# Prepratory work for TTK4115 - Linear System Theory Lab day 1


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October 2022


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# Part I - Monovariable control

## 2.1.1 Problem 1 - Equations of motion

The equations of motion can be stated in the form:

$$J_p\ddot{p} = L_1 V_d \tag{2.2a}$$
$$J_e\ddot{e} = L_2 cos(e) + L_3 V_s cos(p) \tag{2.2b}$$
$$J_\lambda\ddot{\lambda} = L_4 V_s cos(e) sin(p) \tag{2.2c}$$

where the sum and difference of motor voltages are given by:

$$V_s = V_b + V_f \tag{2.3a}$$
$$V_d = V_b - V_f \tag{2.3b}$$

respectively [1].

Equation (2.2a) can be derived as follows, using that $J_\theta\ddot{\theta} = \sum \tau = \sum Fr$:

$$\begin{aligned}\sum Fr &= L_p(-F_f + F_b + F_{b,f} + F_{g,b}) \\ &= L_p(F_b - F_f) \\ &= L_p(K_f V_b - K_f V_f) \\ &= L_p K_f (V_b - V_f) \\ J_\theta\ddot{\theta} &= L_1 V_d \qquad \blacksquare\end{aligned}$$

Equation (2.2b) can be derived as follows, using that $J_e\ddot{e} = \sum \tau = \sum Fr$:

$$\begin{aligned}\sum Fr &= F_{g,c} l_c cos(e) - (F_{g,f} + F_{g,b}) l_h cos(e) + (F_f + F_b)) l_h cos(p) \\ &= (F_{g,c} l_c - F_{g,f} l_h - F_{g,b} l_h) cos(e) + l_h K_f (V_f + V_b) cos(p) \\ &= L_2 cos(e) + L_3 V_s cos(p) \\ J_e\ddot{e} &= L_2 cos(e) + L_3 V_s cos(p) \qquad \blacksquare\end{aligned}$$

Equation (2.2c) can be derived as follows, using that $J_\lambda\ddot{\lambda} = \sum \tau = \sum Fr$:

$$\begin{aligned}\sum Fr &= |r||F| sin(p) \\ &= l_h (F_f + F_b) cos(e) sin(p) \\ &= l_h K_f V_s cos(e) sin(p) \\ J_\lambda\ddot{\lambda} &= L_4 V_s cos(e) sin(p) \qquad \blacksquare\end{aligned}$$

In the derivations for the equations of motion, the positive directions are assumed to be given as shown in the figure in the assignment. From the derivations above we are left with the constants:

$$\begin{aligned}L_1 &= l_p K_f \\ L_2 &= F_{g,c} l_c - F_{g,f} l_h - F_{g,b} l_h \\ L_3 &= l_h K_f = L_4\end{aligned}$$

### 2.1.2 Problem 2 - Linearization

From the assignment we are given the assumption that the moments of inertia are constant and given by [1]:

$$J_p = 2m_p l_p^2 \tag{2.4a}$$
$$J_e = m_c l_c^2 + 2m_p l_h^2 \tag{2.4b}$$
$$J_\lambda = m_c l_c^2 + 2m_p(l_h^2 + l_p^2) \tag{2.4c}$$

With the variable transformation $\tilde{V_s} = V_s - V_{s,0} = 0$ resulting in $e^* = 0$ we can find $V_{s,0}$ as:

$$\begin{aligned}
\sum F &= 0 \\
F_{g,c} l_c + l_h(F_f + F_b - 2F_{g,f}) &= 0 \\
V_{s,0} &= \frac{2l_h F_{g,f} - F_{g,c} l_c}{l_h K_f} \\
V_{s,0} &= -\frac{L_2}{L_3}
\end{aligned}$$

In this problem we use the Jacobian of the A-matrix to find the linearization of the system around the equilibrium point $(e^*, p^*, \lambda^*) = (0, 0, 0)$. We also substitute $V_s^* = \tilde{V}_s^* + V_{s,0}$ input $(\tilde{V}_s^*, V_{s,0}) = (0, -\frac{L_2}{L_3})$:

$$\begin{bmatrix} \Delta\ddot{p} \\ \Delta\ddot{e} \\ \Delta\ddot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{L_3(\tilde{V}_s^* + V_{s,0}) sin(p^*)}{J_e} & -\frac{L_2 sin(e^*)}{J_e} & 0 \\ \frac{L_4(\tilde{V}_s^* + V_{s,0}) cos(e^*) cos(p^*)}{J_\lambda} & -\frac{L_4(\tilde{V}_s^* + V_{s,0}) sin(e^*) sin(p^*)}{J_\lambda} & 0 \end{bmatrix} \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & \frac{L_1}{J_p} \\ \frac{L_3 cos(p^*)}{J_e} & 0 \\ \frac{L_4 cos(e^*) sin(p^*)}{J_\lambda} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix}$$

$$\begin{bmatrix} \Delta\ddot{p} \\ \Delta\ddot{e} \\ \Delta\ddot{\lambda} \end{bmatrix} \Big|_{(e^*,p^*,\lambda^*)=(0,0,0),\ (\tilde{V}_s^*, V_{s,0})=(0,-\frac{L_2}{L_3})} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & \frac{L_1}{J_p} \\ \frac{L_3}{J_e} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix}$$

Leaving us with:

$$\begin{aligned}
\Delta\ddot{p} &= \frac{L_1}{J_p} V_d \\
\Delta\ddot{e} &= \frac{L_3}{J_e} \tilde{V}_s \\
\Delta\ddot{\lambda} &= -\frac{L_2 L_4}{J_\lambda L_3} p
\end{aligned}$$

in which

$$K_1 = \frac{L_1}{J_p}$$
$$K_2 = \frac{L_3}{J_e}$$
$$K_3 = -\frac{L_2 L_4}{J_\lambda L_3} = -\frac{L_2}{J_\lambda}$$

### 2.1.3 Problem 3 - PD Control

The PD controller to be implemented is given as [1]:

$$V_d = K_{pp}(p_c - p) - K_{pd}\dot{p} \tag{2.7}$$

Substituting (2.7) into

$$\ddot{p} = K_1 V_d \tag{2.5a}$$

gives

$$\begin{aligned}\ddot{p} &= K_1(K_{pp}(p_c - p) - K_{pd}\dot{p} \\ &= K_1 K_{pp}(p_c - p) - K_1 K_{pd}\dot{p}\end{aligned}$$

The transfer function $G(s) = \frac{p(s)}{p_c(s)}$ assuming the initial state of $p(0) = 0$, can be found as:

$$\begin{aligned}\mathcal{L}\{\ddot{p}\} &= \mathcal{L}\{K_1 K_{pp}(p_c - p) - K_1 K_{pd}\dot{p}\} \\ s^2 p(s) - sp(0) - \dot{p}(0) &= K_1 K_{pp} p_c(s) - K_1 K_{pp} p(s) - K_1 K_{pd} sp(s) - p(0) \\ p(s)(s^2 + K_1 K_{pp} + sK_1 K_{pd}) &= K_1 K_{pp} p_c(s)\end{aligned}$$

$$\frac{p(s)}{p_c(s)} = \frac{K_1 K_{pp}}{(s^2 + K_1 K_{pp} + sK_1 K_{pd})}$$

### 2.1.4 Problem 4 - Pole placement

An expression for $K_{pp}$ and $K_{pd}$ can be developed by setting the denominator in the transfer function equal to zero ($D_{tf} = 0$). Then two equations can be derived as follows:

$$D_{tf} = (s^2 + K_1 K_{pp} + s K_1 K_{pd}) = 0$$

$$\lambda_1 = \frac{-K_1 K_{pd} - \sqrt{K_1^2 K_{pd}^2 - 4K_1 K_{pp}}}{2}, \tag{I}$$

$$\lambda_2 = \frac{-K_1 K_{pd} + \sqrt{K_1^2 K_{pd}^2 - 4K_1 K_{pp}}}{2} \tag{II}$$

Rearranging (I) gives:

$$\begin{aligned} 2\lambda_1 + \sqrt{K_1^2 K_{pd}^2 - 4K_1 K_{pp}} &= -K_1 K_{pd} \\ \left(\sqrt{K_1^2 K_{pd}^2 - 4K_1 K_{pp}}\right)^2 &= (-K_1 K_{pd} - 2\lambda_1)^2 \\ K_1^2 K_{pd}^2 - 4K_1 K_{pp} &= (-K_1 K_{pd} - 2\lambda_1)^2 \end{aligned}$$

$$\begin{aligned} K_{pp} &= \frac{K_1^2 K_{pd}^2 - \left(K_1^2 K_{pd}^2 + 4\lambda_1 K_1 K_{pd} + 4\lambda_1^2\right)}{4K_1} \\ &= \frac{-\lambda_1 (K_1 K_{pd} + \lambda_1)}{K_1} \end{aligned}$$

Rearranging (II) and substituting $K_{pp}$ gives $K_{pd}$:

$$\begin{aligned} \lambda_2 &= \frac{-K_1 K_{pd} + \sqrt{K_1^2 K_{pd}^2 + 4\lambda_1 K_1 K_{pd} + 4\lambda_1^2}}{2} \\ &= \frac{-K_1 K_{pd} + \sqrt{(-K_1 K_{pd} - 2\lambda_1)^2}}{2} \\ &= -K_1 K_{pd} - \lambda_1 \end{aligned}$$

$$K_{pd} = -\frac{(\lambda_1 + \lambda_2)}{K_1}$$

Substituting (II) into (I) gives $K_{pp}$:

$$K_{pp} = \frac{-\lambda_1 \left( K_1 \left( -\frac{(\lambda_1+\lambda_2)}{K_1} \right) + \lambda_1 \right)}{K_1}$$
$$K_{pp} = \frac{\lambda_1 \lambda_2}{K_1}$$

Now the equations for $K_{pp}$ and $K_{pd}$ can be used to form hypotheses based on different criteria set to develop wanted forms and placements of $\lambda_1$ and $\lambda_2$.

**Case 1.1 - Strictly real and negative equal poles**

We want $\lambda_1 = \lambda_2 < 0$ where $\{\lambda_1, \lambda_2\} \epsilon \mathcal{R}$.

**Case 1.2 - Strictly real and negative unique poles**

We want $\lambda_1 \neq \lambda_2 < 0$ where $\{\lambda_1, \lambda_2\} \epsilon \mathcal{R}$.

**Case 1.3 - Strictly imaginary poles**

We want $\{\lambda_1, \lambda_2\} \epsilon \mathcal{C}$ on the form $\lambda_{1,2} = \pm j\beta$ with $\alpha = 0$.

**Case 1.4 - Complex conjugated poles with negative real part**

We want $\{\lambda_1, \lambda_2\} \epsilon \mathcal{C}$ on the form $\lambda_{1,2} = \alpha \pm j\beta$ with $\alpha < 0$.

**Case 1.5 - Complex conjugated poles with positive real part**

We want $\{\lambda_1, \lambda_2\} \epsilon \mathcal{C}$ on the form $\lambda_{1,2} = \alpha \pm j\beta$ with $\alpha > 0$.

**Case 1.6 - Strictly real and positive unique poles**

We want $\lambda_1 \neq \lambda_2 > 0$ where $\{\lambda_1, \lambda_2\} \epsilon \mathcal{R}$.

**Case 1.7 - Strictly real and unique poles with one being negative**

We want $\lambda_1 \neq \lambda_2$ and $\lambda_1 < 0 < \lambda_2$ where $\{\lambda_1, \lambda_2\} \epsilon \mathcal{R}$.

**Hypotheses**

For each of the seven cases, our hypotheses can be put into a table as follows:

| Case | Poles | | Hypothesis |
|---|---|---|---|
| | $\lambda_1$ | $\lambda_2$ | |
| 1.1.1 | -1 | -1 | No oscillations. Stationary after 5 sec with s.d. |
| 1.1.2 | -3 | -3 | No oscillations. Stationary after 5/3 sec with s.d. |
| 1.2.1 | -10 | -3 | No oscillations. Stationary after 5/3 sec with s.d. |
| 1.2.2 | -7 | -3 | No oscillations. Stationary after 5/3 sec. Less s.d. |
| 1.2.3 | -7 | -0.5 | No oscillations. Stationary after 10 sec. With s.d. |
| 1.2.4 | -15 | -3 | No oscillations. Stationary after 5/3 sec. Less s.d. |
| 1.3.1 | j | -j | Low freq. oscillations around reference. |
| 1.3.2 | j5 | -j5 | High freq. oscillations around reference. |
| 1.4.1 | -3+3j | -3-3j | Stationary after 5/3 sec with oscillations. |
| 1.4.2 | -3+4j | -3-4j | Stationary after 5/3 sec with higher freq. oscillations |
| 1.4.3 | -1+4j | -1-4j | Stationary after 5/8 sec. Higher freq. oscillations, s.d. |
| 1.4.4 | -8+3j | -8-3j | Stationary after 5/8 sec with oscillations. |
| 1.5.1 | 0.1+0.1j | 0.1-0.1j | Unstable. |
| 1.6.1 | 0.01 | 0.03 | Unstable. |
| 1.7.1 | -3 | 0.05 | Unstable. |
| 1.7.2 | -3 | 0.2 | Unstable. |

# References

[1] Department of Engineering Cybernetics. *Helicopter lab preparation.* 2022. (Visited on 09/12/2022).