Prepratory work for
TTK4115 - Linear System Theory
Lab day 2

Group 11
Håvard Olai Kopperstad
Khuong Huynh
Tomas Nils Tellier

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Department of Engineering Cybernetics

Part II - Multivariable control

2.2.1 Problem 1 - State space formulation

The two following equations are given:

$$\ddot{p} = K_1 V_d \tag{2.5a}$$

$$\ddot{e} = K_2 \tilde{V}_s \tag{2.5b}$$

Putting the system of equations given by the relations for pitch and elevation in (2.5a)-(2.5b) in state-space formulation with:

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \end{bmatrix} \text{ and } \boldsymbol{u} = \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix}$$

gives

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{l_p K_f}{J_p} \\ \frac{l_h K_f}{J_e} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix}$$

2.2.2 Problem 2 - Controllability

To examine the system's controllability we use the controllability indices:

$$\mathcal{C} = \left[\boldsymbol{B} | \boldsymbol{AB} | \dots | \boldsymbol{A}^{n-p} \boldsymbol{B}\right]$$

(assuming $\boldsymbol{B}$ has full column rank) to examine $\mathcal{C}$'s rank:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{l_p K_f}{J_p} \\ \frac{l_h K_f}{J_e} & 0 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} B & AB & A^2 B \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \frac{l_p K_f}{J_p} & 0 & 0 \\ 0 & \frac{l_p K_f}{J_p} & 0 & 0 & 0 & 0 \\ \frac{l_h K_f}{J_e} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The row rank of $\mathcal{C}$ is equal to the number of rows, $rank(\mathcal{C}) = 3$, thus the system is controllable.

2.2.3 Problem 3 - Feedback and feedforward

We are given the reference as:

$$\boldsymbol{r} = \begin{bmatrix} p_c, & \dot{e}_c \end{bmatrix}^T$$

along with the state-feedback controller with reference-feed-forward on the following form:

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x}$$

where

$$\boldsymbol{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

Substituting $\boldsymbol{u}$ into the state-space equation gives:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x}) \\ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{r} - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x} \\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{r} \end{aligned}$$

When examining $p(t)$ and $\dot{e}(t)$ as t goes to infinity, they both go to a constant value as:

$$\lim_{t\to\infty} p(t) = p_c \Rightarrow \dot{p} = \ddot{p} = 0$$

$$\lim_{t\to\infty} \dot{e}(t) = \dot{e}_c \Rightarrow \ddot{e} = 0$$

which in turn gives

$$\dot{\boldsymbol{x}} = 0$$

Examining the state-space equation when t goes to infinity gives:

$$\begin{aligned} 0 &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{r} \\ \boldsymbol{B}\boldsymbol{F}\boldsymbol{r} &= (\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})\boldsymbol{x} \\ \boldsymbol{x} &= (\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{F}\boldsymbol{r} \end{aligned}$$

The output $\boldsymbol{y}$ in the MIMO-system can be described as:

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \end{bmatrix}$$

Also when examining $\boldsymbol{y}$ as t goes to infinity,

$$\lim_{t\to\infty} \boldsymbol{y} = y_\infty = \boldsymbol{r}$$

Now we can derive an expression for $\boldsymbol{F}$ by using the output equation inserted the equation for $\boldsymbol{x}$ evaluated when $t \to \infty$:

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{C}((\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{F}\boldsymbol{r}) \\ \boldsymbol{r} &= (\boldsymbol{C}(\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B})\boldsymbol{F}\boldsymbol{r} \\ \boldsymbol{F}\boldsymbol{r} &= (\boldsymbol{C}(\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B})^{-1}\boldsymbol{r} \end{aligned}$$

$\boldsymbol{F}$ when $t \to \infty$ is given as:

$$\boldsymbol{F} = (\boldsymbol{C}(\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{B})^{-1} = \begin{bmatrix} k_{11} & k_{13} \\ k_{21} & k_{23} \end{bmatrix}$$

2.2.4 Problem 4 - Linear Quadratic Regulator (LQR)

When moving from a PID-regulator and choosing values in the state-feedback matrix directly through manual placement of poles, to implementing a LQR-regulator and choosing a $\boldsymbol{K}$ matrix through minimizing the cost function [1]:

$$J = \int_0^\infty (\boldsymbol{x}^{\boldsymbol{T}}(t)\boldsymbol{Q}_{LQR}\boldsymbol{x}(t) + \boldsymbol{u}^{\boldsymbol{T}}(t)\boldsymbol{R}_{LQR}\boldsymbol{u}(t))dt$$

with $\boldsymbol{Q}_{LQR}$ and $\boldsymbol{R}_{LQR}$ being diagonal weighting matrices when applied in the lab, a hypothesis for the behaviour from the helicopter in theory and practice can be given.

How different choices of $\boldsymbol{Q}_{LQR}$ and $\boldsymbol{R}_{LQR}$ affects the helicopter in theory are that scaling the $\boldsymbol{Q}_{LQR}$ matrix up by any factor a leaving the $\boldsymbol{R}_{LQR}$ matrix as is, will prioritize the states of the system. That means the regulator will try to penalize changes in the pitch angle, pitch rate and elevation rate from their respective set points, and thereby be quicker in response to deviations from the set points.

By scaling up only a single pivot element in the $\boldsymbol{Q}_{LQR}$ matrix means that the regulator prioritizes the corresponding single state in the system. If for example the state p is prioritized, the regulator will penalize deviations in the pitch angle, i.e. the helicopter will quickly try to move its pitch angle to the desired set point. Although $\boldsymbol{Q}_{LQR}$ might be a diagonal matrix the scaling of a single element does not make the regulator penalize deviations in that single corresponding state only, since there is a connection between the weight of $\boldsymbol{Q}_{LQR}$ and $\boldsymbol{R}_{LQR}$ that affects the regulator.

Scaling the whole $\boldsymbol{R}_{LQR}$ matrix up means that the regulator prioritizes the inputs of the system. This in turn means the regulator will not prioritize the states of the system, so the helicopter will react slower to deviations in the states. Scaling $\boldsymbol{R}_{LQR}$ up by any factor a and keeping $\boldsymbol{Q}_{LQR}$ unscaled has the same effect as scaling $\boldsymbol{Q}_{LQR}$ down and keeping $\boldsymbol{R}_{LQR}$ unscaled. It is the relationship between the two matrices that matters for the behaviour of the helicopter.

If both matrices are scaled up or down by the same factor a, the behaviour of the system will not change. The scaling only affects the cost function value but gives the same $\boldsymbol{K}$ matrix.

Looking at the mathematical model of the helicopter there are no limits to how great or how small the previously mentioned scaling factor, a, could be. The ideal model would respond quicker and quicker to deviations in the states if a is increased when scaling $\boldsymbol{Q}_{LQR}$. The physical helicopter can not decrease its response time to an infinitely small value when scaling $\boldsymbol{Q}_{LQR}$ by the factor $a \to \infty$ because of physical limitations, i.e. actuator limitations. Therefore there will be a considerable difference in how a theoretical ideal helicopter would respond to a great scaling factor a, compared to the physical helicopter.

A LQR-regulator will always give a stable system if the system is linear. Therefore increasing or decreasing a will never make the theoretical system unstable. However, since the helicopter is a linearized system, there may be situations where the helicopter is unstable if the states are too far from their respective operating points.

A test-plan for different choices in values on the diagonal $\boldsymbol{Q}_{LQR}$ and $\boldsymbol{R}_{LQR}$ that are to be tested:

Case	Scaling		Hypothesis
	$\boldsymbol{Q}_{LQR}$	$\boldsymbol{R}_{LQR}$	
1.1.1	1	1	Default. The system is stable and goes to reference.
1.2.1	$2Q_{33}$	1	Quicker response to deviations in $\dot{e}$.
1.2.2	$4Q_{33}$	1	Even quicker than 1.2.1
1.2.3	$\frac{1}{2}Q_{33}$	1	Slower than 1.2.1
1.2.4	$\frac{1}{4}Q_{33}$	1	Even slower than 1.2.3
1.3.1	$2Q_{11}$	1	Quicker response to deviations in p.
1.3.2	$4Q_{11}$	1	Even quicker than 1.3.1
1.3.3	$\frac{1}{2}Q_{11}$	1	Slower than 1.3.1
1.3.4	$\frac{1}{4}Q_{11}$	1	Even slower than 1.3.3
1.4.1	$4Q_{22}$	1	No visible effect. $\dot{p}$ not an input.
1.4.2	$10Q_{22}$	1	Same as 1.4.1
1.4.3	$100Q_{22}$	1	Same as 1.4.1
1.4.4	$\frac{1}{100}Q_{22}$	1	Same as 1.4.1
1.5.1	$100Q$	1	Quicker than 1.2.1 and 1.3.1
1.5.2	1	$100R$	Slower than 1.2.4 and 1.3.4
1.5.3	1	$20Q$	Faster than 1.5.2
1.5.4	$100Q$	$100R$	Equal to 1.1.1
1.5.5	10^6Q	10^6R	Equal to 1.1.1
1.5.6	$10^{-6}Q$	$10^{-6}R$	Equal to 1.1.1

In addition, ***Bryson's rule of thumb*** was applied in an attempt to find optimal values for the weighting matrices $\boldsymbol{Q}_{LQR}$ and $\boldsymbol{R}_{LQR}$. Bryson's rule calculates these optimal values based on the maximum acceptable values for all states and outputs, respectively:

$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2}$$
$$R_{jj} = \frac{1}{\text{maximum acceptable value of } u_j^2}$$

As one can observe in the equations, Bryson's rule only calculates the diagonal values of the weighting matrices. The remaining values are set to zero.

The maximal acceptable values of the states and inputs were rationally chosen to:

$$x_{max} = \pi * \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{16} \end{bmatrix} \qquad u_{max} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Applying Bryson's rule results in the following weighting matrices:

$$Q_{LQR} = \frac{1}{\pi^2} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 256 \end{bmatrix} \qquad R_{LQR} = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$$

2.2.5 Problem 5 - Integral action

We introduce two additional states, γ and ζ, for which the differential equations are given by:

$$\dot{\gamma} = p_c - p$$
$$\dot{\zeta} = \dot{e}_c - \dot{e}$$

It follows that the augmented state vector is given as:

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \\ \gamma \\ \zeta \end{bmatrix}$$

The system takes the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{G}\boldsymbol{r}$$
$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x}$$

The augmented $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{G}$ matrices can be found as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{l_p K_f}{J_p} \\ \frac{l_h K_f}{J_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since integral action now has been introduced, we expect stationary deviations in the states p and $\dot{e}$ to be counteracted and removed. We do not expect the state $\dot{p}$ to be affected by the integral action, since $\dot{\gamma}$ and $\dot{\zeta}$ are introduced as functions of p_c and p, $\dot{e}_c$ and $\dot{e}$ respectively. The weight of γ and ζ can be scaled in the $\boldsymbol{Q}_{LQR}$ similar to the three other states, and thereby affecting how quick the integrator affects the stationary deviations that might occur.

It is possible to see from equation (2.10) in the "Helicopter lab preparation" document that $\boldsymbol{F}$ must take the same dimention (2x2) as in problem 3 where no integral action was introduced, even though the state vector $\boldsymbol{x}$ is now of dimention 5x1.

Case	Scaling		Hypothesis
	$\boldsymbol{Q}_{LQR}$	$\boldsymbol{R}_{LQR}$	
2.1.1	1	1	The system is stable and goes to reference. No s.d.
2.2.1	$10Q_{44}$, $10Q_{55}$	1	Quicker integral action in state p and $\dot{e}$ than 2.1.1
2.2.2	$1000Q_{44}$, $1000Q_{55}$	1	Even quicker integral action than 2.2.1
2.2.3	$\frac{1}{10}Q_{44}$, $\frac{1}{10}Q_{55}$	1	Slower integral action than 2.2.2
2.2.4	$10Q_{11}$	1	Quicker response than case 1.3.1, now with no s.d.
2.2.5	$10Q_{22}$	1	No visible effect such as in case 1.4.1
2.2.6	$10Q_{11}$, $10Q_{22}$	1	Similar behaviour to case 2.2.4
2.2.7	$\frac{1}{10}Q_{11}$	1	Slower response than case 1.3.4, now with no s.d.
2.2.8	1	$10R_{11}$	Similar to reducing $\boldsymbol{Q}_{LQR}$ with $\frac{1}{2}$
2.2.9	$10Q_{33}$	1	Quicker response than case 1.2.2, no s.d.
2.2.10	$20Q_{33}$	1	Even quicker than case 2.2.9, no s.d.
2.2.11	$\frac{1}{20}Q_{33}$	1	Slower response than case 1.2.4, no s.d.

References

[1] Department of Engineering Cybernetics. *Helicopter lab preparation.* 2022. (Visited on 09/12/2022).