Algorithmic Aspects of Game Theory

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Hex, choquet: 2 players.
We say a game is **determined** if either of players has a winning strategy, If σ is a winning strategy of P, then $\forall_\pi G(\sigma, \pi) \leftarrow$ wins P.

1. Is the choquet game determined if we replace $\mathcal{R}$ with $\mathcal{Q}$ (and its topology)?
If so, who has a winning strategy?

2. Let's consider a variant of choquet games on topological spaces. We have a property: If X is not a Baire space[1] $\implies$ E has a winning strategy (E means Empty, not Eve!).
Example with rational numbers:
$G^q \leftarrow$ set $\mathcal{Q} \setminus q$ dense, open.
Q is countable.
$F \subset Q$
$\cap_{q \in F} G^q = G^F$
$|F| < \mathfrak{c}$
Our strategy:

- we start with set G^{q_0}
- opponent plays a set, say S_1
- we play a set $S1 \cap G^{q_0} \cap G^{q_1}$

3. If X is complete then NE has w.s.
A complete space is also a Baire space.

4. Consider Nim game.
Setup: n heaps with tokens $h_1, h_2, ..., h_n$.
Move: choose a heap and remove $r > 0$ tokens.
Win: The last move.
We have two players: E and $\forall$, Eve move first. Q: Who has a winning strategy? When is the game determined?

[1] X is Baire if:
$G_i \leftarrow$ are dense and open for $i \in \mathcal{N}$ then $\cap_{i>0} G_i \neq \emptyset$

$n = 1 \leftarrow$ Eve always wins
$n = 2 \leftarrow$ ((1, 1) wins Adam, (2, 1) wins Eve, (2, 2) wins Adam)
$(h_1, h_2) \rightarrow$ equalise them if possible
Eve has a winning strategy iff $h_1 \neq h_2$
General case: Eve wins if the xor of stack heaps is non-zero. Proof: The winning configuration has xor 0. From a situation with xor $\neq 0$ is always able to produce a situation with xor $= 0$ and if xor $= 0$, it's impossible to make a move such that xor $= 0$ after the move.

1. $(0, ..., 0, h_j, 0, ..., 0)$ is a winning position for Eve.
2. if $h_1 \otimes h_2 ... \otimes h_n = 0$ then the position is *balanced.* Balanced positions are winning positions.
3. Show strategy (next tutorials)

3 Lecture 2 (6 III 2019)

Determinacy

If we have a **game of finite duration** with 2 players, we can expand the game in a tree, where a leaf signifies the end of the game. A leaf maps to one of three possible situations:

- existential player ($\exists$) wins
- universal player ($\forall$) wins
- draw

If we map those situations to values accordingly: $1, -1, 0$, the existential player aims to maximize (and universal to minimize) the outcome value.

Let's consider **infinite** games now. Suppose we have 2 players and draw is not possible in the game. If the player does not know the winning strategy, it is possible that they may "loop" in a position with winning strategy but never proceed with it.

There **exist** indeterminate perfect information games.

Infinite XOR game: E and A alternately play words $w_0, w_1, w_2, ... \in \{0, 1\}^+$ which are concatenated to $w_0 w_1 w_2 ...$.

Infinite XOR: any function $f : \{0, 1\}^{|\mathbb{N}|} \rightarrow \{0, 1\}$ such that if v, w differ by one bit then $f(v) \neq f(w)$.
$v \sim w$ iff differ by a finite number of bits.
We can choose set S s.t. $\{0, 1\}^{|\mathbb{N}|} \supseteq S$ has $\exists!$ element for each equivalence class (from *Axiom of Choice*).
Each equivalence class of $\sim$ is countable, thus there is continuum of equivalence classes.
Eve wins iff $f(w_0 w_1 ...) = 0$, Adam otherwise. No player has a winning strategy in this game.
<u>1. Suppose Adam wins.</u> In the first play:

```
E  0      w_2      w_4
A    w_1      w_3
```

Then in the next game Eve can steal his strategy:

```
E  1 w_1       w_3      w_5
A        w_2       w_4
```

2. Suppose Eve wins. In the first play:

```
E  w_0     w_2       w_4
A       0        w_3
```

Then in the next play:

```
E  w_0           w_3
A       1 w_2         w_4
```

Game on graph

An *arena* is a directed graph, consisting of:

- the set of *positions* Pos
- the set of *moves* $Moves \subseteq Pos \times Pos$

$Pos = Pos_\exists \cup Pos_\forall, Pos_\exists \cup Pos_\forall \neq \emptyset$.
A *play* is a finite or infinite sequence $q_0 \to q_1 \to q_2 \to ... \to q_k (\to ...)$
Game equation

$$X = (E \cap \diamond X) \cup (A \cap \Box X) = Eve(X)$$
$$Y = (E \cap \Box Y) \cup (A \cap \diamond Y) = Adam(Y)$$

$E = Pos_\exists, A = Pos_\forall, X, Y \in \mathcal{P}(Pos)$
"Modal logic" symbols here:
$\diamond Z = \{p : (\exists_q) Moves(p,q) \wedge q \in Z\}$ [2]
$\Box Z = \{p : (\forall_q)(p \to q) \Rightarrow q \in Z\}$
Knaster-Tarski Theorem: $\langle L, \leqslant \rangle$ complete lattice[3], $f : L \to L$ monotonic, then there exists a least fixed point $\mu x.f(x) = \bigwedge\{d : f(d) \leqslant d\}$ and a greatest fixed point: $\gamma y.f(y) = \bigvee\{d : d \leqslant f(d)\}$.
Proof: We show the proof for the greatest fixed point. Let $a = \bigvee A, A = \{z : z \leqslant f(z)\}$
$Z \subseteq Pos$ is a trap for Adam if $Z \subseteq Eve(Z)$
$Z \subseteq Pos$ is Garden of Eden for Eve if $Eve(Z) \subseteq Z$

Proposition: Pos can be divided to three disjoint sets: $\mu X.Eve(X)$, $\mu X.Adam(X), (\gamma Y.Eve(Y)) \cap (\gamma Y.Adam(Y))$

Definition: strategy

A strategy (for Eve) is a set of finite plays s.t.:

- if $last(w) \in Pos_\exists$ then $\exists! q$ s.t. $last(W) \to q$ and wq is in S
- if $last(w) \in Pos_\forall$ then $\forall(q)(last(w) \to q) \Rightarrow wq \in S$

[2] $p \to q$ also denotes $Moves(p,q)$ below. A position p, such that $(\forall_p) p \not\to q$ is called *terminal*, which we also write $p \not\to$.
[3] A *complete lattice* is a partially ordered set $\langle L, \leqslant \rangle$, such that each subset $Z \subseteq L$ has the least upper bound $\bigvee Z$, and the greates lower bound $\bigwedge Z$. In particular, $\bigvee \emptyset$ is the least element denoted $\perp$, and $\bigwedge \emptyset$ is the greatest element denoted $\top$.