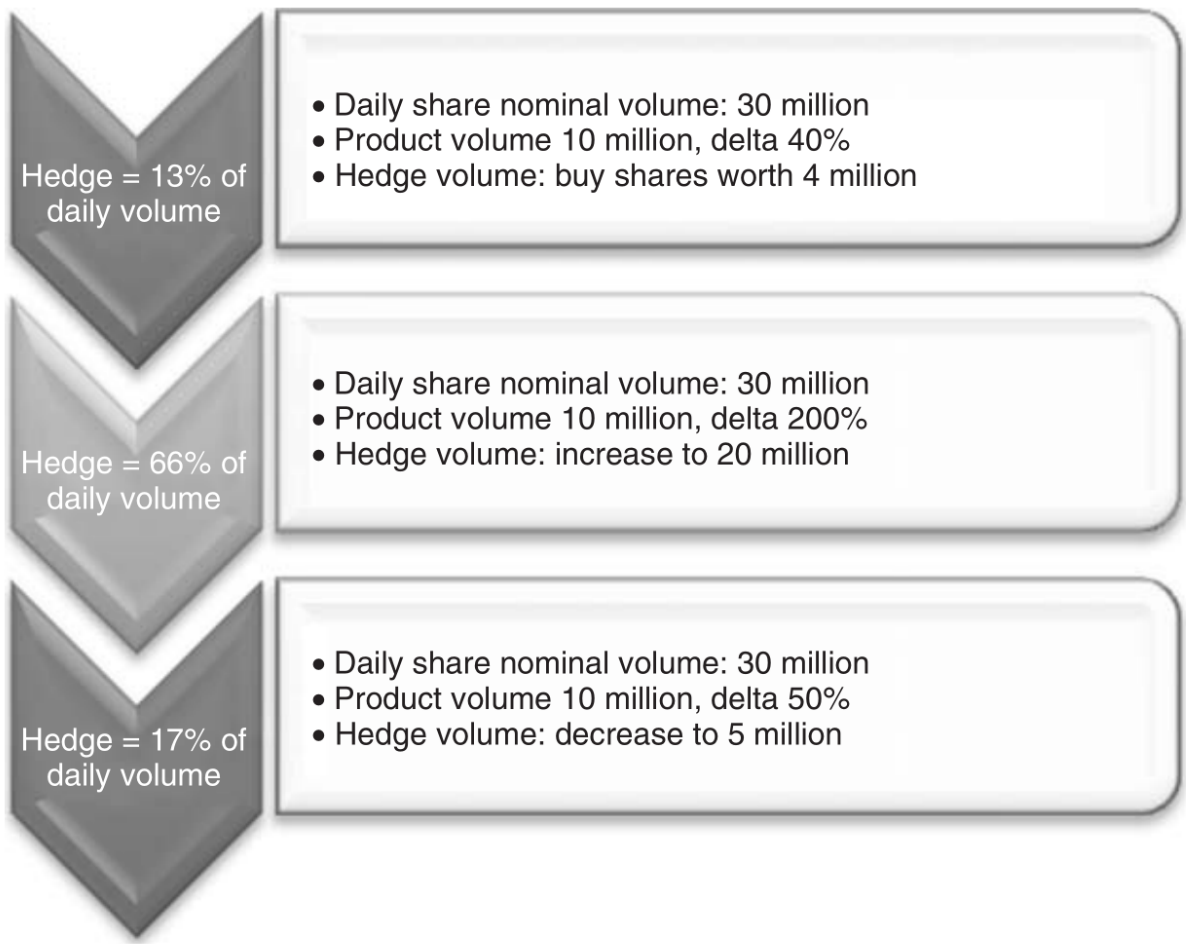


Figure 13.5: Reverse convertible hedging example

13.4.3.3 Liquidity

- **Trading Spreads:** Vary significantly by region and issuer; narrower spreads in markets like Germany enhance trading viability for smaller sizes.
- **Secondary Market:** Liquidity is typically provided by the issuer, making it essential for investors to consider the availability of market quotes, especially if they plan to trade the product before maturity.

13.5 Practice Questions and Problems

1. What is the value of the reverse convertible at maturity if the value of the underlying asset ends at 100, the strike price of the certificate is 80, and the coupon is 12%? The

strike price is equal to the initial spot price of the underlying.

Solution

89.6

2. What is the value of the reverse convertible at maturity if the value of the underlying asset ends at 45, the strike price of the certificate is 60, and the coupon is 9%? The strike price is equal to the initial spot price of the underlying.

Solution

50.4

3. What is the value of the barrier reverse convertible at maturity if the value of the underlying asset ends at 90, the strike price of the certificate is 100, the coupon is 7%, and the knock-out barrier is 70? The lowest value of the underlying asset during maturity was 65. The strike price is equal to the initial spot price of the underlying.

Solution

97

4. What is the value of the barrier reverse convertible at maturity if the value of the underlying asset ends at 90, the strike price of the certificate is 100, the coupon is 7%, and the knock-out barrier is 70? The lowest value of the underlying asset during maturity was 75. The strike price is equal to the initial spot price of the underlying.

Solution

107

5. What is the value of the barrier reverse convertible at maturity if the value of the underlying asset ends at 123, the strike price of the certificate is 100, the coupon is 7%, and the knock-out barrier is 70? The lowest value of the underlying asset during maturity was 65. The strike price is equal to the initial spot price of the underlying.

Solution

107

6. What is the value of the barrier reverse convertible at maturity if the value of the underlying asset ends at 137, the strike price of the certificate is 100, the coupon is 7%, and

the knock-out barrier is 70? The lowest value of the underlying asset during maturity was 75. The strike price is equal to the initial spot price of the underlying.

> **Solution**
>
> 107

7. What is the value of the autocallable note at maturity if the value of the invested capital was 1000, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 4600, and the coupon is 6%? The price of the underlying each year ended at [4583, 4309, 5128, 5630].

> **Solution**
>
> 1180

8. What is the value of the autocallable note at maturity if the value of the invested capital was 1000, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 4600, and the coupon is 4%? The price of the underlying each year ended at [4583, 4309, 3925, 4528].

> **Solution**
>
> 1000

9. What is the value of the autocallable note at maturity if the value of the invested capital was 1000, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 4600, and the coupon is 8%? The price of the underlying each year ended at [4672, 4982, 5023, 5628].

> **Solution**
>
> 1080

10. What is the value of the express certificate at maturity if the value of the invested capital was 500, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 2500, barrier is 1500, and the coupon is 14%? The price of the underlying each year ended at [2403, 2625, 2039, 2901]. The lowest value of the underlying asset during maturity was 1958.

> **Solution**
>
> 640

11. What is the value of the express certificate at maturity if the value of the invested capital was 500, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 2500, barrier is 1500, and the coupon is 14%? The price of the underlying each year ended at [2200, 2350, 2194, 1830]. The lowest value of the underlying asset during maturity was 1820.

> **Solution**
>
> 500

12. What is the value of the express certificate at maturity if the value of the invested capital was 500, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 2500, barrier is 1500, and the coupon is 14%? The price of the underlying each year ended at [2200, 1800, 1630, 1775]. The lowest value of the underlying asset during maturity was 1420.

> **Solution**
>
> 355

13. What is the value of the express certificate at maturity if the value of the invested capital was 500, autocall occurs if the underlying price at the anniversary date is higher then initial asset price 2500, barrier is 1500, and the coupon is 14%? The price of the underlying each year ended at [2200, 2431, 2700, 2398]. The lowest value of the underlying asset during maturity was 1420.

> **Solution**
>
> 710

14. What is the value of the long constant leverage certificate if the value of the underlying asset moves from 100 to 120, and the leverage was 3?

> **Solution**
>
> 160

15. What is the value of the long constant leverage certificate if the value of the underlying asset moves from 100 to 90, and the leverage was 5?

> **Solution**
>
> 50

16. What is the leverage of the mini-future certificate if the value of the underlying asset is 245 and financing level is 195?

Solution

4.9

17. What is the leverage of the mini-future certificate if the value of the underlying asset is 350 and financing level is 290?

Solution

5.83

A Time Value of Money

References

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Learning Outcomes:

- Understand the concept of the time value of money (TVM).
- Calculate the present value (PV) of both single and multiple future cash flows using appropriate discount rates.
- Calculate the future value (FV) of both single and ongoing investments using given interest rates.
- Apply TVM formulas to various financial scenarios, including loans, savings, and investments, to make informed decisions.
- Understand the effects of compounding frequency on FV and PV calculations.
- Critically analyze TVM problems, taking into account the impact of rate, time, and cash flows on financial decisions.

A.1 Notation and Terminology

A.1.1 Basic Notation and Terminology

- P = Principal (i.e., value of initial deposit)
- A = Accumulated amount (i.e., sum of the principal and interest)
- r = Nominal interest rate
- m = Number of conversion periods per year, (a conversion period is the interval of time between successive interest payments)

Annually	Semiannually	Quarterly	Monthly	Weekly	Daily
$m = 1$	$m = 2$	$m = 4$	$m = 12$	$m = 52$	$m = 365$

- t = Term of investment (in years)

A.1.2 Simple Interest

Interest is always computed based on the original principal.

Interest Earned	Accumulated Amount
$I = Prt$	$A = P(1 + rt)$

A.1.3 Discrete Compound Interest

Interest payments are added to the principal at the end of each conversion period and therefore earn interest during future conversion periods.

Accumulated Amount	Present Value Formula
$A = P\left(1 + \frac{r}{m}\right)^{mt}$	$P = A\left(1 + \frac{r}{m}\right)^{-mt}$

A.1.4 Continuous Compound Interest

Continuous compounding of interest is equivalent to a discrete compounding of interest where m, the number of conversion periods per year, goes to infinity.

Accumulated Amount	Present Value Formula
$A = Pe^{rt}$	$P = Ae^{-rt}$

A.1.5 Effective Rate of Interest

The effective interest rate, r_{eff}, is the simple interest rate that produces the same accumulated amount in 1 year as the nominal rate, r, compounded m times a year.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

A.2 Future Value Examples

A.2.1 Example 1

Suppose \$1,000 is deposited into an account with an interest rate of 16% compounded annually. How much money is in the account after 3 years?

Step 1: Since interest is compounded annually, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 1$, and $t = 3$.

$$\begin{aligned} A &= 1000\left(1 + \frac{0.16}{1}\right)^{1\cdot 3} \\ &= 1000\left(1 + 0.16\right)^3 \\ &= 1000\left(1.16\right)^3 \approx \$1,560.90 \end{aligned}$$

Therefore, after 3 years of accumulating interest, the original investment of \$1,000 is worth \$1,560.90.

A.2.2 Example 2

Suppose \$1,000 is deposited into an account with an interest rate of 16% compounded quarterly. How much money is in the account after 3 years?

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 4$, and $t = 3$.

$$
\begin{aligned}
A &= 1000\left(1+\frac{0.16}{4}\right)^{4\cdot 3} \\
&= 1000\,(1+0.04)^{12} \\
&= 1000\,(1.04)^{12} \approx \$1,601.03
\end{aligned}
$$

Therefore, after 3 years of accumulating interest, the original investment of \$1,000 is worth \$1,601.03.

> **Observation**
>
> Compare the accumulated amounts in the above two examples. Both examples have the same principal, interest rate, and term. But since interest is compounded more frequently in Example 2 (4 times a year) than in Example 1 (1 time a year), the accumulated amount is higher in Example 2.

A.2.3 Example 3

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded quarterly.

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$
A = P\left(1+\frac{r}{m}\right)^{mt}
$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), $m = 4$, and $t = 5$.

$$
\begin{aligned}
6000 &= 3000\left(1+\frac{r}{4}\right)^{4\cdot 5} \\
&= 3000\left(1+\frac{r}{4}\right)^{20}
\end{aligned}
$$

Step 3: Solve for the interest rate, r.

Method 1

Divide both sides by 3000

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Take the natural logarithm of both sides.

$$\begin{aligned}
\ln(2) &= \ln\left[\left(1 + \frac{r}{4}\right)^{20}\right] \\
&= 20\ln\left(1 + \frac{r}{4}\right) && \text{since } \ln(m^n) = n\ln(m)
\end{aligned}$$

Divide both sides by 20.

$$\ln(2)/20 = \ln\left(1 + \frac{r}{4}\right)$$

Take the exponential of both sides.

$$\begin{aligned}
e^{\ln(2)/20} &= e^{\ln(1+\frac{r}{4})} \\
&= 1 + \frac{r}{4} && \text{since } e^{\ln(x)} = x
\end{aligned}$$

Subtract 1 from both sides.

$$e^{\ln(2)/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(e^{\ln(2)/20} - 1) \approx 0.1411.$$

Method 2

Here is an alternate method for solving for the interest rate r. We start with the following equation.

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Instead of taking the natural logarithm of both sides as we did before, now take the 20th root of both sides (i.e., raise both sides to the power of 1/20).

$$2^{1/20} = 1 + \frac{r}{4}$$

Subtract 1 from both sides.

$$2^{1/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(2^{1/20} - 1) \approx 0.1411$$

Note that this value of r is numerically equal in both methods since

$$\begin{aligned} e^{\ln(2)/20} &= e^{\ln(2^{1/20})} && \text{since } n\ln(m) = \ln(m^n) \\ &= 2^{1/20} && \text{since } e^{\ln(x)} = x \end{aligned}$$

Therefore, an interest rate of approximately 14.11% compounded quarterly is required for an investment of \$3,000 to double in value in 5 years.

A.2.4 Example 4

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded continuously.

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), and $t = 5$.

$$6000 = 3000e^{5r}$$

Step 3: Solve for the interest rate, r.

Divide both sides by 3000.

$$2 = e^{5r}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(2) &= \ln(e^{5r}) \\ &= 5r & \text{since } \ln(e^x) = x \end{aligned}$$

Divide both sides by 5.

$$r = \ln(2)/5 \approx 0.1386$$

Therefore, an interest rate of approximately 13.86% compounded continuously is required for an investment of $3,000 to double in value in 5 years.

> **Observation**
>
> Compare the last two examples. Since continuous compounding of interest earns interest faster than discrete compounding, a lower interest rate is needed for an investment to double in value over a fixed term if interest is compounded continuously. In our examples, an interest rate of 13.86% was needed for the investment with continuous compound interest to double in value in 5 years, while an interest rate to 14.11% was needed for the investment with quarterly compound interest.

A.2.5 Example 5

How long will it take for $5,000 to grow to $8,000 if the investment earns interest at 6% per year compounded monthly?

Step 1: Since interest is compounded monthly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, $m = 12$, and $r = 0.06$.

$$\begin{aligned} 8000 &= 5000\left(1 + \frac{0.06}{12}\right)^{12 \cdot t} \\ &= 5000\left(1 + 0.005\right)^{12t} \end{aligned}$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = 1.005^{12t}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(8/5) &= \ln(1.005^{12t}) \\ &= 12t\ln(1.005) \qquad\qquad \text{since } \ln(m^n) = n\ln(m) \end{aligned}$$

Divide both sides by $12\ln(1.005)$.

$$t = \frac{\ln(8/5)}{12\ln(1.005)} \approx 7.85$$

Therefore, it will take approximately 7.85 years for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded monthly.

A.2.6 Example 6

How long will it take for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded continuously?

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, and $r = 0.06$.

$$8000 = 5000e^{0.06t}$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = e^{0.06t}$$

Take the natural logarithm of both sides.

$$
\begin{aligned}
\ln(8/5) &= \ln(e^{0.06t}) \\
&= 0.06t && \text{since } \ln(e^x) = x
\end{aligned}
$$

Divide both sides by 0.06.

$$t = \frac{\ln(8/5)}{0.06} \approx 7.83$$

Therefore, it will take approximately 7.83 years for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded monthly.

> **Observation**
>
> Compare the last two examples. Both examples have the same principal, accumulated amount, and interest rate. But since continuous compounding of interest earns interest faster than discrete compounding, it should take less time for the investment to grow to \$8,000 if interest is compounded continuously.

A.2.7 Example 7

Find the effective interest rate corresponding to a nominal interest rate of 10% compounded semiannually.

Step 1: Recall the formula for effective interest rate, r_{eff}.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

Step 2: Plug in the given values: $r = 0.1$ and $m = 2$.

$$
\begin{aligned}
r_{\text{eff}} &= \left(1 + \frac{0.1}{2}\right)^2 - 1 \\
&= 1.05^2 - 1 \\
&= 0.1025
\end{aligned}
$$

Therefore, an investment earning interest compounded semiannually at 10% earns the same amount of interest after 1 year as an investment earning simple interest at 10.25%.

A.2.8 Example 8

Suppose you have \$12,000 in the bank earning interest at a rate of 12% compounded quarterly. Your cousin calls you and needs \$12,000 to buy a new car. You are willing him to loan him the money, but you'd hate to lose out on the interest you would gather by simply leaving your money alone. If you charge your cousin an interest rate compounded continuously, what rate should you charge in order to earn the same amount of interest you otherwise would have?

Step 1: Assume your cousin is prepared to pay you back after t years.

We'll use t as the term in each of the following calculations. Eventually, we'll see that the interest rate you charge does not depend on the specific value of t.

Step 2: Compute the accumulated amount of the \$12,000 after t years assuming you leave your money in the bank.

$$\begin{aligned} A &= P\left(1+\frac{r}{m}\right)^{m\cdot t} \\ &= 12000\left(1+\frac{0.12}{4}\right)^{4t} \\ &= 12000\,(1.03)^{4t} \end{aligned}$$

Step 3: Compute the accumulated amount of the \$12,000 after t years assuming you let your cousin borrow the money.

This would be the amount that your cousin repays you after t years.

$$\begin{aligned} A &= Pe^{rt} \\ &= 12000e^{rt} \end{aligned}$$

Step 4: Equate the two accumulated amounts and solve for r.

$$12000\,(1.03)^{4t} = 12000e^{rt}$$

Divide both sides by 12000.

$$1.03^{4t} = e^{rt}$$

Take the natural logarithm of both sides.

$$\ln(1.03^{4t}) = \ln(e^{rt})$$

Simplify using properties of logarithms ($\ln(m^n) = n\ln(m)$ and $\ln(e^x) = x$).

$$4t\ln(1.03) = rt$$

And finally, divide both sides by t. Here is where we see that the time it would take your cousin to repay you does not affect the interest rate you would charge.

$$r = 4\ln(1.03) \approx 0.1182$$

Therefore, charging your cousin 11.82% interest compounded continuously earns the same amount of interest as leaving your money in the bank earning interest at a rate of 12% compounded quarterly.

A.3 Present Value Examples

A.3.1 Example 1

How much money should be deposited in a bank paying a yearly interest rate of 6% compounded monthly so that after 3 years, the accumulated amount will be $20,000?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded monthly, we'll use the present value formula for discrete compounding of interest.

$$P = A\left(1 + \frac{r}{m}\right)^{-mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$P = 20000\left(1 + \frac{0.06}{12}\right)^{-(12)(3)}$$

$$= 20000(1.005)^{-36} \approx \$16,712.90$$

Therefore, $16,712.90 invested at 6% interest compounded monthly will be worth $20,000 in 3 years.

A.3.2 Example 2

Use the accumulated amount for discrete compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for discrete compounding of interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$\begin{aligned} 20000 &= P\left(1 + \frac{0.06}{12}\right)^{(12)(3)} \\ &= P(1.005)^{36} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{20000}{1.005^{36}} \approx \$16,712.90$$

A.3.3 Example 3

Parents wish to establish a trust fund for their child's education. If they need \$170,000 in 7 years, how much should they set aside now if the money is invested at 20% compounded continuously?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded continuously, we'll use the present value formula for continuous compounding of interest.

$$P = Ae^{-rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$P = 170,000e^{-(0.2)(7)}$$

$$= 170,000e^{-1.4} \approx \$41,921.48$$

Therefore, \$41,921.48 invested at 20% interest compounded continuously will be worth \$170,000 in 7 years.

A.3.4 Example 4

Use the accumulated amount for continuous compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for continuous compounding of interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$\begin{aligned} 170000 &= Pe^{(0.2)(7)} \\ &= Pe^{1.4} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{170000}{e^{1.4}} \approx \$41,921.48$$

A.4 Try It Yourself

A.4.1 Exercise 1

If \$6,00 is invested at 7% compounded continuously, what will be the accumulated amount after 6 years?

Show answer

$A = 6000e^{0.42}$

A.4.2 Exercise 2

If \$7,000 is invested at 16% compounded quarterly, what will be the accumulated amount after 3 years?

Show answer

$A = 7000(1.04)^{12}$

A.4.3 Exercise 3

Find the interest rate r needed for an investment of \$2,000 to grow to \$8,000 in 7 years if compounded continuously.

Show answer

$r = \ln(4)/7$

A.4.4 Exercise 4

Find the interest rate r needed for an investment of \$7,000 to grow to \$12,000 in 21 years if compounded monthly.

Show answer

$r = 12\left[(12/7)^{1/252} - 1\right]$

A.4.5 Exercise 5

Find the time it would take for an investment of \$1,000 to grow to \$100,000 if interest is compounded quarterly at an annual rate of 8%.

Show answer

$t = \frac{\ln(100)}{4\ln(1.02)}$

A.4.6 Exercise 6

Find the time it would take for an investment of \$2,500 to grow to \$6,000 if interest is compounded continuously at an annual rate of 24%.

Show answer

$t = \frac{25}{6}\ln(12/5)$

A.4.7 Exercise 7

Calculate the effective rate of interest corresponding to a nominal interest rate of 52% compounded weekly.

Show answer

$r_{eff} = 1.01^{52} - 1$

A.4.8 Exercise 8

Your grandma would like to establish a trust fund for your education. How much should she set aside now if she wants \$50,000 in 9 years and interest is compounded monthly at an annual rate of 12%?

Show answer

$P = 50000(1.01)^{-108}$

A.4.9 Exercise 9

You are preparing to run for president and want to have \$100,000 in 6 years to start your campaign. How much money do you need now if interest is compounded continuously at an annual rate of 15%?

Show answer

$P = 100000e^{-0.9}$

A.4.10 Exercise 10

You have \$50,000 in the bank earning 7% interest compounded quarterly. However, your cousin needs a \$50,000 investment to start up his new financial consulting business. In order to get the same total return as leaving your money in the bank, what interest rate r should you request from your cousin if interest is compounded continuously?

Show answer

$r = 4\ln(1 + 0.07/4)$

B Interest Rates

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 4 - Interest Rates

Learning Outcomes:

- Understand the different types of interest rates.
- Define the risk-free rate and its significance in financial derivatives.
- Explain the concept of continuous compounding and its importance in pricing financial derivatives.

B.1 Types of Rates

B.1.1 Treasury Rate

- Rate on instrument issued by a government in its own currency.

B.1.2 The U.S. Fed Funds Rate

- Unsecured interbank overnight rate of interest.
- Allows banks to adjust the cash (i.e., reserves) on deposit with the Federal Reserve at the end of each day.
- The effective fed funds rate is the average rate on brokered transactions.
- The central bank may intervene with its own transactions to raise or lower the rate.
- Similar arrangements in other countries.

B.1.3 Repo Rate

- Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them for X and buy them bank in the future (usually the next day) for a slightly higher price, Y.
- The financial institution obtains a loan.
- The rate of interest is calculated from the difference between X and Y and is known as the repo rate.

B.1.4 LIBOR (ICE LIBOR)

- Detailed information about LIBOR: https://www.theice.com/iba/libor
- LIBOR is the rate of interest at which a AA bank can borrow money on an **unsecured** basis from another bank.
- Based on **submissions** from a panel of contributor banks (16 for each of USD and GBP).
- It is calculated daily for 5 currencies and 7 maturities.
- There have been some suggestions that banks manipulated LIBOR during certain periods.
- Why would they do this?

B.2 Alternative Reference Rates

Country/Currency/CODE	IBOR Rate	New Reference Rate
USA/Dollars/USD	USD ICE LIBOR	SOFR
UK/Pounds Sterling/GBP	GBP ICE LIBOR	SONIA
Switzerland/Swiss Francs/CHF	CHF ICE LIBOR	SARON
Japan/Yen/JPY	JPY ICE LIBOR, Tibor	TONAR
EU/Euro/EUR	Euribor	ESTER

B.2.1 SOFR (Secured Overnight Financing Rate)

- CME Group Education
- Administered by Federal Reserve Bank of New York (link)
- Transaction-based, calculated from overnight US Treasury repurchase (repo) activity.
- SOFR is a broad measure of the cost of borrowing USD cash overnight, collateralized by U.S. Treasury securities.
- SOFR is a good representation of general funding conditions in the overnight Treasury repo market.

- As such, it will reflect an economic cost of lending and borrowing relevant to the wide array of market participants active in the market.

B.2.2 SONIA (Sterling Overnight Index Average)

- CME Group Education
- Administered by Bank of England (link)
- Unsecured transaction-based index, wholesale based (beyond Interbank)
- It has been endorsed by the Sterling Risk-Free Reference Rate Working Group (Working Group) as the preferred risk-free reference rate for Sterling Overnight Indexed Swaps (OIS).
- In January 2018, the Working Group added banks, dealers, investment managers, non-financial corporates, infrastructure providers, trade associations and professional services firms.
- In April 2018, the BOE introduced a series of reforms of the SONIA benchmark.

B.2.3 €STR (or ESTER, Euro Short-Term Rate)

- Administered by European Central Bank (link)
- It is based on the unsecured market segment.
- The ECB developed an unsecured rate, because it is intended to complement the EONIA.
- Furthermore, a secured rate would be affected by the type of the collaterals.
- The money market statistical reporting covers the 50 largest banks in the euro area in terms of balance sheet size.
- While the EONIA (link) reflects the interbank market, the €STR extends the scope to money market funds, insurance companies and other financial corporations because banks developed significant money market activity with those entities.

B.3 OIS Rate

- An **overnight indexed swap** is swap where a fixed rate for a period (e.g. 3 months) is exchanged for the geometric average of overnight rates (or overnight rate compounded over the term of the swap).
- The underlying floating rate is typically the rate for overnight lending between banks, either non-secured or secured (SOFR, SONIA, €STR).
- For maturities up to one year there is a single exchange (swap term is not overnight).
- For maturities beyond one year there are periodic exchanges, e.g. every quarter.
- The OIS rate is a continually refreshed overnight rate.
- The fixed rate of OIS is typically an interest rate considered less risky than the corresponding interbank rate (LIBOR) because there is limited counterparty risk.

B.3.1 The Risk-Free Rate

- The Treasury rate is considered to be artificially low because:
  - Banks are not required to keep capital for Treasury instruments
  - Treasury instruments are given favorable tax treatment in the US
- OIS rates are now used as a proxy for risk-free rates in derivatives valuation.

B.4 Time Value of Money

B.4.1 Compounding Frequency

- When we compound m times per year at rate r an amount P grows to $P(1 + r/m)^m$ in one year.
- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
- Effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	Value of $100 at end of year ($)
Annually m = 1	110.00
Semiannually m = 2	110.25
Quarterly m = 4	110.38
Monthly m = 12	110.47
Weekly m = 52	110.51
Daily m = 365	110.52

B.4.2 Continuous Compounding

- **Rates used in option pricing are nearly always expressed with continuous compounding.**
- In the limit as we compound more and more frequently we obtain continuously compounded interest rates.
- Notation:
  - r: continuously compounded annual interest rate
  - T: time to maturity in years
  - e: Euler's number (mathematical constant)

$$\text{Future value} = P \times e^{rT}$$

$$\text{Present value} = P \times e^{-rT}$$

- USD 100 grows to $100 \times e^{rT}$ when invested at a continuously compounded rate r for time T.
- USD 100 received at time T discounts to $100 \times e^{-rT}$ at time zero when the continuously compounded discount rate is r.

B.4.3 Conversion Formulas

- r_c: continuously compounded rate
- r_m: same rate with compounding m times per year

$$r_c = m \ln(1 + \frac{r_m}{m})$$

$$r_m = m(e^{r_c/m} - 1)$$

Examples:

- 10% with semiannual compounding is equivalent to $2 \ln(1.05) = 9.758\%$ with continuous compounding.
- 8% with continuous compounding is equivalent to $4(e^{0.08/4} - 1) = 8.08\%$ with quarterly compounding.