Financial Derivatives

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Preamble

The course Financial Derivatives delves into the intricacies of derivative instruments, focusing on both fixed and conditional contracts, including forwards, futures, swaps, and options. It explores the utilization of derivatives within financial markets, particularly for hedging against various risks. Students will gain a comprehensive understanding of the essence and valuation of financial derivatives, along with detailed insights into the exchange trading mechanisms for futures and options.

Upon completing this course, the student will be able to:

- define financial derivatives and understand their specific characteristics
- explain the differences between fixed future contracts and options
- price financial derivatives
- suggest their use

References:

- HULL, John. Options, futures, and other derivatives. Global edition. Harlow: Pearson, 2018. ISBN 9781292212890.
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. xix, 597. ISBN 9781119381815.
- BLÜMKE, Andreas. How to invest in structured products : a guide for investors and investment advisors. Chichester: Wiley, 2009. xvi, 374. ISBN 9780470746790.

1 Introduction to Financial Derivatives

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 4 - Interest Rates
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Define and explain the concept of derivatives in financial markets.
- Analyze the importance and role of derivatives in finance.
- Distinguish between derivative markets, including their structure and function.
- Compare and contrast forward contracts with contingent claims, highlighting their differences and uses.
- Identify the various types of derivatives.
- Discuss the applications of derivatives in hedging, speculation, and arbitrage.
- Evaluate criticisms and potential misuses of derivatives in financial markets.

1.1 What Are Derivatives?

1.1.1 Understanding Derivatives

- A derivative is a financial contract whose value depends on the performance of **an underlying asset**.
- This underlying asset, referred to as the underlying, may be a stock, currency, interest rate, commodity, or any other marketable item. Its market price is known as the cash or spot price.

- The range of underlying assets encompasses a wide array, including stocks, currencies, commodities, debt instruments, and even non-traditional metrics like weather conditions or insurance claims.

1.1.2 The Significance of Derivatives

Derivatives are pivotal in the financial ecosystem due to their ability to:

- Facilitate risk management by transferring risks between parties.
- Craft unique investment strategies and returns not feasible with direct investments.
- Serve as a source of market insight and future price expectations.
- Offer cost efficiencies in transactions.
- Minimize required capital investment.
- Simplify the process of taking short positions compared to dealing with the underlying directly.
- Enhance the liquidity and operational efficiency of the primary markets they relate to.

1.2 Derivative Markets

Derivatives are traded in two main venues: **organized exchanges** and **over-the-counter (OTC)** markets.

1.2.1 Exchange-Traded Derivatives

These derivatives are **standardized** contracts with fixed features such as contract size, expiration date, and underlying assets. Their trading is **guaranteed by a clearing house**, which requires participants to post margins. This arrangement ensures transparency and regulation, safeguarding the integrity of the market.

1.2.2 Over-The-Counter (OTC) Derivatives

In contrast, OTC derivatives offer **customization** in terms of contract size, asset specification, and expiration dates. This flexibility, however, comes with a higher risk of default (credit risk) for the parties involved.

1.2.2.1 The OTC Market Evolution

- **Prior to 2008**: The OTC market was largely unregulated, with banks serving as the primary market makers. Transactions were governed by master agreements, and some were cleared through central counterparties (CCPs), similar to clearing houses.
- **Since 2008**: Significant regulatory changes have been implemented to reduce systemic risk and enhance transparency. Standardized OTC transactions must now be cleared through CCPs, and all trades are required to be reported to a central repository. These measures aim to bolster market stability and integrity.

> The Lehman Bankruptcy
>
> The bankruptcy of Lehman Brothers on September 15, 2008, stands as the largest in U.S. history. Lehman Brothers was heavily involved in the over-the-counter (OTC) derivatives market, engaging in high-risk financial activities. The firm's inability to refinance its short-term debt ultimately led to its downfall. At the time of its bankruptcy, Lehman Brothers had an extensive network of transactions, with hundreds of thousands outstanding across approximately 8,000 counterparties. The process of unwinding these transactions has posed significant challenges for both the Lehman liquidators and the involved counterparties, illustrating the complex and interconnected nature of modern financial markets.

1.3 Forward Contracts vs. Contingent Claims

Both forward contracts and contingent claims are essential financial instruments that derive their value from the performance of an underlying asset, playing pivotal roles in the global financial markets for hedging, speculation, and arbitrage.

1.3.1 Forward Commitments

Forward commitments are agreements to buy or sell an asset at a predetermined future date and price. They include:

- **Forward Contracts**: Private, ***non-standardized*** agreements between two parties to buy or sell an asset at a specified future time at a price agreed upon at the contract's initiation.
- **Futures Contracts**: ***Standardized*** forward contracts traded on organized exchanges that require the posting of a margin. Futures contracts are marked to market daily, which mitigates credit risk.

- **Swaps**: Agreements between two parties to exchange sequences of cash flows for a set period based on a specified principal amount. Swaps often involve the exchange of a fixed interest rate for a floating rate, or vice versa.

1.3.2 Contingent Claims

Contingent claims are financial instruments that ***offer the holder the right, but not the obligation***, to buy or sell an asset at a predetermined price within a specified timeframe. The primary form of contingent claims is:

- **Options**: Contracts that give the buyer the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a specified strike price on or before a certain date. Options are of two types: *European options*, which can be exercised only at expiration, and *American options*, which can be exercised at any time before expiration.

1.3.3 Key Differences and Risks

- **Obligation vs. Right**: A forward contract represents an obligation to carry out the transaction, potentially leading to large losses depending on the market's movement. In contrast, a contingent claim like an option provides a right, limiting the buyer's potential loss to the premium paid for acquiring this right.
- **Risk Profile**: The potential loss on a long forward contract can equal the full contract price, while for a short position, the loss can be theoretically infinite due to the asset's price potential to rise indefinitely. Conversely, the risk of a contingent claim is limited to the premium paid, providing a risk-averse strategy for speculators and hedgers.
- **Uses and Applications**: Forward contracts and futures are commonly used for hedging against price movements in commodities, currencies, and interest rates. Swaps are utilized for managing interest rate risk and currency exposure. Options are employed for hedging, speculative trading, and income generation through premium collection.

Incorporating these instruments into investment and risk management strategies requires an understanding of their mechanics, risks, and market behavior. Their usage reflects the financial goals, risk tolerance, and market outlook of the participants, illustrating the complexity and diversity of modern financial markets.

1.4 Types of Derivatives

1.4.1 Forward Contract

Definition

A **forward contract** is a customized, over-the-counter derivative agreement between two parties, where the buyer agrees to purchase, and the seller agrees to sell, an underlying asset at a predetermined future date and price established at the contract's inception.

- **Long Position**: The party committing to purchase the asset.
- **Short Position**: The party committing to sell the asset.

Characteristics of Forward Contracts

- **Underlying Asset**: Specifies the type and quantity of the asset to be traded.
- **Settlement Method**: Describes how the contract will be executed or settled upon expiration.
- **Forward Price**: The agreed-upon price for the underlying asset exchange, designed to make the contract's initial value zero.

Key Points

- **Popularity in Foreign Exchange**: Forward contracts are frequently used for hedging in the foreign exchange markets.
- **OTC Markets**: Typically involves at least one financial institution, allowing for customization but lacking centralized regulation.
- **Flexibility and Risk Management**: Forward contracts offer tailored solutions for specific hedging needs, particularly in markets lacking standardized instruments. This customization enables precise risk management tailored to the parties' unique requirements.
- **Market Dynamics and Pricing**: The determination of forward prices is influenced by various factors, including the underlying asset's current price, interest rates, and the asset's expected future price volatility. This dynamic pricing mechanism reflects the market's consensus on future price movements, adjusted for the time value of money.

1.4.2 Futures Contract

- A futures contract is a **standardized** derivative traded on **futures exchanges**, like the CME Group or Intercontinental Exchange, facilitating the buying and selling of underlying assets at future dates.
- **Futures and Liquidity**: Futures contracts provide liquidity and price discovery in a regulated environment, offering a transparent and efficient means for market participants to hedge against price volatility or speculate on future price movements.

Forwards vs. Futures

Forwards	Futures
Customized terms, traded over-the-counter.	Standardized terms, traded on regulated exchanges.
Counterparty risk, with less regulatory oversight.	Mitigated counterparty risk through clearinghouses.
Settlement occurs at contract maturity.	Daily mark-to-market settlement.

1.4.3 Swap Contract

- A **swap** is an OTC derivative where two parties exchange cash flow series, addressing multi-period risks. Unlike a forward contract focusing on a single-period risk, swaps often manage interest rate, currency, or commodity exposure over extended periods.
- Swaps can be structured in numerous ways to suit different types of risk management strategies. For example, interest rate swaps exchange fixed for floating interest rate payments to manage interest rate risk, while currency swaps exchange cash flows in different currencies to hedge against currency risk.

1.4.4 Options

Definition

Options are versatile financial derivatives allowing the holder to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specific timeframe. The buyer of the option pays a premium to the seller or writer for this right, without the obligation to execute the transaction.

Types of Options

- **Call Options**: Grant the holder the right to **purchase** the underlying asset at the strike price. Investors buy calls when they anticipate the underlying asset's price will increase.
- **Put Options**: Provide the holder the right to **sell** the underlying asset at the strike price. Puts are purchased when an investor expects the underlying asset's price to decline.

Exercise Styles

- **American Options**: These can be exercised at any point up to and including the expiration date, offering maximum flexibility to the holder.
- **European Options**: Can only be exercised on the expiration date itself, limiting the timing of execution to this single moment.

1.5 Applications of Derivatives

1.5.1 Hedging

Hedging is a strategic financial practice aimed at reducing potential risks and mitigating financial exposure. Its core purpose is to **minimize risk**, though it does not assure a more favorable outcome. This strategy employs various financial instruments, such as forward and option contracts, each with distinct characteristics and uses.

Forward contracts are pivotal in risk management, enabling the hedger to lock in a future transaction price for an underlying asset, thereby neutralizing the risk of adverse price fluctuations. This fixed-price agreement ensures predictability in financial planning.

Option contracts, on the other hand, serve as a form of financial insurance. They grant investors the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specific timeframe. This mechanism allows investors to safeguard against potential adverse price movements while preserving the opportunity to capitalize on favorable price changes. Unlike forward contracts, acquiring an option requires paying an upfront premium, which represents the cost of the protection offered.

Examples to Illustrate Hedging

1. Consider a U.S. company that anticipates a payment of GBP 10 million for imports from Britain in three months. To hedge against the risk of GBP appreciating against USD, the company can enter into a long position in a forward contract. By doing so, it secures a fixed exchange rate for the future payment, thus avoiding the uncertainty of currency fluctuations.

2. An investor holding 1,000 shares of Microsoft, valued at $28 per share, faces the risk of a potential decrease in share value. To mitigate this risk, the investor can purchase a put option, which provides the right to sell the shares at a strike price of $27.50. For example, if a two-month put option costs $1 per share, the investor would buy 10 contracts (each contract typically covers 100 shares), paying a total premium of $1,000. This strategy protects the investor's portfolio from a decline below the strike price, while still allowing participation in any upward price movements.

1.5.2 Arbitrage

Arbitrage represents a foundational concept in financial markets, involving the exploitation of price differentials across different markets or forms of an asset to secure risk-free profits. This practice underscores the **Law of One Price**, which posits that in efficient markets with minimal transaction costs and unrestricted information flow, identical assets should uniformly price.

When disparities arise, arbitrageurs can purchase the undervalued asset in one market and sell it at a higher price in another, leveraging these discrepancies to generate a risk-free return. This activity not only capitalizes on the price differentials but also plays a crucial role in driving the prices of identical assets towards convergence, thus contributing to market efficiency.

Arbitrage serves as a relative valuation method, providing insights into the correct pricing of one asset or derivative in relation to another. It operates under the principle that no two identical assets should exist with different pricing or that equivalent asset combinations yielding the same returns should not vary in price.

The presence and subsequent elimination of arbitrage opportunities are indicative of market efficiency, where markets compensate investors for risks appropriately. However, when arbitrage allows for returns above the risk-free rate without exposure to risk, it challenges the notions of market efficiency by presenting an anomaly of abnormal returns.

Example to Illustrate Arbitrage

Imagine a scenario where a stock is priced at GBP 100 in London and USD 150 in New York, with the current exchange rate being 1.5300 GBP/USD. Purchase the stock for USD 150 in

New York, then sell it in London for GBP 100. Convert GBP 100 to USD 153 to secure a profit of USD 3.

1.5.3 Speculation

Speculation in financial markets is driven by the intention to profit from anticipated market movements. Speculators engage in market positions with the expectation that prices will either rise or fall, aiming to capitalize on these predicted changes.

When engaging in futures contracts, speculators face significant risk and reward potential. Futures contracts obligate the purchase or sale of an asset at a predetermined future date and price, exposing the speculator to potentially unlimited losses or gains, depending on market movements.

Conversely, options contracts offer a different risk profile. By purchasing options, a speculator gains the right, but not the obligation, to buy (call option) or sell (put option) an asset at a specified price within a certain period. The maximum loss in this scenario is limited to the premium paid for the options, providing a safety net against adverse market movements.

Examples to Illustrate Speculation

Trading strategies for an investor with $2,000 anticipating a stock price increase:

1. **Buying Shares Outright:** If the investor is confident in their prediction and prefers direct exposure, they could use their $2,000 to buy shares of the stock directly. This approach offers unlimited upside potential if the stock price increases but also exposes the investor to potential losses if the price falls.

2. **Buying Call Options:** The investor could purchase call options on the stock they believe will increase in value. This strategy allows the investor to control a larger amount of stock with a smaller investment (the premium paid for the options), with the potential for significant gains if the stock price rises as anticipated. The risk is limited to the premium paid.

3. **Futures Contracts:** If the investor has access to futures trading and is knowledgeable about its risks, they could engage in a futures contract that bets on the stock price increasing. This strategy requires careful risk management due to the potential for large losses.

1.6 Criticisms and Misuses of Derivatives

Derivatives, while fundamental to risk management and price discovery in financial markets, have faced criticisms and concerns regarding their potential for misuse and the risks they can introduce to the financial system. These criticisms encompass several key areas:

1. **Speculation and Gambling:** Derivatives are often utilized for speculative purposes, where investors bet on the direction of market prices with the aim of generating profits. This speculative use can resemble gambling when trades are made based on predictions rather than informed decisions, potentially leading to significant losses.

2. **Systemic Risk and Destabilization:** Derivatives can add systemic risk to the financial system, especially when used in large volumes or in complex combinations that are difficult to understand and manage. The interconnectedness of market participants through derivatives can lead to contagion effects, where the failure of one entity can trigger a chain reaction affecting others.

3. **Complexity:** The inherent complexity of many derivative products makes them difficult to value and understand, even for sophisticated investors. This complexity can obscure the real risks involved and lead to mispricing or inappropriate use of these instruments.

4. **Role Shifting:** Traders and institutions may shift roles between being hedgers, who use derivatives to manage risk, to speculators, who seek to profit from market movements, or from being arbitrageurs, who exploit price discrepancies, to speculators. This shifting can blur the lines between risk management and speculative activities, increasing the potential for losses and systemic risks.

5. **Need for Controls:** Given the potential for misuse and the complex nature of derivatives, it is crucial to establish comprehensive controls and regulatory frameworks. These controls should ensure that derivatives are used for their intended purposes, such as hedging risk, rather than for unchecked speculation. Proper oversight, transparent reporting requirements, and clear guidelines for risk assessment and management are essential components of a robust regulatory framework.

To address these criticisms and mitigate the risks associated with derivatives, financial markets and regulatory bodies have implemented measures such as:

- **Central Clearing Counterparties (CCPs):** These entities act as intermediaries for derivative transactions, reducing counterparty risk and increasing transparency.

- **Margin Requirements:** These requirements ensure that parties in derivative contracts have sufficient capital to cover potential losses, reducing the risk of default.

- **Regulatory Reforms:** Post-2008 financial crisis, reforms like the Dodd-Frank Act in the United States and EMIR in Europe have increased oversight and regulation of derivative markets, aiming to improve transparency, reduce systemic risk, and ensure that derivatives fulfill their role in financial markets responsibly.

1.7 Practice Questions and Problems

1.7.1 Time Value of Money

1. A bank quotes an interest rate of 7% per annum with quarterly compounding. How much you will earn from \$100 investment after (a) 1 year and (b) 3 years? What is the equivalent rate with (a) continuous compounding and (b) annual compounding? Verify your results.

1.7.2 Theoretical Foundations of Derivatives

1. Explain carefully the difference between hedging, speculation, and arbitrage.
2. What is the difference between the over-the-counter market and the exchange-traded market?
3. "Options and futures are zero-sum games." What do you think is meant by this statement?
4. What is the difference between a long forward position and a short forward position?
5. What is the difference between entering into a long forward contract when the forward price is \$50 and taking a long position in a call option with a strike price of \$50?
6. Explain carefully the difference between selling a call option and buying a put option.
7. When first issued, a stock provides funds for a company. Is the same true of an exchange-traded stock option? Discuss.

1.7.3 Practical Applications of Forwards and Futures

1. A trader enters into a short cotton futures contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound; (b) 51.30 cents per pound?
2. An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.5000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.4900 and (b) 1.5200?

1.7.4 Practical Applications of Options

1. A trader buys a call option with a strike price of $30 for $3. Does the trader ever exercise the option and lose money on the trade. Explain.
2. Suppose that you write a put contract with a strike price of $40 and an expiration date in three months. The current stock price is $41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?
3. Suppose you own 5,000 shares that are worth $25 each. How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?
4. You would like to speculate on a rise in the price of a certain stock. The current stock price is $29, and a three-month call with a strike of $30 costs $2.90. You have $5,800 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving investment in the option. What are the potential gains and losses from each?

1.7.5 Hedging Strategies

1. Explain why a futures contract can be used for either speculation or hedging.
2. A US company expects to have to pay 1 million Canadian dollars in six months. Explain how the exchange rate risk can be hedged using (a) a forward contract and (b) an option.
3. The CME Group offers a futures contract on long-term Treasury bonds. Characterize the investors likely to use this contract.

2 Forwards and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 1 - Introduction
  - Chapter 2 - Mechanics of futures markets
  - Chapter 3 - Hedging Strategies Using Futures
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Understand forwards and futures, including their characteristics, payoff structures, and key differences.
- Gain knowledge of exchange and over-the-counter (OTC) markets, focusing on their functionalities and distinctions.
- Learn the basics of hedging with futures, managing basis risk, and applying cross hedging techniques.
- Explore the role of stock index futures in portfolio risk management and speculation.

2.1 Forwards and Futures Characteristics

Definition

Forwards and futures are derivative contracts **obligating** two parties to exchange an asset at a predetermined future date and price.

- **Forward/Futures Price:** This is the agreed-upon price at which the underlying asset will be exchanged in the future. It's important to note that this price is determined at the contract's inception and may vary across contracts with different expiration dates, reflecting the market's expectations of future price movements.

- **Positioning:**
  - A **long position** signifies the buyer's **commitment to purchase** the underlying asset. The buyer stands to benefit from a rise in the asset's price over time but also bears the risk of a potential decrease.
  - A **short position** represents the seller's **obligation to sell** the asset. While the seller can profit from a decline in the asset's price, there is also the risk of unlimited loss if the asset's price increases substantially.
- **Risk Exposure:**
  - The potential loss for the holder of a **long position** can extend up to the full contract price, emphasizing the risk of a total loss if the asset's value drops to zero.
  - Conversely, the **short position** holder faces potentially unlimited loss, as there is no upper limit to how high an asset's price can climb.
- **Contract Specifications:**
  - **Deliverable Assets:** Clearly defines the asset or assets that can be delivered under the contract, including any standards or grades if applicable.
  - **Delivery Location:** Specifies where the asset will be delivered, which can significantly impact logistics and costs.
  - **Delivery Time:** Outlines when delivery of the asset is expected, providing a timeframe within which the contract must be settled.
- **Settlement:** Futures contracts often settle daily based on market price changes, a process known as "marking to market." Forwards, however, usually settle at the end of the contract term, with the final payment reflecting the difference between the forward price and the underlying asset's price at maturity.

```python
import plotly.graph_objects as go
import numpy as np

# Parameters
F = 100  # Futures price
Q = 1  # Quantity of the asset
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices

# Payoff calculations
long_payoff = (spot_prices - F) * Q
short_payoff = (F - spot_prices) * Q

# Create the figure
fig = go.Figure()
```

```
# Add traces for long and short positions
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_payoff,
        mode="lines",
        name="Long Position",
        line=dict(width=3),
        hovertemplate="Long Position<br>Spot Price: %{x:.0f}<br>Payoff: %{y:.0f}<extra></ext
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_payoff,
        mode="lines",
        name="Short Position",
        line=dict(width=3),
        hovertemplate="Short Position<br>Spot Price: %{x:.0f}<br>Payoff: %{y:.0f}<extra></ex
    )
)
fig.add_hline(y=0, line_dash="solid", line_color="black", line=dict(width=0.7))

# Layout
fig.update_layout(
    title="Payoff from a Futures Contract",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Payoff",
    legend_title="Position",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

```
Unable to display output for mime type(s): text/html
```

2.2 Payoff From a Forward Contract

When discussing forward contracts, it's essential to understand the setup:

- **Initiation Time:** The contract begins at $t = 0$
- **Expiration Time:** The contract ends at $t = T$
- **Initial Spot Price:** The price of the underlying asset at start, denoted as S_0
- **Final Spot Price:** The price of the underlying asset at expiration, S_T
- **Forward Price:** Agreed upon price for the asset, represented as F_0

Forward contracts involve two parties with contrasting positions:

- The **long position** commits to purchasing the asset at F_0, seeking to profit from an increase in S_T.
- The **short position** commits to selling the asset at F_0, benefiting if S_T decreases.

Payoff from a forward contract is then calculated as:

- **Long Party Payoff at Expiration:** $S_T - F_0$
- **Short Party Payoff at Expiration:** $-[S_T - F_0]$

Example: Gold Purchase Agreement

Barbara Nix agreed to buy 1 kilogram of gold at a price of $38,000 per kilo from Metals Inc. in 90 days. After 90 days, the spot price of gold reached $38,500 per kilo. What is the payoff for each party?

- **For Nix (Long Party):** The gain is $500, calculated as

$$S_T - F_0 = \$38,500 - \$38,000 = \$500$$

- **For PM Metals Inc. (Short Party):** The loss is $500

2.3 Exchange Markets

Exchange markets facilitate the trading of futures contracts, allowing participants to hedge against price changes or speculate on market movements. Key characteristics include the standardized nature of contracts and the regulatory oversight provided to ensure market integrity.

- **Regulation:** In the United States, the Commodity Futures Trading Commission (CFTC) oversees the futures markets, aiming to protect market participants and maintain market integrity.

- **Futures Contract Specifications:** Futures contracts are highly standardized, with detailed specifications, e.g., CME Group's Gold Futures.

2.3.1 Delivery Mechanisms

- **Contract Settlement:** Most futures contracts are closed out before maturity, avoiding physical delivery. Instead, positions are settled by entering into an offsetting transaction.
- **Delivery Options:** For contracts requiring physical delivery, specifications outline the deliverable commodities, delivery locations, and timelines. The short position holder usually determines the delivery specifics.
- **Cash Settlement:** Certain contracts, such as those for stock indices, settle in cash instead of physical delivery.

2.3.2 Understanding Market Quotes

- **Settlement Price:** This is the closing price used for the daily settlement process.
- **Open Interest:** Represents the total number of open contracts, equal to either the number of long or short positions.
- **Trading Volume:** Indicates the total number of contracts traded in a day, which can exceed the open interest.

> **i** Example of Open Interest and Trading Volume
>
> Imagine a futures market for wheat. At the start of the trading day, there are 1,000 open contracts for December Wheat futures. Throughout the day, traders conduct the following transactions:
>
> 1. **Transaction 1:** A new buyer and a new seller enter the market, agreeing on a contract. This transaction increases the open interest by 1 contract, bringing the total open interest to 1,001.
> 2. **Transaction 2:** An existing holder of a long position sells to a new participant. This transaction does not change the open interest since one new party replaces another in the open position count, keeping the open interest at 1,001.
> 3. **Transaction 3:** Two new participants (a buyer and a seller) enter the market with 5 new contracts. This increases the open interest by 5, making it 1,006.
> 4. **Transaction 4:** An existing buyer and an existing seller close out their position by entering into an offsetting contract. This transaction decreases the open interest by 1, bringing it down to 1,005.
>
> By the end of the day, the open interest in December Wheat futures is 1,005, indicating the total number of active contracts.

The **trading volume** for the day is calculated by adding up all the new contracts traded, regardless of whether they increase, decrease, or leave open interest unchanged. For the day's transactions:

- **Transaction 1:** Adds 1 to the trading volume.
- **Transaction 2:** Adds 1 to the trading volume (although it doesn't affect open interest).
- **Transaction 3:** Adds 5 to the trading volume.
- **Transaction 4:** Adds 1 to the trading volume (as it involves the exchange of contracts, despite decreasing open interest).

Therefore, the total trading volume for the day is 8 contracts.
This example illustrates how open interest and trading volume provide different insights into market activity. Open interest shows the total number of outstanding contracts, indicating market liquidity and potential future activity. Trading volume measures the number of contracts traded within a day, offering insights into the market's current activity level.

2.3.3 Daily Settlement Process and Margins

The clearinghouse calculates the settlement price daily, adjusting traders' margin accounts based on market movements:

- **Mark-to-Market:** Daily gains or losses are reflected in the margin account, ensuring that the account balance accurately represents the market value of the position.

Margins are crucial for managing credit risk in futures trading:

- **Initial Margin:** The upfront deposit required to open a position.
- **Maintenance Margin:** A lower threshold that must be maintained; falling below triggers a margin call, requiring the account to be topped up to the initial margin level again.
- **Margin Calls:** If an account falls below the maintenance margin, additional funds must be deposited to meet the initial margin requirement.

Example of Margin, Maintenance Margin, and Margin Call

Let's consider an investor, Alex, who wants to enter into a futures contract for crude oil. The price of one contract (representing 1,000 barrels) is $60 per barrel, making the total value of the contract $60,000.

2.3.3.1 Initial Margin

The exchange requires an **initial margin** of 5% to open a position. Therefore, Alex must deposit:

$$5\% \times \$60,000 = \$3,000$$

This deposit is a security against potential losses on the position.

2.3.3.2 Maintenance Margin

The exchange sets a **maintenance margin** at 3% of the contract value, which is the minimum amount that must be maintained in the margin account. For Alex, this is:

$$3\% \times \$60,000 = \$1,800$$

2.3.3.3 Margin Call

Assume the price of crude oil drops to $58 per barrel, decreasing the value of Alex's futures contract to:

$$1,000 \text{ barrels} \times \$58 = \$58,000$$

This decrease results in a loss of $2,000 on Alex's position, reducing his margin account to $1,000 ($3,000 initial margin - $2,000 loss), which is below the maintenance margin of $1,800. As a result, Alex receives a **margin call** from his broker, requiring him to deposit additional funds to bring his margin account back up to the initial margin level of $3,000.

To meet the margin call, Alex must deposit an additional $2,000 into his margin account. If Alex fails to meet the margin call, his position may be closed out by the broker to limit further losses.

This example illustrates the concepts of initial margin, maintenance margin, and margin calls within the context of futures trading. These mechanisms protect both the investor and the broker from the volatility and potential losses associated with leveraged positions in the futures market.

Some Useful Links

- Margin: Know What's Needed
- The Benefits of Futures Margins
- Understanding Futures Margin

2.4 OTC Markets

Over-the-Counter (OTC) markets facilitate the trading of forward contracts and other financial instruments directly between two parties without the intermediation of exchanges. These markets offer flexibility and customization in the contracts traded but historically lacked the transparency and regulation of their exchange-traded counterparts.

2.4.1 Regulatory Evolution

- **Pre-Crisis Era:** Prior to the 2007–2008 financial crisis, the OTC market operated with minimal regulatory oversight. This lack of regulation contributed to systemic risks, as evidenced by the crisis.
- **Post-Crisis Reforms:** In response to the financial crisis, significant regulatory measures were introduced globally to increase transparency, reduce systemic risk, and improve market stability. These include the Dodd-Frank Act in the United States and the European Market Infrastructure Regulation (EMIR) in the EU, mandating the reporting of OTC transactions and the use of central clearing parties for certain classes of derivatives.

2.4.2 Collateral Requirements

To mitigate counterparty risk, collateral requirements were introduced, including:

- **Initial Margin:** An upfront deposit required to enter into a contract, intended to cover potential future exposure in the period immediately following a default.
- **Variational Margin (or Variation Margin):** Additional funds that must be deposited to cover adverse price movements, ensuring the value of the collateral matches the exposure at the end of each trading day.

2.4.3 Clearing Mechanisms

- **Bilateral Clearing:** Traditionally, OTC markets relied on bilateral clearing, where each transaction is a separate agreement between two parties. This method allows for customization but lacks centralized risk management, making it harder to monitor and mitigate systemic risk.

- **Central Clearing:** The introduction of a **Central Counterparty (CCP)** for clearing OTC trades represents a significant shift. The CCP acts as a buyer to every seller and a seller to every buyer, reducing counterparty risk and increasing market transparency. Central clearing houses also enforce margin requirements and perform regular mark-to-market adjustments to manage risk. While OTC contracts are customized, CCPs have facilitated a move towards some level of standardization, especially in the processes for managing risk and settling trades.

2.5 Forward vs. Futures Contracts Summary

Characteristics	**Forward Contracts**	**Futures Contracts**
Market Type	Over-the-Counter (OTC)	Exchange-Traded
Standardization	Customized features to suit parties' needs	Standardized in terms of contract size and expiration
Delivery Date	Typically one specified date	Range of specified dates allowing flexibility
Settlement of Gains and Losses	At contract maturity	Daily, via marking to market
Settlement Method	Physical delivery or cash settlement	Often closed out before maturity, usually cash settled
Credit Risk	Present, due to lack of central clearing	Mitigated by the clearinghouse's guarantee
Regulation and Transparency	Limited, due to private nature	High, due to regulatory oversight and transparency
Liquidity and Market Depth	Varies, often lower due to customization	Higher, facilitated by standardization and exchange

2.6 Introduction to Hedging with Futures

Hedging using futures contracts is a risk management strategy that allows individuals and corporations to stabilize the price of assets they intend to purchase or sell in the future. This technique is critical for managing the volatility associated with commodity prices, foreign exchange rates, interest rates, and other financial variables.

- **Long Futures Hedge:** Used when anticipating the purchase of an asset in the future. A long hedge allows the buyer to lock in a purchase price, mitigating the risk of rising prices.
- **Short Futures Hedge:** Employed when planning to sell an asset in the future. It enables the seller to secure a selling price, protecting against falling prices.

2.6.0.1 Arguments in Favor of Hedging

- **Business Focus:** Hedging allows companies to concentrate on their core business activities by reducing exposure to financial market risks.
- **Cost Predictability:** It provides predictability in costs and revenues, which is beneficial for budgeting and financial planning.
- **Risk Management:** Effectively manages risks related to fluctuations in interest rates, exchange rates, and commodity prices.

2.6.0.2 Arguments Against Hedging

- **Shareholder Autonomy:** Critics argue that shareholders can diversify their portfolios independently and undertake personal hedging strategies if desired.
- **Competitive Risk:** There's a belief that hedging could introduce additional risk, especially if competitors choose not to hedge, potentially affecting market dynamics and competitive positioning.
- **Complexity and Perception:** Hedging can be complex to implement and explain, especially in scenarios where the hedge results in a loss while the underlying asset gains in value, which might confuse stakeholders.

2.7 Hedging and Basis Risk

Basis risk is a critical concept in the realm of financial hedging, referring to the risk that the difference between the spot price and the futures price (the basis) will not behave as anticipated when the hedge is unwound. This discrepancy can lead to less effective hedges and unexpected financial outcomes.

Basis risk arises due to several factors:

1. **Asset Mismatch:** Often, the asset being hedged does not perfectly match the asset underlying the futures contract, introducing a discrepancy in price movements.
2. **Timing Uncertainty:** The exact timing of when an asset will be bought or sold can add uncertainty, affecting the effectiveness of the hedge.
3. **Early Contract Closure:** Sometimes, it is necessary to close out a futures contract before its delivery month, which can influence the basis and the outcome of the hedge.

To mitigate basis risk, selecting an appropriate futures contract is vital:

- **Delivery Month:** Opt for a contract with a delivery month as close as possible to but after the anticipated end of the hedge to minimize the time gap's impact on the basis.

- **Cross Hedging:** When direct futures contracts for the asset are unavailable, choose a contract with a price highly correlated to the asset's price. This approach, known as cross hedging, can help manage the risk despite the asset mismatch.

Long Hedge Example for Asset Purchase

- Initial Futures Price (F_1): 88.0
- Futures Price at Purchase (F_2): 89.1
- Spot Price at Purchase (S_2): 90.0
- Basis at Purchase (b_2): 0.9 (calculated as $S_2 - F_2$)

Hedging Outcome:

- The asset's cost is 90.0.
- The gain from futures is 1.1 ($F_2 - F_1$).
- The net amount paid effectively becomes 88.9, considering the gain from futures and the basis at purchase.

Short Hedge Example for Asset Sale

- Initial Futures Price (F_1): 0.98
- Futures Price at Sale (F_2): 0.925
- Spot Price at Sale (S_2): 0.92
- Basis at Sale (b_2): -0.005 (calculated as $S_2 - F_2$)

Hedging Outcome:

- The asset's sale price is 0.92.
- The gain from futures is 0.055 ($F_1 - F_2$).
- The net amount received effectively becomes 0.975, factoring in the gain from futures and the basis at sale.

2.8 Cross Hedging

Cross hedging involves using futures contracts of a commodity or asset that is different from, but correlated to, the asset being hedged. It is particularly useful when no futures market exists for the specific asset in question. The effectiveness of a cross hedge depends on the correlation between the price movements of the hedged asset and the futures contract used for hedging.

The **optimal hedge ratio** (proportion of exposure to be hedged), denoted as h^*, is calculated using the formula:

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

- ρ is the coefficient of correlation between ΔS and ΔF, indicating how closely the futures prices move in relation to the spot prices of the hedged asset.
- σ_S is the standard deviation of ΔS, representing the change in the spot price of the asset being hedged over the hedging period.
- σ_F is the standard deviation of ΔF, representing the change in the futures price over the hedging period.

To determine the **optimal number of futures contracts** (N^*) required for the hedge, **without adjusting for daily settlement**, is:

$$N^* = h^* \times \frac{Q_A}{Q_F}$$

- Q_A represents the size of the position being hedged, measured in units.
- Q_F denotes the size of one futures contract, also in units.

Example: Hedging Jet Fuel with Heating Oil Futures

Consider an airline that anticipates purchasing 2 million gallons of jet fuel in one month and decides to hedge the price risk using heating oil futures, based on their historical price movements and correlation:

- Historical data provides $\sigma_F = 0.0313$, $\sigma_S = 0.0263$, and $\rho = 0.928$.
- Applying the formula for h^*:

$$h^* = 0.928 \times \frac{0.0263}{0.0313} = 0.78$$

- Given that one heating oil futures contract covers 42,000 gallons, the optimal number of contracts to hedge the airline's exposure is calculated as:

$$0.78 \times \frac{2,000,000}{42,000} \approx 37$$

Thus, to optimally hedge against the price risk of jet fuel, the airline should purchase approximately 37 heating oil futures contracts.

2.8.1 Daily Settlement and Tailing Adjustment in Hedging

The daily settlement process and the concept of tailing adjustments are crucial for refining the hedging strategy, particularly in the context of futures markets. These adjustments are necessary to account for the nuances of daily price changes and the mechanics of futures contracts.

The optimal hedge ratio, **adjusted for daily settlement**, is given by:

$$\hat{h} = \hat{\rho}\frac{\hat{\sigma_S}}{\hat{\sigma_F}}$$

- $\hat{\rho}$ represents the correlation between percentage daily changes in the spot and futures prices. This measures how closely the futures prices move in relation to the spot prices on a day-to-day basis.
- $\hat{\sigma_S}$ is the standard deviation of the percentage daily changes in the spot price, quantifying the daily price volatility of the spot asset.
- $\hat{\sigma_F}$ is the standard deviation of the percentage daily changes in the futures price, indicating the daily volatility in the futures market.

The **optimal number of futures contracts**, incorporating tailing adjustments for daily settlement, can be calculated as follows:

$$N^* = \hat{h} \times \frac{V_A}{V_F}$$

- V_A represents the total value of the position being hedged, calculated as the spot price multiplied by the quantity of the asset (Q_A).
- V_F denotes the value of one futures contract, determined by the futures price multiplied by the quantity specified in one futures contract (Q_F).

This adjustment, often referred to as "tailing the hedge," modifies the hedge ratio to account for the impact of daily settlements on the futures position. It ensures that the hedge remains effective in offsetting the price risk of the underlying asset over the hedging period.

i Example: Hedging with Tailing Adjustments

A transportation company, "TransCo," needs to purchase 500,000 gallons of diesel fuel in one month. However, there are no futures contracts available for diesel. TransCo decides to use heating oil futures as a hedge because heating oil prices are highly correlated with diesel prices. Each heating oil futures contract covers 42,000 gallons.

- Current diesel price (spot price): $3.00 per gallon
- Current heating oil futures price for next month's delivery: $2.90 per gallon

- Correlation between daily percentage changes in diesel and heating oil prices ($\hat{\rho}$): 0.9
- Standard deviation of daily percentage changes in diesel prices ($\hat{\sigma_S}$): 1.5%
- Standard deviation of daily percentage changes in heating oil futures prices ($\hat{\sigma_F}$): 1.8%

2.8.1.1 Solution

- Calculate the optimal hedge ratio

$$\hat{h} = \hat{\rho}\frac{\hat{\sigma_S}}{\hat{\sigma_F}} = 0.9 \times \frac{0.015}{0.018} = 0.75$$

- Determine the V_A and V_F

$$V_A = \$3.00 \times 500,000 = \$1,500,000$$
$$V_F = \$2.90 \times 42,000 = \$121,800$$

- Calculate the optimal number of futures contracts

$$N^* = \hat{h} \times \frac{V_A}{V_F} = 0.75 \times \frac{1,500,000}{121,800} \approx 9.23$$

Given that partial contracts cannot be purchased, TransCo would need to round to the nearest whole number. In practice, the company might choose to hedge with 9 contracts to avoid over-hedging, understanding that this leaves a slight unhedged exposure.
By purchasing 9 heating oil futures contracts, TransCo effectively hedges against the risk of rising diesel prices. The chosen hedge ratio of 0.75, derived from the correlation and volatility (standard deviation) differences between diesel and heating oil prices, optimizes the hedge despite the imperfect match between the two commodities. This example illustrates how cross-hedging can be applied when direct hedging instruments are not available, using tailing adjustments to fine-tune the hedge for daily settlement impacts.

2.9 Stock Index Futures

Stock index futures are a powerful tool for hedging the risk in a portfolio. By taking positions in these futures, investors can manage exposure to market fluctuations without the need to alter the composition of their portfolio.

2.9.1 Hedging Portfolio

The formula to determine the number of contracts to short for hedging purposes is:

$$\beta \times \frac{V_A}{V_F}$$

- V_A is the value of the portfolio being hedged.
- β measures the portfolio's sensitivity to market movements. A beta of 1 indicates that the portfolio moves in line with the market.
- V_F is the value of one futures contract, calculated as the futures price multiplied by a specified multiplier (e.g., $250 times the index for S&P 500 futures).

> **Example: Stock Index Futures Hedging**
>
> Consider a portfolio with the following characteristics:
>
> - S&P 500 futures price: 1,000 points
> - Portfolio value: $5 million
> - Portfolio beta: 1.5
> - Value of one futures contract: $250,000 ($250 times the index)
>
> Necessary position in S&P 500 futures to hedge the portfolio:
>
> $$1.5 \times \frac{5,000,000}{250,000} = 30$$
>
> Thus, to hedge the portfolio against market movements, it would be necessary to short 30 S&P 500 futures contracts.

2.9.2 Adjusting Portfolio Beta

To modify the beta of a portfolio, the formula is:

$$(\beta^* - \beta) \times \frac{V_A}{V_F}$$

- β^* is the desired new beta level.
- A negative outcome suggests shorting futures to decrease beta, while a positive value indicates going long on futures to increase beta.

Example: Adjusting Portfolio Beta

A portfolio manager seeks to adjust the beta of a $10 million portfolio. The current beta of the portfolio is 1.2, but due to a bearish market outlook, the manager wishes to decrease the portfolio's beta to 0.8 to reduce exposure to market volatility. The futures contract used for hedging is based on an index that is currently priced at 3,000 points, with each contract representing $250 times the index value.

- Current portfolio value (V_A): $10,000,000
- Current portfolio beta (β): 1.2
- Desired portfolio beta (β^*): 0.8
- Index futures price: 3,000 points
- Value of one futures contract (V_F): $250 ×3,000 = $750,000

Calculate the number of futures contracts needed to adjust the portfolio's beta to 0.8.

$$N^* = -0.4 \times \frac{10,000,000}{750,000} = -0.4 \times 13.33 \approx -5.33$$

Since we cannot trade a fraction of a contract, we'll need to round to the nearest whole number. In this case, rounding to -5 suggests that the portfolio manager should short 5 futures contracts to achieve the desired beta adjustment.

2.9.3 Additional Notes About Hedging

2.9.3.1 Why Hedge Equity Returns?

Hedging with stock index futures is advantageous for several reasons:

- **Market Timing:** Allows investors to effectively "exit" the market without selling assets, ideal for avoiding transaction costs and capital gains taxes.
- **Risk Management:** Enables precise control over the portfolio's exposure to market risk.
- **Performance:** Ensures returns are based on the specific selection and performance of portfolio assets, beyond the general market movements.

Tip

Imagine your portfolio's stocks have an average beta of 1.0, aligning their performance with the market. Yet, you're confident these stocks will surpass market returns in any scenario. By hedging, you secure returns at the risk-free rate plus any outperformance of your stocks over the market. This strategy minimizes market volatility's impact, ensuring your portfolio's gains are primarily due to your stock selection skills.

2.9.3.2 Stack and Roll Strategy

This approach involves rolling futures contracts forward to manage future exposures:

- Start with futures contracts hedging up to a certain time horizon.
- Before these contracts mature, close them out and replace them with new contracts for the next period.
- This method allows continuous hedging over time, adjusting as market conditions and portfolio needs change.

However, it's crucial to be mindful of the liquidity risks and potential for realized losses.

- **Realized vs. Unrealized Gains/Losses:** Hedging can lead to scenarios where losses on the hedge are realized while corresponding gains in the underlying assets are unrealized, potentially causing liquidity issues.

Warning

The Metallgesellschaft (MG) case in the early 1990s serves as a cautionary example of how rolling forward hedges can lead to significant cash flow problems. MG sold long-term fixed-price contracts for heating oil and gasoline, hedging its exposure with short-dated long futures that were regularly rolled over. When oil prices dropped, MG faced substantial margin calls, creating severe short-term liquidity issues. Despite the hedging strategy's long-term rationale, the immediate financial strain alarmed the company's management and bankers. Ultimately, MG had to close out its hedge positions and cancel the fixed-price contracts with customers, resulting in a staggering loss of $1.33 billion.

2.10 Practice Questions and Problems

2.10.1 Fundamentals of Futures Trading

1. What are the most important aspects of the design of a new futures contract?
2. The party with a short position in a futures contract sometimes has options as to the precise asset that will be delivered, where delivery will take place, when delivery will take place, and so on. Do these options increase or decrease the futures price? Explain your reasoning.
3. What do you think would happen if an exchange started trading a contract in which the quality of the underlying asset was incompletely specified?

4. "Speculation in futures markets is pure gambling. It is not in the public interest to allow speculators to trade on a futures exchange." Discuss this viewpoint.

2.10.2 Open Interest and Trading Volume

1. Distinguish between the terms open interest and trading volume.
2. "When a futures contract is traded on the floor of the exchange, it may be the case that the open interest increases by one, stays the same, or decreases by one." Explain this statement.
3. Why does the open interest usually decline during the month preceding the delivery month? On a particular day, there were 2,000 trades in a particular futures contract. This means that there were 2000 buyers (going long) and 2000 sellers (going short). Of the 2,000 buyers, 1,400 were closing out positions and 600 were entering into new positions. Of the 2,000 sellers, 1,200 were closing out positions and 800 were entering into new positions. What is the impact of the day's trading on open interest?

2.10.3 Margin Mechanics in Futures Trading

1. Explain how margin accounts protect investors against the possibility of default.
2. Suppose that you enter into a short futures contract to sell July silver for \$17.20 per ounce. The size of the contract is 5,000 ounces. The initial margin is \$4,000, and the maintenance margin is \$3,000. What change in the futures price will lead to a margin call? What happens if you do not meet the margin call?

> **Solution**
>
> $S = \$17.40$

3. A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is \$6,000 per contract, and the maintenance margin is \$4,500 per contract. What price change would lead to a margin call? Under what circumstances could \$2,000 be withdrawn from the margin account?

> **Solution**
>
> $S = 166.67$ cents

4. A company enters into a short futures contract to sell 5,000 bushels of wheat for 750 cents per bushel. The initial margin is \$3,000 and the maintenance margin is \$2,000. What price change would lead to a margin call? Under what circumstances could \$1,500 be withdrawn from the margin account?

> **Solution**
>
> Margin call: $S = 770$ cents; Withdraw \$1500: $S = 720$ cents

2.10.4 Basics of Hedging with Futures

1. Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?
2. Explain what is meant by basis risk when futures contracts are used for hedging.
3. Explain what is meant by a perfect hedge. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.
4. Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.
5. "For an asset where futures prices for contracts on the asset are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

2.10.5 Advanced Hedging Scenarios and Strategies

1. Sixty futures contracts are used to hedge an exposure to the price of silver. Each futures contract is on 5,000 ounces of silver. At the time the hedge is closed out, the basis is \$0.20 per ounce. What is the effect of the basis on the hedger's financial position if (a) the trader is hedging the purchase of silver and (b) the trader is hedging the sale of silver?
2. In the corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in a) June, b) July, and c) January.
3. Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a three-month contract? What does it mean?

> **Solution**
>
> $h = 64.2\%$

4. The standard deviation of monthly changes in the spot price of live cattle is 1.2 (in cents per pound). The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

Solution

Long 3 contracts.

2.10.6 Practical Concerns and Risk Management

1. Give three reasons why the treasurer of a company might not hedge the company's exposure to a particular risk.
2. A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather." Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?
3. Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.
4. A futures contract is used for hedging. Explain why the daily settlement of the contract can give rise to cash flow problems.

2.10.7 Hedging with Stock and Commodity Futures

1. A company has a $20 million portfolio with a beta of 1.2. It would like to use futures contracts on a stock index to hedge its risk. The index futures is currently standing at 1080, and each contract is for delivery of $250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6? What should the company do if it wants to increase the beta of the portfolio to 1.5?

Solution

- short 89 contracts
- short 44 contracts to reduce beta
- long 22 contracts to increase beta

2. On July 1, an investor holds 50,000 shares of a certain stock. The market price is $30 per share. He decides to use the September Mini S&P 500 futures contract. The index is currently 1,500 and one contract is for delivery of $50 times the index. The beta of the stock is 1.3. What strategy should the investor follow? Under what circumstances will it be profitable?

Solution

Short 26 contracts.

3. It is now June. A company knows that it will sell 5,000 barrels of crude oil in September. It uses the October CME Group futures contract to hedge the price it will receive. Each contract is on 1,000 barrels of "light sweet crude". What position should it take? What price risks is it still exposed to after taking the position?

3 Determination of Forward and Futures Prices

References

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  - Chapter 2 - Basics of Derivative Pricing and Valuation
  - Chapter 3 - Pricing and Valuation of Forward Commitments

Learning Outcomes:

- Understand the principles of pricing and valuation for forward and futures contracts.
- Apply pricing models to specific assets, including commodities and financial instruments.
- Analyze the relationship between futures prices and expected spot prices.
- Understand the structure and valuation of Forward Rate Agreements (FRAs).

3.1 Pricing and Valuation of Forward/Futures Contracts

- **Pricing** of forward/futures contracts entails determining the fair market price or rate at the **initiation** of the contract. This process ensures that the contract's initial value is neutral, aligning the interests of both parties without giving an undue advantage to either side.
- **Valuation**, on the other hand, is the process of assessing the contract's current value **after its initiation**. The value fluctuates based on market conditions, reflecting the gain or loss to the contract holder.

Example: Forward Price vs. Value

- An investor enters a long position in a December Gold futures contract on 2 November, agreeing to a price of $1,250/oz. Initially, the contract's value is set to $0, indicating a fair agreement based on current market expectations.
- Post-initiation, the contract's value is subject to change due to market dynamics. For instance, if on 3 November, the market price for December Gold futures falls to $1,225/oz, the value of the investor's long position declines, rendering a **negative value** to their investment.

3.2 Pricing and Valuation Methodology

- **Risk Aversion:** Investors require compensation for taking on additional risk, reflecting the principle that higher risk should be rewarded with higher potential returns.
- **Risk-Neutral Pricing:** Under this approach, it's assumed that investors are indifferent to risk. The pricing of derivatives through arbitrage opportunities ensures that the portfolio, combining the derivative and its underlying asset, yields the risk-free rate of return.
- **Arbitrage-Free Pricing:** This methodology prices derivatives based on the assumption that the market operates under risk-neutral conditions and is free of arbitrage opportunities, ensuring no risk-free profits can be made from market inefficiencies.

The framework for pricing and valuation operates on the principle that market prices adjust to preclude arbitrage profits. The **Law of One Price** asserts that identical cash flows must have the same price, irrespective of future outcomes.

Arbitrageurs follow two cardinal rules:

1. Do not use your own money.
2. Do not take any price risk.

3.2.1 Short Selling

Short selling entails the sale of securities not owned by the seller, typically facilitated by borrowing these securities from a broker. The seller aims to buy back the securities at a lower price to profit from the difference. Obligations include covering dividends and possibly a borrowing fee.

Example: Short Selling

Assuming you short sell 100 shares at \$100 each and close out the short position three months later at \$90, while a \$3 per share dividend is issued during this period:

- Your profit would be the difference in share prices minus the dividends paid, highlighting the risks and rewards of short selling.

$$100 \times (100 - 90 - 3) = \$700$$

- Conversely, buying 100 shares would result in a loss if the share price decreases.

$$100 \times (90 - 100 + 3) = \$ - 700$$

3.2.2 Consumption vs. Investment Assets

- **Investment Assets:** These are assets like gold, stocks, and bonds, held primarily for their potential to appreciate in value over time.
- **Consumption Assets:** Assets such as copper or oil are classified as consumption assets, held mainly for their utility in production or consumption rather than for investment.

3.3 Forward Price for an Investment Asset

3.3.1 Assumptions and Notation

3.3.1.1 Assumptions

1. Market participants incur no transaction costs when trading.
2. All net trading profits are taxed at the same rate for market participants.
3. Market participants can borrow and lend money at the identical risk-free interest rate.
4. Arbitrage opportunities are exploited immediately by market participants.

3.3.1.2 Notation

- S_0: Current spot price of the asset.
- F_0: Today's futures or forward price of the asset.
- T: Time to the delivery date of the contract (in years).

- r: Annualized risk-free interest rate over the period T.

3.3.2 Arbitrage Example

Consider an asset that provides no income, with a current price of \$40, subject to an annual interest rate of 5%, and associated with a forward contract that has a maturity of 3 months.

3.3.2.1 Case 1: Forward Price = \$43

- **Action now:** Borrow \$40 at 5% annual interest for 3 months. Purchase one unit of the asset and enter into a forward contract to sell it in 3 months at \$43.
- **Action in 3 months:** Sell the asset at the forward price of \$43. Repay the loan, which amounts to \$40.50 with interest.
- **Profit realized:** \$2.50.

3.3.2.2 Case 2: Forward Price = \$39

- **Action now:** Short sell one unit of the asset for \$40. Invest the proceeds at 5% annual interest for 3 months. Enter into a forward contract to buy the asset in 3 months at \$39.
- **Action in 3 months:** Buy back the asset at the forward price of \$39. Close the short position and retrieve \$40.50 from the investment.
- **Profit realized:** \$1.50.

3.3.3 The Forward Price Formula

The forward price F_0 for an investment asset that does not provide income and is deliverable in T years is given by the following formula, where r is the annualized risk-free interest rate for the period T.

$$F_0 = S_0 e^{rT}$$

In the given example, with $S_0 = 40$, $T = 0.25$ (3 months), and $r = 0.05$ (5%), the forward price is calculated as:

$$F_0 = 40e^{0.05\times 0.25} = 40.50$$

This calculation demonstrates that the no-arbitrage forward price aligns with the theoretical value, preventing arbitrage opportunities under these conditions.

3.3.4 Adjustments for Known Income or Yield

When the asset provides a known income or yield, the forward price adjusts to account for this. For an asset generating a known income I (expressed as the present value of the income) over the contract's life, the formula modifies to:

$$F_0 = (S_0 - I)e^{rT}$$

For assets providing a yield q (expressed as a continuous compounding rate), the forward price formula adjusts to:

$$F_0 = S_0 e^{(r-q)T}$$

3.4 Valuing a Forward Contract

Initially, a forward contract's value is zero, except for potential impacts from the bid-offer spread. This neutrality in value reflects the agreement's fairness based on current market conditions.

As time progresses, the value of the forward contract can become positive or negative, reflecting changes in market conditions and the underlying asset's price relative to the contract's terms.

Let K represent the agreed-upon delivery price in the contract, and F_0 denote the forward price for a contract negotiated today for future delivery.

The valuation of forward contracts can be understood by comparing contracts with different delivery prices. This comparison reveals:

- The value of being in a long position in a forward contract, where one has agreed to buy the asset, is $(F_0 - K)e^{-rT}$. This formula represents the present value of the profit from the contract if the forward price F_0 set today is higher than the contract's delivery price K.
- Conversely, the value of a short forward contract, where one has agreed to sell the asset, is $(K - F_0)e^{-rT}$. This expresses the present value of the profit from the contract if the delivery price K exceeds today's forward price F_0.

These valuations underscore the importance of the risk-free rate r and the time to delivery T in determining the present value of future contract profits or losses.

3.5 Forward vs. Futures Prices

While forward and futures contracts are similar in their function of agreeing to buy or sell an asset at a future date, their pricing can diverge under certain conditions, despite the common assumption that they are equal when the maturity and asset prices coincide. One notable exception is Eurodollar futures, which often exhibit pricing anomalies due to their specific market characteristics.

The theoretical distinction in pricing between forwards and futures arises primarily from the volatility and uncertainty of interest rates:

- When there is a strong positive correlation between interest rates and the asset's price, futures prices tend to be slightly higher than forward prices. This is because the daily settlement of futures can lead to a net gain in environments where rising interest rates accompany rising asset prices, due to the reinvestment of gains at higher rates.
- Conversely, a strong negative correlation between interest rates and the asset's price suggests that forward prices may exceed futures prices. In such scenarios, the absence of daily settlement in forward contracts avoids the potential loss from having to reinvest at lower interest rates.

3.6 Forward Price for Specific Assets

3.6.1 Stock Index

A stock index can be regarded as an investment asset that effectively pays a dividend yield, analogous to the income generated by holding the underlying stocks.

The relationship between the futures price (F_0) and the spot price (S_0) of the index is captured by the formula:

$$F_0 = S_0 e^{(r-q)T}$$

Here, q represents the average dividend yield of the portfolio reflected by the index over the contract's life.

This formula assumes the index acts as a tradable investment asset, with changes in the index mirroring changes in the value of a corresponding tradable portfolio. For instance, the Nikkei 225, when viewed purely as a numerical value without considering its underlying assets, does not qualify as an investment asset in this context.

Index Arbitrage

Index arbitrage exploits discrepancies between the futures price and the spot price adjusted for the dividend yield and risk-free rate. When F_0 exceeds $S_0 e^{(r-q)T}$, arbitrageurs purchase the underlying stocks of the index while selling futures. Conversely, if $F_0 < S_0 e^{(r-q)T}$, they buy futures and short sell the index's underlying stocks.
This strategy requires executing trades in futures and various stocks simultaneously, often facilitated by computer algorithms to manage the complexity and timing. However, real-world frictions sometimes prevent perfect execution, leading to temporary deviations from the theoretical no-arbitrage relationship.

3.6.2 Exchange Rates

A foreign currency functions similarly to a security that yields interest, where the yield is the foreign country's risk-free interest rate.

Given r_f as the foreign risk-free interest rate, the forward exchange rate formula is:

$$F_0 = S_0 e^{(r-r_f)T}$$

Exchange Rate Arbitrage

Consider two strategies for converting 1,000 units of a foreign currency into dollars by time T, where S_0 represents the current spot exchange rate, F_0 the forward exchange rate, and r and r_f the domestic and foreign risk-free interest rates, respectively.

- **Strategy 1:** Convert the 1,000 units of foreign currency into dollars at the future time T by initially investing these units at the foreign risk-free rate r_f. Then, agree to a forward contract to sell the resulting amount for dollars at time T.
- **Strategy 2:** Immediately exchange the 1,000 units of foreign currency for dollars at the current spot rate S_0, then invest this dollar amount at the domestic risk-free rate r until time T.

The principle of no arbitrage implies that both strategies must yield the same outcome in dollar terms to prevent free profit opportunities. This condition leads to the following equation:

$$1,000 \times e^{r_f \times T} \times F_0 = 1,000 \times S_0 \times e^{r \times T}$$

Simplifying this equation provides the formula for the forward exchange rate:

$$F_0 = S_0 e^{(r-r_f)T}$$

3.6.3 Commodities

For **consumption assets** like commodities, storage costs represent a form of negative income, impacting the forward price:

$$F_0 \leq S_0 e^{(r+u)T}$$

Here, u stands for the storage cost per unit time as a percentage of the asset's value. Alternatively, incorporating the present value of storage costs U:

$$F_0 \leq (S_0 + U)e^{rT}$$

These formulas accommodate the costs associated with holding and storing physical commodities, from agricultural products to metals, affecting their forward pricing.

3.6.4 The Cost of Carry

The cost of carry (c) combines storage costs, interest expenses, and any income earned. For investment assets, the forward price formula simplifies to:

$$F_0 = S_0 e^{cT}$$

In contrast, for consumption assets, the formula accounts for potential lower bounds due to additional costs, hence:

$$F_0 \leq S_0 e^{cT}$$

The concept of a **convenience yield** (y) is introduced for consumption assets, acknowledging the benefits from holding the physical asset as opposed to a derivative position. It's defined such that:

$$F_0 = S_0 e^{(c-y)T}$$

3.7 Futures Prices and Expected Spot Prices

When analyzing futures contracts, we consider the expected return required by investors, denoted as k. This expected return plays a critical role in determining the futures price F_0 relative to the expected spot price at maturity $E(S_T)$.

An investment strategy involving futures contracts can be described as follows:

- By investing an amount $F_0 e^{-rT}$ at the risk-free rate r and entering a long futures contract, an investor can secure a cash inflow of S_T at maturity.
- The present value (PV) of this investment strategy is calculated as $-F_0 e^{-rT} + E(S_T)e^{-kT}$, where E denotes the expected value.

In efficient markets, investments are typically priced to yield a zero net present value (NPV). This principle leads to the fundamental pricing relation for futures contracts:

$$F_0 = E(S_T)e^{(r-k)T}$$

The relationship between futures prices and expected future spot prices varies with the underlying asset's systematic risk:

- **No Systematic Risk**: When the asset carries no systematic risk, the expected return from the asset k equals the risk-free rate r, leading to $F_0 = E(S_T)$.
- **Positive Systematic Risk**: For assets with positive systematic risk, such as stock indices, $k > r$ implies that futures prices will be lower than the expected spot prices $F_0 < E(S_T)$.
- **Negative Systematic Risk**: Conversely, assets like gold, which may exhibit negative systematic risk during certain periods, have $k < r$, resulting in futures prices exceeding expected spot prices $F_0 > E(S_T)$.

Contango and Backwardation

- CME Institute
- **Backwardation**: This market condition occurs when futures prices are below the expected future spot prices, often observed in markets expecting lower future prices.
- **Contango**: In contrast, a contango market condition is characterized by futures prices that exceed expected future spot prices, typically occurring in markets where future prices are anticipated to rise.

3.8 Forward Rate Agreement (FRA)

An FRA is a financial contract that allows parties to exchange a fixed interest rate for a floating rate (often based on LIBOR, SOFR, or SONIA) at a future date, applied to a specified principal amount. The agreement does not involve the exchange of principal but rather the difference in interest payments based on the agreed rates. The value of an FRA at inception is zero, as the fixed rate equals the forward rate. The value changes as the forward rate changes over time.

Example: FRA

Consider an FRA where Party A and Party B agree to exchange a fixed rate of 3% for three-month SOFR on a principal of $100 million in two years. The interest rates are compounded quarterly.

- **Position of Parties:** Party A will pay the floating SOFR rate and receive a fixed 3%. Party B takes the opposite position.
- **Outcome:** If the three-month SOFR rate is 3.5% in two years, Party A receives a net payment from Party B calculated as follows:

$$\$100,000,000 \times (0.035 - 0.030) \times 0.25 = \$125,000$$

This amount would ideally be paid at 2.25 years. However, considering the advance determination of SOFR, the actual payment is adjusted to its present value and made at the 2-year mark (discounted for three months at 3.5%).

- **Zero Rates**: The n-year zero rate (or spot rate) is the interest rate for an investment lasting n years, with interest and principal paid at the end of the term. No intermediate payments are made.

Example: Zero Rate

A 5-year zero rate of 5% per annum, under continuous compounding, means $100 grows to $128.40 over 5 years, calculated as $100 \times e^{0.05 \times 5}$.

- **Forward Rates**: Forward rates represent the future interest rates implied by current zero rates for future periods. Derived from zero rates for various maturities, indicating the cost of borrowing or lending money in the future, as implied by today's rates.

For calculating a forward rate between times T_1 and T_2, given zero rates R_1 and R_2 for these times, the forward rate, R_F, is:

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

Example: Forward Rate

Year	Zero rate for an n-year investment (% per annum)	Forward rate for nth year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

To illustrate, consider calculating the forward rate for the fourth year using the data provided in the Table. Assuming $T_1 = 3$, $T_2 = 4$, $R_1 = 4.6\%$, and $R_2 = 5.0\%$, the forward rate, R_F, is calculated as follows:

$$R_F = \frac{0.05 \times 4 - 0.046 \times 3}{4 - 3} = 6.2\%$$

This result indicates the forward rate for the fourth year is 6.2% per annum, which is higher than the zero rate for the same period due to the upward sloping yield curve.

The value of an FRA at inception is zero since the fixed rate typically equals the forward rate for the reference rate at that time. However, the value can change over time as the forward rate fluctuates. The valuation of an FRA can be determined by comparing the difference between the agreed fixed rate (R_K) and the current forward rate (R_F) over the contract period (t), discounted back to the present value.

3.9 Practice Questions and Problems

3.9.1 Basics of Forward and Futures Contracts

1. Explain what happens when an investor shorts a certain share.
2. What is the difference between the forward price and the value of a forward contract?
3. What is meant by (a) an investment asset and (b) a consumption asset. Why is the distinction between investment and consumption assets important in the determination of forward and futures prices?
4. Explain carefully why the futures price of gold can be calculated from its spot price and other observable variables whereas the futures price of copper cannot.

5. Explain carefully the meaning of the terms convenience yield and cost of carry. What is the relationship between futures price, spot price, convenience yield, and cost of carry?
6. What is the cost of carry for (a) a non-dividend-paying stock, (b) a stock index, (c) a commodity with storage costs, and (d) a foreign currency?

3.9.2 Calculation of Forward and Futures Prices

7. Suppose that you enter into a three-month forward contract on a non-dividend-paying stock when the stock price is $108 and the risk-free interest rate (with continuous compounding) is 4% per annum. What is the forward price?

> **Solution**
>
> $F_0 = 109.085$

8. A four-months long forward contract on a non-dividend-paying stock is entered into when the stock price is $150 and the risk-free rate of interest is 5.7% per annum with continuous compounding.

   - a) What are the forward price and the initial value of the forward contract?
   - b) Two months later, the price of the stock is $168 and the risk-free interest rate is still 5.7%. What are the forward price and the value of the forward contract?

> **Solution**
>
> Forward price $F_0 = 152.88$; initial value 0; final value 169.6

9. The risk-free rate of interest is 4.1% per annum with continuous compounding, and the dividend yield on a stock index is 6.2% per annum. The current value of the index is 2445. What is the one-month futures price?

> **Solution**
>
> $F_0 = 2440.725$

10. A stock index currently stands at 725. The risk-free interest rate is 7.6% per annum (with continuous compounding) and the dividend yield on the index is 1.8% per annum. What should the futures price for a three-month contract be?

> **Solution**
>
> $F_0 = 735.589$

11. An index is 550. The three-month risk-free rate is 4.60% per annum and the dividend yield over the next three months is 5.80% per annum. The six-month risk-free rate is 5.34% per annum and the dividend yield over the next six months is 4.93% per annum. Estimate the futures price of the index for three-month and six-month contracts. All interest rates and dividend yields are continuously compounded.

> **Solution**
>
> 3-month $F_0 = 548.352$; 6-month $F_0 = 551.129$

12. The spot price of silver is $11 per ounce. The storage costs are $0.25 per ounce payable quarterly in advance. Assuming that interest rates are 1.80% per annum for all maturities, calculate the futures price of silver for delivery in nine months.

> **Solution**
>
> $F_0 = 11.91$

13. The spot price of oil is $39 per barrel and the cost of storing a barrel of oil for one year is $1.2, payable at the end of the year. The risk-free interest rate is 8.60% per annum, continuously compounded. What is an upper bound for the one-year futures price of oil?

> **Solution**
>
> $F_0 = 43.70$

3.9.3 Arbitrage and Risk Management

14. Suppose that the risk-free interest rate is 3.00% per annum with continuous compounding and that the dividend yield on a stock index is 0.60% per annum. The index is standing at 3646, and the futures price for a contract deliverable in ten months is 3701. What arbitrage opportunities does this create?

> **Solution**
>
> $F_0 = 3719.654$

15. The eight-month interest rates in Switzerland and the United States are, respectively, 3.60% and 5.40% per annum with continuous compounding. The spot price of the Swiss franc is $0.9. The futures price for a contract deliverable in two months is also $0.9. What arbitrage opportunities does this create?

Solution

$F_0 = 0.911$

16. When a known future cash outflow in a foreign currency is hedged by a company using a forward contract, there is no foreign exchange risk. When it is hedged using futures contracts, the daily settlement process does leave the company exposed to some risk. Explain the nature of this risk. Assume that the forward price equals the futures price. In particular, consider whether the company is better off using a futures contract or a forward contract when

    - a) The value of the foreign currency falls rapidly during the life of the contract
    - b) The value of the foreign currency rises rapidly during the life of the contract
    - c) The value of the foreign currency first rises and then falls back to its initial value
    - d) The value of the foreign currency first falls and then rises back to its initial value

4 Swaps

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  - Chapter 7 - Swaps
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  - Chapter 3 - Pricing and Valuation of Forward Commitments

Learning Outcomes:

- Understand the fundamental principles and nature of swaps, including their purpose and functionality in financial markets.
- Analyze the mechanics and applications of interest rate swaps, identifying the roles of comparative advantage and valuation methods.
- Explore currency swaps, including their types, uses, and the valuation process for fixed-for-fixed currency swaps.
- Examine additional swap arrangements outside of interest rate and currency swaps, recognizing their unique features and applications.

4.1 Nature of Swaps

- **Definition and Nature**: A swap is an OTC contract, inherently subject to default risk due to its bilateral negotiation and customization. It represents a mutual agreement to exchange cash flows at designated times, adhering to agreed-upon rules.
- **Comparison with Forward Contracts**: Unlike forward contracts, which culminate in a single cash flow exchange on a future date, swaps facilitate multiple cash-flow exchanges over time. This difference highlights swaps' flexibility and applicability in various financial strategies.

- Conceptually, a swap can be viewed as a series of forward contracts bundled together. This perspective underscores its utility in hedging and financing operations, where the timing and value of cash flows are critical.
- **Value Dynamics at Inception**: At the initiation of a swap, its market value is zero. This initial neutrality reflects a balance within the swap's structure, where some forward components might hold positive value, and others, negative. Over time, the value of these components evolves, influenced by market conditions and the underlying assets' performance.

4.2 Interest Rate Swaps

This section delves into a classic example of a "Plain Vanilla" Interest Rate Swap, illustrating a transaction between two prominent entities: Apple and Citigroup. Over a three-year period, they engage in an exchange that highlights the fundamental mechanics of interest rate swaps.

- **Transaction Overview**: Apple commits to paying Citigroup a fixed interest rate of 3% per annum. These payments occur bi-annually, based on a hypothetical principal amount of $100 million. Conversely, Citigroup is responsible for paying Apple based on the six-month SOFR rate, calculated against the same principal value.
- **Roles Defined**: In this arrangement, Apple assumes the role of the **fixed-rate payer**, while Citigroup takes on the position of the **floating-rate payer**.

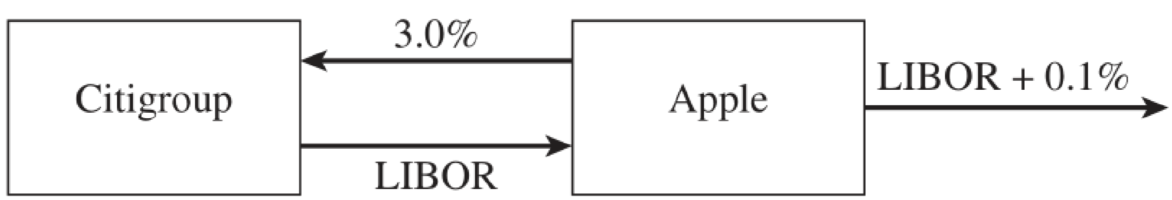
- **Cash Flow Dynamics**: The table provided outlines potential outcomes for a modified scenario where the swap duration is two years, retaining the $100 million notional principal. It details the exchange rates, corresponding cash flows, and net outcomes for Apple across several dates. This table exemplifies the variability in net cash flow resulting from fluctuations in the SOFR rate, with Apple's position oscillating between receiving and paying funds based on these rates.

Date	SOFR rate (%)	Floating cash flow received ($'000s)	Fixed cash flow paid ($'000s)	Net cash flow ($'000s)
June 8, 2022	2.20	550	750	-200
Sept. 8, 2022	2.60	650	750	-100
Dec. 8, 2022	2.80	700	750	-50
Mar. 8, 2023	3.10	775	750	+25
June 8, 2023	3.30	825	750	+75
Sept. 8, 2023	3.40	850	750	+100
Dec. 8, 2023	3.60	900	750	+150
Mar. 8, 2024	3.80	950	750	+200

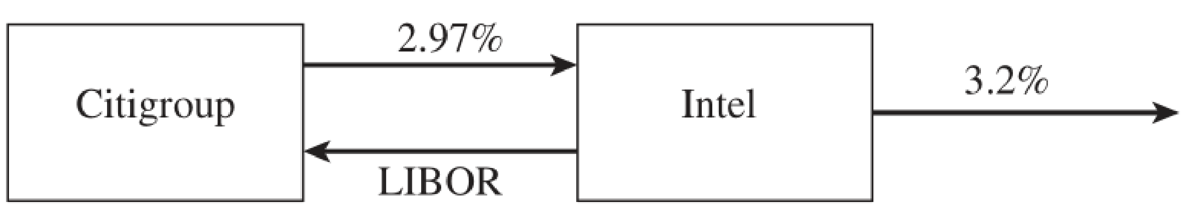
4.2.1 Typical Applications of Interest Rate Swaps

Interest rate swaps serve as versatile financial instruments, facilitating strategic adjustments to both liabilities and assets. Entities utilize these swaps to modify their interest rate exposure, transitioning between fixed and floating rates as per their financial strategies or market outlook.

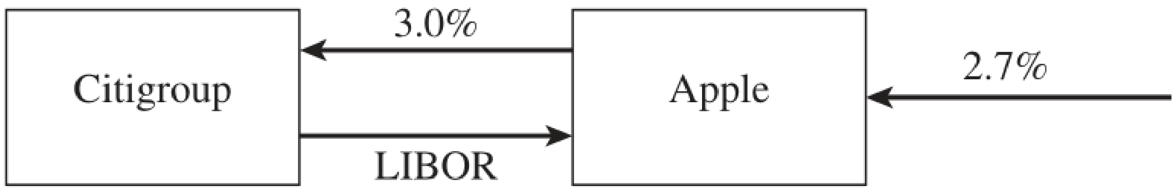
- Apple Transforms a **Liability** from Floating to Fixed:



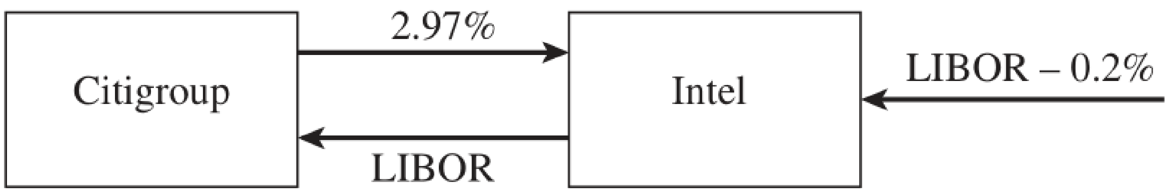
- Intel Transforms a **Liability** from Fixed to Floating:



- Apple Transforms an **Asset** from Fixed to Floating:



- Intel Transforms an **Asset** from Floating to Fixed:



4.2.2 Insights from a Swap Market Maker (Citigroup)

- **Market Quotations**: This section offers a glimpse into the swap market's dynamics, presenting Citigroup's bid and offer rates for fixed-rate swaps with varying maturities. The bid-offer spread, typically ranging from three to four basis points, reflects the market maker's pricing strategy and the liquidity of swap contracts across different terms.

Maturity (years)	Bid	Ask	Swap rate
2	2.97	3.00	2.985
3	3.05	3.08	3.065
4	3.15	3.19	3.170
5	3.26	3.30	3.280
7	3.40	3.44	3.420
10	3.48	3.52	3.500

4.3 The Comparative-Advantage Argument

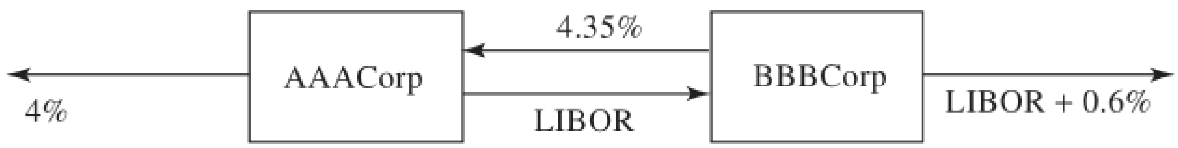
The comparative-advantage argument forms a foundational concept in the realm of financial swaps, particularly illuminating why and how companies engage in interest rate swaps to optimize their borrowing costs. At the heart of this argument is the strategic alignment of borrowing preferences with market conditions to leverage comparative advantages.

4.3.1 The Case Study: AAACorp and BBBCorp

Consider two hypothetical entities: AAACorp, which prefers to borrow using floating interest rates, and BBBCorp, which opts for fixed-rate borrowing. Their respective borrowing rates are tabled below, illustrating the differential costs of borrowing in fixed versus floating rate markets:

Entity	Fixed Rate	Floating Rate
AAACorp	4.0%	Floating - 0.1%
BBBCorp	5.2%	Floating + 0.6%

- **Direct Swap**: Initially, let's explore a scenario where AAACorp and BBBCorp engage in a direct swap agreement. This entails AAACorp borrowing at its preferred floating rate and BBBCorp at its preferred fixed rate, followed by an exchange of their respective financial obligations. The direct swap mechanism is visually represented as follows:



- **Indirect Swap via Financial Institution**: In a more complex structure, a financial institution plays an intermediary role, facilitating the swap between AAACorp and BBBCorp. This arrangement often introduces efficiencies and reduces the direct negotiation complexities between the companies:

4.3.2 Critical Analysis of the Comparative Advantage Argument

While the comparative-advantage argument underpins the financial logic for interest rate swaps, it is not without criticism, particularly when scrutinizing the underlying rates' term structure:

- **Rate Term Disparity**: The fixed rates quoted (4.0% for AAACorp and 5.2% for BBBCorp) pertain to 5-year borrowing, whereas the floating rates (Floating - 0.1% for

AAACorp and Floating + 0.6% for BBBCorp) are based on 6-month terms. This discrepancy in rate terms introduces complexities in directly comparing and assessing the advantages of one type of rate over the other.

- **Future Rate Uncertainty for BBBCorp**: BBBCorp's future borrowing costs, especially its fixed rate, are contingent upon the spread above the floating rate at which it borrows subsequently. This introduces an element of uncertainty and risk, particularly in fluctuating interest rate environments.

4.4 Valuation of Interest Rate Swaps

Interest rate swaps are sophisticated financial instruments utilized to manage interest rate exposure. Initially, their valuation is approximately zero, reflecting an equitable initiation of terms. As market conditions evolve, their value can fluctuate significantly. This dynamic nature requires a detailed understanding of valuation techniques, notably through the lens of Forward Rate Agreements (FRAs).

4.4.1 Valuation Framework

The valuation process encompasses several steps, designed to ascertain the present value of the swap's anticipated cash flows:

1. **Floating Forward Rates Calculation**: This initial step involves projecting future interest rates based on current market data, which will determine the floating rate payments of the swap.
2. **Swap Cash Flows Determination**: By applying the calculated forward rates, one can estimate the future cash flows for both fixed and floating legs of the swap.
3. **Cash Flow Discounting**: The future cash flows are then discounted back to their present value using risk-free rates, typically Overnight Indexed Swap (OIS) rates, to reflect the time value of money and credit risk.

Example: Swap Valuation

4.4.1.1 Swap Agreement Overview

- **Nature of Swap**: This transaction entails an exchange where one party pays a fixed annual rate of 3% and receives the SOFR semi-annually on a principal of $100 million.
- **Remaining Duration**: The swap has a remaining term of 1.2 years with scheduled exchanges occurring at 0.2, 0.7, and 1.2 years.

- **Risk-Free Rates**: The Overnight Indexed Swap (OIS) zero rates applicable for maturities of 3, 9, and 15 months are respectively set at 2.8%, 3.2%, and 3.4%.
- **Historical SOFR Rate**: The SOFR rate for the upcoming exchange in 3 months has been previously fixed at 2.9%.
- **Forward SOFR Rates**: Projected forward SOFR rates for exchanges at 0.2, 0.7, and 1.2 years, considering both continuous and semi-annual compounding, are as follows:
  - Continuous compounding rates: 2.50%, 3.36%, 3.68%
  - Semi-annual compounding rates: 2.516%, 3.388%, 3.714%

4.4.1.2 Cash Flow Valuation

The valuation process meticulously calculates the fixed and floating cash flows, their net impact, and their present values as detailed in the table below. These calculations employ the forward SOFR rates and discount them using the corresponding OIS rates to ascertain the swap's current value.

Time (years)	Fixed Cash Flow (USD million)	Floating Cash Flow (USD million)	Net Cash Flow (USD million)	Discount Factor	Present Value of Net Cash Flow (USD million)
0.2	-1.500	+1.258	-0.242	0.9944	-0.241
0.7	-1.500	+1.694	+0.194	0.9778	+0.190
1.2	-1.500	+1.857	+0.357	0.9600	+0.343
Total					**+0.292**

- **Fixed Cash Flow Calculation**: At 0.7 years, the fixed payment is calculated as $0.5 \times 0.03 \times 100 = -\$1.5\text{million}$.
- **Floating Cash Flow Projection**: The floating cash flow, based on the semi-annual compounding forward rate for 0.7 years (3.388%), is $0.5 \times 0.03388 \times 100 = \$1.694\text{million}$.
- **Net Cash Flow and Present Value**: The resulting net cash flow of \$0.194 million is then discounted using the exponential factor derived from the OIS rate for 0.7 years ($e^{-0.032\times 0.7} = 0.9778$), leading to a present value of \$0.190 million.

The swap's value, a summation of the present values of net cash flows across all time points, is calculated to be \$0.292 million.
It's crucial to acknowledge the approximation inherent in these calculations due to the exclusion of factors such as holiday calendars and day count conventions, which can affect precise cash flow timings and amounts.

4.5 Currency Swaps

Currency swaps are intricate financial instruments that facilitate the exchange of principal and interest payments in different currencies between two parties. A fixed-for-fixed currency swap involves exchanging fixed interest rate payments in one currency for fixed interest rate payments in another currency, alongside the exchange of principal amounts at both the start and end of the agreement.

4.5.1 Case Study: British Petroleum and Barclays Swap Agreement

- **Agreement Overview**: British Petroleum enters into a five-year currency swap with Barclays, where British Petroleum agrees to pay a fixed interest rate of 3% in U.S. dollars and receives a fixed interest rate of 4% in British pounds sterling.
- **Principal Exchange**: The principal amounts involved are USD 15 million and GBP 10 million, exchanged at the inception and conclusion of the swap term.
- **Interest Payment Schedule**: Interest is paid annually, reflecting the fixed rates agreed upon in both currencies.

The table below outlines the yearly cash flows resulting from this swap, highlighting the exchange of principal and interest payments between British Petroleum and Barclays:

Date	Dollar Cash Flow (millions)	Sterling Cash Flow (millions)
February 1, 2022	+15.00	-10.00
February 1, 2023	-0.45	+0.40
February 1, 2024	-0.45	+0.40
February 1, 2025	-0.45	+0.40
February 1, 2026	-0.45	+0.40
February 1, 2027	-15.45	+10.40

4.5.2 Typical Applications of Currency Swaps

Currency swaps serve versatile functions in global finance, including but not limited to:

- **Liability Management**: They enable entities to convert liabilities from one currency to another, aligning with their currency exposure preferences or expectations.
- **Investment Optimization**: Similarly, currency swaps can convert investments from one currency to another, facilitating global investment strategies that match investors' risk and return profiles.

4.5.3 Comparative Advantage in Currency Swaps

Comparative advantage in currency swaps can emerge from various factors, including the differential impact of taxation on borrowing costs across different currencies.

- **Scenario**: General Electric seeks to borrow in Australian dollars (AUD), while Qantas Airways aims to borrow in U.S. dollars (USD).
- **Borrowing Costs**: After tax adjustments, the borrowing costs for each entity in their non-preferred currencies reveal a potential for mutual benefit through a currency swap.

	USD	AUD
General Electric	5.0%	7.6%
Qantas Airways	7.0%	8.0%

4.6 Valuation of Fixed-for-Fixed Currency Swaps

Fixed-for-fixed currency swaps are complex financial instruments that involve exchanging fixed interest payments in two different currencies. The essence of valuing these swaps lies in understanding that each payment exchange represents a forward contract on currencies. The valuation process hinges on the assumption that forward exchange rates, as determined by the market's expectations of future currency values, will be realized.

The valuation of a fixed-for-fixed currency swap can be methodically approached by considering the swap as a series of forward foreign exchange contracts. Each of these contracts can be valued based on the premise that the forward exchange rates, agreed upon at the initiation of the contract, accurately predict future exchange rates.

i Example: Valuation of Currency Swap

Consider a swap agreement involving the exchange of interest payments in Japanese yen and U.S. dollars, with the following parameters:

- **Interest Rates**: Japanese yen interest rates are fixed at 1.5% per annum with continuous compounding, while U.S. dollar interest rates are fixed at 2.5% per annum with continuous compounding.

- **Swap Payments**: Annually, 3% interest is received in yen, and 4% interest is paid in dollars.
- **Principal Amounts**: The principal amounts involved are $10 million and 1,200 million yen.
- **Swap Duration**: The swap has a remaining life of 3 years.
- **Current Exchange Rate**: The spot exchange rate is 110 yen per dollar.

Time (years)	Dollar Cash Flow (million)	Yen Cash Flow (million)	Forward Exchange Rate	Dollar Value of Yen Cash Flow (million)	Net Cash Flow (million)	Present Value (million)
1	-0.4	+36	0.009182	0.3306	-0.0694	-0.0677
2	-0.4	+36	0.009275	0.3339	-0.0661	-0.0629
3	-10.4	+1236	0.009368	11.5786	+1.1786	+1.0934
Total						**+0.9629**

- **Annual Payments**: The institution commits to an annual payment of \$0.4 million in U.S. dollars (calculated as 0.04×10) while receiving 36 million yen (calculated as $1,200 \times 0.03$).
- **Principal Exchange**: At the end of the third year, a principal payment of \$10 million is made in dollars, and a reciprocal receipt of 1,200 million yen is recorded.
- **Spot Exchange Rate**: The initial exchange rate stands at $1/110 = 0.009091$ dollars per yen, serving as the basis for forward rate calculations.

The valuation intricately relies on the forward exchange rates, derived from the differential in continuous compounding interest rates between the two currencies:

- **Interest Rates**: The domestic (USD) and foreign (JPY) interest rates are set at 2.5% (r) and 1.5% (r_f), respectively.
- **Forward Rate for Year 1**: The one-year forward exchange rate is calculated as $0.009091 \times e^{(0.025-0.015)\times 1} = 0.009182$, reflecting the expected exchange rate in one year based on interest rate differentials.
- **Subsequent Years' Rates**: Forward rates for years two and three are similarly derived, adjusting for the respective time periods.

The forward exchange rates are pivotal in valuing the yen-denominated cash flows in dollars, subsequently determining the net cash flow and its present value:

- **Year 1 Valuation**: The yen cash flow's conversion at the one-year forward rate yields $36 \times 0.009182 = 0.3306$ million dollars. The net cash flow, after accounting for the dollar payment, stands at $0.3306 - 0.4 = -0.0694$ million dollars.
- **Present Value Calculation**: The present value of the year 1 net cash flow is calculated as $-0.0694 \times e^{-0.025\times 1} = -0.0677$ million dollars, applying the domestic interest rate for discounting.
- Similar methodologies are applied for the cash flows in years two and three, ensuring that each forward contract's value is accurately captured.

The aggregate value of the forward contracts, representing the net present value of all future cash flows under the swap agreement, culminates in a total of \$0.9629 million. This valuation encapsulates the financial institution's gain from the swap, assuming that the calculated forward rates are realized as projected.

4.7 Other Currency Swaps

4.7.1 Fixed-for-Floating Currency Swaps

A fixed-for-floating currency swap is a hybrid financial instrument that combines elements of a fixed-for-fixed currency swap with a fixed-for-floating interest rate swap. This type of swap involves the exchange of fixed interest payments in one currency for floating interest payments in another currency.

> **Example**
>
> Consider an agreement where a party pays a floating interest rate on a GBP 7 million principal and receives a fixed 3% interest rate on a USD 10 million principal, with semi-annual payments over a period of 10 years. This can be further dissected into:
>
> 1. A currency swap component where 3% fixed interest on a USD 10 million principal is received in exchange for paying a 4% fixed interest on a GBP 7 million principal.
> 2. An interest rate swap component where a 4% fixed interest is received, and a sterling floating interest rate is paid on a notional GBP 7 million principal.

4.7.2 Floating-for-Floating Currency Swaps

Floating-for-floating currency swaps involve the exchange of floating interest payments in two different currencies. This type of swap is essentially a combination of a fixed-for-fixed currency swap and two floating interest rate swaps.

> **Example**
>
> An example of this swap could involve exchanging sterling floating interest payments on a GBP 7 million principal for dollar floating interest payments on a USD 10 million principal. This arrangement can be broken down into:
>
> 1. A base swap where 3% fixed interest on a USD 10 million principal is exchanged for 4% fixed interest on a GBP 7 million principal.
> 2. An interest rate swap where 4% fixed interest is exchanged for sterling floating interest on a GBP 7 million principal.
> 3. An additional interest rate swap where 3% fixed interest is paid in exchange for receiving USD floating interest on a USD 10 million principal.

4.8 Other Types of Swaps

Beyond the conventional currency and interest rate swaps, the derivatives market offers a variety of specialized swaps designed to meet diverse financial needs and strategies. These include:

- **Amortizing/Step-Up Swaps**: These involve gradually increasing or decreasing notional principal amounts, typically to match the amortization of an underlying asset or liability.
- **Compounding Swap**: Involves the reinvestment of periodic interest payments to compound over the life of the swap.
- **Constant Maturity Swap (CMS)**: The swap's interest rate is reset periodically based on the rate of a constant maturity instrument, such as a 10-year Treasury note.
- **LIBOR-in-Arrears Swap**: The interest rate is determined at the end of the payment period rather than the beginning, introducing additional risk and potential reward.
- **Accrual Swap**: Interest accruals are contingent on the performance of a benchmark rate or index, potentially pausing under certain conditions.
- **Equity Swap**: Involves the exchange of the returns of an equity asset with those of another financial instrument, which can be fixed or floating rate interest.
- **Cross Currency Interest Rate Swap**: A variant of currency swaps where both legs of the swap are in different currencies and at least one leg is a floating interest rate.
- **Floating-for-Floating Currency Swap**: Both parties exchange floating interest rate payments in different currencies.
- **Diff Swap**: An interest rate differential swap, where payments are based on the difference between two reference rates.
- **Commodity Swap**: Involves exchanging a fixed price for a commodity for its floating market price over the term of the swap.
- **Variance Swap**: A forward contract on future volatility; the payments are based on the variance of a specified underlying asset.

The diversity of swap instruments in financial markets allows for tailored risk management and investment strategies. Whether through modifying payment structures, leveraging different types of rates (fixed, floating, or equity-based), or engaging in swaps based on commodities or variances, these instruments offer significant flexibility and potential for hedging, speculation, or arbitrage. Understanding the nuances and applications of each type is essential for effective financial management and strategy development.

4.9 Practice Questions and Problems

4.9.1 The Basic Usage of Swaps in Risk Management

1) A bank finds that its assets are not matched with its liabilities. It is taking floating-rate deposits and making fixed-rate loans. How can swaps be used to offset the risk?

2) Explain the difference between the credit risk and the market risk in a financial contract.

3) Explain why a bank is subject to credit risk when it enters into two offsetting swap contracts.

4) Why is the expected loss from a default on a swap less than the expected loss from the default on a loan to the same counterparty with the same principal?

4.9.2 Comparative Advantage and Swap Structuring

5) A corporate treasurer tells you that he has just negotiated a five-year loan at a competitive fixed rate of interest of 5.2%. The treasurer explains that he achieved the 5.2% rate by borrowing at six-month LIBOR plus 150 basis points and swapping LIBOR for 3.7%. He goes on to say that this was possible because his company has a comparative advantage in the floating-rate market. What has the treasurer overlooked?

6) Companies A and B have been offered the following rates per annum on a $20 million five-year loan:

	Fixed Rate	Floating Rate
Company A	5.0%	Floating + 0.1%
Company B	6.4%	Floating + 0.6%

Company A requires a floating-rate loan; company B requires a fixed-rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies.

7) Company X wishes to borrow U.S. dollars at a fixed rate of interest. Company Y wishes to borrow Japanese yen at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes:

	Yen	Dollars
Company X	5.0%	9.6%
Company Y	6.5%	10.0%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

8) Companies X and Y have been offered the following rates per annum on a \$5 million 10-year investment:

	Fixed Rate	Floating Rate
Company X	8.0%	Floating
Company Y	8.8%	Floating

Company X requires a fixed-rate investment; company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and will appear equally attractive to X and Y.

4.9.3 Swap Valuation

9) A \$100 million interest rate swap has a remaining life of 10 months. Under the terms of the swap, six-month LIBOR is exchanged for 7% per annum (compounded semiannually). The average of the bid-offer rate being exchanged for six-month LIBOR in swaps of all maturities is currently 5% per annum with continuous compounding (5.063% with semi-annual compounding). The six-month LIBOR rate was 4.6% per annum two months ago. What is the current value of the swap to the party paying floating? What is its value to the party paying fixed?

10) A currency swap has a remaining life of 15 months. It involves exchanging interest at 10% on 20 million GBP for interest at 6% on 30 million USD once a year. The term structure of interest rates in both the United Kingdom and the United States is currently flat, and if the swap were negotiated today the interest rates exchanged would be 4% in dollars and 7% in sterling. All interest rates are quoted with annual compounding (the continuously compounded interest rates in sterling and dollars are 6.766% per annum and 3.922% per annum). The current exchange rate (dollars per pound sterling) is 1.5500. What is the value of the swap to the party paying sterling? What is the value of the swap to the party paying dollars?

5 Options

References

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  - Chapter 10. Mechanics of Options Markets
  - Chapter 11. Properties of Stock Options
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

- Comprehend the fundamentals and definitions of options, including the mechanics behind option payoffs and profits.
- Identify and understand other option characteristics, adjustments, and the relationship with related assets.
- Grasp the principles underlying option pricing dynamics, along with the factors influencing option values.
- Analyze the upper and lower bounds for option prices and the concept of put-call parity in option trading strategies.
- Evaluate the specific considerations for American options, including the impact of early exercise decisions and dividends on option valuation.

5.1 Understanding Options

Definition

Options are financial derivatives that offer the buyer the **right, but not the obligation**, to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) on or before a specified date. The seller of the option, also known as the writer, receives a premium from the buyer in exchange for this right.

5.1.0.1 Types of Options

- **Call Option**: Grants the holder the right to purchase an asset at the strike price by the expiration date. It is a bullish bet, with the buyer anticipating an increase in the asset's price.

```
import plotly.graph_objects as go
import numpy as np

# Parameters
S = 100  # Strike price of the option
Q = 1  # Quantity of the asset
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
premium = 5  # Option premium

# Payoff calculations for call options
long_call_profit = np.maximum(spot_prices - S, 0) * Q - premium
short_call_profit = -np.maximum(spot_prices - S, 0) * Q + premium

# Create the figure
fig = go.Figure()

# Add traces for long and short call option positions
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call_profit,
        mode="lines",
        name="Long Call",
        line=dict(width=3),
        hovertemplate="Long Call<br>Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>"
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call",
        line=dict(width=3),
        hovertemplate="Short Call<br>Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>"
    )
```

```python
)
fig.add_hline(y=0, line_dash="solid", line_color="black", line=dict(width=0.7))

# Layout
fig.update_layout(
    title="Profit from Call Option, K = 100, c = 5",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Position",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

```
Unable to display output for mime type(s): text/html
```

- **Put Option**: Provides the holder the right to sell an asset at the strike price by the expiration date. It represents a bearish outlook, where the buyer expects the asset's price to decline.

```python
# Parameters
S = 100  # Strike price of the option
Q = 1  # Quantity of the asset
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
premium = 5  # Option premium

# Payoff calculations for call options
long_put_profit = np.maximum(S - spot_prices, 0) * Q - premium
short_put_profit = -np.maximum(S - spot_prices, 0) * Q + premium

# Create the figure
fig = go.Figure()

# Add traces for long and short call option positions
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put_profit,
        mode="lines",
```

```
        name="Long Put",
        line=dict(width=3),
        hovertemplate="Long Put<br>Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put",
        line=dict(width=3),
        hovertemplate="Short Put<br>Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>"
    )
)
fig.add_hline(y=0, line_dash="solid", line_color="black", line=dict(width=0.7))

# Layout
fig.update_layout(
    title="Profit from Put Option, K = 100, p = 5",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Position",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

5.1.0.2 Execution Styles

- **American Option**: Characterized by the flexibility it offers, an American option can be exercised at any point up until its expiration. This feature provides the holder with the opportunity to respond to market movements and exercise the option when it is most advantageous.
- **European Option**: This option type can only be exercised on its expiration date, not before. The restriction on early exercise makes European options more predictable in terms of their valuation, but it limits the holder's flexibility in responding to market changes.

5.2 Option Payoffs and Profits

5.2.1 Understanding Option Payoffs

- S_T: Represents the price of the underlying asset at the expiration date T.
- K: Denotes the exercise or strike price of the option.

The payoff to the option buyer, whether for a call or put, depends on the relationship between the strike price K and the underlying asset's price at expiration S_T:

- **Call Option Payoff**: $c_T = \max(0, S_T - K)$
- **Put Option Payoff**: $p_T = \max(0, K - S_T)$

> Example: Option Payoff
>
> Given an underlying asset price at expiration S_T of \$28 and a strike price K of \$25, the payoffs for call and put options are calculated as follows:
>
> - **Call Buyer Payoff**: $c_T = \max(0, \$28 - \$25) = \$3$
> - **Put Buyer Payoff**: $p_T = \max(0, \$25 - \$28) = \$0$

- The call option is **in the money** as it has a positive payoff of \$3.
- The put option is **out of the money**, resulting in zero payoff, indicating that exercising the option is not advantageous.

5.2.2 Calculating Option Profit

The profit from an option trade must account for the option premium paid upfront (c_0 for calls and p_0 for puts). Thus, profit formulas are adjusted as follows:

- **Profit for Call Buyer**: $\Pi_{call} = \max(0, S_T - K) - c_0$
- **Profit for Put Buyer**: $\Pi_{put} = \max(0, K - S_T) - p_0$

> Example: Option Profit
>
> Considering CBX stock options with a strike price $K = \$30$, where the call and put premiums are \$1 and \$2 respectively, and the stock price at expiration S_T is \$27.50:
>
> - **Call Option Profit**: $\Pi_{call} = \max(0, \$27.5 - \$30) - \$1 = -\$1$
> - **Put Option Profit**: $\Pi_{put} = \max(0, \$30 - \$27.5) - \$2 = \$0.5$

- The call option buyer incurs a loss of \$1, as the call's intrinsic value does not offset the premium paid.
- The put option buyer realizes a profit of \$0.5, indicating that the intrinsic value exceeds the premium paid, making the option trade beneficial despite the underlying asset's price movement.

5.3 Other Option Characteristics, Adjustments, and Related Assets

5.3.1 Other Option Characteristics

Options are versatile financial instruments with various characteristics that cater to different investment strategies and risk profiles. They can be based on a wide range of underlying assets and come with specific features that define their contractual terms.

- Selling options requires posting margin due to the potential obligation to fulfill the contract.

5.3.1.1 Specification of Exchange-Traded Options

Options can be written on **various assets**, providing flexibility and diversity in investment choices:

- Stocks
- ETFs (Exchange-Traded Funds) and ETPs (Exchange-Traded Products)
- Foreign Currency
- Stock Indices
- Futures

Key specifications include:

- **Expiration Date**: The last date the option can be exercised.
- **Strike Price**: The price at which the underlying asset can be bought or sold.
- **Option Style**: European (exercisable only at expiration) or American (exercisable any time before expiration).
- **Option Class**: Call (right to buy) or Put (right to sell).

5.3.1.2 Option Value

- **Intrinsic Value**: The profit that could be realized if the option were exercised immediately.
- **Time Value**: The additional value reflecting the potential for the option to gain intrinsic value before expiration.

Moneyness describes the intrinsic value of an option in its current state:

- **At-the-Money**: The option's strike price is equal to the underlying asset's current price.
- **In-the-Money**: The option would have positive intrinsic value if exercised now.
- **Out-of-the-Money**: The option has no intrinsic value (only time value).

5.3.2 Dividends and Stock Splits Adjustments

- **Cash Dividends**: Typically, option contracts are not adjusted for cash dividends. The reasoning behind this is that cash dividends are expected events, and their potential impact is already factored into the option's price through market mechanisms.
- **Stock Splits and Stock Dividends**: These corporate actions directly affect the company's stock price and, consequently, the value of options on that stock. To preserve the value of existing options, adjustments are made to both the strike price (K) and the number of options (N) held.

For an n-for-m stock split, adjustments are made as follows:

1. **Strike Price Adjustment**: The new strike price is calculated using the formula $\frac{m}{n} \times K$. This adjustment ensures that the economic value of the strike price relative to the stock price remains constant.
2. **Number of Options Adjustment**: The quantity of options held is recalibrated to $\frac{n}{m} \times N$. This change ensures that the overall exposure of the option holder remains unchanged.

Examples: Option Adjustments

5.3.2.1 Adjustments for a 2-for-1 Stock Split

A call option to buy 100 shares at \$20 per share.

- **Strike Price**: Becomes $\frac{1}{2} \times \$20 = \$10$.
- **Number of Options**: Increases to $\frac{2}{1} \times 100 = 200$ options.
- **2-for-1 Stock Split**: This split essentially doubles the number of shares in circu-

lation, halving the stock price. To maintain the option's value, the strike price is halved, and the number of options is doubled.

5.3.2.2 Adjustments for a 5% Stock Dividend

A call option to buy 100 shares at \$20 per share.

- **Strike Price**: Adjusts to $\frac{1}{1.05} \times \$20 = \$19.05$.
- **Number of Options**: Increases to $\frac{1.05}{1} \times 100 = 105$ options.
- **5% Stock Dividend**: This dividend increases the number of shares by 5%, slightly reducing the stock price. The option's strike price is adjusted to reflect the new price, and the number of options is increased by 5% to maintain the holder's position value.

5.3.3 Related Assets

5.3.3.1 Warrants

- Issued by corporations or financial institutions, warrants give the right to buy the issuer's stock at a specific price before expiration.
- The exercise of warrants typically leads to the issuance of new stock.

5.3.3.2 Employee Stock Options

- A form of compensation for employees, usually at-the-money when issued.
- Exercise leads to the issuance of new company stock at the strike price.

5.3.3.3 Convertible Bonds

- Bonds that can be converted into a predetermined amount of the company's equity at certain times during its life.
- Often callable, allowing the issuer to force conversion under specified conditions.

5.4 Understanding the Dynamics of Option Pricing

In the realm of financial derivatives, stock options are pivotal instruments whose valuation intricately hinges on multiple underlying factors. The valuation dynamics of these options can be dissected into several key variables, each exerting a distinct influence under the ceteris paribus (all else equal) assumption. Herein, we delineate the impact of these variables on the prices of European and American options, using the following notations to denote directional influences: a positive (+) impact suggests that an increase in the variable elevates the option's price, a negative (−) impact denotes a price decrease, and an uncertain (?) relationship indicates an ambiguous effect.

Variable	European Call	European Put	American Call	American Put
Current Stock Price (S_0)	+	−	+	−
Strike Price (K)	−	+	−	+
Time to Expiration (T)	?	?	+	+
Volatility (σ)	+	+	+	+
Risk-free Rate (r)	+	−	+	−
Amount of Future Dividends (D)	−	+	−	+

To ensure clarity in discourse, we employ the following notations throughout:

- c: Price of a European call option
- p: Price of a European put option
- C: Price of an American call option
- P: Price of an American put option
- S_0: Current stock price
- S_T: Stock price at option maturity
- K: Strike price
- T: Option's life span
- σ: Volatility of the stock price
- D: Present value of dividends disbursed during the option's life
- r: Risk-free interest rate with continuous compounding over maturity T

5.4.1 Comparative Analysis of American and European Options

A pivotal aspect of option theory is the intrinsic value comparison between American and European options. American options, characterized by their flexibility of exercise prior to expiration, inherently command a value that is not less than their European counterparts, which are exercisable only at maturity. This valuation principle is succinctly encapsulated in the following inequalities:

$$C \geq c$$

$$P \geq p$$

These relations underscore a fundamental valuation floor for American options, driven by their enhanced exercise flexibility. This comparative analysis not only enriches our understanding of option pricing dynamics but also accentuates the critical role of exercise timing in option valuation.

5.5 Upper and Lower Bounds for Option Prices

5.5.1 Upper Boundaries for Option Prices

5.5.1.1 Call Options

For both American and European call options, the principle that an option's value cannot exceed the current price of the underlying stock is fundamental. Mathematically, this is represented as:

$$c \leq S_0 \text{ and } C \leq S_0$$

This ceiling on call option prices prevents the possibility of arbitrage profits that could arise from buying the stock outright and selling the call option, thereby exploiting price discrepancies.

5.5.1.2 Put Options

The valuation cap for put options varies between American and European styles due to their exercise terms.

- **American Put Options:** The value is naturally capped at the strike price, K, because the option grants the right but not the obligation to sell the stock at K. Thus, $P \leq K$.
- **European Put Options:** The maximum value is the present value of the strike price, $p \leq Ke^{-rT}$, considering that it can only be exercised at maturity. This prevents arbitrage opportunities involving writing the option and investing the proceeds at the risk-free rate.

5.5.2 Lower Boundaries for Option Prices

5.5.2.1 European Call Options

For a European call option on a stock that does not pay dividends, the price floor is determined by the difference between the stock's current price and the present value of the strike price:

$$c \geq S_0 - Ke^{-rT}$$

This lower bound highlights the intrinsic value of the option, beyond which arbitrage becomes viable.

> **Example**
>
> What is a lower bound for the following European call option?
>
> - $S_0 = 20$, $T = 1$, $r = 10\%$, $K = 18$, $D = 0$
>
> $$\text{Lower bound} = S_0 - Ke^{-rT} = 20 - 18e^{-0.1} = 3.71$$
>
> What if the European call price is \$3?
>
> - An arbitrageur can short the stock, buy the call, and invest proceeds at 10%.

5.5.2.2 European Put Options

The lower bound for a European put option, similarly on a non-dividend-paying stock, is the present value of the strike price minus the current stock price:

$$p \geq Ke^{-rT} - S_0$$

This calculation ensures that the option's price reflects its minimum economic value.

> **Example**
>
> What is a lower bound for the following European put option?
>
> - $S_0 = 37$, $T = 0.5$, $r = 5\%$, $K = 40$, $D = 0$
>
> $$Ke^{-rT} - S_0 = 40e^{-0.05\times 0.5} - 37 = 2.01$$
>
> What if the European put price is \$1?
>
> - An arbitrageur can borrow \$38 for 6 months to buy both the put and the stock.

5.5.3 American Options and Early Exercise Decision

5.5.3.1 American Call Options on Non-dividend Paying Stocks

The conventional wisdom suggests that **it is suboptimal to exercise an American call option on a non-dividend paying stock before expiration**. Consider an American call option with the following parameters: $S_0 = 100$, $T = 0.25$, $K = 60$, and $D = 0$. The dilemma of whether to exercise the option immediately hinges on the anticipated utility from holding the stock versus the option.

- **If intending to hold the stock:** Exercising early forfeits the call option's time value, providing no additional benefit over holding the option.
- **If intending to close the stock position:** Selling the option is preferable, capturing both its intrinsic and time value, unlike exercising, which yields only the intrinsic value.
- **Justifications Against Early Exercise:**
  - **Preservation of Capital:** Delaying the exercise preserves liquidity by deferring the payment of the strike price.
  - **Insurance Benefit:** The option acts as a hedge against the stock's depreciation below the strike price.
  - **Maximization of Value:** Selling the option rather than exercising it realizes both intrinsic and time value.
- Because it is not optimal to exercise an American stock option on a non-dividend-paying stock early, the **upper and lower bounds will be the same as those for European options**.

5.5.3.2 American Put Options on Non-dividend Paying Stocks

Contrary to calls, American put options on non-dividend paying stocks **may warrant early exercise**, especially when deeply in the money, due to the immediate gain realization and the time value of money.

Imagine a scenario where the strike price is \$10, and the stock price plummets near zero. Immediate exercise yields a \$10 gain, maximizing the investor's return as stock prices cannot become negative and due to the preference for current versus future value.

- **Justifications for Early Exercise:**
  - **Immediate Value Realization:** Exercising deep in-the-money puts captures the maximum possible gain immediately.
  - **Time Value of Money:** Receiving proceeds today is financially more beneficial than an identical future payment due to potential investment returns.

- **Decreased Attractiveness with Stock Price Decline:** As S_0 diminishes, the attractiveness of early exercise increases, especially under higher interest rates and lower volatility.

- Because it may be optimal in some cases to exercise an American put option early, the **lower bound** changes accordingly:

$$P \geq \max(K - S_0, 0)$$

5.5.4 Summary of Price Boundaries

The upper and lower price limits for options serve as critical indicators for arbitrage strategies and investment analysis. They are succinctly summarized as follows:

5.5.4.1 Call Options

The upper bound is the stock's current price, while the lower bound is either the difference between the stock's price and the discounted strike price or zero, whichever is greater.

- Upper bound for European and American call options:

$$c \leq S_0 \text{ and } C \leq S_0$$

- Lower bound for European and American call option:

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$
$$C \geq \max(S_0 - Ke^{-rT}, 0)$$

5.5.4.2 Put Options

The upper bound for European options is the discounted strike price, while for American options, it's the strike price itself. The lower bound is the greater of the discounted strike price minus the stock's price or zero.

- Upper bound for European and American put options:

$$p \leq Ke^{-rT} \text{ and } P \leq K$$

- Lower bound for European and American put option:

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

$$P \geq \max(K - S_0, 0)$$

5.6 Put-Call Parity

The principle of put-call parity is a cornerstone in the theoretical framework of financial derivatives, providing a fundamental relationship between the prices of European call and put options. This parity underlines the equilibrium that must exist between these options when they have identical strike prices and expiration dates. Below, we delve deeper into the concept, illustrating its implications for both non-dividend and dividend-paying stocks, and extend the discussion to American options.

5.6.1 Put-Call Parity for Non-dividend Paying Stocks

Consider two distinct portfolios, A and B, each composed of different financial instruments but constructed to have equivalent values at the expiration of the options involved:

- **Portfolio A** consists of a European call option and a zero-coupon bond paying K (the strike price) at time T (expiration).
- **Portfolio B** includes a European put option and the underlying stock itself.

The table below presents the values of these portfolios at option expiration under two scenarios: when the stock price at expiration, S_T, is above or below the strike price, K.

		$S_T > K$ (Above Strike)	$S_T < K$ (Below Strike)
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio B	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

Given their identical payouts at maturity, both portfolios must have the same present value, leading to the foundational put-call parity equation:

$$c + Ke^{-rT} = p + S_0$$

5.6.2 Arbitrage Opportunities via Put-Call Parity

Consider an example where $S_0 = \$31$, $r = 10\%$, the call option price $c = \$3$, and the strike price $K = \$30$. The table outlines potential arbitrage strategies based on discrepancies in put option pricing, illustrating the mechanism for securing risk-free profits by leveraging the put-call parity principle.

Three-month put price = \$2.25	Three-month put price = \$1
Action now:	*Action now:*
Buy call for \$3	Borrow \$29 for 3 months
Short put to realize \$2.25	Short call to realize \$3
Short the stock to realize \$31	Buy put for \$1
Invest \$30.25 for 3 months	Buy the stock for \$31
...	
Action in 3 months if $S_T > 30$:	*Action in 3 months if* $S_T > 30$:
Receive \$31.02 from investment	Call exercised: sell stock for \$30
Exercise call to buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27
...	
Action in 3 months if $S_T < 30$:	*Action in 3 months if* $S_T < 30$:
Receive \$31.02 from investment	Exercise put to sell stock for \$30
Put exercised: buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27

5.6.3 Extension to American Options

While put-call parity directly applies to European options, its principles offer insight into the valuation boundaries of American options, which can be exercised at any time before expiration. In the absence of dividends, the relationship between American call and put prices can be expressed as:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

This range provides a basis for evaluating the relative pricing of American calls and puts, emphasizing the influence of early exercise rights and the absence of dividend payments on option valuation.

5.7 Effect of Dividends on Options

The presence of dividends within the lifespan of an option introduces significant nuances to the valuation and strategic exercise decisions of options. This section delves into how known dividends impact the valuation of options, adjusting the traditional models to accommodate the dividend factor.

When dividends are anticipated during the option's term, they must be factored into the option's present value. We denote the present value of expected dividends during the option's life as D. The ex-dividend date marks the occasion for these adjustments, impacting the strategic exercise decisions for American call options.

The prospect of dividends alters the conventional wisdom that American call options should not be exercised early. Specifically, it may become optimal to exercise these options just before the ex-dividend date to capture the dividend payout.

Crucial Point: Apart from the period just before the ex-dividend date, early exercise of a call option remains suboptimal.

5.7.1 Adjusting Lower Bound Valuations for Dividends

The introduction of dividends necessitates a revision of the lower bound calculations for both call and put option prices:

- **For Call Options:** The lower bound formula adjusts to account for the dividend's negative impact on the option's value, reflecting the loss of dividend income upon early exercise:

$$c \geq S_0 - D - Ke^{-rT}$$

- **For Put Options:** Conversely, the lower bound for put options incorporates dividends positively, indicating an increase in value due to the potential decrease in the underlying stock's price upon dividend payout:

$$p \geq D + Ke^{-rT} - S_0$$

5.7.2 Revising Put-Call Parity with Dividends

The presence of dividends also modifies the put-call parity relationship, a fundamental principle in options pricing:

- **European Options with Dividends:** The parity formula integrates D to balance the equation, highlighting the direct impact of dividends on the call option's lower valuation compared to its put counterpart:

$$c + D + Ke^{-rT} = p + S_0$$

- **American Options with Dividends:** When dividends are present, the valuation bounds for American options adjust to reflect the diminished value of the call option due to potential early exercise for dividend capture:

$$S_0 - D - K < C - P < S_0 - Ke^{-rT}$$

Dividends play a pivotal role in the strategic exercise and valuation of options, particularly affecting American call options' early exercise decisions. Adjusting option valuation models to account for dividends is essential for accurate pricing and effective investment strategy formulation.

5.8 Practice Questions and Problems

5.8.1 Option Profitability and Exercise Conditions

1. Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.
2. An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.
3. An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.
4. Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

5.8.2 Margin Requirements, Market Choices, and Contract Adjustments

5. Explain why margin accounts are required when clients write options but not when they buy options.
6. A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the NASDAQ OMX and (b) the over-the-counter market for trading?
7. The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.
8. Consider an exchange-traded call option contract to buy 500 shares with a strike price of \$40 and maturity in four months. Explain how the terms of the option contract change when there is

    1. A 10% stock dividend
    2. A 10% cash dividend
    3. A 4-for-1 stock split

5.8.3 Option Pricing Bounds

9. Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.
10. Explain why an American option is always worth at least as much as its intrinsic value.
11. What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?
12. What is a lower bound for the price of a one-month European put option on a non-dividend paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

5.8.4 Early Exercise and Put-Call Parity

13. Give at least two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal.
14. The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put. Explain this statement.
15. The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of \$20?
16. List and explain the six factors affecting stock option prices.

6 Options Trading Strategies and Hedging

> **References**
>
> - HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
>   - Chapter 12 - Trading Strategies Involving Options
> - PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
>   - Chapter 5 - Derivatives Strategies

Additonal sources on option trading strategies:

- tastylive
- options playbook
- investopedia

Learning Outcomes:

- Understand the structure and purpose of Principal Protected Notes, including how they safeguard the principal amount while offering potential investment gains.
- Analyze strategies that combine positions in the underlying asset with options to manage risk and enhance potential returns, focusing on protective puts and covered calls.
- Explore various option spread strategies, such as bull spreads, bear spreads, calendar spreads, and butterfly spreads, to understand their risk/reward profiles and market outlook implications.
- Examine option combination strategies like straddles, strangles, strips, and straps, highlighting their use in volatile markets to capitalize on significant price movements in either direction.

```python
# Load python libraries
import plotly.graph_objects as go
import numpy as np
```

6.1 Principal Protected Note

Principal Protected Notes (PPNs) enable investors to engage in potentially high-reward investments without the fear of losing their principal amount. This dual-feature mechanism is facilitated through a blend of a zero-coupon bond and a derivative instrument, typically a call option.

> **Example: PPN**
>
> Consider an investment in a PPN valued at $1,000, structured as follows:
>
> 1. **Zero-Coupon Bond Component**: A 3-year zero-coupon bond with a face value of $1,000 ensures the principal protection. Given a continuous compounding interest rate of 6%, the present value of this bond is calculated as:
>
> $$PV = \$1,000 \times e^{-0.06\times 3} = \$835.27$$
>
> This calculation confirms that an initial investment of $835.27 will mature to $1,000 over 3 years, effectively protecting the principal.
>
> 2. **Call Option Component**: The remaining funds, amounting to $164.73 ($1,000 - $835.27), are utilized to purchase a 3-year at-the-money call option on a stock portfolio. This option provides the upside potential.

The feasibility of structuring a PPN profitably hinges on several market conditions:

- **Dividend Levels**: Higher dividends reduce the attractiveness of the call option component.
- **Interest Rates**: The zero-coupon bond's cost is inversely related to interest rates; higher rates make principal protection cheaper.
- **Portfolio Volatility**: Higher volatility increases the price of the call option, but enhancing the investment's upside potential.

To cater to diverse investor preferences, PPNs can be customized with various features:

- **Strike Price Adjustments**: Options may be set out of the money to increase potential returns.
- **Return Caps**: Limits may be imposed on the maximum return to reduce the cost of the call option.
- **Structural Innovations**: Features like knock-outs, averaging mechanisms, etc., can be incorporated to tailor risk and return profiles.

6.2 Combining Underlying and Option

Combining positions in the underlying stock with options can create tailored risk/reward profiles that suit various investment strategies and market views. Two popular strategies that illustrate this principle are the protective put and the covered call.

6.2.1 Protective Put

A protective put strategy involves buying a stock (or holding a currently owned stock) and buying a put option for the same stock. This strategy is used to protect against a decline in the stock's price while allowing for participation in any upside. It guarantees a minimum selling price (the strike price of the put) for the stock until the put's expiration, effectively setting a floor on the potential losses without capping the potential gains.

```
# Parameters for the protective put strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
S0 = 100  # Initial stock price
K = 100  # Strike price of the put option
premium_put = 5  # Premium paid for the put option
initial_investment = S0 + premium_put  # Total initial investment including the put premium

# Profit calculations for protective put
stock_profit = spot_prices - S0  # Profit from the stock alone
put_option_profit = np.maximum(K - spot_prices, 0) - premium_put  # Profit from the long put
protective_put_profit = stock_profit + put_option_profit  # Total profit from the protective

# Create the figure
fig = go.Figure()

# Add traces for the individual components' profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=stock_profit,
        mode="lines",
        name="Stock Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
```

```
        x=spot_prices,
        y=put_option_profit,
        mode="lines",
        name="Put Option Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the protective put net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=protective_put_profit,
        mode="lines",
        name="Protective Put",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference, adjusted for the initial investment
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Protective Put Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
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```

```
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```

6.2.2 Covered Call

A covered call strategy involves owning a stock and selling a call option on that stock. This strategy is aimed at generating additional income from the option premium, which can enhance the overall returns on the stock, especially in flat or slightly bullish markets. It limits the upside potential to the strike price of the sold call but provides premium income that offers some protection against a decline in the stock's price.

```
# Parameters for the covered call strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
S0 = 100  # Initial stock price
K = 100  # Strike price of the call option
premium_call = 5  # Premium received for the call option

# Profit calculations for covered call
stock_profit = spot_prices - S0  # Profit from the stock alone
call_option_profit = np.where(spot_prices > K, K - spot_prices, 0) + premium_call  # Profit f
covered_call_profit = stock_profit + call_option_profit  # Total profit from the covered cal

# Create the figure
fig = go.Figure()

# Add traces for the individual components' profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=stock_profit,
        mode="lines",
        name="Stock Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_option_profit,
        mode="lines",
        name="Call Option Profit",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
```

```
# Add trace for the covered call net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=covered_call_profit,
        mode="lines",
        name="Covered Call",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Covered Call Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3 Option Spreads

Option spread strategies involve holding two or more options of the same type simultaneously to capitalize on movements in the underlying asset's price. These strategies can be tailored to reflect various market outlooks, from bullish to bearish or even neutral.

6.3.1 Bull Spread Strategy

A bull spread is a strategy used when an investor expects a moderate increase in the price of the underlying asset. It can be constructed using either calls or puts.

6.3.1.1 Using Calls (Call Bull Spread)

- **Position**: Buy a call option with a lower strike price and sell another call option with a higher strike price. Both options share the same expiration date.
- **Profit Potential**: Maximum profit is limited to the difference between the two strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Point**: Lower strike price + net premium paid.

```
# Parameters for the bull spread strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the long call option
K2 = 105  # Strike price of the short call option
premium_long = 5  # Premium paid for the long call
premium_short = 2  # Premium received for the short call

# Profit calculations
long_call_profit = np.maximum(spot_prices - K1, 0) - premium_long
short_call_profit = -(np.maximum(spot_prices - K2, 0) - premium_short)
bull_spread_profit = long_call_profit + short_call_profit  # Net profit of the bull spread

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call_profit,
        mode="lines",
        name="Long Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call",
        line=dict(width=3, dash='dot'),
```

```
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bull spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bull_spread_profit,
        mode="lines",
        name="Bull Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bull Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.1.2 Using Puts (Put Bull Spread)

- **Position**: Buy a put option with a higher strike price and sell another put option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the difference between the strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Point**: Higher strike price - net premium paid.

```
# Parameters for the bull spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the long put option (lower strike)
K2 = 105  # Strike price of the short put option (higher strike)
premium_long = 2  # Premium paid for the long put
premium_short = 5  # Premium received for the short put

# Profit calculations
long_put_profit = np.maximum(K1 - spot_prices, 0) - premium_long
short_put_profit = -(np.maximum(K2 - spot_prices, 0) - premium_short)
bull_spread_profit = long_put_profit + short_put_profit  # Net profit of the bull spread usi

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put_profit,
        mode="lines",
        name="Long Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bull spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bull_spread_profit,
```

```
        mode="lines",
        name="Bull Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bull Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.2 Bear Spread Strategy

A bear spread is used when an investor expects a moderate decline in the price of the underlying asset. Similar to the bull spread, it can be executed using calls or puts.

6.3.2.1 Using Calls (Call Bear Spread)

- **Position**: Buy a call option with a higher strike price and sell another call option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the net premium received for the spread.
- **Risk**: Limited to the difference between the strike prices minus the net premium received.
- **Break-even Point**: Lower strike price + net premium received.

```
# Parameters for the bear spread strategy using call options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 90  # Strike price of the short call option (lower strike)
K2 = 105  # Strike price of the long call option (higher strike)
```

```python
premium_short = 5  # Premium received for the short call
premium_long = 2  # Premium paid for the long call

# Profit calculations for bear spread using call options
short_call_profit = -(np.maximum(spot_prices - K1, 0) - premium_short)
long_call_profit = np.maximum(spot_prices - K2, 0) - premium_long
bear_spread_profit = short_call_profit + long_call_profit  # Net profit of the bear spread

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call_profit,
        mode="lines",
        name="Long Call",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bear spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bear_spread_profit,
        mode="lines",
        name="Bear Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
```

```
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bear Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.2.2 Using Puts (Put Bear Spread)

- **Position**: Buy a put option with a higher strike price and sell another put option with a lower strike price, both having the same expiration date.
- **Profit Potential**: Maximum profit is limited to the net premium received for the spread.
- **Risk**: Limited to the difference between the strike prices minus the net premium received.
- **Break-even Point**: Higher strike price - net premium received.

```
# Parameters for the bear spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 105  # Strike price of the long put option (higher strike)
K2 = 95  # Strike price of the short put option (lower strike)
premium_long = 5  # Premium paid for the long put
premium_short = 2  # Premium received for the short put

# Profit calculations for bear spread using put options
long_put_profit = np.maximum(K1 - spot_prices, 0) - premium_long
short_put_profit = -(np.maximum(K2 - spot_prices, 0) - premium_short)
bear_spread_profit = long_put_profit + short_put_profit  # Net profit of the bear spread

# Create the figure
fig = go.Figure()
```

```python
# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put_profit,
        mode="lines",
        name="Long Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put",
        line=dict(width=3, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the bear spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=bear_spread_profit,
        mode="lines",
        name="Bear Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Bear Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
```

```
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.3 Butterfly Spread Strategy

The butterfly spread is a neutral option strategy that is used when an investor expects little to no movement in the underlying asset's price. It can be constructed using either calls or puts and involves both buying and selling options at three different strike prices.

6.3.3.1 Using Calls (Call Butterfly Spread)

- **Position**:
  - Buy one call option at a lower strike price (A).
  - Sell two call options at a middle strike price (B).
  - Buy one call option at a higher strike price (C).
  - All options have the same expiration date, and the strike prices are equidistant.
- **Profit Potential**: Maximum profit is achieved if the underlying asset's price is equal to the middle strike price at expiration. The profit is the difference between the middle and lower (or higher) strike prices minus the net premium paid.
- **Risk**: Limited to the net premium paid to establish the spread.
- **Break-even Points**: There are two break-even points:
  - Lower break-even: Lower strike price (A) + net premium paid.
  - Upper break-even: Higher strike price (C) - net premium paid.

```
# Parameters for the butterfly spread strategy using call options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 95  # Strike price of the first long call option (lower strike)
K2 = 100  # Strike price of the short call options (middle strike)
K3 = 105  # Strike price of the second long call option (higher strike)
premium_long1 = 8  # Premium paid for the first long call
premium_short = 5  # Premium received for each short call (twice for the middle strike)
premium_long2 = 4  # Premium paid for the second long call
```

```
# Profit calculations for butterfly spread using call options
long_call1_profit = np.maximum(spot_prices - K1, 0) - premium_long1
short_call_profit = 2 * (-(np.maximum(spot_prices - K2, 0) - premium_short))
long_call2_profit = np.maximum(spot_prices - K3, 0) - premium_long2
butterfly_spread_profit = long_call1_profit + short_call_profit + long_call2_profit  # Net pr

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call1_profit,
        mode="lines",
        name="Long Call 1",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_call_profit,
        mode="lines",
        name="Short Call (2x)",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_call2_profit,
        mode="lines",
        name="Long Call 2",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the butterfly spread net profit
```

```
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=butterfly_spread_profit,
        mode="lines",
        name="Butterfly Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Butterfly Spread Using Call Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.3.2 Using Puts (Put Butterfly Spread)

- **Position**:
  - Buy one put option at a higher strike price (A).
  - Sell two put options at a middle strike price (B).
  - Buy one put option at a lower strike price (C).
  - All options have the same expiration date, and the strike prices are equidistant.
- **Profit Potential**: Maximum profit is similar to the call butterfly and is achieved if the underlying's price equals the middle strike price at expiration.
- **Risk**: Limited to the net premium paid for the spread.
- **Break-even Points**: Identical in concept to the call butterfly, adjusted for the put options' strike prices.

```python
# Parameters for the butterfly spread strategy using put options
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K1 = 105  # Strike price of the first long put option (higher strike)
K2 = 100  # Strike price of the short put options (middle strike)
K3 = 95  # Strike price of the second long put option (lower strike)
premium_long1 = 8  # Premium paid for the first long put
premium_short = 5  # Premium received for each short put (twice for the middle strike)
premium_long2 = 4  # Premium paid for the second long put

# Profit calculations for butterfly spread using put options
long_put1_profit = np.maximum(K1 - spot_prices, 0) - premium_long1
short_put_profit = 2 * (-(np.maximum(K2 - spot_prices, 0) - premium_short))
long_put2_profit = np.maximum(K3 - spot_prices, 0) - premium_long2
butterfly_spread_profit = long_put1_profit + short_put_profit + long_put2_profit  # Net profi

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_put1_profit,
        mode="lines",
        name="Long Put 1",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_put_profit,
        mode="lines",
        name="Short Put (2x)",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
```

```
        y=long_put2_profit,
        mode="lines",
        name="Long Put 2",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the butterfly spread net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=butterfly_spread_profit,
        mode="lines",
        name="Butterfly Spread",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Butterfly Spread Using Put Options",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
Unable to display output for mime type(s): text/html
```

6.3.4 Key Characteristics of Option Spreads

- **Market Outlook**: Bull spreads are used when expecting a moderate price increase, while bear spreads are for a moderate price decrease. Butterfly spread is best suited for markets expected to be stable or within a specific range.

- **Risk and Reward**: Bull and bear spreads strategies offer limited risk and reward, making them attractive for traders with a specific market view and risk tolerance. Butterfly spreado ffers a high reward-to-risk ratio if the market remains stable but with limited profit potential outside the narrow price range.
- **Cost Efficiency**: Spreads can be cost-effective ways to gain market exposure compared to outright option purchases.
- **Flexibility**: Investors can adjust the width of the spread and the strike prices to manage the risk-reward ratio according to their market outlook and risk appetite.

Calendar Spreads

Calendar spreads, also known as time or horizontal spreads, involve options of the same underlying asset, strike price, but **different expiration dates**. They capitalize on the difference in time decay rates (theta) between options. Typically, a calendar spread is constructed by selling a short-term option and buying a longer-term option of the same type (call or put).
For more information check, e.g., tastylive

6.4 Option Combinations

Option combinations involve two or more options of different types. It allow traders and investors to express complex market views and hedge positions in ways that single option positions cannot. This section covers four popular option combination strategies: straddles, strangles, strips, and straps.

6.4.1 Straddles

A straddle involves buying or selling both a call and a put option with the same strike price and expiration date.

6.4.1.1 Long Straddle

A long straddle is created by buying both a call and a put option. It is a bet on volatility without taking a directional stance. Traders expect the underlying asset to move significantly, but they are unsure in which direction. The maximum loss is limited to the total premium paid for both options, while the profit potential is unlimited if the underlying moves significantly in either direction.

```python
# Parameters for the long straddle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K = 100  # Strike price for both the call and put options
premium_call = 5  # Premium paid for the call option
premium_put = 5 # Premium paid for the put option

# Profit calculations for long straddle
call_profit = np.maximum(spot_prices - K, 0) - premium_call
put_profit = np.maximum(K - spot_prices, 0) - premium_put
long_straddle_profit = call_profit + put_profit  # Net profit of the long straddle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the long straddle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_straddle_profit,
        mode="lines",
```

```
        name="Long Straddle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Long Straddle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
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```

6.4.1.2 Short Straddle

Conversely, a short straddle involves selling a call and a put option. This strategy bets on low market volatility, with the trader expecting the underlying asset to remain stable. The maximum profit is limited to the premiums received, while the risk is theoretically unlimited, as the underlying asset can move significantly in either direction.

```
# Parameters for the short straddle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K = 100  # Strike price for both the call and put options
premium_call = 5  # Premium received for the call option
premium_put = 5  # Premium received for the put option

# Profit calculations for short straddle
call_profit = np.minimum(K - spot_prices, 0) + premium_call
put_profit = np.minimum(spot_prices - K, 0) + premium_put
short_straddle_profit = call_profit + put_profit  # Net profit of the short straddle

# Create the figure
```

```
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the short straddle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_straddle_profit,
        mode="lines",
        name="Short Straddle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Short Straddle Strategy",
```

```
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
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```

6.4.2 Strangles

A strangle is similar to a straddle but involves options with different strike prices. Typically, the call option has a higher strike price than the put.

6.4.2.1 Long Strangle

A long strangle is less expensive than a long straddle due to the options being out of the money. It requires a larger move in the underlying asset's price to be profitable but still benefits from significant volatility without a clear directional bias. The maximum loss is limited to the total premium paid.

```
# Parameters for the long strangle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K_call = 105  # Strike price of the call option (higher strike)
K_put = 95  # Strike price of the put option (lower strike)
premium_call = 2  # Premium paid for the call option
premium_put = 2  # Premium paid for the put option

# Profit calculations for long strangle
call_profit = np.maximum(spot_prices - K_call, 0) - premium_call
put_profit = np.maximum(K_put - spot_prices, 0) - premium_put
long_strangle_profit = call_profit + put_profit  # Net profit of the long strangle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
```

```
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the long strangle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=long_strangle_profit,
        mode="lines",
        name="Long Strangle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Long Strangle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)
```

```
# Show the figure
fig.show()
```

```
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```

6.4.2.2 Short Strangle

Selling a strangle involves selling an OTM call and an OTM put. This strategy profits from low volatility and time decay. The risk is significant if the underlying moves dramatically, as one of the sold options could become deeply in the money.

```
# Parameters for the short strangle strategy
spot_prices = np.linspace(80, 120, 100)  # Range of spot prices
K_call = 105  # Strike price of the call option (higher strike)
K_put = 95  # Strike price of the put option (lower strike)
premium_call = 2  # Premium received for the call option
premium_put = 2  # Premium received for the put option

# Profit calculations for short strangle
call_profit = np.minimum(K_call - spot_prices, 0) + premium_call
put_profit = np.minimum(spot_prices - K_put, 0) + premium_put
short_strangle_profit = call_profit + put_profit  # Net profit of the short strangle

# Create the figure
fig = go.Figure()

# Add traces for the individual option profits
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=call_profit,
        mode="lines",
        name="Call Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=put_profit,
```

```
        mode="lines",
        name="Put Option",
        line=dict(width=2, dash='dot'),
        hovertemplate="Spot Price: %{x:.0f}<br>Profit: %{y:.0f}<extra></extra>",
    )
)

# Add trace for the short strangle net profit
fig.add_trace(
    go.Scatter(
        x=spot_prices,
        y=short_strangle_profit,
        mode="lines",
        name="Short Strangle",
        line=dict(width=4),
        hovertemplate="Spot Price: %{x:.0f}<br>Net Profit: %{y:.0f}<extra></extra>",
    )
)

# Horizontal line at profit = 0 for reference
fig.add_hline(y=0, line_dash="dash", line_color="black")

# Layout adjustments for clarity
fig.update_layout(
    title="Short Strangle Strategy",
    xaxis_title="Spot Price at Expiration",
    yaxis_title="Profit",
    legend_title="Component",
)

# Show the figure
fig.show()
```

```
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```

6.4.3 Strips and Straps

Strips and straps are modifications of the straddle strategy, offering a directional bias while still betting on volatility.

- **Strip**: A strip consists of buying one call option and two put options with the same strike price and expiration. This strategy bets on volatility but with a bearish outlook, as the additional put provides extra profit if the underlying asset's price falls.
- **Strap**: A strap involves buying two call options and one put option with the same strike price and expiration. It is similar to a strip but with a bullish outlook, benefiting from an upward move in the underlying asset's price.

6.4.4 Key Characteristics of Option Combinations

- **Volatility Sensitivity**: All these strategies are sensitive to changes in implied volatility. Long positions in straddles, strangles, strips, and straps benefit from an increase in volatility, while short positions benefit from a decrease.
- **Directional Bias**: Straddles and strangles are primarily non-directional strategies that profit from significant price movements in either direction. Strips and straps introduce a directional bias, offering asymmetric payoffs based on the underlying asset's movement.
- **Risk and Reward**: The risk-reward profile varies significantly between these strategies. Long positions have limited risk and potentially unlimited reward, while short positions offer limited profit potential with significant risk.
- **Breakeven Points**: These strategies have two breakeven points, one on each side of the underlying asset's price at entry. The underlying must move beyond these points for long strategies to profit.

6.5 Practice Questions and Problems

1. What is meant by a protective put? What position in call options is equivalent to a protective put?
2. Explain two ways in which a bear spread can be created.
3. When is it appropriate for an investor to purchase a butterfly spread?
4. What trading strategy creates a reverse calendar spread?
5. What is the difference between a strangle and a straddle?
6. A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?
7. Explain how an aggressive bear spread can be created using put options.
8. Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Draw and explain profit/loss.

9. An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

7 Reading Week

> **Tip**
>
> Take some rest and revise the first six topics!

8 Option Pricing - Binomial Trees

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 13 - Binomial Trees
- PIRIE, Wendy L. Derivatives. Hoboken: Wiley, 2017. CFA institute investment series. ISBN 978-1-119-38181-5.
  - Chapter 4 - Valuation of Contingent Claims
- Cox, J. C., S. A. Ross, and M. Rubinstein. "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (October 1979): 229–64.

Learning Outcomes:

- Understand the basic principles of the one-step binomial model for option pricing, including how to set up a riskless portfolio and calculate the option's value.
- Grasp the concept of risk-neutral valuation and its application in simplifying the pricing of derivatives by focusing on discounting expected payoffs at the risk-free rate.
- Learn how to extend the binomial model to two steps, including calculating the option value at each node and understanding the implications for option pricing.
- Comprehend the methodology for choosing the up (u) and down (d) factors in binomial models, based on the asset's volatility and the length of the time step, as per the Cox, Ross, and Rubinstein (CRR) model.
- Gain insights into how the binomial model can be adapted for different types of assets, including nondividend paying stocks, stock indices, currencies, and futures contracts.

8.1 A One-Step Binomial Model

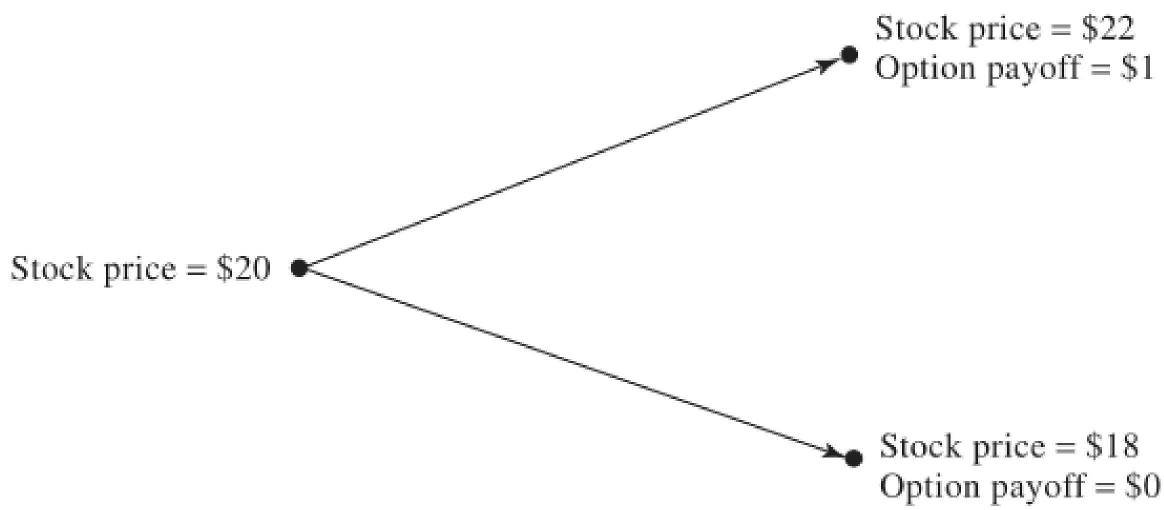
This section provides an overview of a fundamental financial derivative pricing method using a simplified binomial model. The model offers an intuitive approach to option pricing, grounded in the concept of constructing a riskless portfolio and applying arbitrage-free pricing principles.

This generalized model provides a foundational approach for valuing derivatives using discrete-time, binomial frameworks. It not only clarifies the mechanics of risk-neutral valuation but also lays the groundwork for understanding more complex derivatives and pricing methods in continuous-time models.

8.1.1 A Simple Binomial Model for a Call Option

Option Characteristics:

- 3-month call option on a stock.
- The option's strike price is \$21.
- Currently, the stock is priced at \$20.
- In three months, the stock's price can either rise to \$22 or fall to \$18.
- Risk-free rate of interest is 4%.



8.1.1.1 Option Pricing

1. **Constructing a Riskless Portfolio:**

- The portfolio consists of being long Δ shares of the stock and short 1 call option.
- To keep the portfolio risk-free over the option's life, we find Δ such that the portfolio's value is independent of the stock's final price:
  - At stock price \$22: Portfolio value = $\$22\Delta - 1$ (assuming the option is exercised).
  - At stock price \$18: Portfolio value = $\$18\Delta - 0$ (the option expires worthless).
  - Setting these values equal to each other yields $\Delta = 0.25$.

- Thus, a riskless portfolio entails being long 0.25 shares of the stock and short 1 call option.

2. **Valuing the Riskless Portfolio:**

- With a risk-free rate of 4%, we calculate the portfolio's present value.
- Future value of the portfolio in 3 months: $\$22 \times 0.25 - 1 = \$18 \times 0.25 = \$4.50$.
- Discounting this back to the present yields: $4.50 \times e^{-0.04 \times 0.25} = \$4.455$.

3. **Determining the Call Option's Value:**

- The portfolio, consisting of long 0.25 shares and short 1 option, is valued today at \$4.455.
- The current value of 0.25 shares is $0.25 \times \$20 = \$5$.
- Subtracting the portfolio's value from the shares' value gives the option's price: $\$5 - \$4.455 = \$0.545$.

8.1.2 Generalized Framework for Binomial Option Pricing

This section extends the simple binomial model to a more general framework, illustrating how option prices can be systematically derived under varying market conditions. This generalization leverages the fundamental concepts of arbitrage-free pricing and the construction of a synthetic, risk-free portfolio.

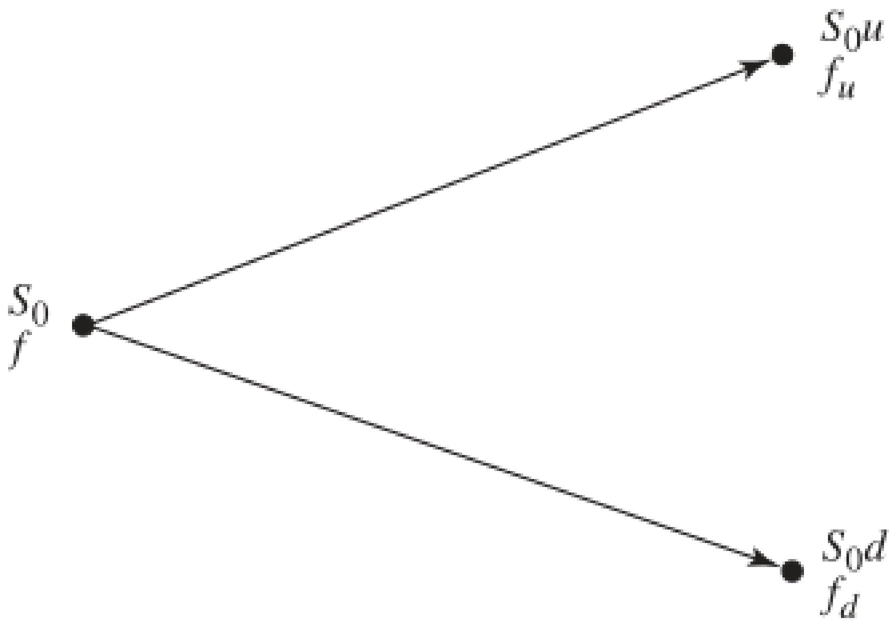
8.1.2.1 Core Principles

- The ending value of a portfolio that contains Δ shares of stock and one short call option can vary based on the stock's final price movement:
  - Following an upward movement, the value is $S_0 u\Delta - f_u$.
  - Following a downward movement, the value is $S_0 d\Delta - f_d$.
- **Creation of a Riskless Portfolio:** To eliminate risk, the values from both upward and downward movements must be equalized, leading to the determination of Δ:

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}$$

  This equation highlights that Δ effectively measures the sensitivity of the option's price to changes in the underlying stock's price, akin to the "delta" in continuous-time models.
- **No-Arbitrage Condition:** The portfolio's risk-free nature implies it should yield returns at the risk-free rate, reinforcing the no-arbitrage principle in financial markets.

8.1.2.2 Analytical Representation



- At time T, the portfolio's value is $S_0u\Delta - f_u$.
- Discounted to present value, this becomes $(S_0u\Delta - f_u)e^{-rT}$.
- The initial cost, or present value, of establishing this portfolio is $S_0\Delta - f$.

By equating the initial investment to the present value of the future payoff, we derive the option's pricing formula:

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

Substituting Δ yields the generalized option pricing equation:

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

where p represents the risk-neutral probability of an upward price movement:

$$p = \frac{e^{rT} - d}{u - d}$$

8.1.2.3 Interpretation of p as a Probability

- The parameter p and its complement $(1-p)$ can be interpreted as the probabilities of the stock's upward and downward price movements, respectively.
- This interpretation leads to a significant insight: the value of an option, or more broadly any derivative, in this model is equivalent to its expected payoff in a "risk-neutral" world, discounted at the risk-free rate. This reflects the fundamental principle that, in an arbitrage-free market, derivative pricing must align with the expected payoff under risk neutrality.

8.1.3 Key Formulas in Binomial Option Pricing

The binomial model provides a straightforward method for determining the price of an option using a discrete-time framework. The central formulas include:

- **Option Price Calculation:** This formula represents the present value of the expected payoff of the option, adjusted for risk neutrality.

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

- **Risk-neutral Probability:** p calculates the risk-neutral probability of the stock price moving up, crucial for determining the expected option payoff.

$$p = \frac{e^{rT} - d}{u - d}$$

8.1.3.1 Variable Definitions:

- f: The current price/premium of the option.
- p: The risk-neutral probability of the stock's price moving up (upward movement).
- $(1-p)$: The risk-neutral probability of the stock's price moving down (downward movement).
- f_u: The payoff from the option in the event of an up movement in the stock price.
- f_d: The payoff from the option in the event of an down movement in the stock price.
- u: The factor by which the stock price increases in the event of an upward movement. This multiplicative factor represents the potential growth of the stock's price in the model's "up" state.
- d: The factor by which the stock price decreases in the event of a downward movement. This multiplicative factor indicates the potential decline of the stock's price in the model's "down" state.
- T: The time to maturity (or expiration) of the option, expressed in years.

- r: The annualized, continuously compounded risk-free interest rate.

Example: Binomial Option Pricing

Consider an option with parameters: $u = 1.1$, $d = 0.9$, $r = 0.04$, $T = 0.25$ (three months), $f_u = 1$ (payoff if the stock price increases), and $f_d = 0$ (payoff if the stock price decreases).

- **Calculating p:**

$$p = \frac{e^{0.04\times0.25} - 0.9}{1.1 - 0.9} = 0.5503$$

This result indicates a 55.03% risk-neutral probability of the stock price increasing.

- **Determining f (Option Price):**

$$f = e^{-0.04\times0.25}(0.5503 \times 1 + 0.4497 \times 0) = 0.545$$

The calculated option price is \$0.545, reflecting the present value of the option's expected payoff, discounted at the risk-free rate.

8.2 Risk-Neutral Valuation Framework

Risk-neutral valuation is a pivotal concept in financial mathematics, offering a streamlined and theoretically robust method for pricing derivatives. This approach is grounded in the notion that the expected return on the underlying asset can be assumed to be the risk-free rate when valuing derivatives. Here, we elaborate on this principle and its practical implications, using a binomial model as the illustrative framework.

8.2.1 Core Concept

- **Expected Stock Price Dynamics:** In a binomial model where the probability of upward and downward movements are denoted by p and $1-p$ respectively, the expected stock price at time T, discounted back to the present, is mathematically equivalent to the stock's initial price compounded at the risk-free rate, S_0e^{rT}. This equivalence underscores the principle that, under risk-neutral valuation, the stock is assumed to grow at the risk-free rate over time.
- **Binomial Trees and Derivative Pricing:** The use of binomial trees in financial modeling demonstrates a general principle: to value a derivative, it is sufficient to assume

that the underlying asset yields a return equal to the risk-free rate, and similarly, the discount rate applied to future payoffs is the risk-free rate. This methodology, known as risk-neutral valuation, simplifies the derivative pricing process by abstracting away from the actual expected return of the underlying asset.

Example Revisited

Consider an option with parameters: $u = 1.1$, $d = 0.9$, $r = 0.04$, $T = 0.25$ (three months), $f_u = 1$ (payoff if the stock price increases), and $f_d = 0$ (payoff if the stock price decreases).
The calculation proceeds as follows:

$$22p + 18(1-p) = 20e^{0.04\times0.25} \Rightarrow p = 0.5503$$

- This implies a 55.03% probability of the option being worth \$1 (if the stock price increases) and a 44.97% probability of it being worth \$0 (if the stock price decreases) at the end of 3 months. The expected value of the option, calculated under the risk-neutral probability, is therefore:

$$0.5503 \times 1 + 0.4497 \times 0 = 0.5503$$

- Discounting this expected value at the risk-free rate gives the present value of the option:

$$0.5503e^{-0.04\times0.25} = 0.545$$

- Remarkably, this value aligns with the outcome obtained through no-arbitrage arguments, affirming the congruence between no-arbitrage principles and risk-neutral valuation in deriving the same pricing result for the derivative.

8.2.2 The Irrelevance of the Stock's Expected Real-World Return

- A key insight from risk-neutral valuation is the irrelevance of the real-world probabilities of the underlying asset's future price movements when valuing options.
- Since **the current stock price embeds these real-world probabilities**, derivative pricing can proceed without needing to account for them again.
- This perspective aligns with a broader financial theory that the expected return of the underlying asset, as anticipated in the real world, does not affect the valuation of derivatives based on that asset.

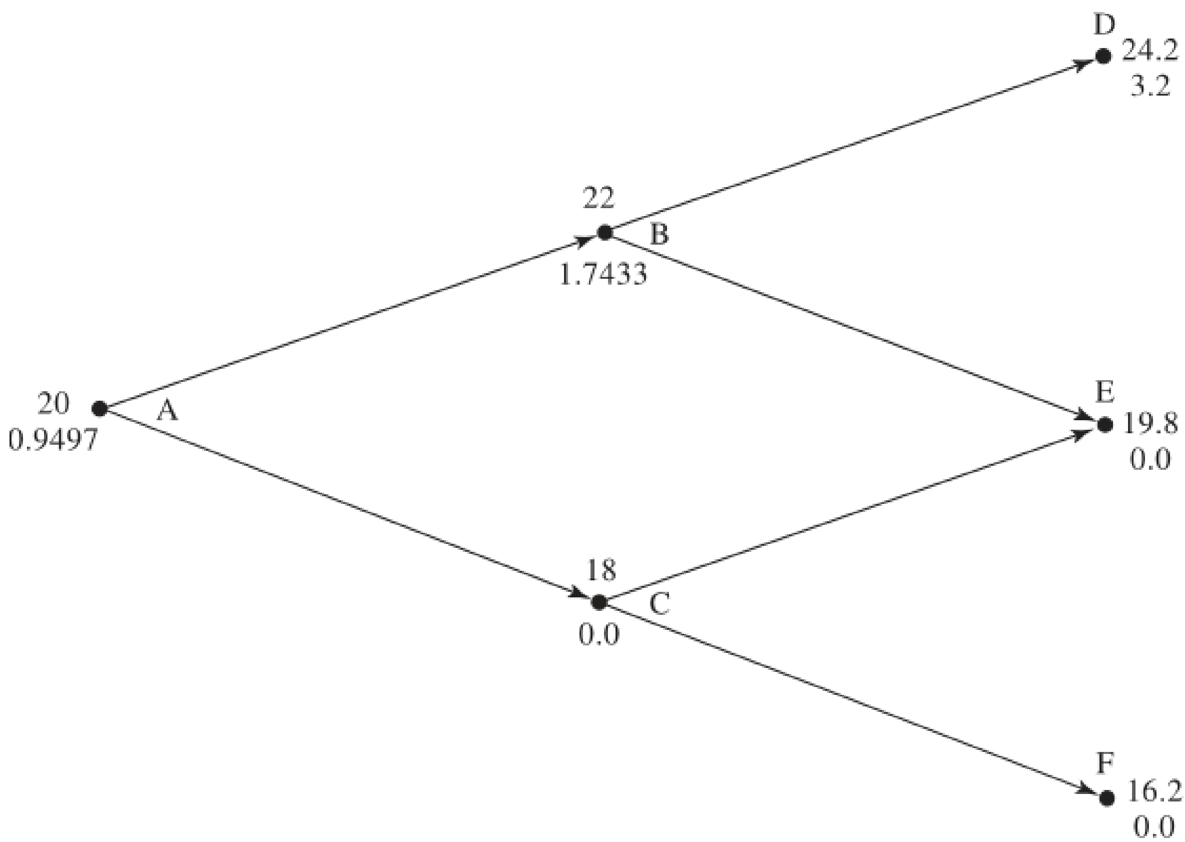
- The valuation hinges instead on the construct of a risk-neutral world where all assets are assumed to grow at the risk-free rate.

8.3 Two-Step Binomial Trees

Two-step binomial trees extend the single-period model to allow for a more detailed examination of option pricing over multiple periods. This method can value both call and put options by simulating possible paths the underlying asset's price might take and then discounting expected payoffs back to the present value.

8.3.1 Valuing a Call Option

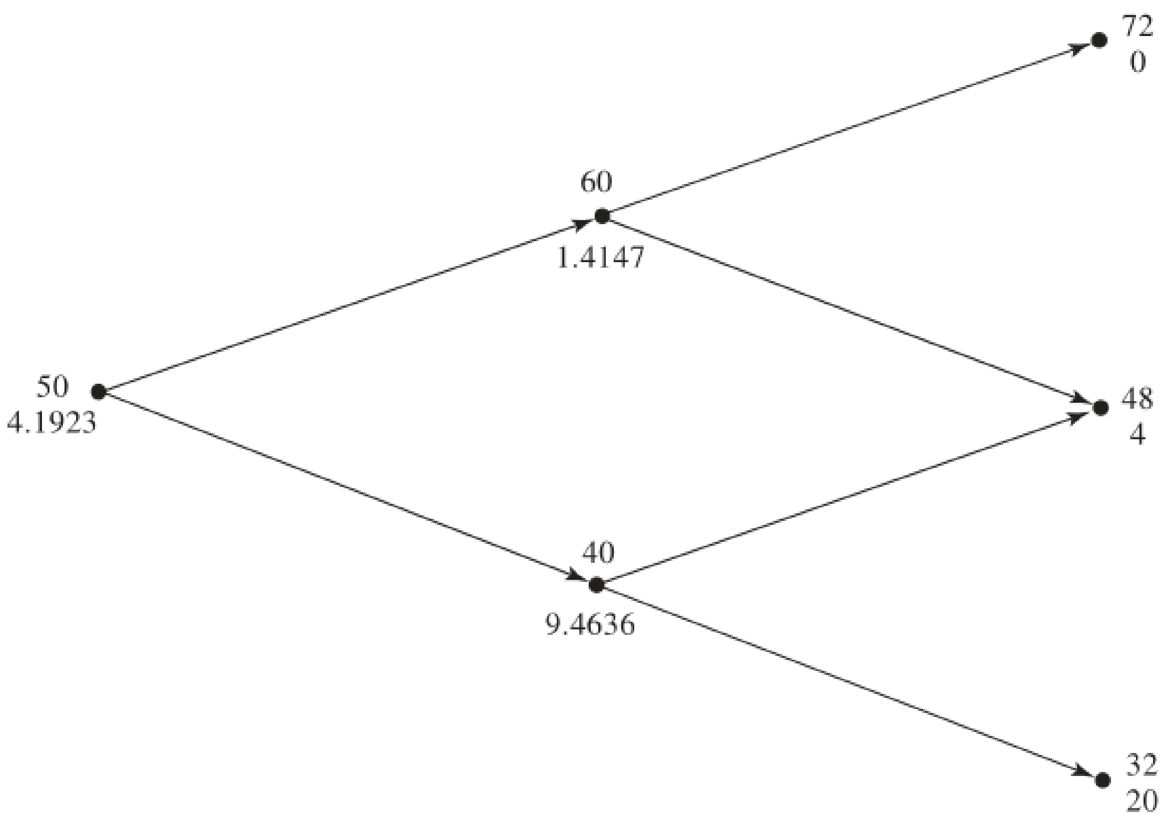
Consider a call option with a strike price (K) of \$21, where each time step represents 3 months. The annual risk-free rate (r) is 4%, with up (u) and down (d) factors of 1.1 and 0.9, respectively, and the risk-neutral probability (p) is 0.5503.



- At the final nodes (D, E, F), the option value simply reflects the option payoff at expiration.
- At nodes B and C, the option's value is calculated as the present value of its expected payoff:
  - Value at node $B = e^{-0.04\times0.25}(0.5503 \times 3.2 + 0.4497 \times 0) = \$1.7433$
  - Value at node $C = 0 + 0 = \$0$
- Moving back to the initial node (A), we compute the present value of the expected option payoff from nodes B and C:
  - Value at node $A = e^{-0.04\times0.25}(0.5503 \times 1.7433 + 0.4497 \times 0) = \$0.9497$

8.3.2 Valuing a Put Option

For a put option with a strike price (K) of \$52 and a time step of 1 year, the parameters are: $r = 5\%$, $u = 1.2$, $d = 0.8$, and $p = 0.6282$.

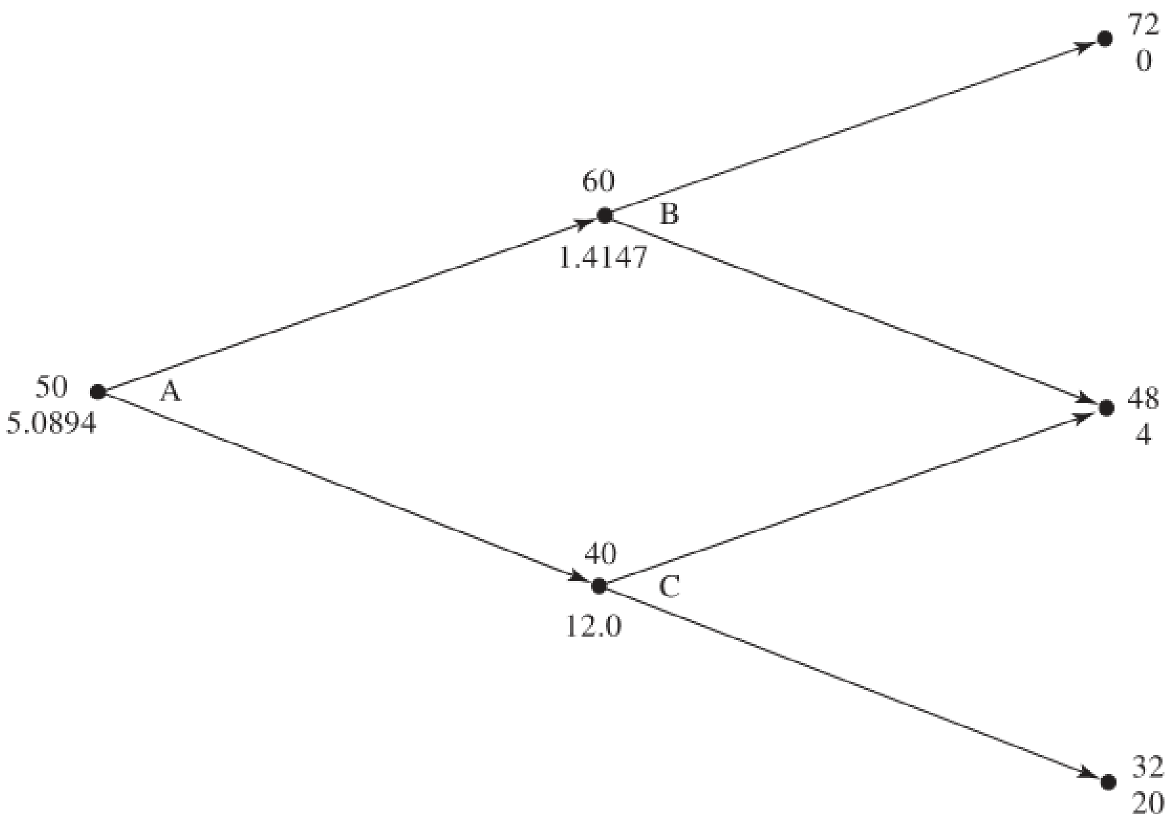


- Following the same approach and determining option values at each node from right to left, we obtain:
  - Value at node $C = e^{-0.05 \times 1}(0.6282 \times 4 + 0.3718 \times 20) = \$9.4636$
  - Value at node $B = e^{-0.05 \times 1}(0.6282 \times 0 + 0.3718 \times 4) = \$1.4147$
  - Value at node $A = e^{-0.05 \times 1}(0.6282 \times 1.4147 + 0.3718 \times 9.4636) = \$4.1923$

8.3.3 What Happens When the Put Option is American?

American options allow for early exercise, affecting their valuation. Specifically, for the previous example with a put option:

- The exercising early increases the option's value at node C to \$12.
- The recalculated value at node A, incorporating the potential for early exercise, is:
  - Value at node $A = e^{-0.05 \times 1}(0.6282 \times 1.4147 + 0.3718 \times 12.0000) = \$5.0894$



- The introduction of the American feature, hence, increases the put option's value from \$4.1923 to \$5.0894, reflecting the added value of early exercise flexibility.
- The same approach applies to call options.

8.4 Choosing u and d for Binomial Models

In binomial option pricing models, accurately modeling the underlying asset's price dynamics is crucial. One effective approach to align these models with the asset's volatility—a key measure of its price fluctuations—is to set the up (u) and down (d) factors based on volatility. This method ensures that the model captures the essence of the asset's risk and return characteristics over the specified time steps.

8.4.1 Cox, Ross, and Rubinstein (CRR) Methodology

Cox, Ross, and Rubinstein's seminal work in 1979 introduced a practical and widely adopted method for setting u and d, which directly ties these parameters to the asset's volatility (σ) and the length of the model's time step (Δt):

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

This approach ensures that the binomial model reflects the underlying asset's volatility, allowing for a more accurate and realistic simulation of its price movements. The choice of u as an exponentiation of volatility times the square root of the time step captures the log-normal distribution of stock prices over time, consistent with empirical observations.

Implications of Girsanov's Theorem

Girsanov's Theorem provides a critical theoretical foundation for applying real-world volatility measurements within a risk-neutral pricing framework. The theorem suggests that while the expected return on an asset may differ between the real world and the risk-neutral world, the asset's volatility (σ) remains consistent across both.

- **Volatility Consistency:** Girsanov's Theorem assures us that volatility—a fundamental input in modeling asset price dynamics—is invariant whether we're considering a real-world or risk-neutral probability measure. This invariance allows us to observe and measure volatility in the real world and then directly apply these measurements to construct a binomial tree for option pricing in the risk-neutral world.

- **Risk-Neutral Valuation:** Despite the potential differences in expected returns between the real and risk-neutral worlds, the theorem supports the use of risk-neutral valuation. By ignoring risk preferences and market expectations, risk-neutral valuation simplifies the pricing of derivatives by focusing solely on the discounting of expected payoffs at the risk-free rate, underpinned by the consistent application of real-world volatility measures.

8.5 The Binomial Tree Model Summary

8.5.1 Core Formulas

8.5.1.1 Price Movement Factors

The magnitude of up (u) and down (d) movements in the asset's price is modeled to reflect the asset's volatility (σ) and the length of the time step (Δt):

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

These factors ensure that the model captures the log-normal distribution of asset price changes, consistent with empirical observations.

8.5.1.2 Probability of an Up Move

The risk-neutral probability (p) of an upward price movement is derived as:

$$p = \frac{a - d}{u - d}$$
$$a = e^{r\Delta t}$$

Here, a represents the asset's expected growth factor over a single time step under the risk-free rate (r), adjusting for any dividends or interest rate differentials.

8.5.1.3 Option Value Calculation

The value of an option in a one-step binomial model is determined by:

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

where f_u and f_d are the option's payoffs in the event of an up or down move, respectively, and T is the total time to maturity.

8.5.2 Application to Various Assets

While the basic binomial tree structure remains constant, adjustments to the calculation of p accommodate different types of underlying assets, reflecting their unique characteristics.

- **Nondividend Paying Stocks:**
  - $a = e^{r\Delta t}$
- **Stock Indices:** For indices that pay dividends, the dividend yield (q) is subtracted from the risk-free rate:
  - $a = e^{(r-q)\Delta t}$
- **Currencies:** For options on currencies, the foreign risk-free rate (r_f) is subtracted from the domestic rate:
  - $a = e^{(r-r_f)\Delta t}$
- **Futures Contracts:** For futures, the growth factor (a) simplifies to 1 since futures prices already embody the cost-of-carry, negating the need for discounting or growth adjustments:
  - $a = 1$

8.6 Practice Questions and Problems

1. A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

> **Solution**
>
> Option price = 1.69

2. A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a six-month European put option with a strike price of \$50?

> **Solution**
>
> Option price = 1.16

3. Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

4. A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

> **Solution**
>
> Option price = 9.6104

5. For the situation considered in Problem 4, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put-call parity.

> **Solution**
>
> Option price = 1.9203

6. Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

> **Solution**
>
> u = 1.0352, d = 0.966, p = 0.4553

7. A stock index is currently 1,500. Its volatilityis 18%. The risk-free rate is 4% per annum (continuously compounded) for all maturities and the dividend yield on the index is 2.5%. Calculate values for u, d, and p when a six-month time step is used. What is the value a 12-month American put option with a strike price of 1,480 given by a two-step binomial tree.

Solution

Option price = 78.41

9 Option Pricing - Black-Scholes Model

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Learning Outcomes:

- Understand the foundational principles of the Black-Scholes-Merton model, including its formulation and applications in financial markets.
- Interpret key parameters within the Black-Scholes-Merton model, explore its properties, assumptions and limitations in real-world scenarios.
- Grasp the concept of risk-neutral valuation and its significance in pricing financial derivatives and managing financial risk.
- Analyze the effect of dividends on option pricing within the Black-Scholes-Merton framework and understand its practical implications for investors and traders.
- Discuss the role of volatility in financial markets, specifically how it influences option pricing and investor strategies.

9.1 Black-Scholes-Merton Model

The Black-Scholes model, developed by Fischer Black and Myron Scholes in 1973, is a groundbreaking mathematical framework for valuing European options. It provides a closed-form solution for the price of a European call or put option based on stock volatility, the risk-free rate, the strike price, and the time to expiration. The model assumes a lognormal distribution of stock prices, continuous trading, and no dividends, among other factors.

The Binomial Option Pricing Model, introduced by Cox, Ross, and Rubinstein in 1979, offers a more flexible approach to option valuation, accommodating American options and dividends. It constructs a discrete-time (binomial) lattice for asset price movements, allowing for an iterative calculation of option prices. This model can be viewed as a simplified, discrete approximation of the continuous processes underlying the Black-Scholes model. As the number of steps in the binomial model increases, its results converge to those of the Black-Scholes formula, illustrating their intrinsic relationship. The binomial model's adaptability to different types of options and its intuitive framework make it a vital complement to the Black-Scholes model in financial theory.

9.1.1 Model Assumptions

The Black-Scholes-Merton model is predicated on several key assumptions to facilitate the derivation of a closed-form solution for pricing European options. These assumptions include:

1. **European-style Options:** The model is applicable to European options, which can only be exercised at expiration.
2. **No Dividends:** It is assumed that the underlying asset does not pay dividends during the life of the option.
3. **No Arbitrage:** The market is efficient, prohibiting the possibility of riskless arbitrage profits.
4. **Short Selling:** Investors are allowed to short sell the underlying asset without restrictions.
5. **No Market Frictions:** The market operates without transaction costs, taxes, or regulatory constraints, implying perfect market conditions.
6. **Constant Risk-Free Rate:** The risk-free interest rate, underpinning the time value of money, is constant over the option's life and the same for all maturities.
7. **Known and Constant Volatility:** The volatility of the underlying asset's returns, a measure of its price fluctuations, is known and remains constant over time.
8. **Geometric Brownian Motion:** The price of the underlying asset follows a geometric Brownian motion, characterizing price changes as continuous and random without sudden jumps.
9. **Continuous Trading and Liquidity:** Trading is possible at any moment during market hours, and the asset is perfectly liquid, allowing for immediate buy and sell transactions.

9.1.2 The Black-Scholes-Merton Formulas for Options Pricing

The model provides explicit formulas to compute the price of call and put options, defined as follows:

- c = price of a European call option
- p = price of a European put option
- $N(x)$ = the standard normal cumulative distribution function, representing the probability that a normally distributed variable with a mean of zero and a standard deviation of one is less than x.
- r = annualized continuously compounded risk-free rate
- σ = annualized constant volatility of the stock returns
- S = current stock price
- K = option strike price
- T = time to expiration (in years)

The core equations are:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

9.1.3 The $N(x)$ Function

The function $N(x)$ is pivotal in calculating the probabilities essential for the Black-Scholes-Merton formula. It quantifies the likelihood that a random draw from a standard normal distribution falls below a specific value x. For practical computations, one may refer to statistical tables or the NORM.DIST function in Excel.

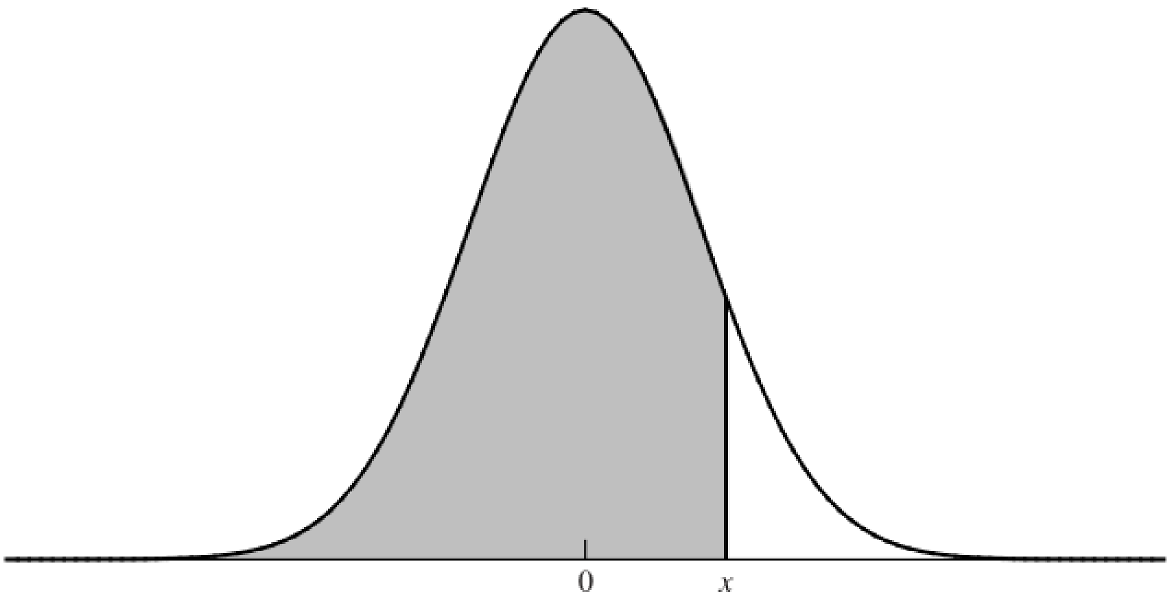

Figure 9.1: image

Example: Black-Scholes-Merton Model

Consider an option with the following parameters:

- $S_0 = 42$ (Current stock price)
- $K = 40$ (Strike price)
- $r = 10\%$ (Risk-free rate)
- $\sigma = 20\%$ (Volatility)
- $T = 0.5$ (Time to expiration in years)

Using these parameters, we calculate:

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = 0.7693 - 0.2\sqrt{0.5} = 0.6278$$

Employing the NORM.DIST function in Excel for d_1 and d_2 yields:

- $N(0.7693) = 0.7791$; $N(0.6278) = 0.7349$;
- $N(-0.7693) = 0.2209$; $N(-0.6278) = 0.2651$.

The discounted strike price is calculated as:

$$Ke^{-rT} = 40e^{-0.1\times 0.5} = 38.049$$

Finally, the call (c) and put (p)

$$c = 42 \times 0.7791 - 38.049 \times 0.7349 = 4.76$$
$$p = 38.049 \times 0.2651 - 42 \times 0.2209 = 0.81$$

9.2 Interpretation of $N(d_1)$ and $N(d_2)$

Understanding the terms $N(d_1)$ and $N(d_2)$ within the Black-Scholes formula is fundamental for grasping the model's insights into option pricing dynamics. These terms provide a probabilistic interpretation that enriches the model's application and theoretical comprehension.

9.2.1 Interpretation #1

- **$N(d_2)$:** This term represents the probability that the option will be exercised at maturity. It quantifies the likelihood that the stock price at maturity will exceed the strike price, making it beneficial for the holder to exercise the option.
- **$S_0 e^{rT} N(d_1)$:** This expression estimates the expected stock price at maturity in a risk-neutral world, adjusting for instances when the stock price falls below the strike price, which are considered as zero value. This adjustment mirrors the option's payoff structure, where only in-the-money scenarios contribute to the expected payoff.
- **Expected Payoff and Present Value:** The expected payoff of holding a call option can be expressed as $S_0 e^{rT} N(d_1) - K N(d_2)$. To derive the option's current value, we discount this expected payoff to the present, yielding the Black-Scholes call option pricing formula: $c = S_0 N(d_1) - K e^{-rT} N(d_2)$.

9.2.2 Interpretation #2

Another lens to view the call option pricing formula is:

$$c = e^{-rT} N(d_2) \left(S_0 e^{rT} N(d_1) / N(d_2) - K \right)$$

- **e^{-rT}:** This factor discounts the expected payoff to its present value, reflecting the time value of money.
- **$N(d_2)$:** Represents the probability of the option being exercised, acting as a weight for the expected payoff.
- **$S_0 e^{rT} N(d_1)/N(d_2)$:** Expected stock price in a risk-neutral world if option is exercised.
- **K:** The strike price, indicating the cost to exercise the option.

9.3 Properties of the Black-Scholes Formula

The behavior of the call (c) and put (p) option prices relative to changes in the underlying stock price (S_0) is critical for understanding the model's implications:

- **As S_0 increases significantly:**
  - The call option price (c) converges to $S_0 - Ke^{-rT}$, because both d_1 and d_2 grow large, making $N(d_1)$ and $N(d_2)$ approach 1. This reflects the option's intrinsic value, as the benefit from exercise becomes almost certain.
  - The put option price (p) approaches zero, as $N(-d_1)$ and $N(-d_2)$ approach zero, indicating a negligible chance of the option being in-the-money.
- **As S_0 decreases significantly:**
  - The call option price (c) trends towards zero, mirroring the diminishing likelihood of the option being exercised.
  - The put option price (p) converges to $Ke^{-rT} - S_0$, reflecting the increasing intrinsic value of the option as the likelihood of exercise grows.

9.4 Risk-Neutral Valuation

The concept of Risk-Neutral Valuation is a cornerstone in the field of financial derivatives, offering a powerful framework for option pricing, particularly within the Black-Scholes-Merton model. This approach simplifies valuation by assuming that all investors are indifferent to risk, thereby pricing assets solely based on the risk-free rate rather than the expected return under actual market conditions.

In the derivation of the Black-Scholes-Merton differential equation, an intriguing observation is that the expected return of the underlying asset, denoted by μ, does not influence the equation. This omission signifies that the equation—and by extension, the option's price—is not affected by investors' risk preferences. The remarkable outcome is that the solution to this differential equation remains consistent whether we consider a risk-neutral world or the real, risk-averse world. This observation leads to the principle of risk-neutral valuation, which posits that under certain conditions, the market can be modeled as if all investors were risk-neutral.

This methodological approach allows the valuation of options and other derivatives in a theoretically consistent manner, abstracting away from the complexities introduced by varying risk preferences among investors. The elegance and simplicity of risk-neutral valuation underpin its widespread application in financial economics and the pricing of derivatives.

9.4.1 Key Points of Risk-Neutral Valuation

- **Indifference to Risk:** In a risk-neutral world, investors expect to earn the risk-free rate on all investments, regardless of the risk involved. This assumption is purely theoretical but facilitates the practical application of option pricing models.
- **Exclusion of μ:** The fact that the differential equation does not incorporate the expected return μ indicates the irrelevance of personal risk preferences in pricing derivatives through this model.
- **Universal Pricing Principle:** The risk-neutral valuation framework suggests that the price of a derivative is determined by the expected payoff under risk neutrality, discounted at the risk-free rate.

9.4.2 Application of Risk-Neutral Valuation

Applying risk-neutral valuation to price options involves a straightforward three-step process:

1. **Assumption of Risk-Free Returns:** Initially, it is assumed that the expected return on the stock (or the underlying asset) is equal to the risk-free rate. This simplification aligns the future stock price dynamics with a risk-neutral perspective, where all assets are expected to grow at this rate.
2. **Calculation of Expected Payoff:** The next step involves computing the expected payoff of the option at expiration. This calculation is performed under the assumption that the stock prices follow a risk-neutral probability distribution, reflecting a world where the expected rate of return on all assets is the risk-free rate.
3. **Discounting at the Risk-Free Rate:** Finally, the expected payoff is discounted back to the present using the risk-free rate. This discounting reflects the principle that future cash flows must be adjusted to present value at a rate that reflects their time value of money, devoid of risk premiums.

> Example: Risk-Neutral Valuation of a Call Option
>
> Consider a call option with a strike price of K, and let's assume that under the risk-neutral world, the expected stock price at expiration is $E[S_T]$. The expected payoff from holding this call option can be represented as $E[max(S_T - K, 0)]$, where S_T is the stock price at expiration. By discounting this expected payoff at the risk-free rate r, we can calculate the present value of the call option.

9.5 The Effect of Dividends

The inclusion of dividends in the valuation of options introduces additional complexity, as dividends can significantly affect option prices. The adjustment for dividends in the Black-Scholes model and considerations for American call options are pivotal for accurate pricing.

9.5.1 Valuing European Options on Dividend-Paying Stocks

To accommodate dividends in the Black-Scholes model, **the stock price is adjusted by subtracting the present value of dividends expected to be paid during the life of the option**. This adjustment reflects the reduction in stock price that typically occurs on the ex-dividend date, thereby affecting the option's intrinsic value. It's crucial to include only those dividends that are expected to be paid before the option expires, as these are the ones that will impact the option holder.

For a dividend-paying stock, the adjusted stock price S_{adj} in the Black-Scholes formula becomes:

$$S_{adj} = S_0 - PV(\text{dividends})$$

where $PV(\text{dividends})$ is the present value of dividends paid during the option's life. This adjustment aims to capture the expected drop in the stock price due to dividend payouts, ensuring the option valuation accurately reflects this anticipated decrease.

9.5.2 American Calls on Dividend-Paying Stocks

The traditional wisdom that American call options should not be exercised early due to their time value does not always hold in the presence of dividends. The possibility of early exercise becomes relevant when the stock pays dividends.

9.5.2.1 Criteria for Early Exercise

For an American call option on a dividend-paying stock, early exercise might be optimal just before an ex-dividend date if the dividend amount exceeds the loss of time value from early exercise, specifically if the dividend is greater than:

$$K[1 - e^{-r(t_{i+1} - t_i)}]$$

Here, K is the strike price, r is the risk-free rate, and t_i to t_{i+1} represents the interval between dividend payments. This condition evaluates whether the immediate gain from capturing the dividend outweighs the potential benefits of holding the option for its time value.

9.5.3 Black's Approximation for American Call Options with Dividends

Black's Approximation offers a pragmatic method for approximating the price of American call options on dividend-paying stocks. This approximation involves comparing two European call option prices:

1. **First European Price:** This is calculated for a European call option with the same expiration date as the American option, incorporating the adjusted stock price for dividends.
2. **Second European Price:** This is calculated for a European call option expiring just before the final ex-dividend date within the American option's life.

The value of the American call option is approximated as the maximum value derived from these two European call options. This method acknowledges the critical influence of dividends on option pricing and provides a practical solution for incorporating this factor into American option valuation.

Black's Approximation simplifies the valuation process by sidestepping the complex optimization problem of determining the exact optimal early exercise strategy in the presence of dividends. It offers a balance between accuracy and computational efficiency, making it a valuable tool for practitioners in the financial markets.

9.6 Volatility in Financial Markets

Volatility is a pivotal concept in finance, encapsulating the degree of variation in the price of a financial instrument over time. It is central to the valuation of options, where it measures the extent of uncertainty or risk associated with the price change of the underlying asset.

- **Definition:** Volatility, denoted as σ, quantifies the expected variability in the returns of a stock or any financial asset. It reflects the degree of uncertainty or risk about the magnitude of changes in an asset's value.
- **Range:** Typically, stocks exhibit annual volatilities in the range of 15% to 60%. This range indicates the expected yearly change in the stock price, highlighting the variability in asset prices across different markets and conditions.
- **Impact on Options:** The value of an option is exceptionally sensitive to changes in volatility. Since options are derivatives based on the underlying asset, increased volatility translates to higher option premiums, all else being equal.

- **Observability:** Unlike direct observables like stock prices, volatility is not directly observable and must be estimated. Historical volatility is derived from past market prices, while implied volatility is inferred from current market prices of options.

9.6.1 Historical Volatility

- **Calculation:** Historical volatility is the standard deviation of the continuously compounded returns of an asset over a specified period, typically annualized. This measure provides a backward-looking estimate of volatility based on actual market data.
- **Daily Volatility Example:** For a stock priced at \$30 with an annual volatility of 25%, the standard deviation of the stock's price change over one day (Δt) can be approximated as $25\% \times \sqrt{\frac{1}{252}} = 1.57\%$, assuming 252 trading days in a year. This calculation aids in understanding daily price variability.
- **Annualization:** By annualizing the standard deviation of daily returns (multiplied by $\sqrt{252}$), investors can compare volatility across assets with different time frames.

9.6.2 Nature of Volatility

- **Market Hours:** Volatility tends to be higher during trading hours compared to when the market is closed, reflecting the immediate impact of news and events on asset prices.
- **Trading Days:** For options valuation, time is measured in trading days, not calendar days, acknowledging the fact that price changes are primarily driven by market activity. This convention ensures that the time to maturity for options reflects actual market exposure.

9.6.3 Implied Volatility

- **Definition:** Implied volatility is the volatility implied by the market price of an option, based on the Black-Scholes-Merton model. It represents the market's expectation of future volatility over the life of the option.
- **VIX Index:** The VIX S&P 500 Volatility Index is a widely recognized measure of market expectation of near-term volatility conveyed by S&P 500 stock index option prices. The VIX S&P 500 Volatility Index.
- **Application:** Implied volatility is crucial in options trading, serving as a standard measure that reflects the market's view on future volatility. It is instrumental in pricing, trading strategies, and risk management.
- **Volatility Smile and Surface:** The volatility smile and surface graphically represent implied volatility across different strike prices and maturities. Deviations from a flat surface—expected under the BSM model assumptions—reveal market anomalies, such as varying risk preferences and the impact of market events on different options.

- **Market Anomalies:** The observed curvature in the volatility smile and the dynamic topology of the volatility surface challenge the BSM model's assumptions, demonstrating real-market frictions and behaviors not accounted for by the model. These phenomena offer valuable insights into the complexity of market dynamics and the factors influencing option pricing beyond theoretical models.

9.6.3.1 Role of Implied Volatility in Option Trading

- **Market's Perception of Value:** Implied volatility reflects the market's anticipation of future price fluctuations of the underlying asset. Higher implied volatility indicates greater expected price movement and, consequently, a higher option premium.
- **Pricing Medium:** In options markets, implied volatility is often used as a standard metric to quote prices. A call option priced at $10, corresponding to an implied volatility of 25%, illustrates how traders communicate and gauge option value through volatility levels.
- **Valuation of Options:** Implied volatility aids in comparing the relative value of options by standardizing price differences attributable to intrinsic value and time decay. Options with lower implied volatility are considered cheaper relative to those with higher implied volatility, offering insights into potential over- or under-valuation.
- **Revaluation Over Time:** As market conditions and perceptions change, so does implied volatility. Tracking its evolution allows traders and portfolio managers to reassess the value of their positions, adapting strategies accordingly.
- **Market Sentiment Indicator:** The level of implied volatility reflects market participants' expectations about future volatility, serving as a barometer for market sentiment.
- **Communication Tool:** For regulators, banks, and compliance officers, implied volatility serves as a universal language to discuss and assess the risk and valuation of options portfolios.

9.7 Practice Questions and Problems

1. Name some assumptions of the Black-Scholes-Merton model. Compare with the real world and explain potential issues.
2. Explain the principle of risk-neutral valuation.
3. Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. (Tip: Use Excel function NORMDIST to calculate $N(x)$)

Solution

Put option price = 2.37

4. What difference does it make to your calculations in the previous Problem if a dividend of \$1.50 is expected in two months? (Tip: Use Excel function NORMDIST to calculate $N(x)$)

Solution

Put option price = 3.03

5. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

Solution

Call option price = 5.06

6. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

Solution

Put option price = 6.40

7. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

   a. What is the price of the option if it is a European call?
   b. What is the price of the option if it is an American call?
   c. What is the price of the option if it is a European put?
   d. Verify that put–call parity holds.

Solution

European call = 2.52 European put = 1.05

8. What is implied volatility? How can it be calculated?

9. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

Solution

1-day std. $= 1.9\%$

10. What is the actual implied volatility of the S&P 500? Try to interpret the value.
11. A call option on a non-dividend-paying stock has a market price of \$2.50. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% perannum. What is the implied volatility?

Solution

Implied volatility $= 39.6\%$

10 Options on Indices, Currencies, and Futures

References

- HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
  - Chapter 17 - Options on Stock Indices and Currencies
  - Chapter 18 - Futures Options and Black's Model
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  - Chapter 4 - Valuation of Contingent Claims

Learning Outcomes:

- Master the use of index options for hedging and portfolio protection; understand the benefits compared to direct stock transactions.
- Apply models like Black-Scholes for valuing index options.
- Utilize currency options for hedging foreign exchange risks; grasp valuation techniques using both spot and forward prices.
- Evaluate European futures options using Black's model; distinguish between options on futures and physical commodities.

10.1 Options on Stock Indices

Options on stock indices allow investors to speculate on the direction of an entire market index or hedge their portfolio market risk without directly transacting the component stocks. Some of the most widely traded stock index options in the United States are:

- **S&P 100 Index (OEX and XEO):** These options provide a broad market exposure to the top 100 stocks in the S&P index. OEX options are American-style, allowing exercise at any point before the expiration date, whereas XEO options are European-style, exercisable only at expiration.

- **S&P 500 Index (SPX):** A benchmark index for U.S. equities, these European-style options reflect the performance of 500 large-cap stocks and are a key tool for institutional investors for hedging and exposure adjustment.
- **Dow Jones Index (DJX):** Priced at 1/100th of the Dow Jones Industrial Average, these options are a cost-effective way to gain exposure to the movements of the Dow Jones, consisting of 30 significant stocks.
- **Nasdaq 100 Index (NDX):** These European-style options represent the 100 largest non-financial stocks from the Nasdaq, making them crucial for investors focused on technology and innovation sectors.

All mentioned index options are exchange-traded, have their contract size set at 100 times the index value, and are settled in cash, which simplifies the transaction and eliminates the need for physical delivery of the underlying securities.

10.1.1 Using Index Options for Portfolio Insurance

Portfolio insurance is a strategy used to protect against market declines through the use of put options. The strategy involves purchasing puts to cover potential losses in a portfolio's value due to adverse market movements.

- **Initial Setup:**
  - Let S_0 represent the current index value.
  - Let K be the strike price of the put option.
  - The portfolio's beta (β) indicates its sensitivity to market movements relative to the index.
- **Insurance Strategy:**
  - For a portfolio with $\beta = 1.0$, purchase one put option for every $\$100 \times S_0$ of portfolio value to hedge against downturns.
  - If $\beta \neq 1.0$, adjust the number of put options accordingly, purchasing β options per $\$100 \times S_0$.
  - In both cases, K is chosen to give the appropriate insurance level.

Example: Portfolio Insurance, $\beta = 1.0$

- **Portfolio Details:**
  - Portfolio $\beta = 1.0$
  - Portfolio Value = \$500,000
  - Current Index Value = 1,000
- **Objective:** Insure the portfolio so its value does not fall below \$450,000 in the

next three months.

- **Implementation Strategy:**
  - The manager opts to buy put options as a form of insurance. Since each contract covers 100 times the index value, the manager needs one put option for every \$100,000 of the portfolio value (i.e., $500,000/100 \times 1,000 = 5$ put options).
  - These options have a strike price (K) of 900 (10% decrease).

- **Mechanism of Insurance:**
  - If the index drops to 880 in three months, the portfolio's expected value would decrease proportionately to about \$440,000.
  - The put options' payoff can be calculated as follows:

$$\text{Payoff} = 5 \times [(900 - 880) \times 100] = \$10,000$$

  - This payoff compensates for the decrease in the portfolio's value, effectively bringing it up to the insured value of \$450,000.

Example: Portfolio Insurance, $\beta \neq 1.0$

Consider a portfolio with a β of 2.0. This high beta implies that the portfolio's returns are expected to be twice as volatile as the underlying index's returns. Currently, the portfolio is valued at \$500,000, while the index stands at 1,000. With a risk-free rate of 12% per annum and a dividend yield of 4% on both the portfolio and the index.

- **Objective:** Insure the portfolio so its value does not fall below \$450,000 in the next three months.

- **Required Number of Contracts:** $(500,000/100,000) \times 2 = 10$ put option contracts are needed.

10.1.1.1 Determining the Appropriate Strike Price Using CAPM Model

- **Index Return Scenario:** Assume the index rises to 1,040 in three months, reflecting a 4% return.

- **Total Return Including Dividends:** 4% (index return) + 1% (dividends) = 5%.

- **Excess Return Over Risk-Free Rate:** 5% - 3% (equivalent quarterly risk-free rate) = 2%.

- **Excess Return for Portfolio:** $2\% \times 2 = 4\%$ (due to beta of 2.0).
- **Net Portfolio Return Calculation:** Excess return (4%) + risk-free rate over three months (3%) - dividend yield (1%) = 6%.
- **Projected Portfolio Value:** $\$500,000 \times 1.06 = \$530,000$.
- **Result:** Similar calculations can be carried out for other values of the index at the end of the three months. Appropriate strike price for the 10 put option contracts that are purchased is 960 (or 955 when we include dividends).

Value of Index in Three Months	Value of Portfolio in Three Months ($)
1,080	570,000
1,040	530,000
1,000	490,000
960	450,000
920	410,000
880	370,000

10.2 Valuation of Stock Index Options

The valuation of European stock index options hinges on the understanding of asset price dynamics under different conditions. The probability distribution of the asset price at maturity T remains consistent in two distinct scenarios: 1. When the asset originates at a price S_0 with a yield of q. 2. When the asset starts at a discounted price of $S_0 e^{-qT}$, assuming it yields no income.

By utilizing this equivalence, we simplify the valuation process for European options. Instead of accounting for dividends directly, we **adjust the initial stock price** to $S_0 e^{-qT}$ and **proceed with the valuation as if the stock pays no dividends**. This approach effectively streamlines the calculation by isolating the impact of yields and focusing on pure price movements.

10.2.1 Extending Lower Bounds and Put-Call Parity

The intrinsic value forms the lower bound for option prices, ensuring that the option price does not fall below its immediate exercise value. These bounds are critical in preventing arbitrage opportunities.

- **Lower Bound for Call Options:**

$$c \geq \max(S_0 e^{-qT} - Ke^{-rT}, 0)$$

- **Lower Bound for Put Options:**

$$p \geq \max(Ke^{-rT} - S_0 e^{-qT}, 0)$$

Put-call parity establishes a risk-neutral relationship between the prices of European put and call options with identical strike prices and expirations.

- **Put-Call Parity**

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

10.2.2 Extending Black-Scholes Model

The Black-Scholes formula for pricing European options underlies much of modern financial derivatives trading. This extedned version considers the adjusted stock price that accounts for divid yield (q). The formulas for calls and puts are as follows:

- **Call Option Price:**

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

- **Put Option Price:**

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

10.2.3 Alternative Formulas Using Forward Prices

The valuation of European options can also be efficiently conducted using forward prices, which simplifies the treatment of dividends and risk-free rates. This approach directly incorporates the expected future price of the asset, eliminating the need to adjust the current price for dividends and rates separately.

- **Call Option Price:**

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

- **Put Option Price:**

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

Where F_0 is the forward price of the asset, representing the expected price at time T, adjusted for the risk-free rate r and the dividend yield q. The terms d_1 and d_2 in these formulas are modified to incorporate the forward price:

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- The forward price $F_0 = S_0 e^{(r-q)T}$ effectively encapsulates the expected growth of the asset price due to the net effect of interest rates and dividend yields.
- This methodology is particularly useful for valuing options on assets that pay dividends, as it directly uses the futures or forward index price, which corresponds to a contract maturing at the same time as the option.

10.2.4 Implied Forward Prices and Dividend Yields

Forward prices and dividend yields can also be derived from market prices of European calls and puts, enabling a deeper understanding of market expectations and assisting in strategic investment decisions.

- **Implied Forward Price:**
$$F_0 = K + (c - p)e^{rT}$$
This formula calculates the implied forward price from the current prices of European calls and puts with the same strike price and time to maturity. It reflects the market's expectation of the asset price at the option's expiry.
- **Implied Dividend Yield:**
$$q = -\frac{1}{T} \ln \frac{c - p + Ke^{-rT}}{S_0}$$
This expression estimates the average dividend yield expected during the life of the option, derived from the observed prices of calls and puts.
- These calculations are essential for estimating term structures of forward prices and dividend yields, particularly relevant in over-the-counter (OTC) European options.
- For American options, which can be exercised at any point up to and including the date of expiration, understanding the term structure of dividend yields is crucial due to their potential impact on early exercise decisions.

10.3 Currency Options

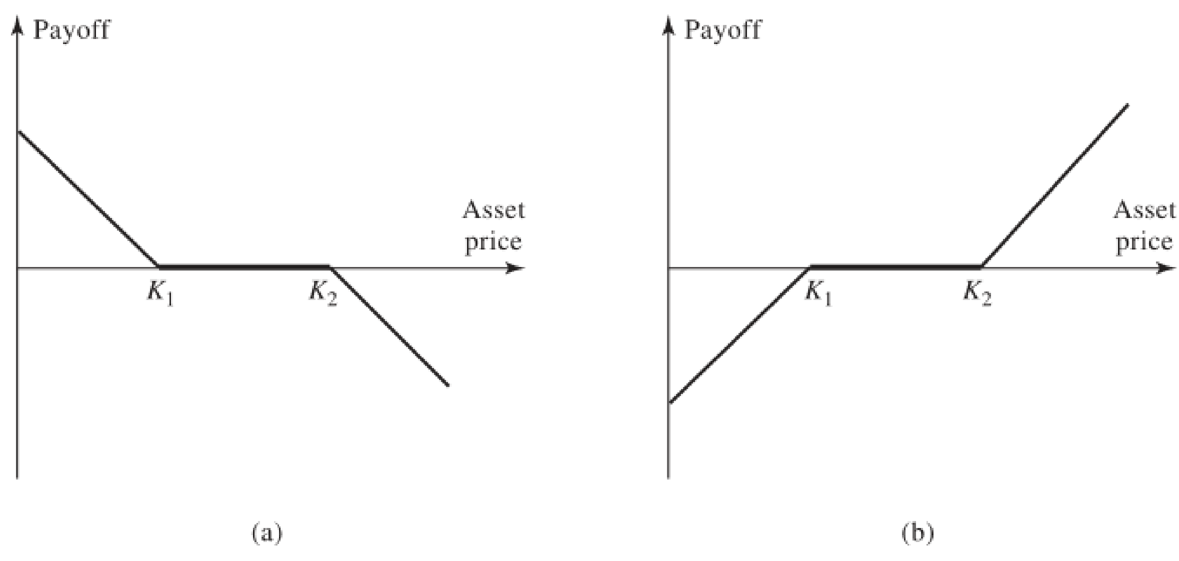
Currency options are financial instruments that provide an essential hedging mechanism against foreign exchange risk, utilized predominantly by corporations with international exposure. These options are traded on structured exchanges such as NASDAQ OMX and extensively in the over-the-counter (OTC) market, reflecting their importance in financial risk management.

Currency as an asset class behaves similarly to stocks that provide a yield, with the yield in this case being the foreign interest rate, r_f. This similarity allows the application of stock option pricing methods to currency options, considering r_f as analogous to the dividend yield in stock options.

10.3.1 Range Forward Contracts

Range forward contracts are bespoke financial agreements that help manage currency risk by setting bounds on the exchange rate fluctuations:

- When expecting to pay currency, one typically sells a put option with a lower strike price, K_1, and buys a call option with a higher strike price, K_2 ($K_2 > K_1$). This structure ensures that the exchange rate remains between K_1 and K_2.
- Conversely, when expecting to receive currency, one buys a put option at K_1 and sells a call option at K_2.
- The cost of the put typically offsets the premium received for the call, balancing the overall cost of the contract.



These contracts are particularly advantageous for businesses that seek to mitigate the risk of adverse currency movements while retaining some potential for benefiting from favorable movements within a predetermined range.

10.3.2 Extending Black-Scholes Model

Using the Black-Scholes model adjusted for currencies, we derive the following formulas:

- **Call Option Price:**
$$c = S_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2)$$

- **Put Option Price:**
$$p = K e^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

10.3.3 Alternative Formulas Using Forward Prices

The use of forward prices further simplifies the valuation by incorporating expectations about future exchange rates:

- **Call Option Price:**
$$c = e^{-rT}[F_0 N(d_1) - K N(d_2)]$$

- **Put Option Price:**
$$p = e^{-rT}[K N(-d_2) - F_0 N(-d_1)]$$

Where:

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where $F_0 = S_0 e^{(r-r_f)T}$ represents the forward exchange rate, adjusted for the interest rate differential between the two currencies. This model encapsulates the market's expectations of future exchange rates, reducing the need for separate adjustments for interest rates and foreign yields in the pricing formulas.

10.4 Options on Futures and Black's Model

Options on futures are derivatives that confer the right, but not the obligation, to enter into a futures contract at a predetermined price before a specified expiration date. These options are categorized by the maturity month of the underlying futures contract. Typically, these options are American-style, meaning they can be exercised at any time up to and including the expiration day, which usually aligns with or occurs a few days before the earliest delivery date of the underlying futures contract.

10.4.1 Mechanics of Futures Options

10.4.1.1 Call Options

When a call option on a futures contract is exercised, the holder assumes a **long position in the futures contract**. Additionally, the holder receives a cash amount equivalent to the difference between the futures price at the most recent settlement and the strike price of the option.

- **Example:**
  - A September call option on copper futures has a strike price of 320 cents per pound.
  - The option is exercised when the futures price is 331 cents, with the most recent settlement price at 330 cents.
  - Each contract represents 25,000 pounds of copper.
  - Upon exercising, the trader receives a long position in the September futures contract and a cash payout of $25,000 \times 10$ cents = \$2,500.
  - If desired, the position in the futures contract can be closed out immediately to get $25,000 \times (331 - 330)$ cents = \$250.
  - The total payoff from exercising the option is \$2,750.

10.4.1.2 Put Options

Conversely, when a put futures option is exercised, the holder gains a **short position in the futures contract**. Similar to the call option, there is a cash payout which in this case is the difference between the strike price and the futures price at the most recent settlement.

- **Example:**
  - A December put option on corn futures has a strike price of 600 cents per bushel.
  - The option is exercised when the futures price is 580 cents per bushel, with the most recent settlement price at 579 cents per bushel.
  - Each contract covers 5,000 bushels.

- The trader receives a short position in the December futures contract and a cash payout of $5,000 \times (600 - 579)$ cents = $\$1,050$.
- If desired, the position in the futures contract can be closed out immediately to get $5,000 \times (579 - 580)$ cents = $-\$50$.
- The total payoff from exercising the option is $1,000.

10.4.2 Payoffs of Futures Options

The payoff for futures options, if the position in the futures contract is closed out immediately after exercising the option, is straightforward:

- **Call Option Payoff:**
$$\text{Payoff} = F - K$$
Where F is the futures price at the time of exercise, and K is the strike price of the option.
- **Put Option Payoff:**
$$\text{Payoff} = K - F$$
Again, F represents the futures price at the time of exercise.

10.4.3 Potential Advantages of Futures Options over Spot Options

Futures options offer several distinct advantages over spot options under certain conditions:

1. **Equivalence with Spot Options:**
   - European futures options and European spot options are equivalent in value when the futures contract matures concurrently with the option's expiration. This equivalence arises because both types of options will reflect the same underlying economic exposures at expiration.
2. **Market Liquidity:**
   - Futures contracts often enjoy greater liquidity compared to many underlying assets, particularly in markets where the underlying assets are large or illiquid by nature (e.g., commodities, certain financial instruments). This increased liquidity generally makes it easier and potentially less costly to trade futures options.
3. **Settlement Characteristics:**
   - Upon exercising a futures option, the holder acquires a position in the futures contract rather than the immediate delivery of the underlying asset. This feature can be particularly advantageous in markets where physical delivery is less desirable or practical.

4. **Unified Trading Platforms:**
    - Futures options and the underlying futures contracts typically trade on the same exchange. This unification can simplify access, monitoring, and execution for traders and investors, enhancing market efficiency.
5. **Cost Efficiency:**
    - The transaction costs associated with trading futures options can be lower than those for trading spot options, largely due to standardized contract terms and centralized trading venues which foster more competitive pricing.

10.4.4 Put-Call Parity and Lower Bounds for European Futures Options

10.4.4.1 Put-Call Parity

Consider the following two portfolios:

1. European call plus Ke^{-rT} of cash.
2. European put plus long futures plus cash equal to F_0e^{-rT}.

Both portfolios must be worth the same at time T. Therfore, for European futures options, the put-call parity condition is expressed as:

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

Here, c and p represent the prices of the European call and put options, respectively, K is the strike price, F_0 is the current futures price, and r is the risk-free interest rate. This equation ensures that no arbitrage opportunities exist between buying a call and selling a put when adjusted for the present value of the strike price and the futures price.

10.4.4.2 Lower Bounds

The lower bounds for the prices of futures options reflect the minimum value these options must hold to prevent arbitrage:

- **Call Option Lower Bound:**

$$c \geq (F_0 - K)e^{-rT}$$

- **Put Option Lower Bound:**

$$p \geq (K - F_0)e^{-rT}$$

10.4.5 Black's Model for Valuing Futures Options

In the pricing of futures contracts, a pivotal assumption is that no initial investment is required, leading to a **zero expected return in a risk-neutral environment**. This implies that the expected growth rate of the futures price is also zero, essentially treating the futures price as equivalent to a stock paying a continuous dividend yield equal to the risk-free rate r. This approach simplifies the valuation by aligning the futures price dynamics with those of a dividend-paying stock.

Black's model extends the classic Black-Scholes framework to futures options, incorporating unique aspects of futures pricing:

- **Model Parameters:**
  - S_0, the current futures price, is denoted by F_0.
  - q, typically representing the dividend yield in stock options, is set to the domestic risk-free rate r in the context of futures. This setting neutralizes the expected growth of the futures price, maintaining the risk-neutral valuation framework.
- **Historical Context:**
  - The model was first proposed by Fischer Black in 1976 and is specifically adapted for European options on futures, providing a practical solution that sidesteps the complexities of estimating income on the underlying asset.

10.4.6 Pricing Formulas

The formulas for valuing European futures options using Black's model are derived as follows:

- **Call Option Price:**
$$c = e^{-rT}[F_0 N(d_1) - K N(d_2)]$$
- **Put Option Price:**
$$p = e^{-rT}[K N(-d_2) - F_0 N(-d_1)]$$

Where:
$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

10.4.7 Futures Option Price vs. Spot Option Price

Comparing futures options to spot options under different market conditions reveals nuanced valuation dynamics:

- **Normal Market Conditions (Contango):**
  - If futures prices are higher than spot prices, an American call option on futures is typically more valuable than a similar call option on the spot market. Conversely, an American put option on futures is less valuable than its spot counterpart.
- **Inverted Market Conditions (Backwardation):**
  - In situations where futures prices are lower than spot prices, the value relationships reverse: American calls on futures become less valuable, whereas puts on futures gain in relative value.

10.5 Practice Questions and Problems

10.5.1 Index Options

1. A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

> Solution
>
> Lower bound = 16.90

2. Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth $10. What is the value of a three-month put option on the index with a strike price of 245?

> Solution
>
> Put option value = 3.84

3. Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

Solution

Call option value = 11.15

4. An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

Solution

Put option value = 40.6

10.5.2 Currency Options

1. A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

Solution

Lower bound European = 0.069 Lower bound American = 0.10

2. Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

Solution

Put option value = 0.0162

3. A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value (a) a European four-month call option with a strike price of 0.79 and (b) an American four-month call option with the same strike price.

Solution

- European Option Value: \$0.0235
- American Option Value: \$0.0250

10.5.3 Futures Options

1. Consider a two-month futures call option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Solution

Lower bound European = 6.88 Lower bound American = 7

2. Consider a four-month futures put option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Solution

Lower bound European = 2.90 Lower bound American = 3

3. Calculate the value of a five-month European futures put option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

Solution

Put option value = 1.50

4. A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

Solution

Call option value = 2.01

5. A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month European

call option on the futures with a strike price of 60. If the call were American, would it ever be worth exercising it early?

Solution

- European Option Value: 4.3155
- American Option Value: 4.4026

6. Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

Solution

Arbitrage profit = $1

10.5.4 Strategic Considerations

1. Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

2. Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

3. Explain how corporations can use range forward contracts to hedge their foreign exchange risk when they are due to receive a certain amount of a foreign currency in the future.

4. An index currently stands at 1,500. Six-month European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The risk-free rate is 5%. What is the implied dividend yield?

Solution

Implied dividend yield = 1.99%

5. Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

11 The Greek Letters

References

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Learning Outcomes:

- Identify and describe the Greek letters used in options trading.
- Explain the concept of Delta and its importance in hedging strategies.
- Understand Gamma and its impact on the curvature of the options price relative to the stock price.
- Comprehend Theta and how time decay affects the value of options.
- Explore Vega and its relationship with volatility in the pricing of options.
- Recognize Rho and its sensitivity to changes in interest rates.
- Develop strategies for managing Delta, Gamma, and Vega to optimize risk and return in options portfolios.

11.1 Case Study

- A bank sells a European call option for \$300,000, covering 100,000 shares of a stock that does not pay dividends.
- **Initial Stock Price** (S_0): \$49
- **Strike Price** (K): \$50
- **Risk-Free Rate** (r): 5%
- **Volatility** (σ): 20%
- **Time to Expiration** (T): 20 weeks
- **Expected Return** (μ): 13%
- **Black-Scholes-Merton Valuation**: \$240,000

- **Objective**: Determine how the bank can hedge its risk to secure a $60,000 profit.

11.1.1 Naked Position

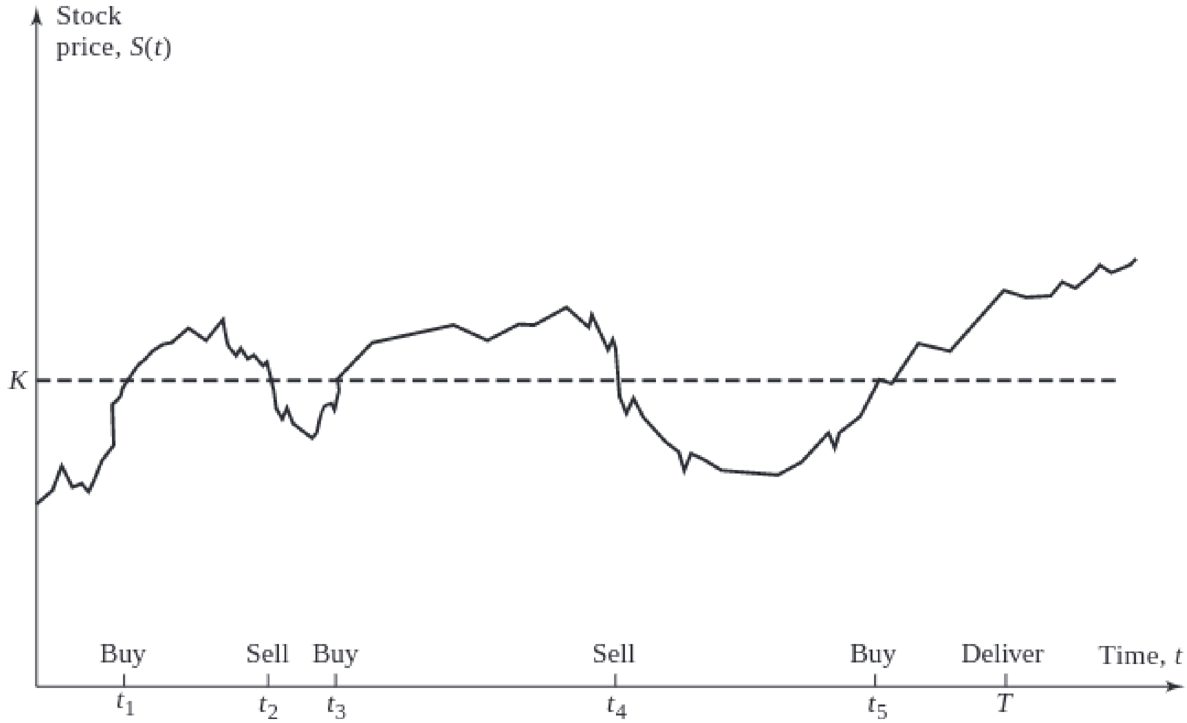
- **Description**: The bank takes no protective action against the position's exposure.
- **Risk Analysis**: This strategy leaves the position fully exposed to market fluctuations, with potential unlimited losses if the stock price rises significantly.

11.1.2 Covered Position

- **Description**: The bank purchases 100,000 shares immediately.
- **Risk Analysis**: Although this strategy can mitigate the risk if the stock's price increases above the strike price, it requires significant capital investment and is exposed to loss if the stock price declines.

11.1.3 Dynamic Hedging: Stop-Loss Strategy

- **Buy**: 100,000 shares are purchased when the stock price hits $50.
- **Sell**: 100,000 shares are sold when the stock price drops below $50.
- **Risk**: This approach attempts to limit losses by dynamically adjusting the position based on stock price movements. However, it may lead to high transaction costs and potential slippage; the actual transaction price may not always align with the trigger price due to market volatility.



The most efficient way to hedge would be to use **Greek letters.**

11.2 The Greek Letters

Greek letters are vital tools in options trading, providing a quantitative understanding of the sensitivity of an option's price to various factors. These metrics are crucial for effective risk management and strategic decision-making in trading portfolios.

- The Greeks are typically calculated using the Black-Scholes-Merton (BSM) model, a foundational tool in financial mathematics for valuing options.
- The Greek letters are the partial derivatives with respect to the model parameters that are liable to change.
- In practice, the volatility parameter in the BSM model is set to the implied volatility of the option, a modification often referred to as the "practitioner Black-Scholes" model.
- The discussion here pertains primarily to European options on non-dividend-paying stocks.

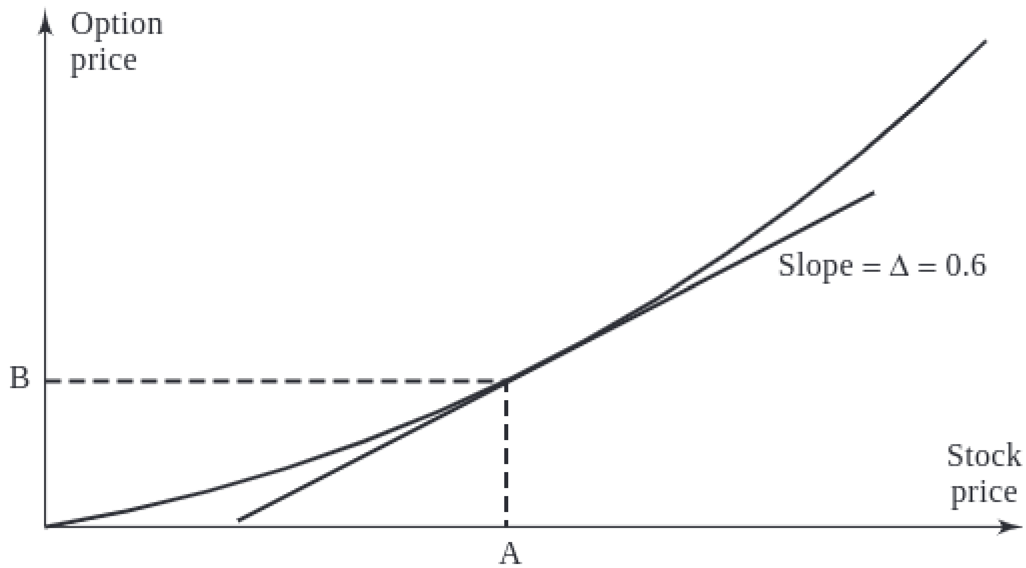
11.2.1 Description of Key Greeks

The Greeks measure the sensitivity of the option's price to one of the underlying parameters, **holding all other parameters constant**. Each Greek addresses a different risk factor:

- **Delta (Δ)**: Measures the rate of **change of the option's price** with respect to changes in the underlying asset's price. Specifically, delta is the first derivative of the option's price with respect to the stock's price. For instance, a delta of 0.6 suggests the option's price will move approximately \$0.60 for every \$1.00 movement in the underlying stock.
- **Gamma (Γ)**: Represents the rate of **change of delta** with respect to changes in the underlying asset's price. Gamma helps assess the stability of an option's delta, providing insight into how delta might change as the stock price varies. High gamma values indicate that delta is highly sensitive to changes in the stock price.
- **Theta (Θ)**: Measures the sensitivity of the option's price to **the passage of time**, known as "time decay." Theta indicates the expected rate of change in the option's price for a one-day decrease in its time to expiration. This is particularly important as the option approaches its expiry date.
- **Vega (ν)**: Quantifies the sensitivity of the option's price to **changes in the volatility** of the underlying asset. A vega of 1.5 suggests that the option's price is expected to change by \$1.50 for every 1% change in the implied volatility of the underlying stock.
- **Rho (ρ)**: Measures the sensitivity of the option's price to **changes in the risk-free interest rate**. For instance, a rho of 0.05 indicates that the option's price will change by \$0.05 for every 1% change in interest rates.

11.3 Understanding Delta (Δ)

Delta is a measure of an option's price sensitivity relative to changes in the price of the underlying asset. It's expressed as the amount an option's price is expected to move for a one-unit change in the price of the underlying asset.



The followint graph showing how delta varies with stock price for call and put options.

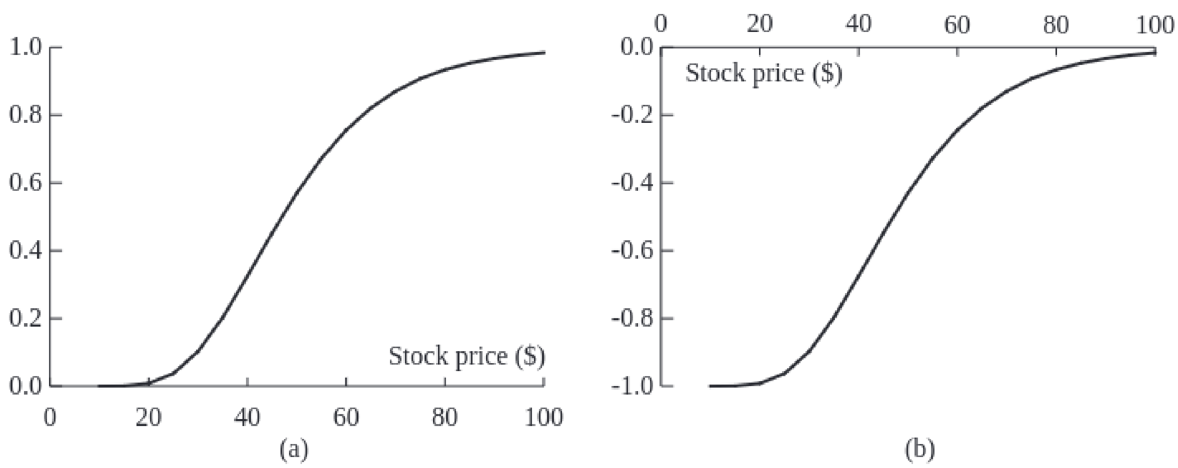


Figure 11.1: Variation of Delta With Stock Price (K=50)

- **Call Option (a)**: Delta increases as the stock price rises, peaking near 1 as the option goes deep in-the-money.
- **Put Option (b)**: Delta decreases (becomes more negative) as the stock price drops, indicating increased sensitivity as the option moves further in-the-money.

The next graph showing how delta changes as time to maturity decreases.

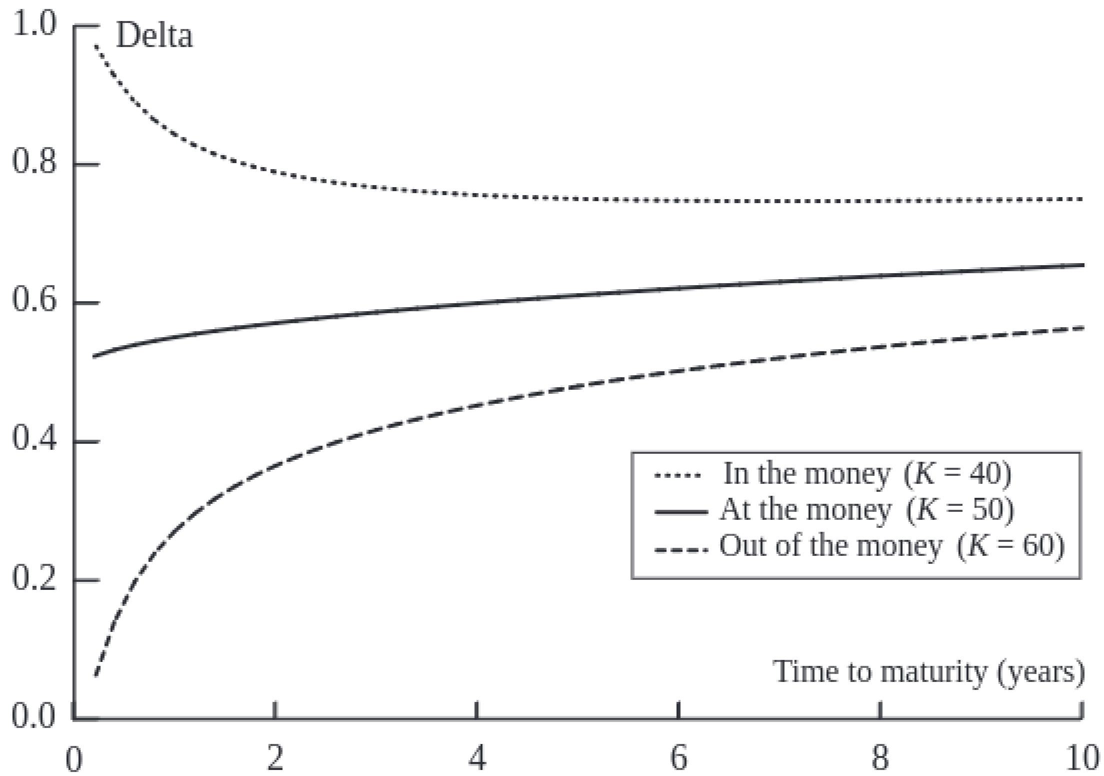


Figure 11.2: Variation of Delta with Time to Maturity

Delta's sensitivity to time decay depends on the moneyness of the option. For options that are exactly at-the-money, delta converges to 0.5. For out-of-the-money options, it approaches zero because such options are likely to expire worthless, whereas for in-the-money options, delta converges to 1.

11.3.1 Delta Hedging: Principles and Practices

Delta hedging is a strategy used to mitigate risk in option trading by setting up a position in the underlying asset. The position's size is determined by the option's delta, aiming to neutralize the effect of small price movements of the underlying asset.

To achieve delta neutrality, the required number of units to hedge, N_H, is calculated using the formula:

$$N_H = -\left(\frac{\text{Portfolio delta}}{\text{Delta of the hedging instrument}}\right)$$

A delta-neutral portfolio implies a total delta of zero, indicating no sensitivity to small price movements in the underlying stock. Note that the **delta of the stock itself is always 1**.

- **Rebalancing**: Delta values change as the stock price fluctuates and as time passes, necessitating periodic rebalancing of the hedge (buy high, sell low).
- **Delta for Calls and Puts**: The delta of a European call option on a non-dividend paying stock is represented by $N(d_1)$, whereas for a put, it's $N(d_1) - 1$.

Example: Delta Hedging I

Consider an investor who holds a short position in 1,000 call options, each with a delta of -0.6. Create a delta-neutral portfolio to shield against small price movements.

- **Current Delta**: $1,000 \times -0.6 = -600$
- **Hedging Action**: Purchase 600 shares of the stock (since each stock has a delta of 1).

This strategy ensures that any gain or loss on the options is countered by a corresponding loss or gain in the stock holdings.

Example: Delta Hedging II

Consider a portfolio comprising 1,000 shares of a non-dividend-paying stock. Each share contributes a delta of 1 to the portfolio, resulting in a total portfolio delta of 1,000.
To achieve delta neutrality using call options with a delta of 0.40, we calculate the required number of calls to sell:

$$N_H = -\left(\frac{1,000}{0.40}\right) = -2,500$$

This means selling 2,500 call options ensures that the positive delta from the stock holdings is exactly offset by the negative delta contribution from the short call positions.

Example: Delta Hedging III

Consider a scenario where you have a short position in 10,000 shares of a non-dividend-paying stock:

11.3.1.1 Using Call Options for Hedging:

- **Option Delta** (Δ_c): 0.668
- To neutralize the negative delta of -10,000 from the short stock position, you would need to purchase call options to increase delta:

$$N_H = -\left(\frac{-10,000}{0.668}\right) = 14,970$$

Purchasing 14,970 call options will add a positive delta, balancing out the negative delta from the short stock position.

11.3.1.2 Using Put Options for Hedging:

- **Option Delta** (Δ_p): -0.332
- To balance the negative delta, you can use put options which inherently have a negative delta:

$$N_H = -\left(\frac{-10,000}{-0.332}\right) = -30,120$$

Selling 30,120 put options would counteract the negative delta from the short stock position, achieving neutrality.

11.3.2 Hedging Cost Scenarios from Case Study

The following table presents a simulation of hedging dynamics when the option expires in-the-money, with total hedging costs amounting to $263,300:

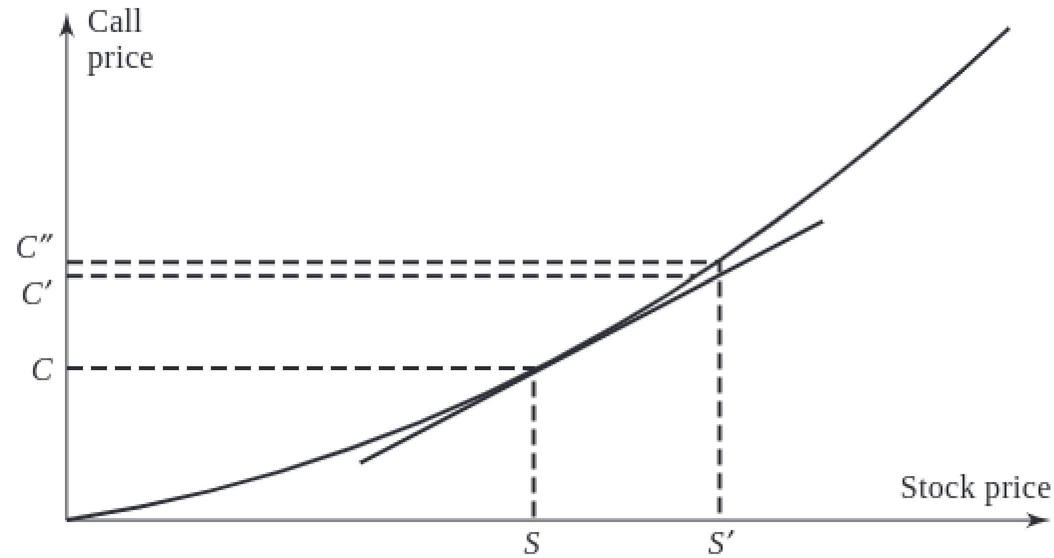
Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

- The cost reflects the higher intrinsic value and sensitivity (delta) as the option gains worth.
- If the option expires out-of-the-money, the costs should be slightly lower, as reduced delta correlates with a decreased likelihood of the option finishing in-the-money.

11.4 Understanding Gamma (Γ)

Gamma (Γ) is the second derivative of the option's price with respect to the underlying asset's price. It measures the rate of change of delta (Δ) and is crucial for understanding the curvature or the convexity of the option's value in relation to the stock price. It's crucial for assessing the stability of an option's delta, thus influencing how often a delta-hedged portfolio needs rebalancing. Here are some key characteristics of gamma:

- **Zero for Stocks**: Gamma for a long or short position in a single share of stock is zero because a stock's delta does not change.
- **Symmetry for Calls and Puts**: Gamma is identical for both call and put options.
- **Non-negativity**: Gamma is always non-negative. It reaches its highest value when an option is at-the-money, highlighting increased sensitivity at this point.
- **Risk Measurement**: Gamma quantifies the non-linearity risk—risk that remains in a delta-neutral portfolio due to price movements of the underlying asset.



11.4.1 Gamma and Hedging Dynamics

Gamma plays a critical role in addressing potential errors in delta hedging, particularly when stock prices exhibit significant movement:

- **Small Changes**: For minor fluctuations in stock price, delta hedging generally performs well.
- **Larger Shifts**: For more substantial stock price movements, delta-plus-gamma hedging provides greater accuracy, accommodating the curvature in the relationship between an option's price and the stock price.
- **Gamma Risk**: This term refers to the risk arising from sudden, significant movements in stock prices—often referred to as jumps—which can leave a previously well-hedged position exposed. Gamma risk is particularly relevant in markets characterized by high volatility or discontinuous price movements.
- **Theta and Gamma Interaction**: Typically, when theta is large and positive, indicating substantial time decay benefits, gamma tends to be large and negative. This inverse relationship highlights the trade-offs between potential profitability from time decay and the risk from larger price movements.

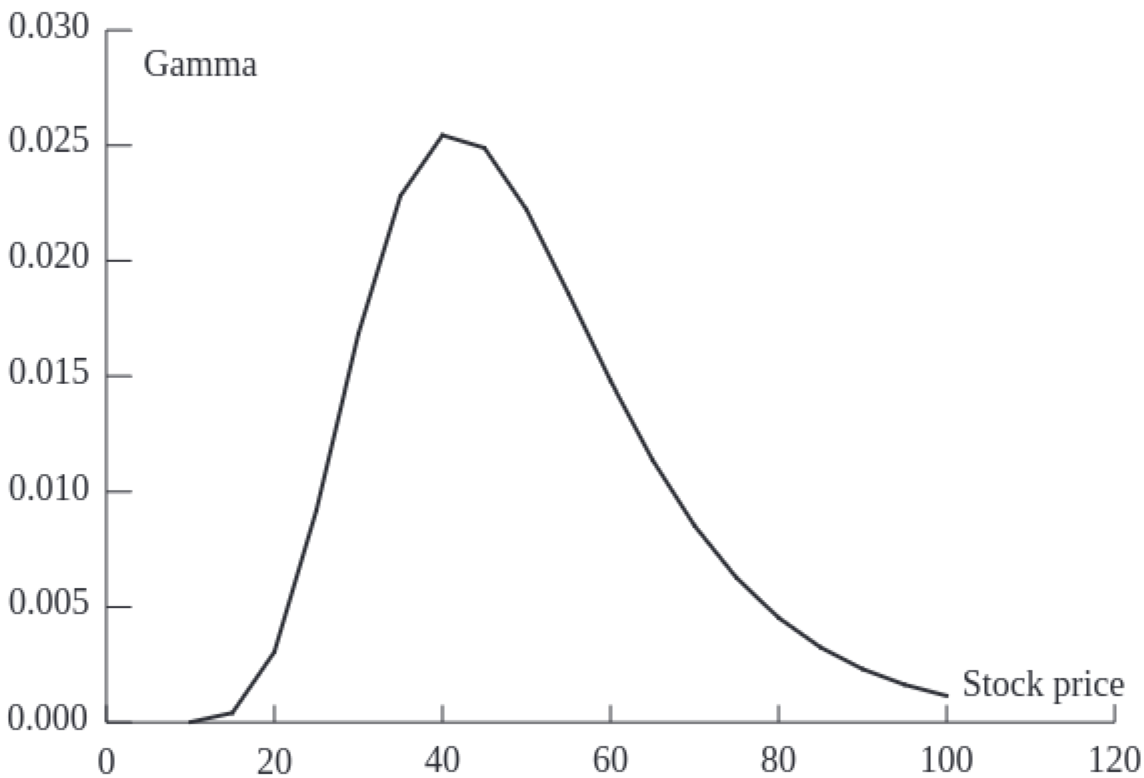


Figure 11.3: Gamma variation with stock price

This chart demonstrates how gamma peaks when an option is at-the-money and diminishes as the option moves deeper into or out of the money. This pattern underscores the heightened sensitivity and potential risk/reward near the at-the-money point.

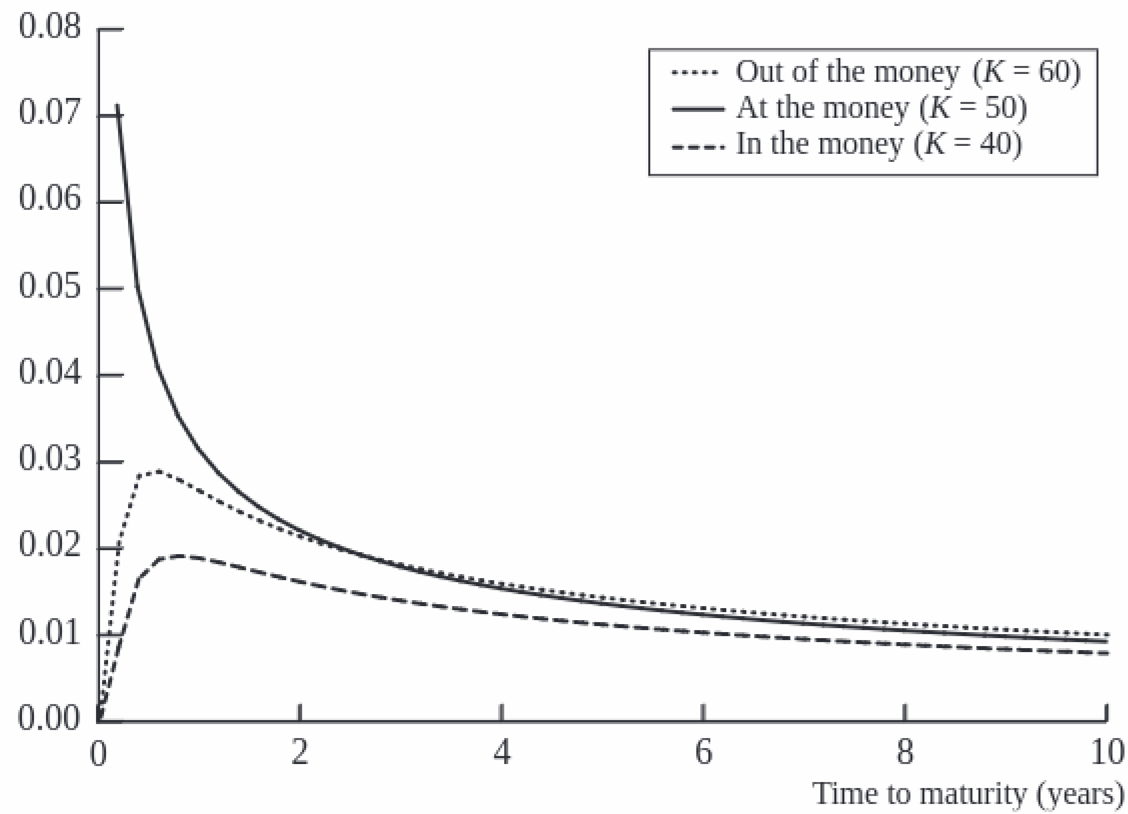


Figure 11.4: Gamma variation over time

Gamma increases as the option approaches expiration, particularly for at-the-money options. This increase reflects growing sensitivity to stock price movements as time to exercise decreases, emphasizing the importance of precise hedging strategies in the final days before an option's expiration.

11.5 Understanding Theta (Θ)

Theta (Θ) represents the sensitivity of the price of an option or a portfolio of derivatives to the passage of time, assuming all other factors remain constant. It is a critical component in options pricing, reflecting the temporal decay of an option's value.

- **Directionality**: Theta is typically negative for both calls and puts. This reflects the loss in time value as options approach their expiration date.
- **Non-applicability to Stocks**: Since stocks do not have an expiration date, they inherently have a theta of zero.

- **Impact on Options**: Each day closer to expiration decreases the option's time value, reducing its price if other conditions remain unchanged.

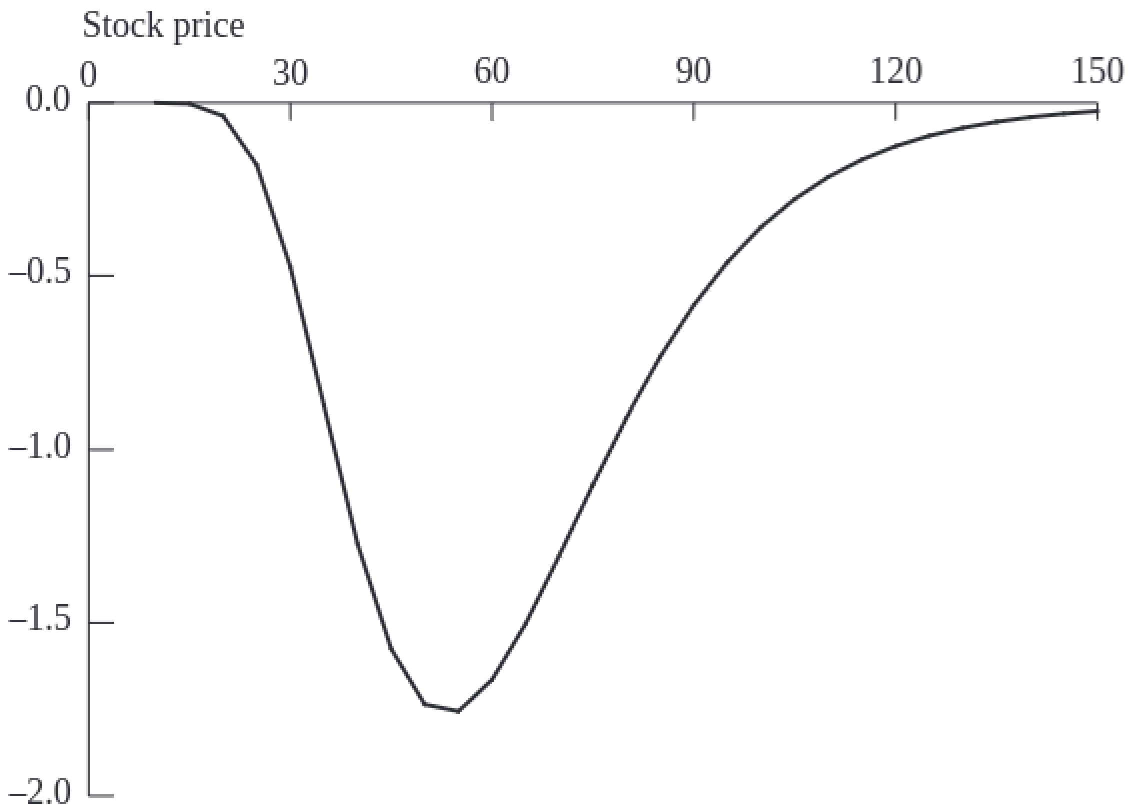


Figure 11.5: Theta variation with stock price

The graph illustrates how theta behaves relative to the stock price for a call option. Notably, theta becomes increasingly negative as the option moves around at-the-money.

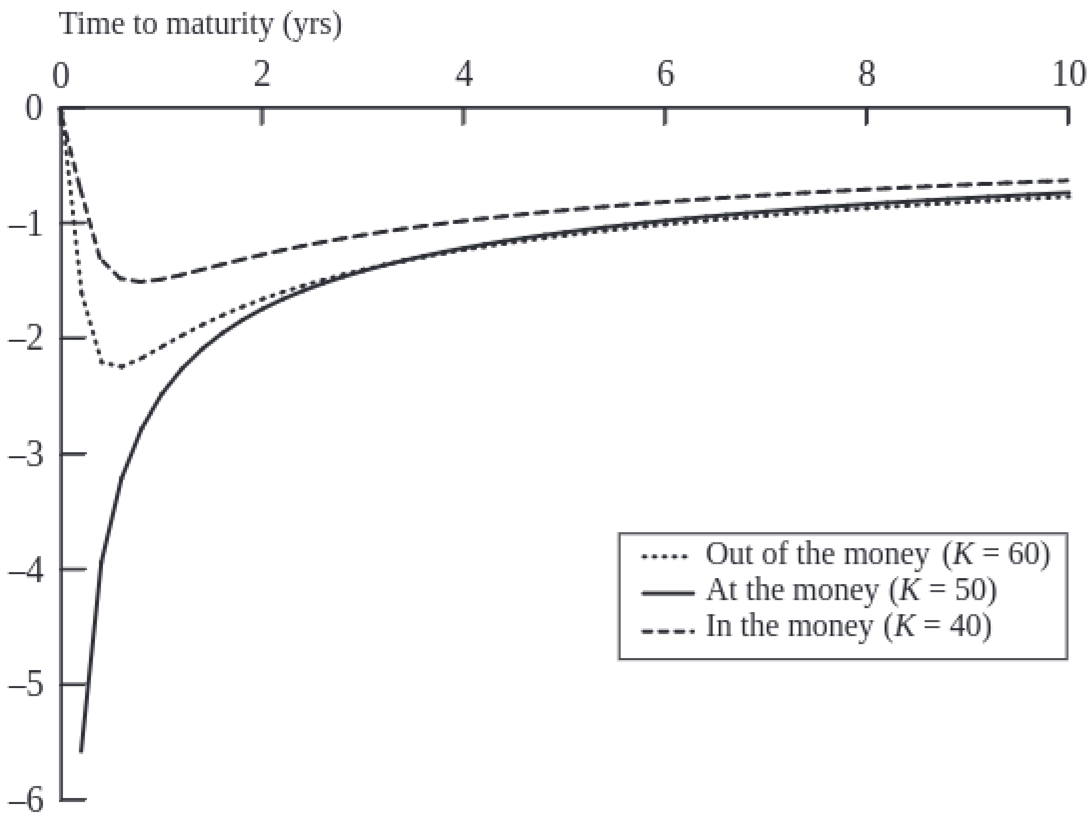


Figure 11.6: Theta variation over time

This visualization shows theta's behavior as the option nears its expiry. Theta typically becomes more negative as expiration approaches, indicating an acceleration in the rate of time value decay. This is most pronounced for at-the-money options, where the uncertainty about finishing in-the-money is highest.

- **Theta vs. Hedge Parameters**: Unlike delta, which can be hedged, theta represents a guaranteed decline in value over time and is not a hedgeable risk. However, its predictability and inevitability make it a significant factor in options strategy.
- **Descriptive Utility**: Despite its non-hedgeable nature, theta is considered a useful descriptive statistic for portfolios, particularly in strategies aimed at earning through time decay, such as selling options.
- In a delta-neutral setting, where the portfolio is insulated against small price movements in the underlying asset, theta can serve as an indirect measure of gamma exposure. This relationship arises because high gamma values, which indicate greater sensitivity to price changes, usually accompany high rates of time decay (theta). Thus, monitoring theta provides insights into potential gamma risks in the portfolio.

11.6 Understanding Vega (ν)

Vega represents the rate of change in the value of an option or a derivatives portfolio with respect to changes in volatility. This Greek is crucial in options trading, reflecting how the price of options reacts to fluctuations in the underlying asset's volatility.

Vega highlights the impact of volatility, an essential but unobservable market factor that significantly influences option pricing. Future volatility, being an estimation, adds a layer of complexity and risk in predicting option values.

- **Directionality**: Vega is always positive. An increase in the implied volatility of the underlying asset generally leads to an increase in the value of both call and put options.
- **Symmetry**: The vega of a call option is equal to the vega of a put option with the same terms.
- **Calculation**: For a portfolio, vega can be understood as a weighted average of the vegas of the individual positions, reflecting the aggregate sensitivity to a uniform shift in implied volatilities across the portfolio.
- **Volatility Sensitivity**: Of all the Black-Scholes-Merton (BSM) model variables (stock price, strike price, time to expiration, risk-free rate, and volatility), an option's price is most sensitive to changes in volatility. This sensitivity makes vega a key focus in risk management and trading strategies.

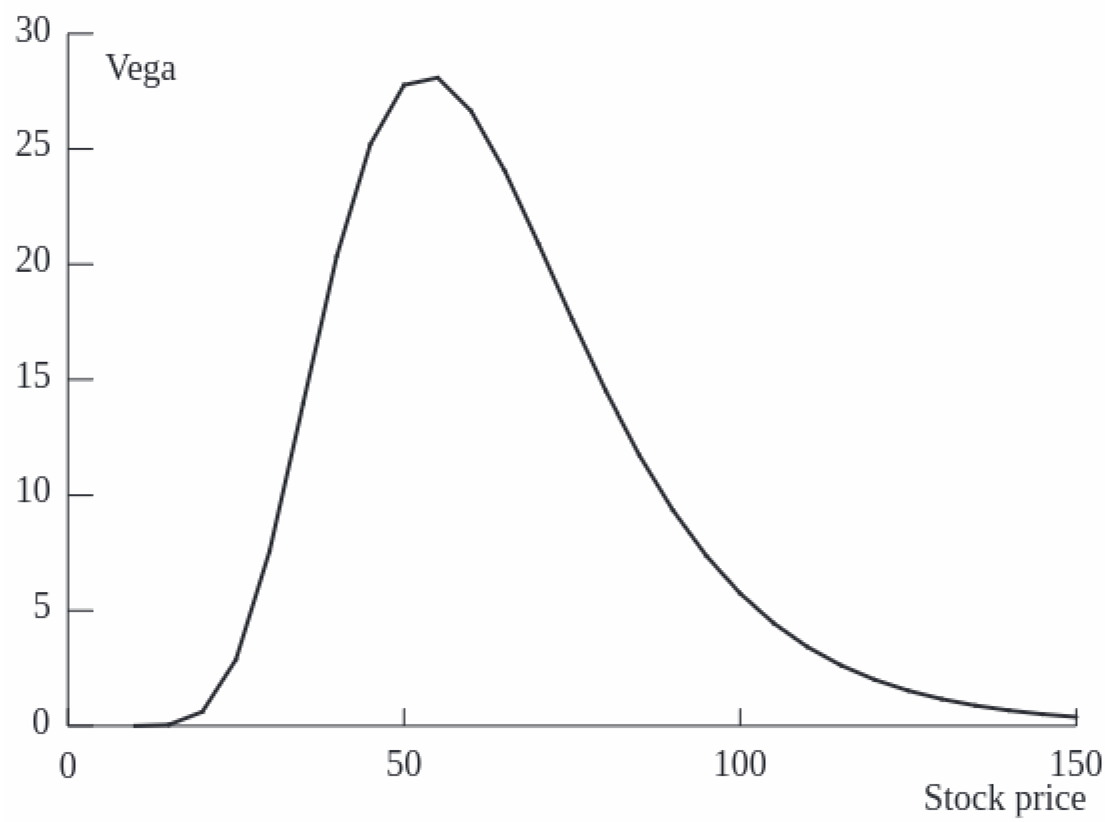


Figure 11.7: Vega variation with stock price

The chart illustrates how vega changes in relation to the stock price. Vega tends to be higher when an option is at-the-money and decreases as the option moves deeper into or out of the money. This pattern reflects the increased sensitivity to volatility changes when the option's strike price is near the current stock price, where the uncertainty about the outcome at expiration is greatest.

11.7 Understanding Rho (ρ)

Rho represents the sensitivity of an option or a derivatives portfolio's value to changes in the risk-free interest rate. It is one of the lesser-focused Greeks but plays a crucial role in environments where interest rate fluctuations are significant.

- **Directionality**: Rho is typically positive for call options and negative for put options. This reflects the different financial implications of options:

- **Call Options**: A positive Rho for call options indicates that their value increases as interest rates rise. This is because higher rates reduce the present value of the exercise price paid at expiration, making it cheaper in present terms to buy the stock at a future date.
- **Put Options**: Conversely, a negative Rho for put options means their value decreases as interest rates rise. Higher rates increase the present value of the proceeds received from selling the stock in the future, thus decreasing the attractiveness of holding a put.

The sensitivity to interest rates through Rho might seem minor compared to other Greeks like Delta or Vega, but it can become significant in certain financial environments:

- **High Interest Rate Volatility**: In periods of significant interest rate changes, Rho becomes a more critical factor for traders, especially those dealing with long-dated options where the cumulative impact of rate changes can be substantial.
- **Portfolio Management**: For portfolios that include a mix of long-dated options and bonds, understanding and managing Rho is vital to hedge interest rate risks effectively.

11.8 Strategic Management of Delta, Gamma, and Vega

Proper management of the Greek values—Delta, Gamma, and Vega—is essential for maintaining balanced options portfolios that align with specific risk management objectives. Here, we outline techniques to achieve neutrality in these Greeks through strategic positioning in options and the underlying asset.

- **Delta**: This Greek can be adjusted directly by taking positions in the underlying asset. For instance, buying or selling stock can increase or decrease the portfolio's delta.

- **Gamma and Vega**: Adjustments to gamma and vega generally require engaging in positions in options or other derivatives, as these Greeks measure sensitivity to changes in the underlying asset's price and volatility, respectively.

The general approach involves **first hedging vega and/or gamma** using options, and then as a **final step, using underlying assets** to reduce delta to zero. This sequence is important because hedging vega and gamma can alter delta, whereas hedging using underlying assets does not affect vega or gamma.

Example: Achieving Greek Neutrality

	Delta	Gamma	Vega
Portfolio	0	-5000	-8000

Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

1. **Achieving Delta and Gamma Neutrality**:
   - **Required Position**: Long 10,000 units of Option 1 and short 6,000 units of the underlying asset.
   - **Rationale**: This positioning uses the delta and gamma from Option 1 to offset the existing negative gamma in the portfolio while also balancing delta.
2. **Achieving Delta and Vega Neutrality**:
   - **Required Position**: Long 4,000 units of Option 1 and short 2,400 units of the underlying asset.
   - **Rationale**: This configuration uses Option 1's vega to counteract the portfolio's negative vega while adjusting the delta with a corresponding position in the underlying asset.
3. **Achieving Delta, Gamma, and Vega Neutrality Across Multiple Options**:
   - **System of Equations**:
     - For Gamma:
       $$-5000 + 0.5w_1 + 0.8w_2 = 0$$
     - For Vega:
       $$-8000 + 2.0w_1 + 1.2w_2 = 0$$
   - **Solution**: Long 400 units of Option 1 and 6,000 units of Option 2, with a short position of 3,240 units in the underlying asset to achieve complete neutrality in delta, gamma, and vega.

11.8.1 Hedging in Practice

- **Daily Delta Neutrality**: Traders typically adjust their portfolios to be delta-neutral at least once per day to manage risk effectively against price movements in the underlying asset.
- **Opportunistic Gamma and Vega Adjustments**: While daily adjustments for delta are common, adjustments for gamma and vega are made as opportunities arise, allowing traders to better manage the curvature and volatility risks associated with their positions.
- **Economies of Scale**: As portfolios increase in size, the relative cost of hedging per option decreases, providing cost efficiencies in larger portfolios.
- **Scenario Analysis**: This involves testing different market scenarios and their effects on the portfolio, considering various assumptions about asset prices and volatilities. This

analysis helps in understanding potential impacts and preparing for various market conditions.

11.9 Practice Questions and Problems

1. Explain how a stop-loss trading rule can be implemented for the writer of an out-of-the money call option. Why does it provide a relatively poor hedge?
2. What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?
3. What does it mean to assert that the theta of an option position is -0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?
4. What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?
5. A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of 80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?
6. A financial institution has the following portfolio of stock options:

Type	Position	Delta	Gamma	Vega
Call	-1,000	0.5	2.2	1.8
Call	-500	0.8	0.6	0.2
Put	-2,000	-0.4	1.3	0.7
Call	-500	0.7	1.8	1.4

   A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

   1. What position would make the portfolio delta neutral?
   2. What position in the traded option and in stocks would make the portfolio both gamma neutral and delta neutral?
   3. What position in the traded option and in stocks would make the portfolio both vega neutral and delta neutral?

4. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

12 Structured Products I

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  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

13 Structured Products II

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  - Chapter 1 - Derivative Markets and Instruments

Learning Outcomes:

A Time Value of Money

References

This material was originaly published HERE by Department of Mathematics, Penn State University Park.

Learning Outcomes:

- Understand the concept of the time value of money (TVM).
- Calculate the present value (PV) of both single and multiple future cash flows using appropriate discount rates.
- Calculate the future value (FV) of both single and ongoing investments using given interest rates.
- Apply TVM formulas to various financial scenarios, including loans, savings, and investments, to make informed decisions.
- Understand the effects of compounding frequency on FV and PV calculations.
- Critically analyze TVM problems, taking into account the impact of rate, time, and cash flows on financial decisions.

A.1 Notation and Terminology

A.1.1 Basic Notation and Terminology

- $P =$ Principal (i.e., value of initial deposit)
- $A =$ Accumulated amount (i.e., sum of the principal and interest)
- $r =$ Nominal interest rate
- $m =$ Number of conversion periods per year, (a conversion period is the interval of time between successive interest payments)

Annually	Semiannually	Quarterly	Monthly	Weekly	Daily
$m = 1$	$m = 2$	$m = 4$	$m = 12$	$m = 52$	$m = 365$

- $t = $ Term of investment (in years)

A.1.2 Simple Interest

Interest is always computed based on the original principal.

Interest Earned	Accumulated Amount
$I = Prt$	$A = P(1 + rt)$

A.1.3 Discrete Compound Interest

Interest payments are added to the principal at the end of each conversion period and therefore earn interest during future conversion periods.

Accumulated Amount	Present Value Formula
$A = P\left(1 + \frac{r}{m}\right)^{mt}$	$P = A\left(1 + \frac{r}{m}\right)^{-mt}$

A.1.4 Continuous Compound Interest

Continuous compounding of interest is equivalent to a discrete compounding of interest where m, the number of conversion periods per year, goes to infinity.

Accumulated Amount	Present Value Formula
$A = Pe^{rt}$	$P = Ae^{-rt}$

A.1.5 Effective Rate of Interest

The effective interest rate, r_{eff}, is the simple interest rate that produces the same accumulated amount in 1 year as the nominal rate, r, compounded m times a year.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

A.2 Future Value Examples

A.2.1 Example 1

Suppose $1,000 is deposited into an account with an interest rate of 16% compounded annually. How much money is in the account after 3 years?

Step 1: Since interest is compounded annually, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 1$, and $t = 3$.

$$\begin{aligned} A &= 1000\left(1 + \frac{0.16}{1}\right)^{1 \cdot 3} \\ &= 1000\left(1 + 0.16\right)^{3} \\ &= 1000\left(1.16\right)^{3} \approx \$1,560.90 \end{aligned}$$

Therefore, after 3 years of accumulating interest, the original investment of $1,000 is worth $1,560.90.

A.2.2 Example 2

Suppose $1,000 is deposited into an account with an interest rate of 16% compounded quarterly. How much money is in the account after 3 years?

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 1000$, $r = 0.16$, $m = 4$, and $t = 3$.

$$A = 1000\left(1 + \frac{0.16}{4}\right)^{4 \cdot 3}$$

$$= 1000\,(1 + 0.04)^{12}$$

$$= 1000\,(1.04)^{12} \approx \$1,601.03$$

Therefore, after 3 years of accumulating interest, the original investment of \$1,000 is worth \$1,601.03.

> **Observation**
>
> Compare the accumulated amounts in the above two examples. Both examples have the same principal, interest rate, and term. But since interest is compounded more frequently in Example 2 (4 times a year) than in Example 1 (1 time a year), the accumulated amount is higher in Example 2.

A.2.3 Example 3

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded quarterly.

Step 1: Since interest is compounded quarterly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), $m = 4$, and $t = 5$.

$$\begin{aligned} 6000 &= 3000\left(1 + \frac{r}{4}\right)^{4 \cdot 5} \\ &= 3000\left(1 + \frac{r}{4}\right)^{20} \end{aligned}$$

Step 3: Solve for the interest rate, r.

Method 1

Divide both sides by 3000

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(2) &= \ln\left[\left(1 + \frac{r}{4}\right)^{20}\right] \\ &= 20 \ln\left(1 + \frac{r}{4}\right) & \text{since } \ln(m^n) = n\ln(m) \end{aligned}$$

Divide both sides by 20.

$$\ln(2)/20 = \ln\left(1 + \frac{r}{4}\right)$$

Take the exponential of both sides.

$$\begin{aligned} e^{\ln(2)/20} &= e^{\ln(1+\frac{r}{4})} \\ &= 1 + \frac{r}{4} & \text{since } e^{\ln(x)} = x \end{aligned}$$

Subtract 1 from both sides.

$$e^{\ln(2)/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(e^{\ln(2)/20} - 1) \approx 0.1411.$$

Method 2

Here is an alternate method for solving for the interest rate r. We start with the following equation.

$$2 = \left(1 + \frac{r}{4}\right)^{20}$$

Instead of taking the natural logarithm of both sides as we did before, now take the 20th root of both sides (i.e., raise both sides to the power of 1/20).

$$2^{1/20} = 1 + \frac{r}{4}$$

Subtract 1 from both sides.

$$2^{1/20} - 1 = \frac{r}{4}$$

And finally, multiply both sides by 4.

$$r = 4(2^{1/20} - 1) \approx 0.1411$$

Note that this value of r is numerically equal in both methods since

$$\begin{aligned} e^{\ln(2)/20} &= e^{\ln(2^{1/20})} && \text{since } n\ln(m) = \ln(m^n) \\ &= 2^{1/20} && \text{since } e^{\ln(x)} = x \end{aligned}$$

Therefore, an interest rate of approximately 14.11% compounded quarterly is required for an investment of \$3,000 to double in value in 5 years.

A.2.4 Example 4

Find the interest rate required for an investment of \$3,000 to double in value after 5 years if interest is compounded continuously.

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 3000$, $A = 6000$ (since the investment is to double in value), and $t = 5$.

$$6000 = 3000e^{5r}$$

Step 3: Solve for the interest rate, r.

Divide both sides by 3000.

$$2 = e^{5r}$$

Take the natural logarithm of both sides.

$$
\begin{aligned}
\ln(2) &= \ln(e^{5r}) \\
&= 5r && \text{since } \ln(e^x) = x
\end{aligned}
$$

Divide both sides by 5.

$$r = \ln(2)/5 \approx 0.1386$$

Therefore, an interest rate of approximately 13.86% compounded continuously is required for an investment of $3,000 to double in value in 5 years.

> **Observation**
>
> Compare the last two examples. Since continuous compounding of interest earns interest faster than discrete compounding, a lower interest rate is needed for an investment to double in value over a fixed term if interest is compounded continuously. In our examples, an interest rate of 13.86% was needed for the investment with continuous compound interest to double in value in 5 years, while an interest rate to 14.11% was needed for the investment with quarterly compound interest.

A.2.5 Example 5

How long will it take for $5,000 to grow to $8,000 if the investment earns interest at 6% per year compounded monthly?

Step 1: Since interest is compounded monthly, use the accumulated amount for discrete compound interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, $m = 12$, and $r = 0.06$.

$$
\begin{aligned}
8000 &= 5000\left(1 + \frac{0.06}{12}\right)^{12\cdot t} \\
&= 5000\,(1 + 0.005)^{12t}
\end{aligned}
$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = 1.005^{12t}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln(8/5) &= \ln(1.005^{12t}) \\ &= 12t\ln(1.005) && \text{since } \ln(m^n) = n\ln(m) \end{aligned}$$

Divide both sides by $12\ln(1.005)$.

$$t = \frac{\ln(8/5)}{12\ln(1.005)} \approx 7.85$$

Therefore, it will take approximately 7.85 years for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded monthly.

A.2.6 Example 6

How long will it take for \$5,000 to grow to \$8,000 if the investment earns interest at 6% per year compounded continuously?

Step 1: Since interest is compounded continuously, use the accumulated amount for continuous compound interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $P = 5000$, $A = 8000$, and $r = 0.06$.

$$8000 = 5000e^{0.06t}$$

Step 3: Solve for the unknown term t.

Divide both sides by 5000.

$$8/5 = e^{0.06t}$$

Take the natural logarithm of both sides.

$$
\begin{aligned}
\ln(8/5) &= \ln(e^{0.06t}) \\
&= 0.06t && \text{since } \ln(e^x) = x
\end{aligned}
$$

Divide both sides by 0.06.

$$t = \frac{\ln(8/5)}{0.06} \approx 7.83$$

Therefore, it will take approximately 7.83 years for $5,000 to grow to $8,000 if the investment earns interest at 6% per year compounded monthly.

> **⚠ Observation**
>
> Compare the last two examples. Both examples have the same principal, accumulated amount, and interest rate. But since continuous compounding of interest earns interest faster than discrete compounding, it should take less time for the investment to grow to $8,000 if interest is compounded continuously.

A.2.7 Example 7

Find the effective interest rate corresponding to a nominal interest rate of 10% compounded semiannually.

Step 1: Recall the formula for effective interest rate, r_{eff}.

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

Step 2: Plug in the given values: $r = 0.1$ and $m = 2$.

$$
\begin{aligned}
r_{\text{eff}} &= \left(1 + \frac{0.1}{2}\right)^2 - 1 \\
&= 1.05^2 - 1 \\
&= 0.1025
\end{aligned}
$$

Therefore, an investment earning interest compounded semiannually at 10% earns the same amount of interest after 1 year as an investment earning simple interest at 10.25%.

A.2.8 Example 8

Suppose you have \$12,000 in the bank earning interest at a rate of 12% compounded quarterly. Your cousin calls you and needs \$12,000 to buy a new car. You are willing him to loan him the money, but you'd hate to lose out on the interest you would gather by simply leaving your money alone. If you charge your cousin an interest rate compounded continuously, what rate should you charge in order to earn the same amount of interest you otherwise would have?

Step 1: Assume your cousin is prepared to pay you back after t years.

We'll use t as the term in each of the following calculations. Eventually, we'll see that the interest rate you charge does not depend on the specific value of t.

Step 2: Compute the accumulated amount of the \$12,000 after t years assuming you leave your money in the bank.

$$\begin{aligned} A &= P\left(1+\frac{r}{m}\right)^{m\cdot t} \\ &= 12000\left(1+\frac{0.12}{4}\right)^{4t} \\ &= 12000\left(1.03\right)^{4t} \end{aligned}$$

Step 3: Compute the accumulated amount of the \$12,000 after t years assuming you let your cousin borrow the money.

This would be the amount that your cousin repays you after t years.

$$\begin{aligned} A &= Pe^{rt} \\ &= 12000e^{rt} \end{aligned}$$

Step 4: Equate the two accumulated amounts and solve for r.

$$12000\left(1.03\right)^{4t} = 12000e^{rt}$$

Divide both sides by 12000.

$$1.03^{4t} = e^{rt}$$

Take the natural logarithm of both sides.

$$\ln(1.03^{4t}) = \ln(e^{rt})$$

Simplify using properties of logarithms ($\ln(m^n) = n\ln(m)$ and $\ln(e^x) = x$).

$$4t\ln(1.03) = rt$$

And finally, divide both sides by t. Here is where we see that the time it would take your cousin to repay you does not affect the interest rate you would charge.

$$r = 4\ln(1.03) \approx 0.1182$$

Therefore, charging your cousin 11.82% interest compounded continuously earns the same amount of interest as leaving your money in the bank earning interest at a rate of 12% compounded quarterly.

A.3 Present Value Examples

A.3.1 Example 1

How much money should be deposited in a bank paying a yearly interest rate of 6% compounded monthly so that after 3 years, the accumulated amount will be $20,000?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded monthly, we'll use the present value formula for discrete compounding of interest.

$$P = A\left(1 + \frac{r}{m}\right)^{-mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$P = 20000\left(1 + \frac{0.06}{12}\right)^{-(12)(3)}$$

$$= 20000(1.005)^{-36} \approx \$16,712.90$$

Therefore, $16,712.90 invested at 6% interest compounded monthly will be worth $20,000 in 3 years.

A.3.2 Example 2

Use the accumulated amount for discrete compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for discrete compounding of interest.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Step 2: Plug in the given values: $A = 20000$, $r = 0.06$, $m = 12$, and $t = 3$.

$$\begin{aligned} 20000 &= P\left(1 + \frac{0.06}{12}\right)^{(12)(3)} \\ &= P(1.005)^{36} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{20000}{1.005^{36}} \approx \$16,712.90$$

A.3.3 Example 3

Parents wish to establish a trust fund for their child's education. If they need \$170,000 in 7 years, how much should they set aside now if the money is invested at 20% compounded continuously?

Step 1: Notice that this is a present value problem since we're given the accumulated amount and we're asked to find the principal. And since interest is compounded continuously, we'll use the present value formula for continuous compounding of interest.

$$P = Ae^{-rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$\begin{aligned} P &= 170,000e^{-(0.2)(7)} \\ &= 170,000e^{-1.4} \approx \$41,921.48 \end{aligned}$$

Therefore, \$41,921.48 invested at 20% interest compounded continuously will be worth \$170,000 in 7 years.

A.3.4 Example 4

Use the accumulated amount for continuous compound interest to solve the previous example.

Step 1: Start with the formula for accumulated amount for continuous compounding of interest.

$$A = Pe^{rt}$$

Step 2: Plug in the given values: $A = 170000$, $r = 0.2$, and $t = 7$.

$$\begin{aligned} 170000 &= Pe^{(0.2)(7)} \\ &= Pe^{1.4} \end{aligned}$$

Step 3: Solve for P.

$$P = \frac{170000}{e^{1.4}} \approx \$41,921.48$$

A.4 Try It Yourself

A.4.1 Exercise 1

If \$6,00 is invested at 7% compounded continuously, what will be the accumulated amount after 6 years?

Show answer

$A = 6000e^{0.42}$

A.4.2 Exercise 2

If \$7,000 is invested at 16% compounded quarterly, what will be the accumulated amount after 3 years?

Show answer

$A = 7000(1.04)^{12}$

A.4.3 Exercise 3

Find the interest rate r needed for an investment of \$2,000 to grow to \$8,000 in 7 years if compounded continuously.

Show answer

$r = \ln(4)/7$

A.4.4 Exercise 4

Find the interest rate r needed for an investment of \$7,000 to grow to \$12,000 in 21 years if compounded monthly.

Show answer

$r = 12\left[(12/7)^{1/252} - 1\right]$

A.4.5 Exercise 5

Find the time it would take for an investment of \$1,000 to grow to \$100,000 if interest is compounded quarterly at an annual rate of 8%.

Show answer

$t = \frac{\ln(100)}{4\ln(1.02)}$

A.4.6 Exercise 6

Find the time it would take for an investment of \$2,500 to grow to \$6,000 if interest is compounded continuously at an annual rate of 24%.

Show answer

$t = \frac{25}{6}\ln(12/5)$

A.4.7 Exercise 7

Calculate the effective rate of interest corresponding to a nominal interest rate of 52% compounded weekly.

Show answer

$r_{eff} = 1.01^{52} - 1$

A.4.8 Exercise 8

Your grandma would like to establish a trust fund for your education. How much should she set aside now if she wants $50,000 in 9 years and interest is compounded monthly at an annual rate of 12%?

Show answer

$P = 50000(1.01)^{-108}$

A.4.9 Exercise 9

You are preparing to run for president and want to have $100,000 in 6 years to start your campaign. How much money do you need now if interest is compounded continuously at an annual rate of 15%?

Show answer

$P = 100000e^{-0.9}$

A.4.10 Exercise 10

You have $50,000 in the bank earning 7% interest compounded quarterly. However, your cousin needs a $50,000 investment to start up his new financial consulting business. In order to get the same total return as leaving your money in the bank, what interest rate r should you request from your cousin if interest is compounded continuously?

Show answer

$r = 4\ln(1 + 0.07/4)$

B Interest Rates

> **References**
>
> - HULL, John. Options, futures, and other derivatives. Ninth edition. Harlow: Pearson, 2018. ISBN 978-1-292-21289-0.
>   - Chapter 4 - Interest Rates

Learning Outcomes:

- Understand the different types of interest rates.
- Define the risk-free rate and its significance in financial derivatives.
- Explain the concept of continuous compounding and its importance in pricing financial derivatives.

B.1 Types of Rates

B.1.1 Treasury Rate

- Rate on instrument issued by a government in its own currency.

B.1.2 The U.S. Fed Funds Rate

- Unsecured interbank overnight rate of interest.
- Allows banks to adjust the cash (i.e., reserves) on deposit with the Federal Reserve at the end of each day.
- The effective fed funds rate is the average rate on brokered transactions.
- The central bank may intervene with its own transactions to raise or lower the rate.
- Similar arrangements in other countries.

B.1.3 Repo Rate

- Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them for X and buy them bank in the future (usually the next day) for a slightly higher price, Y.
- The financial institution obtains a loan.
- The rate of interest is calculated from the difference between X and Y and is known as the repo rate.

B.1.4 LIBOR (ICE LIBOR)

- Detailed information about LIBOR: https://www.theice.com/iba/libor
- LIBOR is the rate of interest at which a AA bank can borrow money on an **unsecured** basis from another bank.
- Based on **submissions** from a panel of contributor banks (16 for each of USD and GBP).
- It is calculated daily for 5 currencies and 7 maturities.
- There have been some suggestions that banks manipulated LIBOR during certain periods.
- Why would they do this?

B.2 Alternative Reference Rates

Country/Currency/CODE	IBOR Rate	New Reference Rate
USA/Dollars/USD	USD ICE LIBOR	SOFR
UK/Pounds Sterling/GBP	GBP ICE LIBOR	SONIA
Switzerland/Swiss Francs/CHF	CHF ICE LIBOR	SARON
Japan/Yen/JPY	JPY ICE LIBOR, Tibor	TONAR
EU/Euro/EUR	Euribor	ESTER

B.2.1 SOFR (Secured Overnight Financing Rate)

- CME Group Education
- Administered by Federal Reserve Bank of New York (link)
- Transaction-based, calculated from overnight US Treasury repurchase (repo) activity.
- SOFR is a broad measure of the cost of borrowing USD cash overnight, collateralized by U.S. Treasury securities.
- SOFR is a good representation of general funding conditions in the overnight Treasury repo market.

- As such, it will reflect an economic cost of lending and borrowing relevant to the wide array of market participants active in the market.

B.2.2 SONIA (Sterling Overnight Index Average)

- CME Group Education
- Administered by Bank of England (link)
- Unsecured transaction-based index, wholesale based (beyond Interbank)
- It has been endorsed by the Sterling Risk-Free Reference Rate Working Group (Working Group) as the preferred risk-free reference rate for Sterling Overnight Indexed Swaps (OIS).
- In January 2018, the Working Group added banks, dealers, investment managers, non-financial corporates, infrastructure providers, trade associations and professional services firms.
- In April 2018, the BOE introduced a series of reforms of the SONIA benchmark.

B.2.3 €STR (or ESTER, Euro Short-Term Rate)

- Administered by European Central Bank (link)
- It is based on the unsecured market segment.
- The ECB developed an unsecured rate, because it is intended to complement the EONIA.
- Furthermore, a secured rate would be affected by the type of the collaterals.
- The money market statistical reporting covers the 50 largest banks in the euro area in terms of balance sheet size.
- While the EONIA (link) reflects the interbank market, the €STR extends the scope to money market funds, insurance companies and other financial corporations because banks developed significant money market activity with those entities.

B.3 OIS Rate

- An **overnight indexed swap** is swap where a fixed rate for a period (e.g. 3 months) is exchanged for the geometric average of overnight rates (or overnight rate compounded over the term of the swap).
- The underlying floating rate is typically the rate for overnight lending between banks, either non-secured or secured (SOFR, SONIA, €STR).
- For maturities up to one year there is a single exchange (swap term is not overnight).
- For maturities beyond one year there are periodic exchanges, e.g. every quarter.
- The OIS rate is a continually refreshed overnight rate.
- The fixed rate of OIS is typically an interest rate considered less risky than the corresponding interbank rate (LIBOR) because there is limited counterparty risk.

B.3.1 The Risk-Free Rate

- The Treasury rate is considered to be artificially low because:
  - Banks are not required to keep capital for Treasury instruments
  - Treasury instruments are given favorable tax treatment in the US
- OIS rates are now used as a proxy for risk-free rates in derivatives valuation.

B.4 Time Value of Money

B.4.1 Compounding Frequency

- When we compound m times per year at rate r an amount P grows to $P(1+r/m)^m$ in one year.
- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
- Effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	Value of $100 at end of year ($)
Annually m = 1	110.00
Semiannually m = 2	110.25
Quarterly m = 4	110.38
Monthly m = 12	110.47
Weekly m = 52	110.51
Daily m = 365	110.52

B.4.2 Continuous Compounding

- **Rates used in option pricing are nearly always expressed with continuous compounding.**
- In the limit as we compound more and more frequently we obtain continuously compounded interest rates.
- Notation:
  - r: continuously compounded annual interest rate
  - T: time to maturity in years
  - e: Euler's number (mathematical constant)

$$\text{Future value} = P \times e^{rT}$$

$$\text{Present value} = P \times e^{-rT}$$

- USD 100 grows to $100 \times e^{rT}$ when invested at a continuously compounded rate r for time T.
- USD 100 received at time T discounts to $100 \times e^{-rT}$ at time zero when the continuously compounded discount rate is r.

B.4.3 Conversion Formulas

- r_c: continuously compounded rate
- r_m: same rate with compounding m times per year

$$r_c = m \ln(1 + \frac{r_m}{m})$$

$$r_m = m(e^{r_c/m} - 1)$$

Examples:

- 10% with semiannual compounding is equivalent to $2\ln(1.05) = 9.758\%$ with continuous compounding.
- 8% with continuous compounding is equivalent to $4(e^{0.08/4} - 1) = 8.08\%$ with quarterly compounding.