Quantitative Macroeconomics

Hopenhayn Model

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References

- Hopenhayn* (2014, Annual Review of Econ.): Comprehensive review paper. Easy to read.
- Hopenhayn (1992, Econometrica): Original paper. Mostly setting the mathematical foundations behind the model.
- Hopenhayn and Rogerson* (1993, JPE): Famous application of the model.
- Chris Edmond lecture notes are also a good source of information.

Introduction

Goal:

- Present the canonical model of industry/firms dynamics: **Hopenhayn (1992)**.
- Many applications that span over
  - Business cycles: investment, employment, adjustment costs, financial shocks;
  - Development and growth: misallocation, financial development;
  - International trade, labor, etc.
- Start with a static model to build intuition and move to a quantitative model.

Introduction

Three Static Models:

- Lucas (1978) span of control model.
- Hopenhayn's (1992) Industry dynamics model.
- Melitz (2003) monopolistic competition a la Dixit & Stiglitz .

Simple Hopenhayn Economy

- Constant measure M of firms indexed by i (no entry/exit yet).
- Fixed number of workers N.
- Firms (plants) are heterogeneous in their productivity $z \sim G(z)$.
- They use only labor as input and produce according to the production function:

$$y = zn^{\eta} \qquad 0 < \eta < 1$$

- Equilibrium wage w equalizes aggregate demand of labor to (fixed) supply of labor.

Simple Hopenhayn Economy

- Profit Maximization (price of the good is normalized to one):

$$\pi(z) = \max_n \{zn^\eta - wn\}$$

- Optimal demand of firm i satisfies: $\eta z_i n_i^{\eta-1} = w$.

- Since w is the same for all firms, Marginal Product of Labor equalizes across firms:

$$\eta z_j n_j^{\eta-1} = \eta z_i n_i^{\eta-1} \qquad \Leftrightarrow \qquad \frac{z_j}{z_i} = \left(\frac{n_j}{n_i}\right)^{1-\eta}$$

  for two arbitrary firms i and j.

Simple Hopenhayn Economy

- This is also the efficient allocation. Suppose a **benevolent social planner** wants to maximize production in the economy.

- Maximize aggregate output Y subject to the aggregate resource constraint (labor).

$$\max_{n_i} Y = \int y_i di = \int z_i n_i^{\eta} di \qquad \text{s.t.} \qquad N = \int n_i di.$$

- Let μ be the multiplier of the constraint. F.O.C implies for firm i:

$$\eta z_i n_i^{\eta-1} = \mu \qquad \Rightarrow \qquad \frac{z_j}{z_i} = \left(\frac{n_j}{n_i}\right)^{1-\eta}.$$

- **Efficient allocation** implies that MPN should equalize across producers!
    - More productive firms (high z) should hire more labor.

Simple Hopenhayn Economy

- MPN equalization implies that average products are equal across firms:

$$\frac{y_i}{n_i} = z_i n_i^{\eta-1} = \frac{\mu}{\eta}.$$

- We can write the aggregate production function as:

$$Y = \int z_i n^\eta di = \int z_i n_i^\eta di = \frac{\mu}{\eta} \int n_i di = \frac{\mu}{\eta} N$$

- Using the aggregate resource constraint and the FOC:

$$N = \int n_i di = \int \left(\frac{\eta}{\mu} z_i\right)^{\frac{1}{1-\eta}} di \quad \Leftrightarrow \quad \frac{\mu}{\eta} N = \left(\int z_i^{\frac{1}{1-\eta}} di\right)^{1-\eta} N^\eta$$

Simple Hopenhayn Economy

- Aggregate production function has the same form of the individual technology:

$$Y = \left(\int z_i^{\frac{1}{1-\eta}} di \right)^{1-\eta} N^\eta$$

- It is also useful to write the production function as a function of the productivity distribution:

$$Y = \left(\int z_i^{\frac{1}{1-\eta}} dG(z) \right)^{1-\eta} M^{1-\eta} N^\eta.$$

- In this interpretation, the production function has CRS in M and N and TFP is given by the geometric mean of firm-level productivity.

Simple Hopenhayn Economy

- The simple aggregation result provides a useful benchmark.
- Changes in the number of firms or changes in the distribution of productivity impact the aggregate output.
- This result can be generalized for multiple inputs.
- For example, suppose a technology: $y_i = zf(k,n)^\eta = z(k^\alpha n^{1-\alpha})^\eta$. Then:

$$Y = \left(\int z_i^{\frac{1}{1-\eta}} di\right)^{1-\eta} f(K,N)^\eta$$

  where K is aggregate capital.
- Efficiency requires that the marginal product of capital is equalized across producers.

Monopolistic Competition

- An alternative way is to model a la Melitz (2003) using monopolistic competition.
- The final good is produced aggregating a continuum of intermediate inputs (varieties):

$$Y = \left(\int_0^M y_i^\eta di \right)^{\frac{1}{\eta}}, \qquad 0 < \eta < 1 \qquad \text{(gross substitutes).}$$

- The solution implies the usual demand for input and optimal price index:

$$y_i = \left(\frac{p_i}{P} \right)^{1/(\eta-1)} Y \qquad \text{where} \qquad P = \left(\int p_i^{\frac{\eta}{\eta-1}} di \right)^{\frac{\eta-1}{\eta}}.$$

Monopolistic Competition

- Intermediate producers production function: $y_i = \tilde{z}_i n_i$ (where $\tilde{z}_i = z^{1/\eta}$).
- Since intermediates are monopolistic producers, they choose both prices and quantities:

$$\max_{y_i,\ p_i} \quad p_i y_i - w \frac{y_i}{\tilde{z}_i} \qquad \text{s.t.} \qquad y_i = \left(\frac{p_i}{P}\right)^{1/(\eta-1)} Y$$

- The solution implies that firms equalize price to markup over marginal cost:

$$p_i = \frac{1}{\eta} \frac{w}{\tilde{z}_i}$$

- More productive firms can charge lower prices and capture a large share of the market.
  - Which implies higher revenue and profits.

Monopolistic Competition

- After some boring calculations (see Melitz), one can show that

$$
\begin{aligned}
Y &= \left(\int \tilde{z}_i^{\frac{\eta}{1-\eta}} dG(z)\right)^{\frac{1-\eta}{\eta}} M^{\frac{1-\eta}{\eta}} N = \left(\int z_i^{\frac{1}{1-\eta}} dG(z)\right)^{\frac{1-\eta}{\eta}} M^{\frac{1-\eta}{\eta}} N \\
Y^{\eta} &= \left(\int z_i^{\frac{1}{1-\eta}} dG(z)\right)^{1-\eta} M^{1-\eta} N^{\eta}.
\end{aligned}
$$

- Agg. production function in Melitz is just a scaled version of the one in Hopenhayn. Everything that maximizes Y also maximizes Y^{η}.
- **Difference**: in Melitz efficiency requires that the **Marginal Revenue Product of Labor** should be equalized across firms.
  - Hopenhayn: the price is the same for all firms; Melitz: prices are different across firms!
- This distinction will be relevant when connecting to the data.

Entry

- Suppose that to open a new firm, a cost of c_e of workers are needed.
  - Once the firm is created, it draws a z from $G(z)$ (ex-post heterogeneity).
  - We can also model ex-ante heterogeneity (i.e., firm observes productivity and then decides whether to entry) but the choice does matter.

- How does a social planner decide the optimal number of firms in this economy?

- Two steps:

  (i) For a fixed number of firms, choose the optimal labor split between the firms that operate (i.e., what we did before).

  (ii) Choose the optimal number of firms.

Entry

- **Planner's problem**:

$$\max_{M,N_e} ZM^{1-\eta}N_e^{\eta} \qquad \text{s.t.} \qquad c_e M + N_e \leq N.$$

- Solution:

$$N_e = \eta N \qquad \text{and} \qquad M = (1-\eta)N/c_e,$$

  and the multiplier of the constraint is equal to the eq. wage.

- Decreasing returns to scale $(\eta < 1)$ is **essential**: without it, we cannot get a non degenerate distribution!
  - In Melitz, the curvature is generated by the elasticity of substitution in the CES production function instead of DRS.

Entry

- Substituting the solution:

$$Y = Z\eta^{\eta}(1-\eta)^{1-\eta} c_e^{-(1-\eta)} N.$$

- So the elasticity of output per capita with respect to the cost of entry is equal to $(1-\eta)$.
- One can think that aggregate TFP is a function of the geometric mean of the productivities (Z) and the cost of entry.
- Main implication: the **cost of doing business** is a potential source of cross-country disparities in income per capita.

Dynamic Model (Hopenhayn (1992))

- Thus far, the model we have solved is fully static: productivity is fixed and there are no up-and-down dynamics.

- Extend to have stochastic productivity $\Rightarrow$ Workhorse model of industry dynamics.
  - Focus on the **stationary equilibrium**: firms enter, grow and decline, and exit, but the overall distribution of firms is unchanging.
  - Endogenous stationary distribution of firm size.

- The household side will be very simple. We will come back to that later.

Dynamic Model (Hopenhayn (1992))

- Continuum of firms, each measure zero, produce with DRS: $y_i = z_i n_i^{\eta}$
- Idiosyncratic risk: individual firm productivities, z, follow a first-order Markov process with distribution function $F(z'|z)$.
- Entrants draw their initial productivity from a fixed distribution $z_0 \sim G(z)$.
  - Having entrants and incumbents draw productivity from different distributions allows non-trivial firm size distribution.
- Fixed cost to enter, c_e, per-period fixed cost, c_f.
- At the beginning of every period, incumbents decide to stay or exit, entrants decide to enter or not.

Incumbent Firms

- Incumbents maximize per-period profits:

$$\pi(z;p,w) = \max_n \{pzn^\eta - wn - wc_f\}$$

- Usual solution:

$$\eta pzn^{\eta-1} = w \qquad \Rightarrow \qquad n(z;p,w) = \left(\frac{\eta pz}{w}\right)^{\frac{1}{1-\eta}}$$

- Profits:

$$\pi(z;p,w) = (1-\eta)(pz)^{\frac{1}{1-\eta}}\left(\frac{\eta}{w}\right)^{\frac{\eta}{1-\eta}} - c$$

- For a given $c_f > 0$, there is a z such that $\pi = 0$. From now on normalize $w = 1$. We will solve for the equilibrium price.

Incumbent Firm

- At the beginning of every period, **before knowing the realization of z**, the firm decides to exit.

- Firms discount future profits by $1/(1+r) \equiv \beta$, the value of the firm with productivity z is given by:

$$V(z) = \pi(z;p) + \beta \max \left\{ \int V(z')dF(z'|z), 0 \right\}$$

  where the implicit assumption is that the value of exit is zero (no scrap value).

- It may be useful to write a discrete policy function: $\chi(z) = \{0, 1\}$, where 1 represents exit.

Incumbent Firm

- Since profits are increasing in z and F is monotone, the value function is also increasing in z.
- There exists a threshold level $\tilde{z}$ s.t., for all $z < \tilde{z}$ the firm decides to exit.
- We can find the threshold by equalizing the expected value of the firm with its scrap value:

$$\mathbb{E}[V(z')|\tilde{z}] = \int V(z')dF(z'|\tilde{z}) = 0$$

- This does NOT mean that the firms never have negative profits. They may incur negative profits if they expected some mean-reversion of z.

Entrants

- Potential entrants are ex-ante identical.
- An entrant firm must pay the entry cost $c_e > 0$ to set-up the plant and draw $z \sim G(z)$. Start producing next period.
- The value of an entrant is:

$$V_e(z) = -c_e + \beta \int V(z) dG(z)$$

- A firm should enter as long $V_e(z) \geq 0$. If $V_e(z) > 0$ firms enter the industry/market and drive profit to zero (free entry).
    - In equilibrium, we have $V_e(z) \leq 0$.

Free Entry Condition

- Let $M \geq 0$ be the mass of entrants. The **free entry condition** implies that in equilibrium:

$$\beta \int V(z) dG(z) \leq c_e.$$

  with strict equality if $M > 0$.

- **Intuition**: it could be that for some parameters the equilibrium features no entry, i.e.$M = 0$.

- In this case, it should be: $V_e(z) < 0$.

Distribution of Firms

- Let $\mu_t([0, z])$ be the measure of firms over the productivity space.

- The entry and exit rules imply an evolution for the distribution:

$$\mu_{t+1} = \int F(z'|z)(1 - \chi(z))d\mu_t + M_{t+1}G(z').$$

- In the **stationary equilibrium**, we have $\mu_{t+1} = \mu_t = \mu$.

- As usual, the distribution is constant over time, but firms are constantly changing their size (since it is a function of z), and entering/exiting the market.

Demand and Supply

- Demand for goods comes from households.

- For simplicity, just assume that the demand is exogenously given by a function $D(p)$, where $D'(p) < 0$. A simple functional form: $D(p) = \overline{D}/p$.

- Supply of goods is given by operating firms:

$$Y(p) = \int y(z;p)d\mu,$$

  note that the costs (c_e, c_f) are paid in labor so they do not show up here.

- Market clearing requires: $D = Y$.
  - $Y(p)$ is increasing in price; $D(p)$ is decreasing in price.

Equilibrium

- **A stationary recursive competitive equilibrium**: is solving for $(p, M, \tilde{z}, \mu)$ such that:
  - goods market clears;
  - incumbents make optimal exit decisions;
  - no further incentives to enter;
  - distribution μ defined recursively by the law of motion.

- The main difference with respect to the Aiyagari models is that we also need to determine the **endogenous** number of firms.

- Nevertheless, because of the linear properties of the distribution law of motion, we can decouple p from m and solve the model in two steps.
  (i) Solve for the optimal price;
  (ii) Solve for the endogenous mass of entrants M.

Solving for Equilibrium

- Discretize the state space z in n_z grid points. The usual methods apply (i.e, Tauchen).
  - Since there is a discrete choice (exit decision), you should not economize in grid points.
- Assume a positive mass $M > 0$ of entrants. Solve for price p following the steps:
  (i) Guess a price p_0. Compute $\pi(z, p_0)$, $n(z, p_0)$, $y(z, p_0)$ for all grid points.
  (ii) Solve for the Bellman Equation of the firm using value function iteration.
  (iii) Given the value function $V(z)$, check the free entry condition.
  (iv) If the free entry is not satisfied, update the guess and try again.

Solving for Equilibrium

- Let i the grid of the state z and f_{ij} the transition probability from state i to j.
- Guess a value function $V^0(z_i)$ (a vector $n_z \times 1$). Using the guess, compute the $V^1(z_i)$ of the incumbent VF using:

$$V^1(z_i; p_0) = \pi(z_i; p_0) + \beta \max \left\{ \sum_{j=1}^{n_z} f_{ij} V^0(z_j; p_0), 0 \right\} \qquad \forall i = 1, .., n_z.$$

- Check if the distance between the guess and the VF is smaller than a specified tolerance: $\max_i |V^1(z_i; p_0) - V^0(z_i; p_0)| < tol$. If yes, stop it. Otherwise, update the guess $V^0(z_i; p_0) = V^1(z_i; p_0)$ and try again.
- Once the value function converges, collect exit decision in a vector $n_z \times 1$:

$$\chi(z_i; p_0) = 1 \text{ if } \sum_{j=1}^{n_z} f_{ij} V(z_j; p_0) < 0; \qquad \chi(z_i; p_0) = 0 \text{ if otherwise.}$$

Solving for Equilibrium

- Let g_i the discretized PMF of $G(z)$ over the same nodes z_i.
- Given the value function, $V(z_i; p_0)$, compute the value of an entrant V^e:

$$V^e(p_0) = -c_e + \beta \sum_{i=1}^{n_z} g_i V(z_i; p_0).$$

- In equilibrium (with $M > 0$), **free entry** $\Rightarrow V^e(p_0) = 0$.
- Since $V(z_i; p_0)$ is monotone increasing in p, if the free entry condition is not satisfied update the price using a root-finding routine (bisection, Brent):
  - If $V^e(p_0) > 0 \Rightarrow$ reduce price to discourage entry.
  - If $V^e(p_0) < 0 \Rightarrow$ increase price to encourage entry.
- Take the new price guess, p_1, and try again (i.e.compute $\pi(z; p_1)$, $V(z; p_1)$) until $V^e = 0$.

Solving for Equilibrium

- Once we have found the optimal price p, we use the law of motion of μ and the goods market clearing condition to find M.
- Let μ_i denote the mass of firms in state i. Because of the linear law of motion for μ, the stationary distribution is linearly homogeneous in M:

$$\mu = \hat{F}(p)\mu + Mg \qquad \Rightarrow \qquad \mu = M(I - \hat{F}(p))^{-1}g$$

  where $\hat{F}(p)$ is the element-wise multiplication of the transition probability matrix with the exit decision vector: $\hat{F}(p) = F \times (1 - \chi(p))$.
- The stationary distribution is a function of the eq. price p and the mass of entrants M: $\mu(p, M)$.
- Recall: μ and g are $n_z \times 1$ vectors; $\hat{F}$ is $n_z \times n_z$ matrix.

Solving for Equilibrium

- To solve for M, use the market clearing: $D(p) = Y(p, M)$. Aggregate supply is the production of all firms:

$$D(p) = \sum_{i=1}^{n_z} y(z_i; p)\mu(z_i; p, M).$$

- We know p, use the equation to find M.

- **Trick**: because μ is linear in M, we can write: $\mu(p, M) = M \times \mu(p, 1)$. Hence:

$$M = \frac{D(p)}{\sum_{i=1}^{n_z} y(z_i; p)\mu(z_i; p, 1)}.$$

- If $M > 0$, you found an equilibrium $(p, M, \tilde{z}, \mu)$.

Solving for Equilibrium

- What if $M \leq 0$? Then, this is not an equilibrium. The **free entry** condition does not hold and we should have no entrants: $M = 0$.
- The only **stationary equilibrium** consistent with no entry must have **no exit**.
- Stationary distribution of firms just given by stationary distribution of the Markov chain: $\mu(z_i) = \overline{f}_i$.
- You bypass the free entry condition and solve for prices using the goods market clearing:

$$D(p) = \sum_{i=1}^{n_z} y(z_i; p)\overline{f}_i.$$

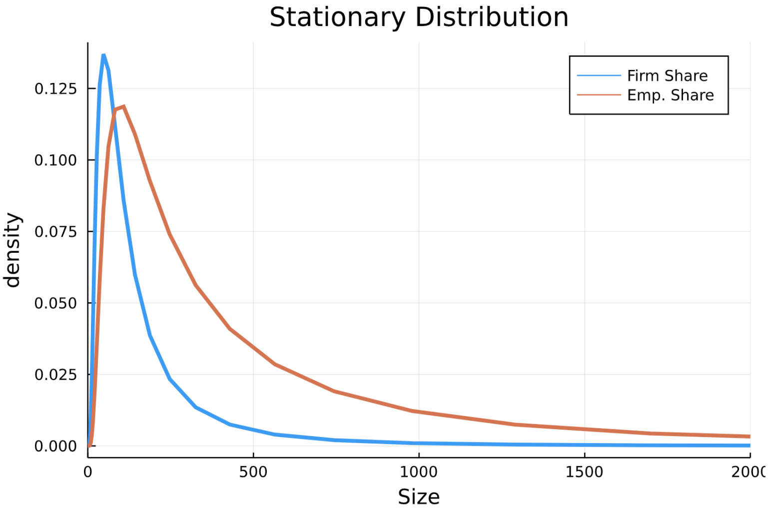
Example

- Calibration: firm's productivity follows an AR(1).

$$\eta = 2/3, \qquad c_e = 40, \qquad c_f = 20, \qquad \beta = 0.8,$$
$$\rho = 0.9, \qquad \sigma = 0.2, \qquad \overline{D} = 100.$$

```
Price: 5.711168128869063
Avg. Firm Size: 78.25684096478187
Exit/entry Rate: 0.15506572450345837
Productivity Cutoff: 0.7911120122611593
Aggregate Output: 17.50955281714009
Aggregate Profits: 16.295419243070263
```

Example



Stationary Distribution
density
Size
0.000
0.025
0.050
0.075
0.100
0.125
0
500
1000
1500
2000
Firm Share
Emp. Share

Comparative Statics

- Increase in entry cost c_e
  - increases prices;
  - decreases exit threshold $\Rightarrow$ less selection, incumbents make more profits, more continue;
  - decreases entry/exit rate $\Rightarrow$increases average age of firms.
  - Ambiguous implications for firm-size distribution and output:
    - ★ price effect $\Rightarrow$ increase output $y(z;p)$ and employment $n(z;p)$.
    - ★ Selection effect (lower threshold) $\Rightarrow$ more incumbent firms are relatively-low productivity firms.

```
Price: 6.1024327084988075
Avg. Firm Size: 87.37988124880879
Exit/entry Rate: 0.12355799954119917
Productivity Cutoff: 0.7217459972375988
Aggregate Output: 16.3869074477021
Aggregate Profits: 18.074290699288674
```

Other Empirical Issues

- Since employment is proportional to productivity, direct connection between productivity and size. A small productivity shock induces reallocation.
- Unconditionally, age of the firm matters:
  - Firms enter small (recall the productivity distribution assumption), then firms survive only if they draw high productivities (and become larger).
  - The model predicts that larger firms are old (and more efficient).
  - However, conditional on size, age is irrelevant.
- Only small firms exit; in the data, some big firms exit as well.

Conclusion

- **Firm Dynamics Model**: open the aggregate production function black box.
- The model presented here is efficient: the welfare theorems hold and the competitive equilibrium is also the solution of the planner's problem.
  - Policies (for instance, taxes) change this result and might affect the employment distribution.
- At this point, we abstract from capital. Introducing capital without some sort of friction does not change the analysis.
- But many papers introduce capital with frictions! Early contributions are:
  - Veracierto (2002, AER) introduces plant-level capital irreversibility to study the aggregate propagation of individual-level investment.
  - Cooley and Quadrini (2001, AER) and Gomes (2001, AER) firms also are subject to financial frictions.