Macroeconomics I

Overlapping Generations Model

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Introduction

- In many situations, assuming a representative agent is not ideal.
- One way to model heterogeneity is to include different “generations” in the economy:
  - Households do not live infinitely;
  - New households are born over time;
  - Old and young live in the same economic space.
- **New economic interactions**: decisions of the elderly affect the prices faced by younger generations.
- Basis of quantitative models for studying pensions, human capital, income/wealth inequality, etc. (N-generation models).
- We will study the analytical case: 2 generations.

What We Learn in this Chapter

- How to write and characterize the equilibrium of the 2-period overlapping generation model with production and endowment.
- Why equilibrium in the OLG model sometimes is inefficient.
- The effect of different types of pension system in the OLG model.

References

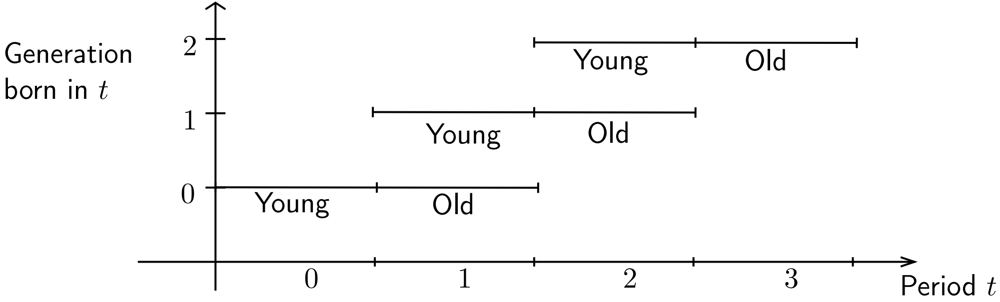
- Acemoglu Ch. 8.
- PhD macrobook Ch. 5 & Ch 6.
- Dirk Krueger notes Ch. 8.

Introduction

- Discrete time; agents live for 2 periods.
- A cohort (or generation) born in t lives for two periods $a = 1,\ 2.$ Their lifetime utility:

$$u(c_t^1, c_{t+1}^2) = u(c_t^1) + \beta u(c_{t+1}^2)$$

- In every period there exist two generations living simultaneously in the economy:



Introduction

Approach:

- **Samuelson:** pure exchange OLG.
  - Used to show the need for money in an inefficient endowment economy.
- **Diamond:** OLG with production.
  - Used to study capital accumulation and growth in an inefficient economy.

Most of our focus will be in the production model, but we should keep in mind the endowment economy for some applications.

Endowment Economy

- Discrete time; agents live for two periods.
- Utility follows the usual preferences (increasing, concave, bounded) and $\beta \in (0, 1)$.
- No population growth (yet).
- When young (period 1), they receive endowment $\omega_{y,t}$ and make consumption-savings decision.
- Denote the price of the bond a that pays one in the next period (used to save) as $q_t = 1/(1 + r_t)$.
- When old (period 2), they receive endowment $\omega_{o,t+1}$ and dissave.

Endowment Economy: Equilibrium

Definition: A sequential markets equilibrium is an allocation for the households $\{c_t^1, c_t^2, a_{t+1}\}_{t=1}^{\infty}$, and prices $\{q_t\}_{t=1}^{\infty}$ such that:

1. Given the prices, $\{c_t^1, c_t^2, a_{t+1}\}_{t=1}^{\infty}$ is the solution of:

$$\max_{c_t^1 \geq 0,\ c_{t+1}^2 \geq 0,\ a_{t+1}} u(c_t^1) + \beta u(c_{t+1}^2)$$

$$s.t. \quad c_t^1 + q_t a_{t+1} \leq \omega_{y,t}, \qquad \text{and} \qquad c_{t+1}^2 \leq \omega_{o,t+1} + a_{t+1}$$

   and the initial old only consumer their endowment $c_1^2 = \omega_{o,1}$.

2. Markets are in equilibrium (at all $t \geq 1$):

$$c_t^1 + c_t^2 = \omega_{y,t} + \omega_{o,t}$$
$$a_{t+1} = 0$$

Endowment Economy: Equilibrium Characterization

- Why $a_{t+1} = 0$? Think about it, in period 1, the initial old consume their endowment, market clearing implies that that the first young generation *must* consume their endowments as well.
  - This implies that $c_1^1 = \omega_{y,1}$ and thus $a_2 = 0$.
  - In the next period, the same thing happen again and again. Hence savings is zero in all periods.

- Fundamentally, if a cohort decides to borrow or save, the other cohort should be in the other side of the transaction.
  - The young cohort can only borrow or lend to the old cohort, since the next young have not been born yet.
  - But the old cohort will not lend to the young, since they will not be alive in the next period be paid back.
  - The young also will not lend to the old, since the old will not be alive to pay back the loan.

Endowment Economy: Euler Equation

- Solving the problem for the generation t:

$$\mathcal{L} = u(c_t^1) + \beta u(c_{t+1}^2) + \lambda_t(\omega_{y,t} + q_t \omega_{o,t+1} - c_t^1 - q_{t+1} c_{t+1}^2)$$

- Implies the usual Euler equation, and substituting consumption by the endowments:

$$q_t u'(\omega_{y,t}) = \beta u'(\omega_{o,t+1}), \quad \forall t$$

- Using a CRRA utility, we get:

$$q_t = \beta \left(\frac{\omega_{y,t}}{\omega_{o,t+1}} \right)^{\sigma}$$

Endowment Economy

$$q_t = \beta \left(\frac{\omega_{y,t}}{\omega_{o,t+1}} \right)^{\sigma}$$

- Note that the price of the bond (or equivalent, the interest rate) depends on the shape of the life-cycle income, even in the balanced growth path.
- It is even possible to have negative interest rates (i.e., $q > 1$) if $\omega_y > \omega_o$ enough.
  - When life-cycle endowments decline over time, consumption is more valuable in the future in the absence of being able to smooth income over time.
- In fact, it would be beneficial to transfer resources from the young to the old when $\omega_y > \omega_o$. The model equilibrium is not efficient!

Endowment Economy: Efficiency

- Suppose $\ln$ utility and $\beta = 1$. Then, $q_t = \omega_{y,t}/\omega_{o,t+1}$.
- Let's say: $\omega_{y,t} = 3$ and $\omega_{o,t+1} = 1$. So $q_t = 3$ and gross interest rate is $1/3$. Is there a redistribution that improves the situation for everybody?
- Yes! Take one unit of the good of the young and give to the old at every t, so $\omega_{y,t} = 2$ and $\omega_{o,t+1} = 2$.
  - Clearly, it is a feasible allocation: total endowment at t is still 4.
  - It improves the situation of the initial old: before he was consuming 1 unit, now she is consuming 3 units.
  - It also improves the situation for everybody else. Because $\ln$ is concave, the household prefers to smooth consumption: $\ln 2 + \ln 2 > \ln 3 + \ln 1$.
- So we have shown the equilibrium is not efficient!

Endowment Economy: Efficiency

- Why the competitive eq. is not Pareto optimal? Is it always like this?
- No. Consider the opposite as before: $\omega_{y,t} = 1$ and $\omega_{o,t+1} = 3$, so $q_t = 1/3$.
- A redistribution that makes $\omega_{y,t} = 2$ and $\omega_{o,t+1} = 2$ is **NOT Pareto improving for the initial old** (for everybody else it is).
- It turns out, in our simple economy with constant endowments, you can always find a transfer sequence that improves the situation for everybody whenever we have $q > 1$ (or negative net interest rates).

Endowment Economy: Efficiency

- In cases the economy does not have constant endowments, it also could have cases where is efficient and not efficient.

- Balasko and Shell (1980) provide the conditions for those general cases. The intuition lies if the net real interest rate converges to a positive or zero when $t \to \infty$.

- Why the economy is not efficient? Basically it is because some mathematical weirdness.

- When we prove the first welfare theorem, we assume a **finite** number of consumers in infinite time. In the OLG model, we have **infinite** set of cohorts of consumers in **infinite** time.
    - This is known as the double infinity problem. It is sort of technical, so if you are interested you can look at some Karl Shell's old papers.

OLG Model with production

Environment, Technology, and Preferences

- Discrete time; agents live for two periods.
- Utility follows the usual preferences (increasing, concave, bounded) and $\beta \in (0, 1)$.
- When young (period 1), they supply one unit of labor to the market and receive wage w_t. They decide consumption and savings.
- When old (period 2), they consume savings (earning interest at rate r_{t+1}).
- Population grows at rate n. Size of the generation born in period t:

$$L_t = (1+n)^t L_0$$

where L_0 is the size of the initial (old) generation.

Environment, Technology, and Preferences

- Production is represented by the aggregate production function with constant returns to scale

$$Y_t = F(K_t, L_t)$$

where $F(.)$ satisfies the usual assumptions (Inada, decreasing marginal returns).

- Define variables per worker: $y_t = Y_t/L_t$ and $k_t = K_t/L_t$:

$$y_t = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) \equiv f(k_t)$$

- Factor markets are competitive:

$$\begin{aligned} r_t =&\ F_k(K_t, L_t) = f'(k_t) \\ w_t =&\ F_l(K_t, L_t) = f(k_t) - k_t f'(k_t) \end{aligned}$$

Consumption Decision

- The problem of a generation born in $t \geq 1$:

$$\max_{c_t^1 \geq 0,\ c_{t+1}^2 \geq 0,\ s_t} u(c_t^1) + \beta u(c_{t+1}^2)$$

$$\begin{aligned} s.t. \quad & c_t^1 + s_t \leq w_t, \\ & c_{t+1}^2 \leq (1 + r_{t+1} - \delta) s_t \end{aligned}$$

- The initial old generation is born with initial capital k_1:

$$\max_{c_1^2 \geq 0} u(c_1^2)$$

$$s.t. \quad c_1^2 \leq (1 + r_1 - \delta) k_1,$$

Consumption Decision

Timing:

1. Beginning of period t: production occurs with the labor of the young and capital of the old. The young receives wage and the old receives interest.

2. End of period t: the young decides consumption and savings. The old consumes savings. Savings occur in the form of capital (the only asset in the economy).

3. Between t and $t+1$: The old dies, the young becomes old, and a new generation is born.

Euler Equation

- Solving the problem for the generation t:

$$\mathcal{L} = u(c_t^1) + \beta u(c_{t+1}^2) + \lambda_t \left(w_t - c_t^1 - \frac{c_{t+1}^2}{(1 + r_{t+1} - \delta)} \right)$$

- Implies the standard Euler equation:

$$u'(c_t^1) = \beta(1 + r_{t+1} - \delta) u'(c_{t+1}^2), \quad \forall t$$

- Substituting c_t^1 and c_{t+1}^2 by the budget constraints, we have an implicit function of savings (as a function of wage and interest rate):

$$s_t = s(w_t, r_{t+1}),$$

- where s is a increasing function in w_t and can be increasing or decreasing in r_{t+1} (depends on the utility).

Competitive Equilibrium

Definition: The sequential markets equilibrium is an allocation for the households $\{c_t^1, c_t^2, s_t\}_{t=1}^{\infty}$, allocation for the firm $\{K_t, L_t\}_{t=1}^{\infty}$ and prices $\{r_t, w_t\}_{t=1}^{\infty}$ such that:

1. Given the prices and k_1, $\{c_t^1, c_t^2, s_t\}_{t=1}^{\infty}$ is the solution of the household problem.
2. Given the prices, $\{K_t, L_t^d\}_{t=1}^{\infty}$ is the solution of the firm problem.
3. Markets are in equilibrium (at all $t \geq 1$):

$$
\begin{aligned}
&c_t^1 L_t + c_t^2 L_{t-1} + K_{t+1} - (1-\delta)K_t = F(K_t, L_t) \\
&K_{t+1} = S_t = s(w_t, r_{t+1})L_t \\
&L_t^d = L_t
\end{aligned}
$$

Note the timing: young generation's savings is tomorrow's capital.

Steady State

- **Steady State**: variables (c^1, c^2, s, k, r, w) are constant over time.
- Remember that $k_t = K_t/L_t$ can be constant over time, aggregate variables grow (via population growth).
- Does the economy have a unique steady state?

Characterizing the Equilibrium

- Using the equilibrium in the asset market, we find the law of motion of an OLG economy:

$$
\begin{aligned}
K_{t+1} &= s(w_t, r_{t+1})L_t \\
k_{t+1} &= \frac{s(w_t, r_{t+1})}{(1+n)} \\
k_{t+1} &= \frac{s(f(k_t) - k_t f'(k_t), f'(k_{t+1}))}{(1+n)}
\end{aligned}
$$

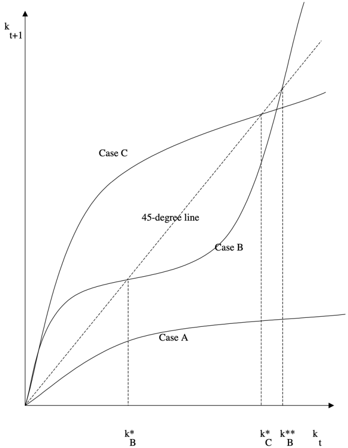
- In principle, we could start from k_1 and iterate to find the sequence of $\{k_{t+1}\}_{t=1}^{\infty}$ that converges to the steady state:

$$
k_{ss} = \frac{s(f(k_{ss}) - k_{ss} f'(k_{ss}), f'(k_{ss}))}{(1+n)}
$$

- **Problem:** Without clearly specifying the utility function there may exist multiple steady states.

Characterizing the Equilibrium

- It is possible to characterize dk_{t+1}/dk_t using the implicit function theorem.
- Depending on the functional forms there can be cases like:
  - No Steady State exists with $k > 0$.
  - A unique Steady State exists.
  - Multiple Steady States exist.



Source: Dirk Krueger's notes.

CRRA and Cobb-Douglas Case

- Consider: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $f(k) = k^{\alpha}$.
- Using the Euler Equation and substituting the budget constraint into it:

$$
\begin{aligned}
c_{t+1}^2 &= c_t^1(\beta(1+r_{t+1}-\delta))^{1/\sigma} \\
s_t(1+r_{t+1}-\delta) &= (w_t - s_t)\beta(1+r_{t+1}-\delta)^{1/\sigma} \\
s_t &= \frac{w_t}{1+\beta^{-1/\sigma}(1+r_{t+1}-\delta)^{-(1-\sigma)/\sigma}}
\end{aligned}
$$

  note that the denominator is > 1 (savings are always less than w_t).
- $\frac{\partial s_t}{\partial w_t} \in (0,1)$.
- $\frac{\partial s_t}{\partial r_{t+1}} < 0$ if $\sigma > 1$, $\frac{\partial s_t}{\partial r_{t+1}} > 0$ if $\sigma < 1$, and $\frac{\partial s_t}{\partial r_{t+1}} = 0$ if $\sigma = 1$.

CRRA and Cobb-Douglas Case

- Capital law of motion:

$$
\begin{aligned}
k_{t+1} &= \frac{s_t}{(1+n)} = \frac{w_t}{(1+n)[1+\beta^{-1/\sigma}(1+r_{t+1}-\delta)^{-(1-\sigma)/\sigma}]} \\
k_{t+1} &= \frac{f(k_t)-k_t f'(k_t)}{(1+n)[1+\beta^{-1/\sigma}(1+f'(k_{t+1})-\delta)^{-(1-\sigma)/\sigma}]} \\
k_{t+1} &= \frac{(1-\alpha)k_t^{\alpha}}{(1+n)[1+\beta^{-1/\sigma}(1+\alpha k_{t+1}^{\alpha-1}-\delta)^{-(1-\sigma)/\sigma}]}
\end{aligned}
$$

- It can be shown that this economy converges to a unique steady state with $k_{ss} > 0$ (ignoring the trivial case $k_0 = 0$).

Canonical Model

- In the special case where $u(c) = \ln c$ ($\sigma = 1$), the income and substitution effects of r_{t+1} on s_t cancel out, and:

$$s_t = \frac{\beta}{(1+\beta)} w_t \quad \text{and} \quad k_{t+1} = \frac{\beta(1-\alpha)k_t^\alpha}{(1+n)(1+\beta)},$$

  so the savings rate is a constant fraction of income (just like in the Solow model!).

- The steady state capital is:

$$k_{ss} = \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{1/(1-\alpha)}$$

- Interest rate

Dynamic Inefficiency

Dynamic Inefficiency

- As in the case of the endowment economy, the production OLG may not be Pareto efficient.
- We saw examples that if every young give to the current old generation, everybody would be better off.
- **Intuition**: some sort of non-existent market. There is no intergenerational lending.
- **Math**: again the double infinity problem.

Planner

- Let μ_t be the Pareto weight that the planner assigns to each generation.
- The planner's problem maximizes consumption utility (per capita):

$$\max_{\{c_t^1, c_t^2\}_{t=1}^{\infty}} \mu_0 u(c_1^2) + \sum_{t=1}^{\infty} \mu_t [u(c_t^1) + \beta u(c_{t+1}^2)]$$

$$\text{s.t.} \quad c_t^1 + \frac{c_t^2}{(1+n)} + (1+n)k_{t+1} - (1-\delta)k_t = f(k_t) \qquad \forall t.$$

- For the problem to be well-defined: $\sum_{t=1}^{\infty} \mu_t < \infty$.

Planner

- Lagrangian

$$
\begin{aligned}
\mathcal{L} = & \mu_0 u(c_1^2) + \sum_{t=1}^{\infty} \mu_t \left(u(c_t^1) + \beta u(c_{t+1}^2) \right) + \dots \\
& \dots \sum_{t=1}^{\infty} \lambda_t \left(f(k_t) + (1-\delta)k_t - c_t^1 - \frac{c_t^2}{(1+n)} - (1+n)k_{t+1} \right)
\end{aligned}
$$

- F.O.Cs for all t:

$$
\begin{aligned}
& \mu_t u'(c_t^1) = \lambda_t \\
& \mu_{t-1} \beta u'(c_t^2) = \lambda_t / (1+n) \\
& \lambda_{t+1}(1 - \delta + f'(k_{t+1})) = \lambda_t (1+n)
\end{aligned}
$$

Planner

- Optimality requires the Euler Equation:

$$\beta(1-\delta+f'(k_{t+1}))u'(c^2_{t+1}) = u'(c^1_t) \qquad \forall t \geq 1$$

  - exactly the same as the households' solution. The planner respects household optimality.

- The planner also have one equation that captures the trade-off of transferring resources between generations:

$$\mu_{t-1}(1+n)\beta u'(c^2_t) = \mu_t u'(c^1_t)$$
$$\mu_{t-1}(1+n)u'(c^1_{t-1}) = \mu_t u'(c^1_t)(1-\delta+f'(k_t)) \qquad \forall t \geq 1$$

- There's no similar condition in competitive equilibrium!

Golden Rule

- Suppose $\mu_t = \omega^t$, with $\omega < 1$, and in the steady state: $\mu_t / \mu_{t-1} = \omega$. Hence:

$$\omega(1 - \delta + f'(k_{ss})) = (1 + n)$$

- This condition is equivalent to the **Modified Golden Rule** of the neoclassical growth model.

- Note that if $\omega = \beta$, this would be equivalent to the discrete time Euler Equation of the neoclassical growth model (with population growth).

- In fact, the condition also says that the planner needs gross interest rates > 1 in the SS:

$$\frac{1 - \delta + f'(k_{ss})}{(1 + n)} = \frac{1}{\omega} > 1$$

Golden Rule

- The previous condition also have a strong connection to the **Golden Rule**.
- To get to the **Golden Rule** (capital that maximizes per capita consumption), we need to use the resource constraint:

$$\underbrace{c^1_{ss} + c^2_{ss}/(1+n)}_{c_{ss}} = f(k_{ss}) + (1-\delta)k_{ss} - (1+n)k_{ss}$$

$$\partial c_{ss}/\partial k_{ss} = 0 \Rightarrow (1 - \delta + f'(k^{GR}_{ss})) = (1+n)$$

- Since $\omega < 1$, the capital chosen by the planner is lower than that of the Golden Rule: $k_{ss} < k^{GR}_{ss}$ (just like in the neoclassical growth model!).
- Pareto Optimal Solution: $f'(k_{ss}) - \delta > n$

Dynamic Inefficiency

- When $f'(k_{ss}) - \delta < n$, we are in the case of **dynamic inefficiency**. This implies overaccumulation of k.
- Is this the case in the decentralized equilibrium?
- Recall the case with Cobb-Douglas and logarithmic utility:

$$k_{ss} = \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)} \right)^{1/(1-\alpha)},$$

- In other words, dynamic inefficiency occurs when:

$$\frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)} - \delta < n.$$

- It's not just a theoretical curiosity. For quite reasonable parameters, we are in the case of **dynamic inefficiency**.

Dynamic Inefficiency

- When we are in **dynamic inefficiency**, we can find a feasible allocation that improves the situation of all generations.
- Suppose: $r_{ss} - \delta < n$. Clearly, $k_{ss} > k_{ss}^{GR}$.
- In t, let's reduce k_{t+1} by $-\Delta k^*$ so that in $t+1, t+2...$ we are in a new SS (this is a feasible allocation).
  - Via resource constraint:

$$c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}$$

  - In t, total consumption, c_t, increases: $\Delta c_t = (1+n)\Delta k^*$.
  - In the new steady state, $\tau = t+1, t+2...$, total consumption increases:

$$\Delta c_\tau = -[\underbrace{f'(k^* - \Delta k^*) - (n+\delta)}_{<0}]\Delta k^* > 0$$

Social Security

Social Security

- One of the first applications of the OLG model is study Social Security.
- Theoretically, social security can prevent capital overaccumulation.
- Two types of Social Security Systems:
  1. **Fully Funded Pension**: the government opens an account in your name and forces savings that will finance your retirement in the future.
  2. **Pay-as-you-go**: the young generation pays the pensions of the current old generation.

Fully Funded System

- The government taxes the agent when young with a lump-sum tax τ, and pays them when old a pension $b = \tau(1 + r_{t+1} - \delta)$.

- Household's problem:

$$\max_{c_t^1 \geq 0,\ c_{t+1}^2 \geq 0,\ s_t} u(c_t^1) + \beta u(c_{t+1}^2)$$

$$\begin{aligned} s.t. \quad & c_t^1 + s_t \leq w_t - \tau, \\ & c_{t+1}^2 \leq (1 + r_{t+1} - \delta)(s_t + \tau) \end{aligned}$$

- In sum, the forced savings were allocated into productive capital. The equilibrium in the asset market: $s_t + \tau = (1 + n)k_{t+1}$

Fully Funded System

- Does it make any difference? Note that the intertemporal budget constraint remains the same:

$$c_t^1 + \frac{c_{t+1}^2}{(1 + r_{t+1} - \delta)} = w_t$$

- The **permanent income** (w_t) of the agent doesn't change, so the optimal allocation c^1 and c^2 is the same.

- If the government forces the household to save, they simply decrease private savings in the same proportion: public saving *crowds out* private saving.

- Asset market: $s_t + \tau = k_{t+1}(1 + n)$.

- Since any increase in τ has an equal decrease in s_t, the equilibrium remains the same!

Fully Funded System

- Prices do not change.
- Households' allocations do not change:
  - F.O.C. of the problem is the same.
  - Permanent income is the same.
- Equilibrium is the same and social security does not generate a Pareto improvement.
- Crucial assumptions for the irrelevance of social security to the equilibrium:
  1. The tax does not distort any individual decision.
  2. There are no restrictions on the choice of private saving s_t (i.e., borrowing constraints).
  3. The tax is the same in present value (τ is equal at t and $t+1$) or the tax generates the same returns as savings.

Pay-as-you-go System

- The government taxes the young with a lump-sum tax τ and pays the old a pension $b = \tau(1+n)$ for the old (and thus keeps the budget balanced).

- Household's problem:

$$\max_{c_t^1 \geq 0,\ c_{t+1}^2 \geq 0,\ s_t} u(c_t^1) + \beta u(c_{t+1}^2)$$

$$\begin{aligned} s.t. \quad & c_t^1 + s_t \leq w_t - \tau, \\ & c_{t+1}^2 \leq (1 + r_{t+1} - \delta)s_t + \tau(1+n) \end{aligned}$$

- From the agent's perspective, the system is a forced saving with return n.

- Unless $r_{t+1} - \delta = n$, the permanent income changes with social security.

Pay-as-you-go System

- The equilibrium usually changes as well.
  - The tax **IS NOT** allocated into productive capital, it is directly transferred to the older generation.
  - This means that the equilibrium condition in the asset market is: $s_t = (1+n)k_{t+1}$.

- Under normal conditions (i.e., without multiple equilibria), the introduction of pensions:
  - Reduces private savings, s_t, and capital k_{t+1}.
  - Increases the interest rate r_{t+1}.

- If we are in the case of **dynamic inefficiency**, private retirement may help achieve the Pareto optimal allocation.
  - On the other hand, if we are in an efficient allocation ($k < k^{GR}$), the disincentive to save may worsen the situation.

Taking Stock

- OLG model: realistic structure that forms the basis of more complex models: human capital, heterogeneity in income/wealth.

- Depending on the functional forms, there may not exist a steady state, just as there may exist multiple ones.

- The decentralized equilibrium may not be efficient (even with usual functional forms).

- We introduced social security in the OLG model. The type of pension system matters for the economy's capital accumulation.