Quantitative Macroeconomics

HH Heterogeneity: Transition Dynamics and Aggregate Fluctuations

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UnB

Introduction

- At this point, we have only focus on the **Stationary Equilibrium**.
- Many questions involve solving the model beyond the Steady-State Stationary Equilibrium.
- **Aggregate Uncertainty:**
  - How the Aiyagari economy reacts to aggregate shocks.
  - Does heterogeneity matters to the business cycles?
- **Transitions Dynamics:**
  - How long it takes to the economy reach a new steady state after an economic reform.
  - How to compute the transition from one steady state to another.

References

- Krueger, Mitman and Perri (2016, Handbook of Macro)*: Application of the model to the great recession.
- Boppart, Krusell and Mitman (2018, JEDC)*: Intuitive paper on how transition dynamics can be used to simulate aggregate shocks (+ history about the MIT shocks).
- Krusell and Smith (1998, JPE): original paper outlining the famous algorithm.
- Heer and Maussner (2009): Ch. 8 and 10; Fehr and Kindermann (2019): Ch. 11: Textbook treatment of the computational methods.
- Algan et al (2014, Handbook of Computational Economics): Entire handbook on how to solve HA economies with aggregate uncertainty. See also their special edition on the JEDC.

Krueger, Mitman and Perri (2016, Handbook of Macro)

- **Question:** How important is household heterogeneity for the amplification and propagation of macroeconomic shocks?

- Focus on the US Great Recession of 2007–09.

- Heterogeneity: earnings, wealth, and household preferences.

- Consequences for cross-sectional inequality in disposable income and consumption expenditures.

Krueger, Mitman and Perri (2016, Handbook of Macro)

Method:

- Summarize empirical facts about the joint distribution of income, wealth, and consumption before and during the great recession.
- Compute various versions of the HA model with aggregate uncertainty and study its cross-sectional and dynamic properties.
  - Simple version of the model "replicates" the results of representative agent model.
  - Model extension with life-cycle, unemployment insurance and social security does a much better job.
- Study the impact of social insurance policies.

Empirical Evidence: Levels

- **Data**: PSID (2004, 2006, 2008, and 2010). New version covers income, wealth, and consumption.

Table 2 PSID Households across the net worth distribution: 2006

	% Share of:			% Expend. Rate		Head's	
NW Q	**Earn.**	**Disp. *Y***	**Expend.**	**Earn.**	**Disp. *Y***	**Age**	**Edu. (yrs)**
Q1	9.8	8.7	11.3	95.1	90.0	39.2	12
Q2	12.9	11.2	12.4	79.3	76.4	40.3	12
Q3	18.0	16.7	16.8	77.5	69.8	42.3	12.4
Q4	22.3	22.1	22.4	82.3	69.6	46.2	12.7
Q5	37.0	41.2	37.2	83.0	62.5	48.8	13.9
	Correlation with net worth						
	0.26	0.42	0.20				

Empirical Evidence: Changes

Table 3 Annualized changes in selected variables across PSID net worth

	Net worth[a]				**Disp. Y (%)**		**Cons. Exp.(%)**		**Exp. Rate (pp)**	
	(1) 04–06		**(2) 06–10**		**(3) 04–06**	**(4) 06–10**	**(5) 04–06**	**(6) 06–10**	**(7) 04–06**	**(8) 06–10**
All	**15.7**	**44.6**	**−3.0**	**−10**	**4.1**	**1.2**	**5.6**	**−1.3**	**0.9**	**−1.6**
NW Q										
Q1	NA	12.9	NA	6.6	7.4	6.7	7.1	0.6	−0.2	−4.2
Q2	121.9	19.5	24.4	3.7	6.7	4.1	7.2	2	0.3	−1.3
Q3	32.9	23.6	4.3	3.3	5.1	1.8	9	0	2.3	−1.1
Q4	17.0	34.7	1.7	3.8	5.0	1.7	5.9	−1.5	0.5	−2
Q5	11.6	132.2	−4.9	−68.4	1.8	−1.2	2.7	−3.5	0.5	−1.4

[a] The first figure is the percentage change (growth rate), the second is the change in 000's of dollars.

- **Last column**: saving rates increase relatively more for wealth-poor households during the recession.

A Business Cycle Model with HH Heterogeneity

Ingredients:

- Idiosyncratic individual shocks + incomplete markets a la Aiyagari-Hugget.
- Aggregate Shocks in the spirit of the Real Business Cycle literature.
- 2 stages life cycle: young (workers) and old (retiree).
- Ex-ante heterogeneity in β.
- Government policy: unemployment insurance and social security.

Production Technology

- Aggregate production function is Cobb-Douglas over capital and labor:

$$Y = Z^* K^\alpha N^{1-\alpha} \tag{1}$$

- $Z^* \equiv ZC^\omega$, where $\omega \geq 0$.
- **Aggregate shock**: Z follows a 2-state Markov with transition matrix $\pi(Z'|Z)$:
  - $Z \in \{Z_l, Z_h\}$. Z_l: recession, Z_h : normal times.
- **Demand externality**: C^ω
  - If $\omega = 0$, standard neoclassical production function.
  - If $\omega > 0$, production is partially determined by demand.

Households

- Standard utility over consumption $u(c)$, they cannot borrow, $a' \geq 0$.
- Ex-ante and fixed heterogeneous discount factor $\beta \in B$.
- Two idiosyncratic states:
  - Employment: $s \in \{e, u\}$. Transition matrix depends on aggregate state: $\pi(s'|s, Z', Z)$.
  - Income: γ. Transition matrix is independent of the aggregate state: $\pi(\gamma'|\gamma)$.
- Stochastic life cycle:
  - households are born as workers and with probability $1 - \theta$ they retire;
  - after retiring, they receive pensions and with probability $1 - \nu$ they die.

Government Policy

- Pensions and unemployment benefits are financed using proportional labor taxes.
- Government runs a balance budget system every period.
- **Unemployment insurance**:
  - Pays a fraction $\rho \in [0, 1)$ of the household potential income: $b = \rho w \gamma$.
  - Financed with tax rate, $\tau(Z)$. The tax adjusts to maintain the budget balanced. In recessions, the tax rate increases.
- **Pension benefits**:
  - Financed with a fixed social security contribution τ_{ss}.
  - Pension benefits, b_{ss}, adjust to maintain the budget balanced. In recessions, the pension decreases.

Household Bellman Equation: Retiree

$$V_R(a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0}\left\{u(c) + \nu\beta \sum_{Z'\in Z} \pi(Z'|Z) V_R(a',\beta;Z',\Phi')\right\}$$

$$\text{s.t} \qquad c + a' = b_{ss}(Z,\Phi) + (1 + r(Z,\Phi) - \delta)a/\nu$$

$$\Phi' = H(Z,\Phi,Z')$$

where Φ is the distribution of agents in the economy.

- Individual state: (a,β), aggregate state: (Z,Φ).
- $H()$: Rational expectations function. The agents correctly forecast the next period distribution, given the state of the economy.

Household Bellman Equation: Worker

$$V_w(s,\gamma,a,\beta;Z,\Phi) = \max_{c,\ a'\geq 0} \Bigg\{ \{u(c) + \beta \sum_{(Z',s',\gamma')\in(Z,s,\gamma)} \pi(Z'|Z)\pi(s'|s,Z',Z)\pi(\gamma'|\gamma)$$

$$\times \left[\theta V_W(s',\gamma',a',\beta;Z',\Phi') + (1-\theta)V_R(s',\gamma',a',\beta;Z',\Phi')\right] \Bigg\}$$

s.t $\quad c + a' = (1-\tau(Z)-\tau_{ss})\gamma w(Z,\Phi)[1-(1-\rho)\mathbf{1}_{s=u}] + (1+r(Z,\Phi)-\delta)a$

$\quad \Phi' = H(Z,\Phi,Z')$

- Individual state: (a,s,γ,β), aggregate state: (Z,Φ).
- Employed earnings: γw; unemployed earnings $\rho\gamma w$.

Equilibrium

- Prices are given by FOCs of firm's problem:

$$w(Z,\Phi) = Z\alpha\left(\frac{K(Z,\Phi)}{N(Z)}\right)^{1-\alpha} \quad \text{and} \quad r(Z,\Phi) = Z(1-\alpha)\left(\frac{N(Z)}{K(Z,\Phi)}\right)^{\alpha}$$

  where aggregate employment $N(Z)$ is given by the distributions of the Markov process (which depends on Z).

- Asset market clears:

$$K(Z,\Phi) = \int a d\Phi$$

- The distribution evolves according to the function: $\Phi' = H(Z, \Phi, Z')$. In equilibrium, this function is *consistent* with the individual decisions.

Digression: Krusell & Smith (1998)

- Because prices are allowed to vary over the cycles and they are needed for the household problem: the aggregate state, (Z, Φ), is part of the state of the HH.

- **Problem:** the distribution, Φ, is a high-dimensional object and the state space increases substantially.

- **Krussel & Smith (1998)**: instead of using the entire distribution, just use some moments of the distribution:
  - Households are "boundedly rational" on how the distribution evolves.
  - In this class of models, the **mean** is enough to correctly forecast prices:

$$\Phi' = H(Z, \Phi, Z') \qquad \Rightarrow \qquad K' = H(Z, K, Z') \tag{2}$$

Digression: Krusell & Smith (1998)

- Substitute Φ by K. Example:

$$
\begin{aligned}
V_R(a, \beta; Z, K) = \max_{c,\ a' \geq 0} & \left\{ u(c) + \nu\beta \sum_{Z' \in Z} \pi(Z'|Z) V_R(a', \beta; Z', K') \right\} \\
\text{s.t} \qquad & c + a' = b_{ss}(Z, K) + (1 + r(Z, K) - \delta) a/\nu \\
& K' = H(Z, K, Z')
\end{aligned}
$$

- **Intuition:** the mean of Φ works well to forecast prices because the savings policy function is approximately linear.

- For more complex models, one may need higher moments.

Back to Krueger, Mitman and Perri: Models

Table 5 Taxonomy of different versions of the model used in the chapter

Name	**Discounting**	**Techn.**	**Soc. Ins.**
KS	$\beta = \overline{\beta}$	$\omega = 0$	$\rho = 1\%$
Het. β	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega = 0$	$\rho = 50\%$
Het. β	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega = 0$	$\rho = 10\%$
Dem. Ext.	$\beta \in [\overline{\beta} - \epsilon \, \overline{\beta} + \epsilon]$	$\omega > 0$	$\rho = 50\%$

- **KS**: Basic model, very similar to Krusell-Smith (1998);
- **Benchmark model**: second row, calibrated to match the US economy.

Calibration

- Model calibrated to quarterly data. $\alpha = 0.36$, $\delta = 0.025$, $u(c) = \log(c)$.
- Z duration and GDP drop of a "severe recession".
- s separation and job-finding rates, γ comes from a persistent-transitory process estimated from PSID.
- θ, ν: a working period of 40 years and a retirement period of 15 years.
- β from uniform distribution: match Gini for the wealth distribution.
- Policy: unemployment benefits of 50%, $\rho = 0.5$. Pension of 40% of avg. wage.

Evaluating the Model: Wealth Distribution

Table 6 Net worth distributions: Data vs models

	Data		Models	
% Share held by:	**PSID, 06**	**SCF, 07**	**Bench**	**KS**
Q1	−0.9	−0.2	0.3	6.9
Q2	0.8	1.2	1.2	11.7
Q3	4.4	4.6	4.7	16.0
Q4	13.0	11.9	16.0	22.3
Q5	82.7	82.5	77.8	43.0
90–95	13.7	11.1	17.9	10.5
95–99	22.8	25.3	26.0	11.8
*T*1%	30.9	33.5	14.2	5.0
Gini	0.77	0.78	0.77	0.35

- Benchmark matches the wealth distribution; but fails in the top 1%. KS fails.

Evaluating the Model: Joint Distribution

Table 8 Selected variables by net worth: Data vs models

	% Share of:						% Expend. rate			
	Earnings		Disp. *Y*		Expend.		Earnings		Disp. *Y*	
NW Q	Data	Mod	Data	Mod	Data	Mod	Data	Mod	Data	Mod
Q1	9.8	6.5	8.7	6.0	11.3	6.6	95.1	96.5	90.0	90.4
Q2	12.9	11.8	11.2	10.5	12.4	11.3	79.3	90.3	76.4	86.9
Q3	18.0	18.2	16.7	16.6	16.8	16.6	77.5	86.0	69.8	81.1
Q4	22.3	25.5	22.1	24.3	22.4	23.6	82.3	87.3	69.6	78.5
Q5	37.0	38.0	41.2	42.7	37.2	42.0	83.0	104.5	62.5	79.6
	Correlation with net worth									
	0.26	0.46	0.42	0.67	0.20	0.76				

- Qualitatively close to the data, but quantitative a bit far; wealth poor consuming too little, and the wealth rich consuming too much.

Evaluating the Model: Dynamics in Normal Times

Table 9 Annualized changes in selected variables by net worth in normal times (2004-06): Data vs model

	Net worth (%)		Disp. Y (%)		Expend (%)		Exp. Rate (pp)	
NW Q	**Data**	**Model**	**Data**	**Model**	**Data**	**Model**	**Data**	**Model**
Q1	NaN	44	7.4	7.2	7.1	6.7	−0.2	−0.4
Q2	122	33	6.7	3.1	7.2	3.6	0.3	0.5
Q3	33	20	5.1	1.6	9	2.5	2.3	0.8
Q4	17	9	5	0.5	5.9	1.7	0.5	1.2
Q5	12	3	1.8	−1.0	2.7	0.5	0.5	1.4
All	16	5	4.1	0.7	5.6	1.8	0.9	0.7

- Slightly too much downward and upward mobility on income, but in general good job.

Evaluating the Model: Dynamics in Recession

Table 10 Annualized changes in selected variables by net worth in a severe recession: Data vs model

	Net worth (%)		Disp. Y (%)		Expend. (%)		Exp. rate (pp)	
NW Q	Data	Model	Data	Model	Data	Model	Data	Model
Q1	NaN	24	6.7	4.9	0.6	4.5	−4.2	−0.4
Q2	24	15	4.1	0.3	2.0	1.2	−1.3	0.8
Q3	4	8	1.8	−2.4	0.8	0.0	−1.1	2.2
Q4	2	4	1.7	−4.0	−1.7	−1.5	−2.0	3.2
Q5	−5	−1	−1.2	−6.4	−3.7	−3.5	−1.4	4.6
All	−3	1	1.2	−3.7	−1.3	−0.8	−1.6	2.0

- Consumption-savings in the recession: $\downarrow$ savings because of consumption smoothing; $\uparrow$ savings because of precautionary savings.
  - In the model: the first is stronger for the richer, but the latter is stronger for the poorer.

Aggregate Shock: Krusell-Smith vs Representative Agent

- The Krusell-Smith economy is remarkably similar to the representative agent in the aggregate.
- **Intuition**: without too many constrained agents, the HA economy behaves as a RA.

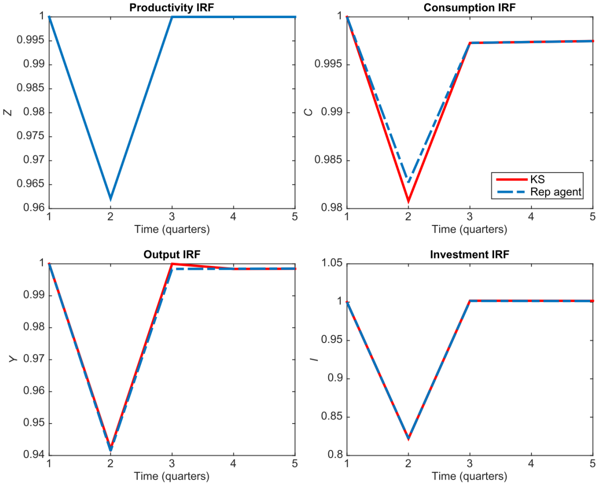
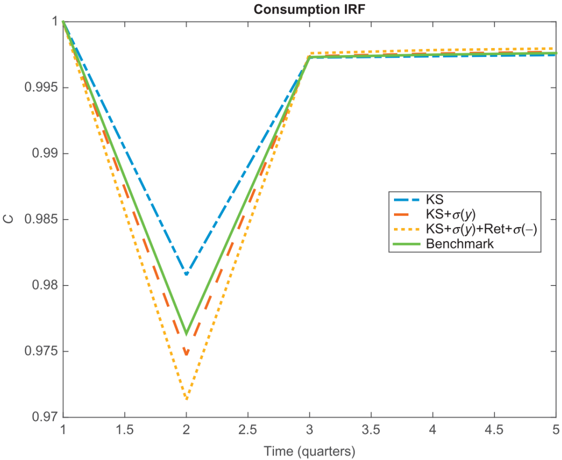


Fig. 3 Impulse response functions (IRF) to aggregate technology shock in KS and RA economies

Aggregate Shock: All models

- Benchmark generates a larger drop in consumption than KS economy.
- Largely accounted by income risk on top of employment risk.
- Recall that benchmark economy has high unemployment benefits.



Differences between KS and Benchmark Economy

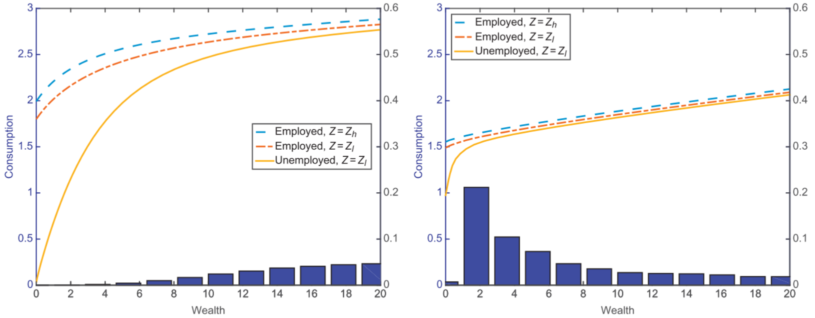


Fig. 5 Consumption function and wealth distribution: Krusell–Smith (left panel) and benchmark (right panel).

- Benchmark generates a larger drop in consumption because it has a larger share of low wealth households.
- The low wealth consumes more in the benchmark because of unemployment benefits.

The Role of Unemployment Insurance

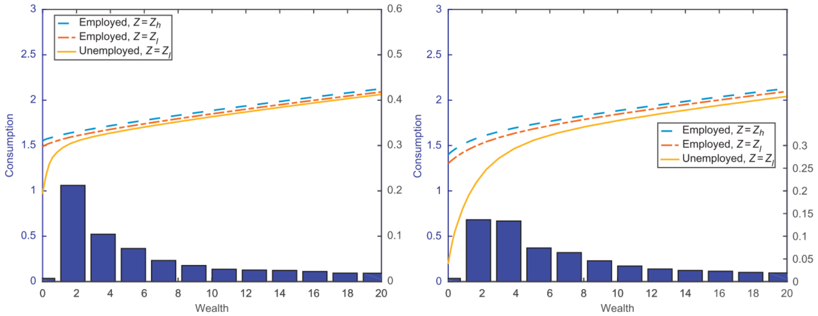


Fig. 10 Consumption function and wealth distribution: Benchmark (left panel) and low UI (right panel).

- Unemployment benefits help the low wealth poor to consume in bad times.
- Aggregate consumption falls much more in recessions without UI.

Demand Externality

- Keynesian flavor increases the size of the recession.
- Lots of wealth poor $\Rightarrow$ large drop in consumption $\Rightarrow$ demand externality $\Rightarrow$ further drops output.

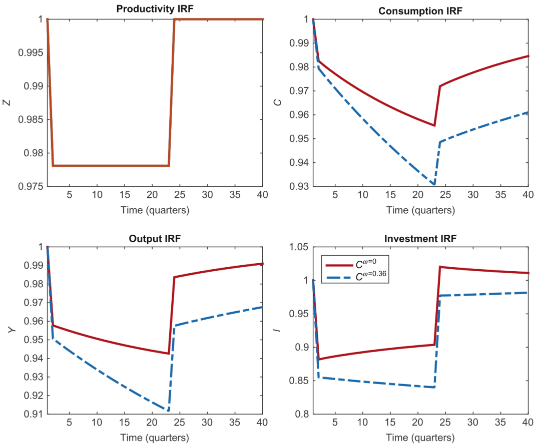


Fig. 14 Impulse response to identical aggregate technology shock: Comparison between economies with and without demand externality.

Conclusion: Krueger, Mitman and Perri

- Simple heterogeneity a la Krusell-Smith/Aiyagari is not enough to generate differences from the representative agent model.
- Other ingredients should be added to get a meaningful wealth distribution.
- Low wealth agents are key to getting the larger fall in consumption.
- Unemployment insurance attenuates the fall of aggregate consumption.
- Demand externality further increases the recession: motivation to include a proper microfoundation of the demand effect.

Computational

- To solve a heterogeneous agent economy with aggregate uncertainty the main methods are:
  - Krusell-Smith (1998, JPE) bounded rationality algorithm.
  - MIT shock (Boppart, Krusell and Mitman, 2018, JEDC).
  - Reiter (2009, JEDC) Method.
- There are others/variations of algorithms. Check Algan et al (2014).

Krusell-Smith Algorithm

- References: Krusell-Smith's original paper is easy to follow. Check also Nakajima's notes.

- Krusell-Smith: use some moments finite moments instead of the entire distribution.
  - In the model we saw before just the mean is enough: $K' = H(Z, K, Z')$.

- Approximate the function forecasting function $H()$ with a log-linear form:

$$\log K' = a_l + b_l \log K \quad \text{if } Z = Z_l$$
$$\log K' = a_h + b_h \log K \quad \text{if } Z = Z_h$$

- We have to find the parameters: (a_l, a_h, b_l, b_h).

Krusell-Smith Algorithm

Discretize the state space: (a, s, K, Z). Recover the prices $r(K, Z)$ and $w(K, Z)$ for each state space using the firm's problem.

(i) Guess the parameters of the forecast function: $(a_l^0, a_h^0, b_l^0, b_h^0)$.

(ii) Given $(a_l^0, a_h^0, b_l^0, b_h^0)$, solve the Bellman Equation of the HH for all the state space (a, s, K, Z).

(iii) Given the household policy functions, simulate T periods:

- Draw a sequence of Z_t for all T. Guess a initial distribution Φ_0.
- Using the policy function and the sequence Z_t, keep updating the distribution Φ_t forward.
- Compute the mean of the distribution K_t (and other moments if necessary).
- Drop the first T_0 periods. Now, we have a sequence $\{Z_t, K_t\}_{t=T_0}^{T}$.

Krusell-Smith Algorithm

(iv) Using the sequence $\{Z_t, K_t\}_{t=T_0}^{T}$, run a linear regression and recover the new coefficients: $(a_l^1, a_h^1, b_l^1, b_h^1)$.

(v) Check the distance between the guess a^0, b^0 and the new parameters a^1, b^1. If it is smaller than tol, we are done. Otherwise, update the guess and start again:

$$a^0 = \lambda a^0 + (1-\lambda)a^1$$
$$b^0 = \lambda b^0 + (1-\lambda)b^1$$

where $\lambda \in (0, 1)$ is a damping parameter.

Krusell-Smith Algorithm: Issues

- After you finish, you must check the R^2 of the forecast regression. If the R^2 is low, you must add more moments or change the function form.
  - In Krusell-Smith, $R^2 = 0.999$, so the perceived law of motion of K is very close to the actual law of motion.

- Poor initial guesses might not converge. One good guess is $a = \log K_{ss}$ and $b = 0$.

- **Good:** KS captures potential non-linearities and large shocks. For instance, asymmetries between the boom and the recession; uncertainty shocks; etc.

- **Bad:** KS can be inaccurate if there are explicitly distributional channels coming from the top of the wealth distribution. Potentially very slow.

Reiter Method

- **Perturbation Methods**:
  - Generalization of the well-known linearization around the steady state.
  - Often used to solve representative agent models.
  - They tend to be fast, but require derivatives and some stability conditions (Blanchard-Kahn).

- Standard software (i.e., dynare) uses this method.

- Reiter (2009) propose to solve for the stationary equilibrium using global methods (projection methods), and then use perturbation methods to solve for the aggregate shock.

- If you need a refresher on Pertubation methods, check Fernandez-Villaverde's notes.

Reiter's Method

- We can write the solution of DSGE models as a nonlinear system of difference equations:

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = 0 \tag{3}$$

  where x is the vector of predetermined variables (state), y is nonpredetermined variables (control).

- Then, we can linearize the system (either numerically or analytically) and use methods to solve the linear system of difference equations:
  - Blanchard and Kahn (1980); Uhlig (1999); Sims (2000); Rendahl (2018).

Reiter's Method

- **Example**: Stochastic Neoclassical Growth model

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} c_t^{-\gamma} - \beta E_t c_{t+1}^{-\gamma}[\alpha k_{t+1}^{\alpha-1} + 1 - \delta] \\ c_t + k_{t+1} - e^{z_t} k_t^{\alpha} - (1-\delta)k_t \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \end{bmatrix} = 0$$

  where $x = [k, z]'$ and $y = [c]$.

- First row is the Euler Equation, second is the feasibility constraint, and the last is the stochastic process of the shock.

Reiter's Method

- **Example**: Krusell-Smith economy.

$$E_t F(x_t, x_{t+1}, y_t, y_{t+1}) = E_t \begin{bmatrix} d\Phi_{t+1} - d\Phi_t \Pi_{g_{a,t}} \\ V_t - (\overline{u}_{g_{a,t}} + \beta \Pi_{g_{a,t}} V_{t+1}) \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \\ \text{ED}(g_{a,t}, d\Phi_t, z_t, P_t) \end{bmatrix}$$

where $x = [d\Phi, z]'$ and $y = [V P]'$.

- $d\Phi$ is the p.d.f of the distribution;
- P_t are the prices;
- ED() is an arbitrary excess demand function (which implicitly includes firm's foc);
- $\Pi_{g_{a,t}}$ is the transition matrix induced by the optimal policy:

$$g_{a,t} = \arg\max u(a(1+r_t) + w_t s - a') + \beta E_t V_{t+1}(a', s', d\Phi', z')$$

Reiter's Method

- Since we discretize both Phi and V_t, the first two rows must hold for ALL the idiosyncratic state.
- The number of equations that we need to linearize is exponentially large.
- Linearization is often done using numerical derivatives. Nowadays people use automatic differentiation to do the job.
- Solution (up to first order) has **certainty equivalence**: no precautionary savings because of aggregate risk.
- The method cannot capture nonlinearities or sign asymmetries.

Transition Dynamics and MIT shocks

- Most of the time, we are interested in simulating an **impulse response function** (IRF).
- A IRF is just the deterministic transition dynamics between two steady states after an unexpected aggregate shock (a MIT shock).
- Boppart, Krusell and Mitman (2018) show that the IRF can be used to compute equilibrium of HA with agg. uncertainty.
- Solving for the transition dynamics is also useful if you are interested in studying the transition to a new steady state after a change in economic policy.

MIT shock

- **MIT shock:** an unpredictable shock to the steady-state equilibrium of an economy without shocks.
  - No shocks are expected to ever materialize but nevertheless a shock now occurs!
- We can now analyze the equilibrium transition along a **perfect-foresight path** until the economy reaches the steady state.
- Some argue that **Tom Sargent** coined the term reflecting that some researchers at MIT used the method.
  - For fresh-water economists, a MIT shock is inconsistent with rational expectations!
  - "A shock of probability zero, only at MIT they can get away with that!".

MIT shock

- Suppose a standard Ayiagari in the steady state at $t = 0$. At $t = 1$, the economy receives an (unexpected) TFP aggregate shock:

$$
\begin{aligned}
Y_t &= Z_t K_t^{\alpha} L_t^{1-\alpha} \\
\log Z_t &= \rho_z \log Z_{t-1} + \varepsilon_t
\end{aligned}
$$

  where $\varepsilon_t = 0.01$ if $t = 1$ and $\varepsilon_t = 0$ otherwise.

- If $0 < \rho < 1$, when $t \to \infty$, the shock vanishes and we are back to the original steady state.

- Our goal is to solve the **transition dynamics** between the two steady states.
  - Because Z_t varies in the transition, aggregate variables (prices, savings, distribution) change during the transition.

Sequential Equilibrium

- Instead of carrying the aggregate state, we index the Bellman Equation by time t.

$$V_t(a, s) = \max_{c,\ a' \geq 0} \left\{ u(c) + \beta \sum_{s' \in S} \pi(s'|s) V_{t+1}(a's') \right\}$$

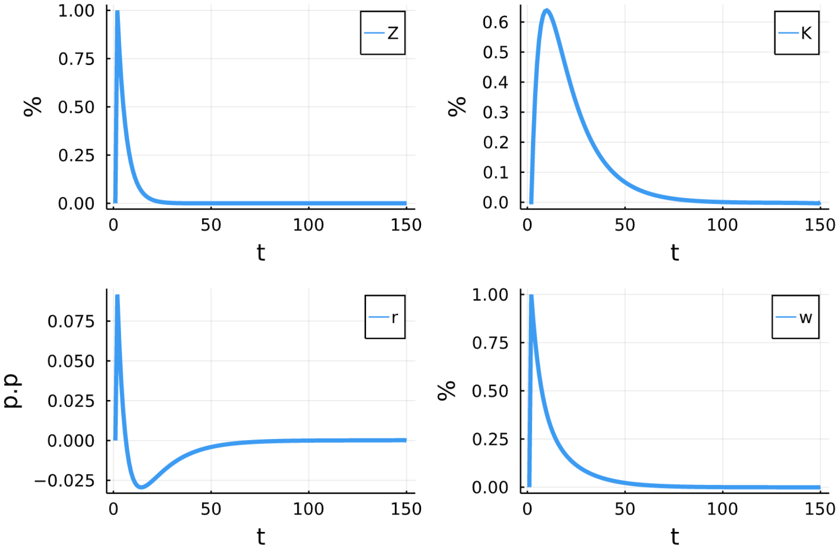
s.t $\quad c + a' = w_t s + (1 + r_t - \delta)a$

- Solve for the transition means solving for the equilibrium in the asset market for all t:

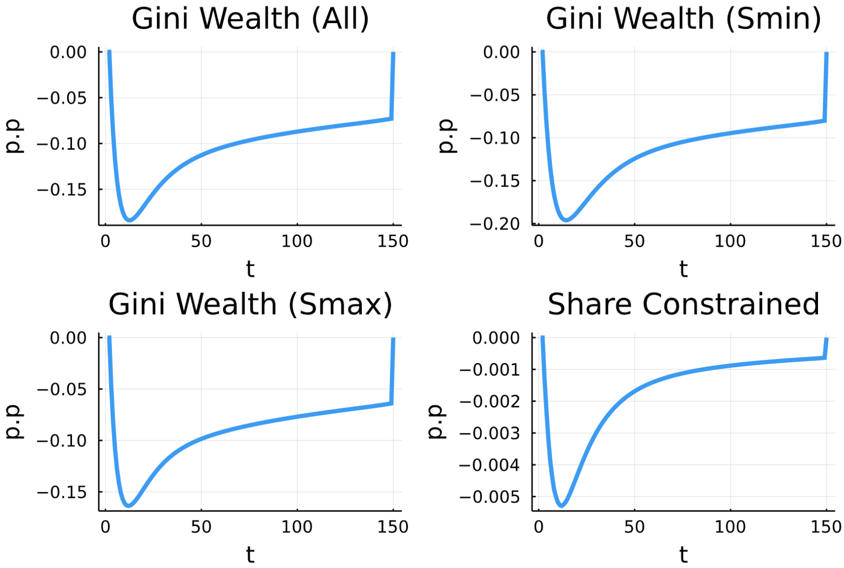
$$\int_{A \times S} a d\Phi_t(a, s; r_t) = K_t(r_t)$$

both the distribution, $\Phi_t(a, s)$, and the aggregate capital, K_t, are indexed by t.

IRF: Standard Aiyagari Economy



IRF: Standard Aiyagari Economy



Transition Dynamics between Steady States

- The method is useful to compute transition between different steady states.

- **Example**: Suppose a labor tax, τ_l, that is used to finance a lump-sum transfer, T_t. The budget constraint:

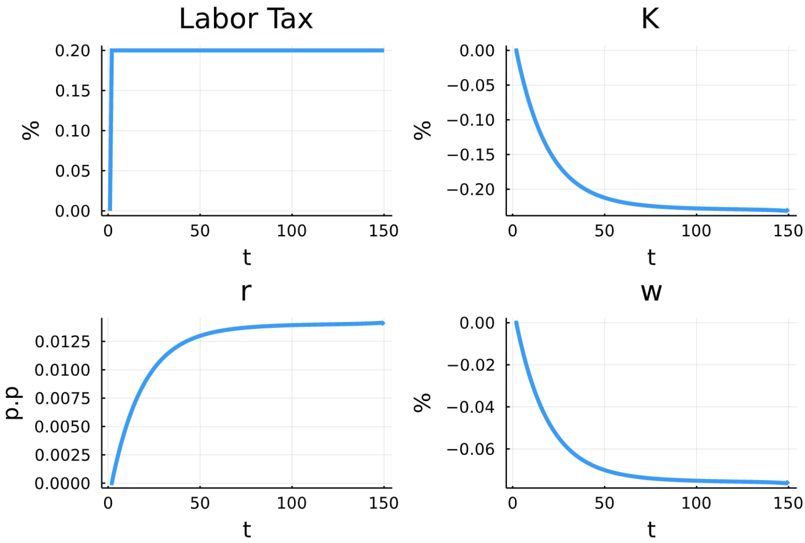
$$c + a' = w_t s(1 - \tau_l) + (1 + r_t - \delta)a + T_t.$$

  The government runs a balanced budget: $T_t = \tau_l w_t L$.

- Suppose the economy is in the SS with $\tau_l = 0$. At $t = 1$, the government decides to raise the tax rate: $\tau_l = 0.2$ (there are no aggregate shocks).

- How long does the economy take to reach the new steady state?

Transition to New SS: Labor Tax



Algorithm

(i) Solve for the initial and the final steady state. Select a large number of periods T.

(ii) Guess a path of $\{K_t^g\}_{t=2}^{T-1}$. K_1 and K_T are given by the initial/final steady state. Recover the prices $\{r_t, w_t\}_{t=2}^{T-1}$ using the firm's problem and the sequence of Z_t.

(iii) Given prices, $\{r_t, w_t\}_{t=2}^{T}$, solve the value function (and policy functions) backwards from $t = T-1, ...2$ starting from the **final steady state value function**.
- Endogenous Grid works well, but careful to use the correct prices!

(iv) Starting from the **initial steady state distribution**, simulate the distribution forward from $t = 1, ..., T-1$ using the policy functions, $g_{a,t}(a, s)$ and the Markov process of s.

Algorithm

(v) Compute aggregate savings (capital) using the distribution for all t: $\{K_t^s\}_{t=2}^{T-1}$.

(vi) Compute the maximum difference between the guess sequence, $\{K_t^g\}$, and the new sequence, $\{K_t^s\}$. If it is smaller than tol, stop. Otherwise, update the guess using the rule:

$$K_t = \lambda K_t^s + (1-\lambda)K_t^g \qquad \text{for } t = 2, .., T-1,$$

where $\lambda \in (0, 1)$ is a dampening parameter, and return to (ii).

Algorithm

- The “shooting algorithm” does not have established convergence properties but tends to work well in practice.
- The damp parameter should not be too large, otherwise, it may not converge.
- T has to be large enough to allow the shock to fade out completely. Always check the last transition between times $T-1$ and T.
- A good initial guess is $K_{ss} = K_t$ for all t.
- If labor supply is endogenous you can guess K/L. If you need to find the eq. in other markets you have to guess an additional sequence.

Boppart-Krusell-Mitman (2018)

- Intuitively, the method uses the impulse response function as a sufficient statistic to compute the eq. of the model.
- In theory, dynamic programming says that any aggregate statistic of the model can be computed as a function of the aggregate state: $x(Z, \Phi)$.
- Instead of using aggregate state, we can also write the aggregate stats as a function of past shocks. For example, the aggregate capital at time t is:

$$K_t = K(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...),$$

where ε_t is the innovation of the aggregate at time t.

Boppart-Krusell-Mitman (2018)

- If we assume that the model response to the shock is approximately linear, we can write K_t as a linear function of past shocks:

$$K_t = \varepsilon_t K(1,0,0,...) + \varepsilon_{t-1} K(0,1,0,...) + \varepsilon_{t-2} K(0,0,1,...) + ...$$

  where $K(0,1,0,...)$ is the (non-linear) response of capital at time t to a shock (scaled to 1) that happened at $t-1$.

- Note that each K is the response of ONLY ONE shock at each point in time.

- In the notation of BKM: $K_0 = K(1,0,0,...)$, $K_1 = K(0,1,0,...)$, etc. Then:

$$K_t = \sum_{s=0}^{\infty} \varepsilon_{t-s} K_s$$

Boppart-Krusell-Mitman (2018)

- When we compute an impulse response function to an MIT shock, we get exactly the response of capital to a 1% shock that happened s periods before!

- That is, we have a sequence of K:

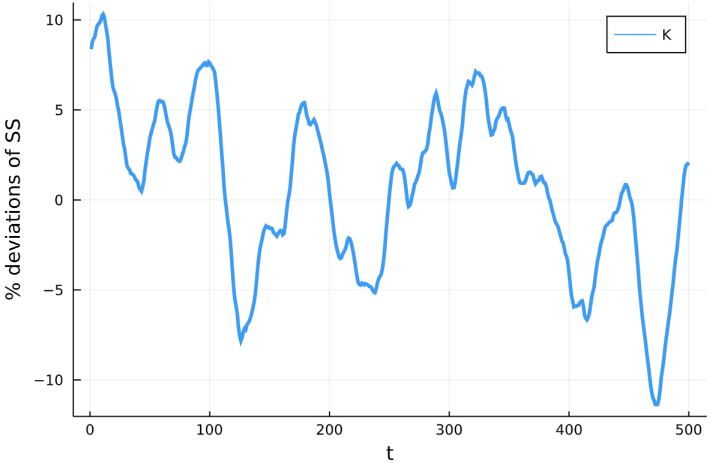
$$[K(1,0,0,...),K(0,1,0,...),K(0,0,1,...),...]$$

- In fact, we have that for all aggregate statistics of the model.

- To simulate the model, we can simply draw a sequence of shocks ε and use the statistics computed by the impulse response.

Boppart-Krusell-Mitman (2018)

Figure: Simulation of Aggregate Capital using BKM



Boppart-Krusell-Mitman (2018)

- **Good:** It is easy to use. The only thing you need is an impulse response function. You can compute using standard dynamic programming methods.
- It is trivial to add more shocks. Because shocks are linear, you just need to simulate two IRF for each shock. Then, the final effect of the shocks is simply additive.
- **Bad:** If the model is highly non-linear or has sign-dependence it can be a poor approximation.
- As every other linear method, it assumes certainty equivalence. No second-order effects from aggregate risk; It may perform poorly if the shock brings you far from the steady state.

State of the Art Methods

- **Auclert, Bardóczy, Rognlie and Straub (2021)**. Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models.
  - As in Boppart-Krusell-Mitman (2018), they use the IRF to solve for the model. Instead of solving for the full transition, they show that the Jacobian of the equilibrium is enough.
  - Python notebooks are available: here.
- **Bayer and Luetticke (2020)**. Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation.
  - Extends Reiter (2009) by applying a step that reduces the dimensionality of the model.
  - The codes are available in their website (Matlab, Python and Julia): https://www.ralphluetticke.com/.