Quantitative Macroeconomics

Bewley-Huggett-Aiyagari-Imrohoroglu Model

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UnB

References

- Ljunqvist-Sargent (Ch. 17 and some parts of Ch. 16): Textbook treatment.
- Aiyagari (1994), Hugget (1993): Original papers. Relatively easy to follow.
- Guvenen (2011, Macroeconomics with Heterogeneity: A Practical Guide): Comprehensive review starting from incomplete markets to model and extensions.
- Heathcote, Storesletten & Violante (2009, Annual Review): Overview paper without equations.

This lecture is a mix of the first two bullet points.

Introduction

Goal:

- Present the canonical dynamic general equilibrium model of incomplete markets with household heterogeneity.
- The framework is used to analyze questions such as:
  - How much of the wealth inequality can be explained by earnings variation across agents?
  - What are the distributional implications of various fiscal policies? How are inequality and welfare affected by such policies?
  - What are the macroeconomic consequences of this heterogeneity in aggregate variables and prices?
- We focus on the stationary equilibrium, the equilibria with constant prices through time.

Introduction

Model Ingredients:

- Typical consumption-savings problem in Infinite horizon.

- Two important features:

  1. **Idiosyncratic Shocks**: Individuals receive exogenous "income shocks": e.g., unemployment shocks, promotions, etc.
  2. **Incomplete Markets**: They cannot trade assets (there is no way to buy insurance in the market).

- There is **NO** aggregate uncertainty.

Introduction

- Individuals are *ex-ante* homogeneous $\Rightarrow$ before birth their expected lifetime utility is the same.
- ...but will be *ex-post* heterogeneous!
- Exogenous earnings distribution, but endogenous wealth distribution.
- **Intuition:**
  - Lucky individuals that receive a sequence of high-income shocks will accumulate assets to insure themselves against future low-income;
  - Unlucky individuals that receive bad shocks will have no assets;
  - Equilibrium will feature a non-degenerate stationary wealth distribution.

Road Map

To fully solve the model, we go through three building blocks:

1. The household consumption-savings problem (asset supply function);
    - Solve the household problem;
    - Solve for the endogenous stationary distribution;
    - Use the distribution and the HH decisions to get the aggregate asset supply.
2. Asset demand function;
    - It can be from the aggregate production function (e.g., firms) or government;
3. Finally, find the equilibrium in the asset market;

Model

Individual's Problem

- Discrete time, infinite horizon, future utility is discounted by $\beta \in (0, 1)$.
- Continuum of individuals with unitary mass.
- Earnings are given by $w_t s_t$, where w_t is the market wage and s_t is a labor endowment, which is idiosyncratic and follows a Markov chain with transition probabilities:

$$\pi(s', \ s) = Pr(s_{t+1} = s' | \ s_t = s). \tag{1}$$

- The individual supplies labor inelastically. The per period utility function is given by: $u(c_t)$, where $u' > 0$, $u'' < 0$ and $c_t \geq 0$.

Individual's Problem

- Agents only have access to a riskless bond that pays an interest rate r.
  - No access to a full set of state-contigent Arrrow-securities. This is the incomplete market.
- They can save and borrow, but there is a borrowing constraint b.
- Full individual problem:

$$
\begin{aligned}
\max_{c_t,\ a_{t+1}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} \quad & c_t + a_{t+1} = w_t s_t + a_t(1 + r_t), \\
& a_{t+1} \geq -\phi \quad \text{and} \quad c_t \geq 0 \quad \text{for } t = 0, 1, ..., \infty \\
& a_0 \text{ is given.}
\end{aligned}
$$

- We will look for a stationary equilibrium so ignore time subscripts in prices for a moment (more on that later).

Borrowing Constraint

- The borrowing constraint can be set exogenously or be bounded by the **natural debt limit**.
- The natural debt limit is the maximum borrowing that the household can pay back (if $c_t = 0$ and s_{min} in all periods).
- Iterating forward:

$$c_t = ws_t + a_t(1+r) - a_{t+1} \geq 0 \Rightarrow a_t \geq -\frac{ws_t}{1+r} + \frac{a_{t+1}}{1+r}$$

$$a_t \geq -\frac{ws_t}{1+r} + \frac{a_{t+1}}{1+r} \geq -\frac{ws_t}{1+r} + \frac{1}{1+r}\left(-\frac{ws_{t+1}}{1+r} + \frac{a_{t+2}}{1+r}\right) \geq \ldots$$

$$a_t \geq -\left(\frac{1}{1+r}\right)\sum_{j=0}^{\infty}\frac{ws_{t+j}}{1+r}$$

note that because $r > 0$ and a_{t+j} bounded, the limit of $a_T/(1+r)^T$ goes to zero as $T \to \infty$.

Borrowing Constraint

- The worst case scenario is when the agent receives the lowest realization in every $t+j$: $s_{min} = s_{t+j}$. Substituting and we get the natural debt limit:

$$a_t \geq \frac{w s_{min}}{r}. \tag{2}$$

- **Inada Conditions**: with Inada conditions ($u(0) = -\infty$), the consumer will never borrow up to the natural debt limit since this implies zero consumption.
- That is NOT true with ad-hoc borrowing limits above the natural one!
- Let us now consider the possibility that the borrowing constraint can bind.

Consumption-Savings Problem

- Consider the Karush-Kuhn-Tucker of the consumption-savings problem and let μ_t be the multiplier of the borrowing constraint.

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \{ \beta^t u(c_t) + \lambda_t (w s_t + a_t(1+r) - c_t - a_{t+1}) + \mu_t (a_{t+1} + \phi) \}$$

  with KKT conditions $\mu_t \geq 0$ and $\mu_t(a_{t+1} + \phi) = 0$.

- The solution implies the Euler Equation for all t:

$$u'(c_t) = \beta(1+r)\mathbb{E}_t[u'(c_{t+1})] + \mu_t$$

- If the constraint does not bind, $a_{t+1} > -\phi \Rightarrow \mu_t = 0$, we have the standard Euler Equation.

Consumption-Savings Problem

- If the borrowing constraint is binding, $a_{t+1} = -\phi$ and $\mu_t > 0$:

$$u'(c_t) > \beta(1+r)\mathbb{E}_t[u'(c_{t+1})].$$

- That means marginal utility of consumption at t is too high (i.e., c_t is too low). The household would like to consume more and smooth consumption but cannot do it.
- In this case, the household will just consume everything and hope for a higher income in the future.
- This situation may arise if the household is too poor (low wealth or low income) and/or the borrowing constraint is too tight. **Aiyagari (1994)** summarizes in a figure.

Consumption-Savings Problem

- Define $\hat{a}_t = a_t + \phi$ with $\hat{a}_t \geq 0$, and the total resources available z_t as

$$z_t = ws_t + \hat{a}_t(1+r) - r\phi$$

  with the associated budget constraint: $c_t + \hat{a}_{t+1} = z_t$.

- Let $\hat{a}_{t+1} = g_a(z_t, s_t)$ be the policy function that characterizes the solution of the problem.
- There will be a cutoff $\hat{z}(s_t)$, such that if $z_t(s_t) \leq \hat{z}(s_t)$, it will be optimal to consume all their resources ($\hat{a}_t = 0$)!
  - Note that in Aiyagari's original paper s_t is iid so $\hat{z}$ does not depend on s_t.

Policy Functions

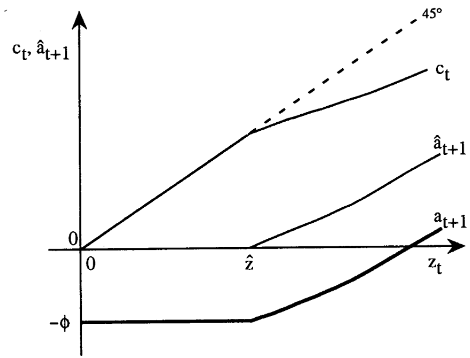

FIGURE Ia
Consumption and Assets as Functions of Total Resources

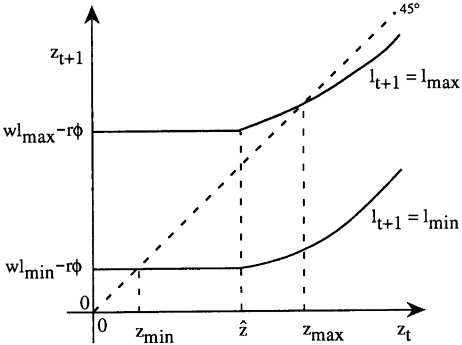

FIGURE Ib
Evolution of Total Resources

Source: Aiyagari (1994). Note: $l_t \equiv s_t$.

Consumption-Savings Problem

- Recall the EE: $u'(c_t) = \beta(1+r)\mathbb{E}_t[u'(c_{t+1})]$.
- What else can we say about optimal savings?
- Three reasons:
    1. Intertemporal substitution: β vs $(1+r)$.
    2. Consumption smoothing: desire of smoothing out contemporaneous income shocks.
    3. Precautionary savings: insurance against future shocks.
- If there is no uncertainty only **1.** is present; with uncertainty **2.** is present, but **3.** depends on the $u()$ or whether the borrowing constraint can bind.

Precautionary Savings

- Suppose 2 periods, $\beta(1+r) = 1$ and $s_1 = \overline{s}$ (deterministic).

$$u'(a_0(1+r) + ws_0 - a_1) = u'(a_1(1+r) + w\overline{s} - a_2)$$

- Only 2 periods: $a_2 = 0$.
- Suppose $s = \overline{s} + \varepsilon$, where $\varepsilon \sim G(\sigma)$ with mean zero and variance σ.
- How the savings behavior changed with the increase in risk?
- If the marginal utility is convex, $u'''(c) > 0$, by Jensen's inequality:

$$\mathbb{E}[u'(a_1(1+r) + w\overline{s} + w\varepsilon)] > u'(a_1(1+r) + w\overline{s})$$

- If the marginal utility is convex, increase in uncertainty implies precautionary savings!

Savings: Risk aversion vs Prudence

- Risk aversion: curvature of $u() \Rightarrow$ consumption smoothing!
- Prudence: curvature of marginal utility $u'() \Rightarrow$ precautionary savings!
- **Example 1:** CRRA: $u'' < 0$ (risk aversion) e $u''' > 0$ (prudence).
- **Example 2:** Quadratic utility:

$$u(c) = -\frac{1}{2}(\bar{c} - c)^2$$

- $u'' < 0$ (risk aversion) but $u''' = 0 \rightarrow$ no prudence!

Precautionary Savings: Borrowing Constraint

- Suppose there is a non-zero probability that in $t+1$ the borrowing constraint will bind.
  - In this case, the individual will NOT be able to smooth consumption.

$$u'(c_t) = \beta(1+r)\mathbb{E}_t[u'(c_{t+1})]$$

- Even if the borrowing constraint cannot bind in $t+1$, it may bind in the future.
  - Precautionary savings depends on how likely the constraint binds (how tight ϕ is, the stochastic process of s_t, etc).

- This motive is present even if $u()$ does not have prudence (quadratic utility).

Consumption-Savings

- To solve the full consumption-savings problem, we can use standard dynamic programming techniques.

- The Bellman equation:

$$V(a,s) = \max_{a' \geq -\phi} \{u((1+r)a + ws - a') + \beta \sum_{s'} \pi(s',s) V(a',s')\}$$

  with the associated policy function $a' = g_a(a,s)$ ($c = g_c(a,s)$ is recovered using the budget constraint).

- Like Aiyagari, if s is iid we can also use a cash-on-hand formulation.

From Partial to General Equilibrium

- At this point, we have taken w and r as given and solved the **partial equilibrium** problem of the consumer.
- Now, we proceed to solve the general equilibrium: we must find the r such that the asset market clears.
- We focus on the **stationary equilibrium**: the aggregates such as total assets, and prices will be constant over time, but the individuals will move up or down the earnings and wealth distribution!
- The equilibrium will feature a **stationary distribution**: a time-invariant distribution that will replicate itself every period.

Stationary Distribution

- The household is characterized by their pair (a, s). Let the joint distribution of types be $\lambda_t(a, s) = Pr(a_t = a, s_t = s)$.

- Given the distribution of agents $\lambda_t(a, s)$, how can we find $\lambda_{t+1}(a, s)$?

- Let $Q((a, s), \mathcal{A} \times \mathcal{S}))$ be the probability that a household with state (a, s) transits to the set $\mathcal{A} \times \mathcal{S}$:

$$Q((a, s), \mathcal{A} \times \mathcal{S})) = \mathcal{I}\{g_a(a, s) \in \mathcal{A}\} \sum_{s' \in \mathcal{S}} \pi(s', s)$$

  where $\mathcal{I}$ is an indicator function.

- Intuitively, a household (a, s) moves to the next state according to the optimal policy function and the exogenous Markov chain.

Stationary Distribution

- To get the next period distribution, we just need to apply the transition function Q to all the points of the distribution:

$$\lambda_{t+1}(\mathcal{A} \times \mathcal{S}) = \int_{A \times S} Q((a, s), \mathcal{A} \times \mathcal{S}) d\lambda_t$$

- The stationary distribution is the distribution that replicates itself for all $(a, s) \in A \times S$: $\lambda(a, s) = \lambda_t(a, s) = \lambda_{t+1}(a, s)$.

- **Intuition**: if we discretize the asset space, Q can be interpreted as a transition probability matrix of a Markov chain with state-space $A \times S$.
  - under certain conditions, the Markov chain admits a unique stationary distribution.

Stationary Distribution

Interpretation of the stationary distribution:

- The fraction of time that an infinitely lived agent spends in the state (a, s).
- Fraction of households in the state (a, s) in a given period in the stationary equilibrium.
- The initial *distribution* of agents remains constant over time even though the state of the individual household is a stochastic process.

Equilibrium

- To close the model, we must define other agents that can demand the assets in the economy:
  - **Hugget (1993)**: Credit economy. Some agents borrow, others will lend. The loan market clears when aggregate demand for loans is zero.

$$\int_{A\times S} g_a(a,s)d\lambda = 0$$

  - **Aiyagari (1994)**: Production economy. Firms demand capital to produce. Market clears when household savings equalize capital demand.

$$\int_{A\times S} g_a(a,s)d\lambda = K$$

- We follow Aiyagari (1994) and assume an aggregate production function.

Firms

- Let the production function be $Y = F(K, N) = K^{\alpha} N^{1-\alpha}$, where $\alpha \in (0, 1)$.
- Capital depreciates at rate δ.
- Markets are competitive and the solution of the firm problem is standard (t is omitted):

$$w = \frac{\partial F(K, N)}{\partial N} = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$
$$r + \delta = \frac{\partial F(K, N)}{\partial K} = \alpha \left(\frac{K}{N}\right)^{-(1-\alpha)}$$

- $\uparrow r \quad \Leftrightarrow \quad \downarrow K/N \quad \Leftrightarrow \quad \downarrow w$.

Equilibrium

- Notice that labor supply is inelastic, so aggregate labor is given by the sum of all labor endowments in the economy.
- Let $\Pi(s)$ be the invariant distribution of the Markov chain. Aggregate labor supply is:

$$N_t = \sum_i s_i \Pi(s_i)$$

- **Example**: two state Markov chain with $s_1 = 1$, $s_2 = 2$ and symmetric transition matrix. $N_t = 1 \times 0.5 + 2 \times 0.5 = 1.5$.

Equilibrium Definition

A **stationary recursive competitive equilibrium** is a value function V; policy functions for the household g_a and g_c; firm's choice K and N; prices w and r; and, a stationary distribution λ such that:

1. Given prices, the V, g_a, and g_c solve the household problem.
2. Given prices, K and N solves the firm's problem:
3. Given the transition function Q, the stationary distribution satisfies:

$$\lambda(\mathcal{A} \times \mathcal{S}) = \int_{A \times S} Q((a, s), \mathcal{A} \times \mathcal{S})) d\lambda$$

4. The labor market clears: $N_t = \sum_i s_i \Pi(s_i)$.
5. The asset market clears: $\int_{A \times S} g_a(a, s) d\lambda = K$.
6. The goods market clears: $\int_{A \times S} g_c(a, s) d\lambda + \delta K = F(K, N)$.

Existence of Equilibrium

- Focus on the asset market: with Cobb-Douglas, it is easy to see that wage is just a function of r.

- To find an equilibrium, we must show that the excess demand function intersects at zero.
  - Technically, we need to show that is continuous and strictly monotone.

- **Capital Demand**: from the firm's problem, capital demand is

$$K(r) = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} N,$$

  if $r \to -\delta \Rightarrow K \to +\infty$; if $r \to +\infty \Rightarrow K \to 0$.

Existence of Equilibrium

- **Asset Supply**: denote the average level of assets as

$$\mathbb{E}a(r) = \int_{A \times S} g_a(a, s; r) d\lambda(a, s; r).$$

- The asset supply is bounded above by: $(1 + r)\beta = 1$.
    - Intuitively, $(1 + r)\beta = 1$ is the complete markets/nonstochastic steady state equilibrium.
    - Because of precautionary savings, for a given r, the asset accumulation must always be higher than the certainty case.
    - With uncertainty, If $(1 + r)\beta = 1$, the agent will accumulate assets to $+\infty$.
    - See Ljungqvist and Sargent for the full argument.

$$r \to \frac{1}{\beta} - 1 \Rightarrow \mathbb{E}a(r) \to +\infty.$$

Existence of Equilibrium

- **Asset Supply**:

$$\mathbb{E}a(r) = \int_{A \times S} g_a(a, s; r) d\lambda(a, s; r).$$

- It is bounded below by $r = -1$. In this case, all households borrow up to the constraint ϕ.
- Using boring dynamic programming arguments one can also show that $\mathbb{E}a(r)$ is continuous.
- However, may not be monotone because w is a function of r and it is very hard to assess what r does to λ.

General Equilibrium

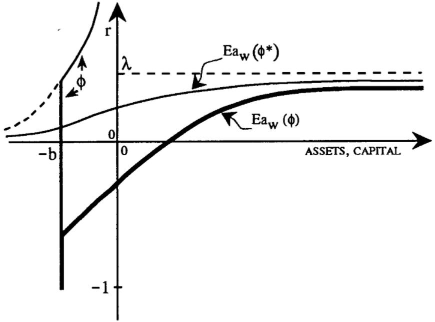

FIGURE IIa
Interest Rate versus Per Capita Assets

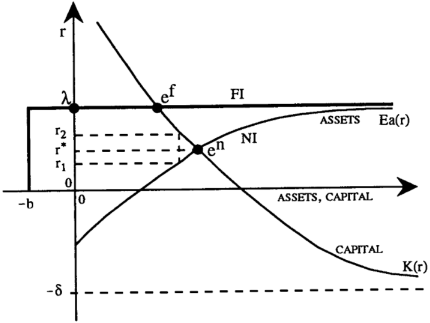

FIGURE IIb
Steady-State Determination

Source: Aiyagari (1994). Note: $\lambda \equiv \frac{1}{\beta} - 1$; $\phi^* \equiv \frac{w s_{min}}{r} > \phi$.

General Equilibrium

- In general equilibrium, r is determined endogenously by: $\mathbb{E}a(r) = K(r)$.

- Because of precautionary savings, aggregate savings will be higher than the case of certainty (and r will be lower).

- The tightness of the borrowing constraint, ϕ, is important. If agents are not allowed to borrow, precautionary savings will be higher and r will be even lower.

Quantitative Exploration

Calibration

- Most of the calibration is standard: $\alpha = 0.33$, utility is CRRA with γ between 1 and 5, $\delta = 0.08$.
- The labor endowment is an estimated AR(1) from a panel-data on labor income:

  $$\log s_t = \rho \log s_{t-1} + \sigma \varepsilon_t,$$

  where $\rho \in (-1, 1)$ and $\varepsilon \sim N(0, 1)$. Then discretize to a discrete Markov chain (more on that later).
- β is calibrated using information on the aggregate wealth-to-income ratio (K/Y).
- ϕ is calibrated to the fraction of agents with negative wealth (or assumed $\phi = 0$).

Quantitative Exploration

Aiyagari (1994):

- For reasonably calibrate parameters the differences between the savings rates with complete and incomplete markets are very small (at most 2%).
  - Although it can be much higher with $\uparrow \sigma$ and $\uparrow \rho$.
- Inequality follows qualitatively the same rank as the data:
  - inequality in wealth $>$ inequality in income $>$ inequality in consumption.
- Moreover, the distributions are right-skewed. Top-inequality is a feature of these models!
  - But the model was still a bit off quantitative. Inequality was lower in the model.

Conclusion

- **Incomplete Markets Model:** new theoretical insights which open the door for old questions (capital taxation, government debt, etc).
- But, where the model shines is to provide a framework to study new questions related to income/wealth inequality.
- A large subsequent literature works so the model matches the distribution of wealth well.
- Then, study policies where inequality is central (progressive taxation, social security, etc).