Macroeconomics I

The Solow Model

Tomás R. Martinez

INSPER

What We Learn in This Chapter

- Some stylized facts about economic growth.
- The basics about production function in macro.
- How to solve the Solow Model.

References

- PhD Macrobook Ch. 2 and 3.
- Acemoglu Ch. 1 and 2.

Introduction

- We are going to build our first macroeconomic model.
- In particular, we will do a basic revision of the Solow model.
- The model is simple, but a lot of the intuition (specially about the production function) carry on to more complicated models.
- Where to start? Look at the data!

Stylized Facts about Economic Growth

Stylized Facts of Kaldor

1. GDP per capita grows at a constant rate over long periods of time.
2. The ratio K/Y has remained almost constant over long periods of time.
3. The growth rate of K/L is approximately constant over long periods of time.
4. The fractions of national income received by labor and capital are approximately constant over time.
5. The real interest rate, or return on capital, has remained stable (but wage is increasing!).
6. There is a large variation in the growth rate, ranging from 2-5%, among the world's fastest-growing countries.

See **Jones (2016)** for details or chapter 2 of the PhD macrobook for a review.

Stylized Facts of Kaldor

Figura: GDP per capita grows at a constant rate over long periods of time

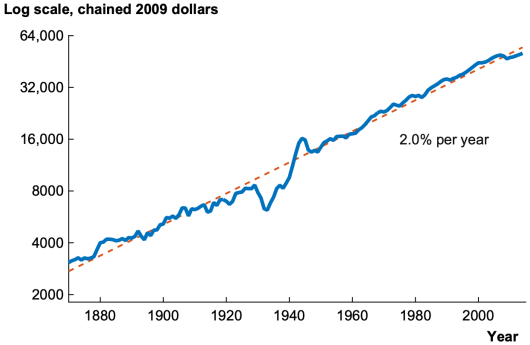


Fig. 1 GDP per person in the United States. Source: *Data for 1929–2014 are from the U.S. Bureau of Economic Analysis, NIPA table 7.1. Data before 1929 are spliced from Maddison, A. 2008. Statistics on world population, GDP and per capita GDP, 1-2006 AD. Downloaded on December 4, 2008 from http://www.ggdc.net/maddison/.*

Source: Jones (2016): The Facts of Economic Growth.

Stylized Facts of Kaldor

Figura: The ratio K/Y has remained almost constant over long periods of time

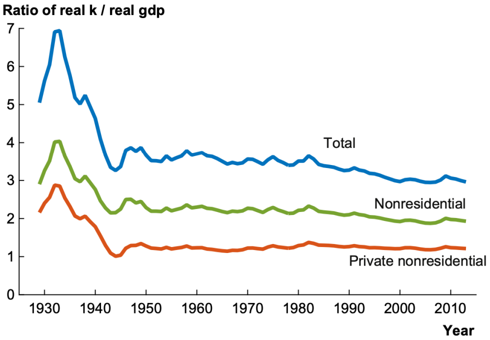


Fig. 3 The ratio of physical capital to GDP. Source: *Burea of Economic Analysis Fixed Assets tables 1.1 and 1.2. The numerator in each case is a different measure of the real stock of physical capital, while the denominator is real GDP.*

Source: Jones (2016): The Facts of Economic Growth.

Stylized Facts of Kaldor

Figura: The fractions of national income received by labor and capital are approximately constant over time

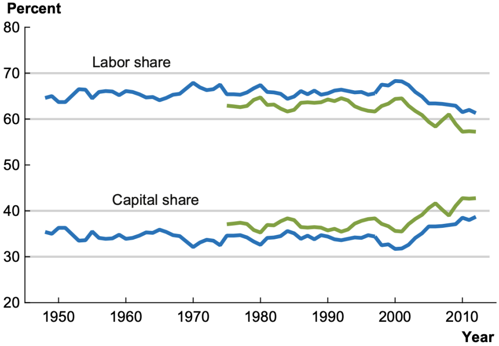


Fig. 6 Capital and labor shares of factor payments, United States. Source: *The series starting in 1975 are from Karabarbounis, L., Neiman, B. 2014. The global decline of the labor share. Q. J. Econ. 129 (1), 61–103. http://ideas.repec.org/a/oup/qjecon/v129y2014i1p61-103.html and measure the factor shares for the corporate sector, which the authors argue is helpful in eliminating issues related to self-employment. The series starting in 1948 is from the Bureau of Labor Statistics* Multifactor Productivity Trends, *August 21, 2014, for the private business sector. The factor shares add to 100%.*

Stylized Facts of Kaldor

Figura: Large variation in growth rate among countries

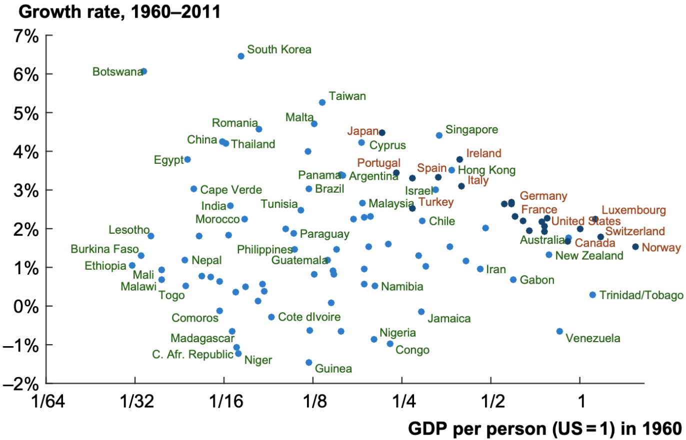


Fig. 26 The lack of convergence worldwide. Source: *The Penn World Tables 8.0.*

Other Stylized Facts

- Kaldor's facts are just the beginning, there are many other "stylized facts"about economic growth (which require more complex models).
  - Large differences in GDP per capita between countries.
  - (Non)-convergence between wealth levels (conditional or not).
  - Human capital: increase in education.
  - Hours worked: reduction in working hours.
  - Structural transformation: Agriculture $\rightarrow$ Industry $\rightarrow$ Services.
  - Inequality, fertility, health.
- Some "facts"are being challenged! (e.g. is the labor income share really constant?)
- See Jones (2016) and Jones and Romer (2010).

Can We Explain Kaldor's Facts with Solow?

- **Solow**: We can explain the facts using an **aggregate** production function subject to some technological changes.
- Key assumption lies on constant returns to scale of production + perfect competitive markets.
- Using a production function of the form $Y = Z_t F(K_t, L_t)$ or, alternatively, $Y = F(K_t, A_t L_t)$, Solow decomposed total growth in

$$\underbrace{Z_t}_{\text{Solow Residual}} \quad F(\quad \underbrace{K_t}_{\text{Capital contribution}} \quad , \quad \underbrace{L_t}_{\text{Labor contribution}} \quad)$$

- This methodology became known as **Growth Accounting**.

Solow Model: Production Function

Aggregate Production Function

$$Y_t = F(K_t, L_t, A_t) = F(K_t, A_t L_t) \tag{1}$$

Where: K: Capital; L: Labor; A: Technological shifter, biased or not towards a certain factor.

(Typical) Neoclassical Assumptions:

(i) $F : \mathbb{R}^3_+ \rightarrow \mathbb{R}_+$ is twice differentiable, strictly increasing and concave in K and L.

- $F_L > 0$; $F_L > 0$;
- $F_L < 0$; $F_L < 0$ (diminishing marginal returns)

(ii) F exhibits constant returns to scale in K and L

- F is homogeneous of degree 1: $cF(K, L, A) = F(cL, cL, A)$ for any $c > 0$.

(iii) Inada conditions.

Solow Model: Production Function

Inada Conditions

F satisfies:

$$\lim_{K \to 0} F_K(K, L, A) = \infty \qquad \lim_{K \to \infty} F_K(K, L, A) = 0$$

$$\lim_{L \to 0} F_L(K, L, A) = \infty \qquad \lim_{L \to \infty} F_L(K, L, A) = 0$$

For all $N > 0$, $K > 0$.

- Also assume: $F(0, L, A) = 0$.

Technology and Production in Macro Models

- In most macro models (but not all!) we assume:
  - there exists a representative firm with a representative (or aggregate) production function that produces a single final good;
  - profits (when they exist) are redistributed equally to all households;
  - it uses capital, K, and/or labor, L, as inputs. These inputs are hired in the spot market (firm's problem is static).

- **Intuition**: The aggregate production function represents the total value added (GDP) of the economy and not a specific good.
  - Capital and labor are inputs used by all sectors in greater or lesser proportion.
  - Intermediate goods are not included, and the cost of basic materials is not quantitatively relevant.
  - In some special case, we could include intermediates M: $F(K, L, M)$. In this case, F represents gross production (not value added).

Technology and Production in Macro Models

- With a single final good Y, if
  - markets are competitive;
  - there are no externalities in production;

  the economy admits a representative firm (even if the firms in the economy have heterogeneous production functions).

- **Theorem and proof**: Acemoglu p. 158.

- Even with multiple goods/sectors, if the production function is the same and factors are completely mobile between sectors, it is possible to show that there is an aggregate function.

Back to the Solow Model

- Households save a fraction $s \in (0,1)$ of their available income $\Rightarrow S_t = sY_t$.

- Productivity, A, grows at rate g. For simplicity, population growth is zero, $n = 0$.

- Capital accumulation follows the motion law:

$$K_{t+1} = K_t(1-\delta) + I_t$$

  where I_t is aggregate investment, and $\delta \in (0,1]$ is the depreciation rate of capital.

- Closed economy without government, thus: $Y_t = C_t + I_t$. In equilibrium, we also have: $S_t = I_t$

Back to the Solow Model

- Households save a fraction $s \in (0,1)$ of their available income $\Rightarrow S_t = sY_t$.
- Productivity, A, grows at rate g. For simplicity, population growth is zero, $n = 0$.
- Capital accumulation follows the motion law:

$$K_{t+1} = K_t(1-\delta) + I_t$$

  where I_t is aggregate investment, and $\delta \in (0,1]$ is the depreciation rate of capital.
- Closed economy without government, thus: $Y_t = C_t + I_t$.
- In equilibrium, we also have: $S_t = I_t$

Solow Model

- **Notation:**
  - Upper case means aggregates (i.e., Y), lower case means per capita, $y \equiv Y/L$.
  - Lower case with tildes means per capita and per efficient labor unit: $\tilde{y} \equiv Y/(AL)$

- Using the fact that the production function is CRS:

$$\tilde{y}_t = \frac{F(K_t, A_t L_t)}{A_t L_t} = F(\tilde{k}_t, 1) = f(\tilde{k}_t) \tag{2}$$

- Then, the capital law motion:

$$K_{t+1} = K_t(1-\delta) + S_t = K_t(1-\delta) + sY_t =$$
$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}}{A_t} \frac{L_{t+1}}{L_t} = \tilde{k}_{t+1}(1+g) = \tilde{k}_t(1-\delta) + sf(\tilde{k}_t)$$

- The **first-order difference equation** describes the equilibrium dynamics of the main object of the model: $\tilde{k}_t$.

Steady State

- We define the **Steady State** as the situation where all the aggregate variables are constant over time: $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}_{ss}$.

$$\tilde{k}_{ss}(1+g) = \tilde{k}_{ss}(1-\delta) + sf(\tilde{k}_{ss}) \tag{3}$$

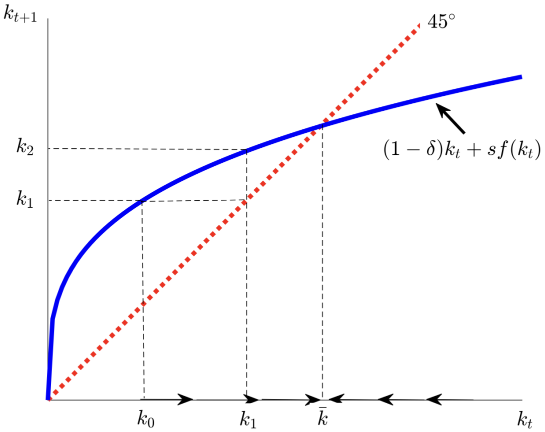
- Assuming a Cobb-Douglas production function: $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$ and $f(k) = k^{\alpha}$.

- Then, steady state capital is:

$$\tilde{k}_{ss} = \left(\frac{s}{\delta + g}\right)^{\frac{1}{1-\alpha}} \tag{4}$$

Solow Model: Dynamics

- Given usual conditions: $\alpha, \delta, s \in (0,1)$, it can be shown that the model has a unique non-zero steady state and that the difference equation converges to $\tilde{k}_{ss}$ for all initial $\tilde{k}_t > 0$.



Solow Model: Dynamics

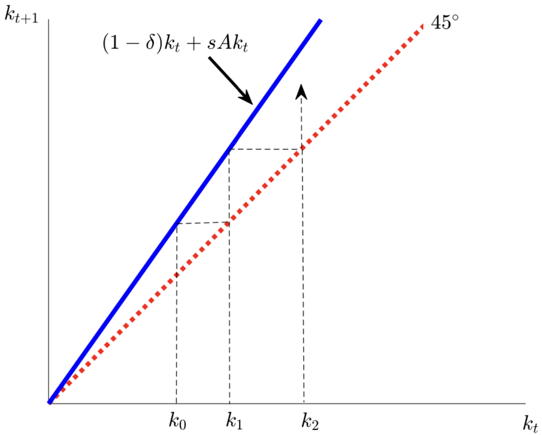
- There are two forces changing the capital stock: investment increases the capital stock, while depreciation decreases.

$$\tilde{k}_{t+1}(1+g) - k_t = \underbrace{i_t}_{sf(\tilde{k}_t)} - \tilde{k}_t\delta$$

- Since $f'(k_t) > 0$ and $f''(k_t) < 0$, the investment effect decreases as k_t increases.

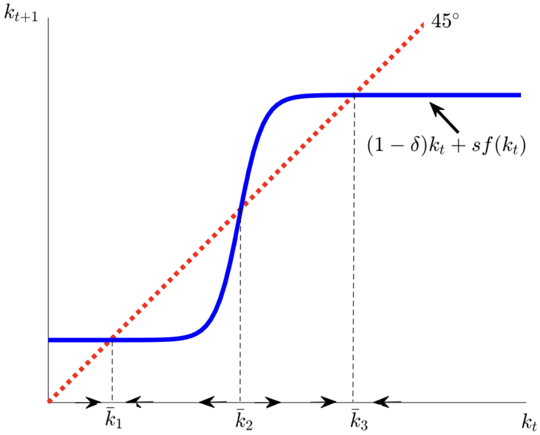
- Clearly, the shape of the production function plays a very important role.

Unbounded (Endogenous) Growth



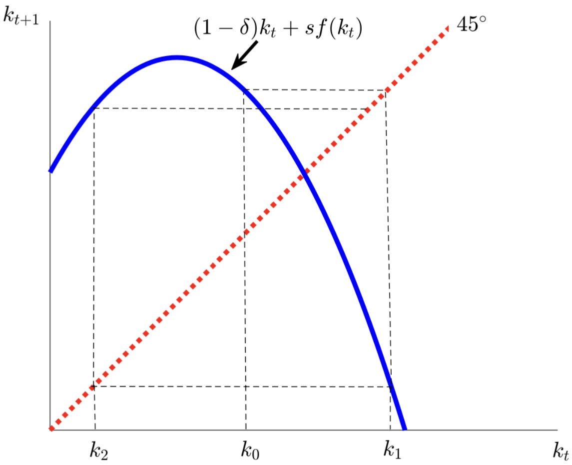
Source: PhD Macrobook.

Poverty Traps



Source: PhD Macrobook.

Chaotic Dynamics



^

Source: PhD Macrobook.

Factor Prices

- In competitive markets, the representative firm solves the following problem:

$$\pi_t = \max_{L_t \geq 0, K_t \geq 0} F(K_t, A_t L_t) - r_t K_t - w_t L_t,$$

where r_t is the return on capital and w_t is the equilibrium wage.

- The solution to the problem implies that the factor prices are given by their marginal products:

$$\begin{aligned} r =& F_K(K, AL) = \alpha (K/AL)^{(\alpha-1)} = \alpha \tilde{k}^{(\alpha-1)} \\ w =& F_L(K, AL) = (1-\alpha) A (K/AL)^{\alpha} = (1-\alpha) A \tilde{k}^{\alpha} \end{aligned}$$

- Where in the last equality, we assume that F is Cobb-Douglas

Digression: Euler's Theorem

- Note that **Constant Returns to Scale** implies a useful property coming from the Euler's Theorem:

Theorem (Euler's Theorem)

Suppose $f : \mathbb{R}^{K+2} \to \mathbb{R}$ *is differentiable in* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$*, with partial derivatives* f_x *and* f_y*, and is homogeneous of degree* m*. Then:*

$$mf(x,y) = f_x(x,y)x + f_y(x,y)y \quad \text{for all } x \in \mathbb{R}\,, y \in \mathbb{R}. \tag{5}$$

Furthermore, f_x *and* f_y *are homogeneous of degree* $m-1$ *in* x *and* y*.*

- Thus, CRS combined with competitive equilibrium (price equals marginal product) implies zero profits for firms.
- In particular, this implies *Production = Total Factor Income*: $Y_t = F(K_t, A_tL_t) = r_tK_t + w_tL_t$.

Can We Explain Kaldor's Facts with Solow?

- Dividing the production function by Y^{α}:

$$Y^{1-\alpha} = \frac{K^{\alpha}(AL)^{1-\alpha}}{Y^{\alpha}} \Leftrightarrow \frac{Y}{L} = A\left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)}$$

- If K/Y is constant (fact 2), then the growth rate of GDP per capita grows at a constant rate g in the balanced growth path (fact 1).
- Note that: $Y/L = A^{1-\alpha}(K/L)^{\alpha}$. This implies:

$$\frac{K}{L} = A\left(\frac{K}{Y}\right)^{1/(1-\alpha)}$$

- K/L grows at the same constant rate as Y/L (fact 3).

Can We Explain Kaldor's Facts with Solow?

- Note that in the steady state:

$$\frac{K}{Y} = \frac{\tilde{k}}{\tilde{y}} = \tilde{k}_{ss}^{1-\alpha} = \frac{s}{g+\delta}$$

- Therefore, in the steady state, K/Y is constant and the Solow model explains stylized facts 1, 2, 3.

- Using the factor prices equation:

$$r = \alpha(K/AL)^{(\alpha-1)} = \alpha(\tilde{k}_{ss})^{(\alpha-1)},$$

we have that r is constant over time (fact 5) (while w grows because of A!).

Stylized Facts of Kaldor

- The fractions of capital and labor income are constant (fact 4):

$$\frac{wL}{Y} = \frac{(1-\alpha)A(K/AL)^{\alpha}L}{K^{\alpha}(AL)^{1-\alpha}} = 1-\alpha$$
$$\frac{rK}{Y} = \frac{\alpha(K/AL)^{\alpha-1}K}{AK^{\alpha}(AL)^{1-\alpha}} = \alpha$$

- Finally, note that the Solow model implies convergence (fact 6).
  - Countries with lower capital per capita (and consequently lower GDP per capita) grow at a higher rate than countries that are in the steady state.

Convergence

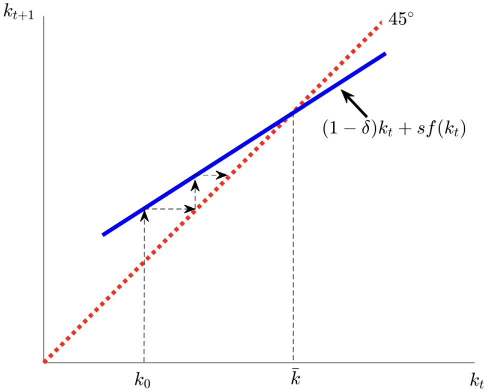
- Set $g = 0$ for simplicity. Approximate the Capital LOM around the steady state (i.e., take a Taylor expansion around $k_s s$):

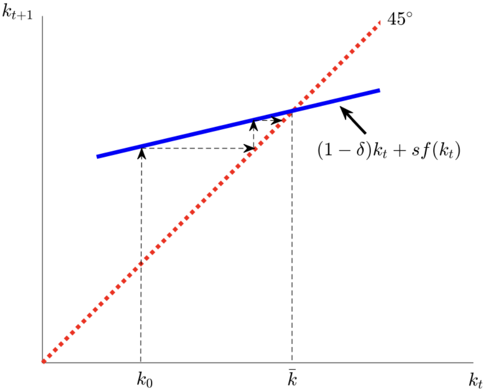
$$\Delta k_{t+1} = (1 - \underbrace{\delta + sf'(k_{ss})}_{\lambda})\Delta k_t$$

where $\Delta k_{t+1} = k_{t+1} - k_{ss}$ and λ represents the **convergence speed**.

- Large λ implies that Δk_{t+1} becomes smaller more quickly, implying faster convergence.
- In particular, Cobb-douglas production + k_{ss} implies: $\lambda = \delta(1 - \alpha)$.
- Convergence is faster when α is small and δ is large.

Slow and Fast Convergence





Source: PhD Macrobook.

Stylized Facts of Kaldor

- The Solow model can explain the main stylized facts about economic growth.
- But it is still subject to Lucas's critique! Exogenous constant saving s is a strong assumption.
  - If there is a change in fiscal policy, households are unlikely to maintain the same savings rate.
- To conduct more complex analyses, we need to endogenize household savings.
  - **Neoclassical growth model**: Basic Solow structure + endogenous capital accumulation.