Extra

1-

a) $\begin{cases} x^2+y^2+z-u^2-v=0 \\ xy^2-uv^2=0 \end{cases} \Leftrightarrow \vec{F}(x,y,z,u,v)=(x^2+y^2+z-u^2-v, xy^2-uv^2)$

$\frac{\partial F_1}{\partial x}=2x \quad \frac{\partial F_1}{\partial y}=2y \quad \frac{\partial F_1}{\partial z}=1 \quad \frac{\partial F_1}{\partial u}=-2u \quad \frac{\partial F_1}{\partial v}=-1$

$\frac{\partial F_2}{\partial x}=y^2 \quad \frac{\partial F_2}{\partial y}=2xy \quad \frac{\partial F_2}{\partial z}=0 \quad \frac{\partial F_2}{\partial u}=-v^2 \quad \frac{\partial F_2}{\partial v}=-2uv$

Todos contínuas porque ... logo $\vec{F}$ é contínua numa vizinhança de $(1,1,0,1,1)$

$F(1,1,0,1,1)=(0,0)$

$JF_{(u,v)}(1,1,0,1,1)=\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow \det=(-2).(-2)-(-1).(-1)=4-1\neq 0=3$

Portanto, podemos aplicar o Teorema da Função Implícita e concluir que as equações definem u e v implicitamente como funções de x, y, z, ou seja, existe $g(x,y,z)=(u,v)$, numa vizinhança de $(1,1,0,1,1)$.

b) $Jg(1,1,0)=\begin{bmatrix} \frac{\partial u}{\partial x}(1,1,0) & \frac{\partial u}{\partial y}(1,1,0) & \frac{\partial u}{\partial z}(1,1,0) \\ \frac{\partial v}{\partial x}(1,1,0) & \frac{\partial v}{\partial y}(1,1,0) & \frac{\partial v}{\partial z}(1,1,0) \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 2/3 & -1/3 \end{bmatrix}$

$\begin{cases} \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F_1}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \\ \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F_2}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \end{cases} \Leftrightarrow \begin{cases} 2x-2u\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=0 \\ y^2-v^2\frac{\partial u}{\partial x}-2uv\frac{\partial v}{\partial x}=0 \end{cases} \overset{(1,1,0,1,1)}{\Leftrightarrow} \begin{cases} 2-2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=0 \\ 1-\frac{\partial u}{\partial x}-2\frac{\partial v}{\partial x}=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \\ 1-\frac{\partial u}{\partial x}-2\left(2-2\frac{\partial u}{\partial x}\right)=0 \end{cases} \Leftrightarrow \begin{cases} \\ -3\frac{\partial u}{\partial x}=-3 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial v}{\partial x}=0 \\ \frac{\partial u}{\partial x}=1 \end{cases}$

$\begin{cases} \frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F_1}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \\ \frac{\partial F_2}{\partial y}+\frac{\partial F_2}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F_2}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \end{cases} \Leftrightarrow \begin{cases} 2-2\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0 \\ 2-\frac{\partial u}{\partial y}-2\frac{\partial v}{\partial y}=0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial v}{\partial y}=\frac{2}{3} \\ \frac{\partial u}{\partial y}=\frac{2}{3} \end{cases}$

$\begin{cases} \frac{\partial F_1}{\partial z}+\frac{\partial F_1}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial F_1}{\partial v}\cdot\frac{\partial v}{\partial z}=0 \\ \frac{\partial F_2}{\partial z}+\frac{\partial F_2}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial F_2}{\partial v}\cdot\frac{\partial v}{\partial z}=0 \end{cases} \Leftrightarrow \begin{cases} 1-2\frac{\partial u}{\partial z}-\frac{\partial v}{\partial z}=0 \\ -\frac{\partial u}{\partial z}-2\frac{\partial v}{\partial z}=0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial v}{\partial z}=-\frac{1}{3} \\ \frac{\partial u}{\partial z}=\frac{1}{3} \end{cases}$

2-

a) $\frac{\partial f}{\partial x} = 2x - 4 = 0$ $\Leftrightarrow \begin{cases} x = 2 \\ y = -3 \end{cases}$ $\Rightarrow$ Logo, Ponto crítico : (2, 3)

$\frac{\partial f}{\partial y} = 2y + 6 = 0$

$\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0$

$H(2,-3) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\left.\begin{matrix} d_1 > 0 \\ d_2 > 0 \end{matrix}\right\}$ mínimo local

$\frac{\partial^2 f}{\partial y^2} = 2$

b) $f(1,-1) = 1 + 1 - 4 - 6 + 5 = -3$

$\frac{\partial f}{\partial x}(1,-1) = -2$

$\frac{\partial f}{\partial y}(1,-1) = 4$

Logo, $f(x,y) \approx -3 - 2h_1 + 4h_2 + \frac{1}{2}(2h_1^2 + 2h_2^2) =$

$= -3 - 2h_1 + 4h_2 + h_1^2 + h_2^2$

3-

a) $\nabla f(x,y,z) = (2x, 2y, 2z)$

b) $div(F(x,y,z)) = 2xy + z^2 + x$

c) $rot(F(x,y,z)) = \left(-2yz, -z, -x^2\right)$

$\frac{\partial F_3}{\partial y} = 0$ $\frac{\partial F_2}{\partial z} = 2yz$

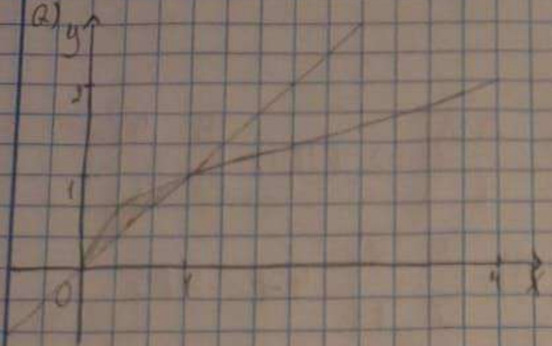
$\frac{\partial F_3}{\partial x} = z$ $\frac{\partial F_1}{\partial z} = 0$

$\frac{\partial F_2}{\partial x} = 0$ $\frac{\partial F_1}{\partial y} = x^2$

d) $Lap(f(x,y,z)) = div(\nabla f(x,y,z)) = 2 + 2 + 2 = 6$

4-

a)

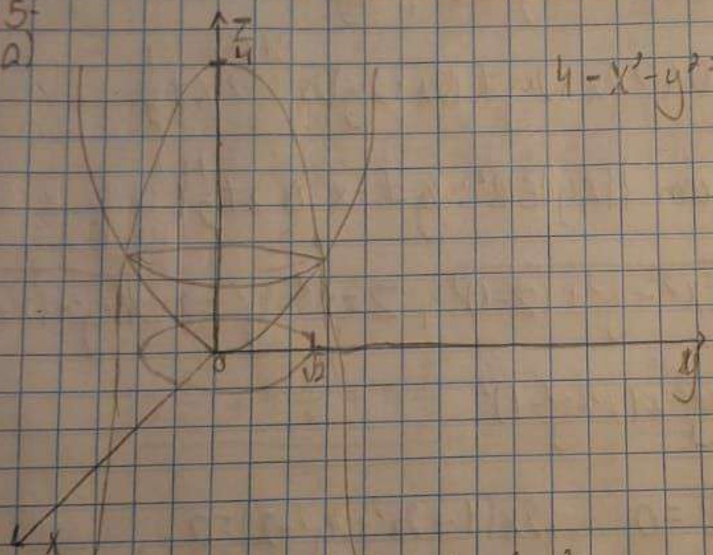


b) $\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$

c) $\int_0^1 \int_{y^2}^{y} dx\,dy = \int_0^1 (y-y^2)\,dy = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

5-

a)



$4 - x^2 - y^2 = x^2 + y^2 \Leftrightarrow x^2 + y^2 = 2$

$$V = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$$

b) $\int_0^{\sqrt{2}} \int_0^{\frac{\pi}{2}} (4-2\rho^2)\rho\,d\theta\,d\rho$

c) $V = \frac{\pi}{2} \int_0^{\sqrt{2}} (4\rho - 2\rho^3)\,d\rho = \frac{\pi}{2}\left(4 - \frac{4}{2}\right) = \pi$

6-

a) $\frac{\partial f_1}{\partial x} = 2x+2y$

$\frac{\partial f_1}{\partial y} = 2x+2y$ $\quad JF(x,y) = \begin{bmatrix} 2x+2y & 2x+2y \\ 2x & 1 \end{bmatrix} \Rightarrow JF(1,1) = \begin{bmatrix} 4 & 4 \\ 2 & 1 \end{bmatrix}$

$\frac{\partial f_2}{\partial x} = 2x$

Logo $\det(JF(1,1)) = \begin{vmatrix} 4 & 4 \\ 2 & 1 \end{vmatrix} = 4-8 = -4 \neq 0$

$\frac{\partial f_2}{\partial y} = 1$

Então f é localmente invertível numa vizinhança de (1,1).

b) $Jf^{-1}(f(1,1)) = [Jf(1,1)]^{-1} = -\frac{1}{4} \operatorname{cof}\left(\begin{bmatrix} 4 & 4 \\ 2 & 1 \end{bmatrix}\right)^T = -\frac{1}{4}\begin{bmatrix} 1 & -2 \\ -4 & 4 \end{bmatrix}^T = \begin{bmatrix} -1/4 & 1 \\ 1/2 & -1 \end{bmatrix}$

c) f é invertível para $\det(JF(x,y)) \neq 0$, logo:

$2x+2y - (2x+2y)2x \neq 0 \Leftrightarrow 2x+2y \neq (2x+2y)2x \underset{y \neq -x}{\Leftrightarrow} x \neq \frac{1}{2}$

Logo f é invertível em $\{(x,y) \in \mathbb{R}^2 : y \neq -x \wedge x \neq \frac{1}{2}\}$

7- $d((0,0,1),(x,y,z)) = \sqrt{x^2+y^2+(z-1)^2}$, $z = x^2+y^2 \Rightarrow \sqrt{x^2+y^2+(x^2+y^2-1)^2}$

$f(x,y) = d^2(x,y) = x^2+y^2+(x^2+y^2-1)^2$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + 2 \cdot 2x(x^2+y^2-1) = 0 \\ \frac{\partial f}{\partial y} = 2y + 2 \cdot 2y(x^2+y^2-1) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x(1+2x^2+2y^2-2) = 0 \\ 2y(1+2x^2+2y^2-2) = 0 \end{cases} \Leftrightarrow$$

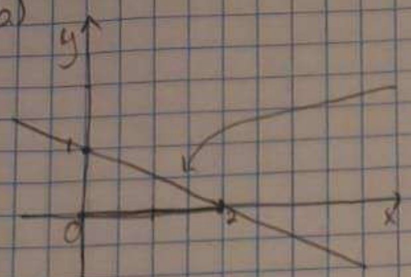
$$\Leftrightarrow \begin{cases} x = 0 \vee x^2+y^2 = \frac{1}{2} \\ y = 0 \vee x^2+y^2 = \frac{1}{2} \end{cases}$$

$z = 0 + 0 \Leftrightarrow z = 0$

$d((0,0,1),(0,0,0)) = \sqrt{0^2+0^2+(-1)^2} = \sqrt{1} = 1$

8.

a)



$p(x,y) = x + y$

$$\int_0^2\int_0^{-\frac{1}{2}x+1}(x+y)\,dy\,dx = \int_0^2 \left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{y=-\frac{1}{2}x+1}dx = \int_0^2\left(-\frac{1}{2}x^2+x+\frac{(-\frac{1}{2}x+1)^2}{2}\right)dx =$$

$$= \left(-\frac{x^3}{6}+\frac{x^2}{2}-\frac{(-\frac{1}{2}x+1)^3}{3}\right)\Big|_{x=0}^{x=2} = -\frac{8}{6}+2+\frac{1}{3} = 1$$

b)

$$m_y = \int_0^2\int_0^{-\frac{1}{2}x+1} x(x+y)\,dy\,dx = \int_0^2 \left(x^2y+x\frac{y^2}{2}\right)\Big|_{y=0}^{y=-\frac{1}{2}x+1}dx =$$

$$= \int_0^2\left(x^2\left(-\frac{1}{2}x+1\right)+\frac{x}{2}\left(-\frac{1}{2}x+1\right)^2\right)dx = \frac{1}{2}\int_0^2\left(-x^3+2x^2+\frac{1}{4}x^3-x^2+x\right)dx =$$

$$= \frac{1}{8}\int_0^2(-4x^3+8x^2+x^3-4x^2+4x)\,dx = \frac{1}{8}\int_0^2(-3x^3+4x^2+4x)\,dx =$$

$$= \frac{-3\frac{2^4}{4}+4\frac{2^3}{3}+4\frac{2^2}{2}}{8} = \frac{-12+\frac{32}{3}+8}{8} = \frac{20}{24} = \frac{5}{6}$$

$$m_x = \int_0^2\int_0^{-\frac{1}{2}x+1} y(x+y)\,dy\,dx = \int_0^2\left(\frac{1}{2}x\left(-\frac{1}{2}x+1\right)^2+\frac{1}{3}\left(-\frac{1}{2}x+1\right)^3\right)dx =$$

$$= \int_0^2\left(\frac{1}{2}x\left(\frac{1}{4}x^2-x+1\right)+\frac{1}{3}\left(-\frac{1}{8}x^3+\frac{3}{4}x^2-\frac{3}{2}x+1\right)\right)dx =$$

$$= \left(\frac{1}{2}\left(\frac{x^4}{16}-\frac{x^3}{3}+\frac{x^2}{2}\right)+\frac{1}{3}\left(-\frac{x^4}{32}+\frac{\cancel{3}}{4}\frac{x^3}{\cancel{3}}-\frac{3}{2}\frac{x^2}{2}+x\right)\right)\Big|_{x=0}^{x=2} =$$

$$= \frac{1}{2}\left(1-\frac{8}{3}+2\right)+\frac{1}{3}\left(-\frac{1}{2}+2-3+2\right) = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

Logo $x_c = \frac{m_y}{m} = \frac{5/6}{1} = \frac{5}{6}$ e $y_c = \frac{m_x}{m} = \frac{1/3}{1} = \frac{1}{3} \Rightarrow CM = \left(\frac{5}{6}, \frac{1}{3}\right)$