Algorithmic Aspects of Game Theory

Tomasz Garbus

2019

Contents

1 Lecture 1 (27 II 2019)

2 Tutorials 1 (28 II 2019)

Hex, choquet: 2 players.
We say a game is **determined** if either of players has a winning strategy, If σ is a winning strategy of P, then $\forall_\pi G(\sigma, \pi) \leftarrow$ wins P.

1. Is the choquet game determined if we replace $\mathcal{R}$ with $\mathcal{Q}$ (and its topology)?
If so, who has a winning strategy?

2. Let's consider a variant of choquet games on topological spaces. We have a property: If X is not a Baire space[1] $\implies$ E has a winning strategy (E means Empty, not Eve!).
Example with rational numbers:
$G^q \leftarrow$ set $\mathcal{Q} \setminus q$ dense, open.
Q is countable.
$F \subset Q$
$\cap_{q \in F} G^q = G^F$
$|F| < \mathfrak{c}$
Our strategy:

- we start with set G^{q_0}
- opponent plays a set, say S_1
- we play a set $S1 \cap G^{q_0} \cap G^{q_1}$

3. If X is complete then NE has w.s.
A complete space is also a Baire space.

4. Consider Nim game.
Setup: n heaps with tokens $h_1, h_2, ..., h_n$.
Move: choose a heap and remove $r > 0$ tokens.
Win: The last move.
We have two players: E and $\forall$, Eve move first. Q: Who has a winning strategy? When is the game determined?

[1] X is Baire <u>if</u>:
$G_i \leftarrow$ are dense and open for $i \in \mathcal{N}$ then $\cap_{i>0} G_i \neq \emptyset$

$n = 1 \leftarrow$ Eve always wins
$n = 2 \leftarrow$ ((1, 1) wins Adam, (2, 1) wins Eve, (2, 2) wins Adam)
$(h_1,\ h_2) \rightarrow$ equalise them if possible
Eve has a winning strategy iff $h_1 \neq h_2$
General case: Eve wins if the xor of stack heaps is non-zero. Proof: The winning configuration has xor 0. From a situation with xor $\neq 0$ is always able to produce a situation with xor $= 0$ and if xor $= 0$, it's impossible to make a move such that xor $= 0$ after the move.

1 $(0, ..., 0, h_j, 0, ..., 0)$ is a winning position for Eve.

2 if $h_1 \otimes h_2 ... \otimes h_n = 0$ then the position is *balanced*. Balanced positions are winning positions.

3 Show strategy (next tutorials)

3 Lecture 2 (6 III 2019)

Determinacy

If we have a **game of finite duration** with 2 players, we can expand the game in a tree, where a leaf signifies the end of the game. A leaf maps to one of three possible situations:

- existential player ($\exists$) wins
- universal player ($\forall$) wins
- draw

If we map those situations to values accordingly: $1, -1, 0$, the existential player aims to maximize (and universal to minimize) the outcome value.

Let's consider **infinite** games now. Suppose we have 2 players and draw is not possible in the game. If the player does not know the winning strategy, it is possible that they may "loop" in a position with winning strategy but never proceed with it.

There **exist** indeterminate perfect information games.

Infinite XOR game: E and A alternately play words $w_0, w_1, w_2, ... \in \{0, 1\}^+$ which are concatenated to $w_0 w_1 w_2 ...$.

Infinite XOR: any function $f : \{0, 1\}^{|\mathbb{N}|} \rightarrow \{0, 1\}$ such that if v, w differ by one bit then $f(v) \neq f(w)$.
$v \sim w$ iff differ by a finite number of bits.
We can choose set S s.t. $\{0, 1\}^{|\mathbb{N}|} \supseteq S$ has $\exists!$ element for each equivalence class (from *Axiom of Choice*).
Each equivalence class of $\sim$ is countable, thus there is continuum of equivalence classes.
Eve wins iff $f(w_0 w_1 ...) = 0$, Adam otherwise. No player has a winning strategy in this game.
<u>1. Suppose Adam wins.</u> In the first play:

```
E  0      w_2      w_4
A    w_1      w_3
```

Then in the next game Eve can steal his strategy:

```
E  1 w_1        w_3       w_5
A        w_2         w_4
```

2. Suppose Eve wins. In the first play:

```
E  w_0    w_2      w_4
A      0      w_3
```

Then in the next play:

```
E  w_0          w_3
A      1 w_2        w_4
```

Game on graph

An *arena* is a directed graph, consisting of:

- the set of *positions* Pos
- the set of *moves* $Moves \subseteq Pos \times Pos$

$Pos = Pos_\exists \cup Pos_\forall$,
$Pos_\exists \cup Pos_\forall \neq \emptyset$.
A *play* is a finite or infinite sequence of moves:
$q_0 \to q_1 \to q_2 \to ... \to q_k (\to ...)$.

Game equation

$$X = (E \cap \diamond X) \cup (A \cap \Box X) = Eve(X)$$
$$Y = (E \cap \Box Y) \cup (A \cap \diamond Y) = Adam(Y)$$

$E = Pos_\exists$,
$A = Pos_\forall$,
$X, Y \in \mathcal{P}(Pos)$
"Modal logic" symbols here:
$\diamond Z = \{p : (\exists_q) Moves(p,q) \wedge q \in Z\}$ [2]
$\Box Z = \{p : (\forall_q)(p \to q) \Rightarrow q \in Z\}$

Knaster-Tarski Theorem: $\langle L, \leqslant \rangle$ complete lattice[3], $f : L \to L$ monotonic, then there exists a least fixed point
$\mu x.f(x) = \bigwedge\{d : f(d) \leqslant d\}$ and a greatest fixed point:
$\gamma y.f(y) = \bigvee\{d : d \leqslant f(d)\}$.
Proof: We show the proof for the greatest fixed point. Let $a = \bigvee A, A = \{z : z \leqslant f(z)\}$
Because f is monotonic, $z \leqslant a$ implies $f(z) \leqslant f(a)$. For $z \in A$, this also means $z \leqslant f(z) \leqslant f(a)$. Hence, $f(a)$ is an upper bound of A, which follows $a \leqslant f(a)$. Using the monotonicity of f once more, we obtain $f(a) \leqslant f(f(a))$. Hence $f(a) \in A$, which follows the converse inequality $f(a) \leqslant a$.

We consider mappings *Eve* and *Adam* defined in the complete lattice $\langle \mathcal{P}, \leqslant \rangle$. $Eve(Z)$ is a set of such positions from which Eve can win, and $Adam(Z)$ is a set of such positions from which Adam can win.

[2] $p \to q$ also denotes $Moves(p,q)$ below. A position p, such that $(\forall_p) p \not\to q$ is called *terminal*, which we also write $p \not\to$.

[3] A *complete lattice* is a partially ordered set $\langle L, \leqslant \rangle$, such that each subset $Z \subseteq L$ has the least upper bound $\bigvee Z$, and the greates lower bound $\bigwedge Z$. In particular, $\bigvee \emptyset$ is the least element denoted $\perp$, and $\bigwedge \emptyset$ is the greatest element denoted $\top$.

Traps and gardens of Eden

A set of positions $Z \subseteq Pos$ is a trap for Adam if $Z \subseteq Eve(Z)$. It means that Adam cannot go out of there.
A set of positions $Z \subseteq Pos$ is Garden of Eden for Eve if $Eve(Z) \subseteq Z$. It means that Adam cannot enter those positions.

The *greatest* trap for Adam is a garden of Eden for Even.
The *least* garden of Eden for Eve is a trap for Adam.

We use the notation: $\overline{Z} = Pos - Z$.

Lemma

$$\overline{Eve(X)} = Adam(\overline{X})$$

Proof. We have:
$\overline{Eve(X)} = \overline{(E \cap \diamond X) \cup (A \cap \Box X)}$
$= \overline{(E \cap \diamond X)} \cap \overline{(A \cap \Box X)}$
$= (\overline{E} \cup \overline{\diamond X}) \cap (\overline{A} \cup \overline{\Box X})$
$= (A \cup \Box \overline{X}) \cap (E \cup \diamond \overline{X})$
$= (A \cap \diamond \overline{X}) \cup (E \cap \diamond \overline{X}) \cup (A \cap E) \cup (\diamond \overline{X} \cap \Box \overline{X}) = Adam(\overline{X})$

Proposition: Pos can be divided to three disjoint sets: $\mu X.Eve(X)$, $\mu X.Adam(X)$, $(\gamma Y.Eve(Y)) \cap (\gamma Y.Adam(Y))$

Definition: strategy

A strategy (for Eve) is a set of finite plays s.t.:

- if $last(w) \in Pos_\exists$ then $\exists! q$ s.t. $last(w) \to q$ and wq is in S
- if $last(w) \in Pos_\forall$ then $\forall(q)(last(w) \to q) \Rightarrow wq \in S$
- S is closed under initial segments, i.e., if $s_0 s_1 ... s_k \in S$, then $s_0 s_1 ... s_i \in S$, for $0 \leqslant i \leqslant k$j

4 Tutorials 1 (7 III 2019)

1. Consider Nim game.
We have n stacks of heights $h_1, h_2 ..., h_n, h \in \{0, 1, 2, ...\}$.
$W_E = \{x \in \mathbb{N}^R : \text{Eve has a w.s.}\}$.
Let's take $x \in \mathbb{N}^R, x = (h_1, ..., h_n)$.
$h_1^b \otimes h_2^b \otimes ... \otimes h_n^b \neq 0 \to$ Eve wins from x.[4]
The proof consists of 3 observations:

- Final position (all empty stacks) has xor equal to 0.
- From a position s.t. xor $\neq 0$, it is always possible to zero the xor. Let xor be equal some y. Let d be the position of leftmost (most important) bit in binary representation of xor. Thus in some stack, the binary representation must have The d-th bit activated. We can then deactivate $d-$th bit and set appropriate values on all less significant bits (to zero the xor), and the resulting stack height will be lower.
- From a position s.t. xor $= 0$, all moves lead to xor $\neq 0$. If we take a non-zero number of tokens from a stack, it means we alter a non-zero number of binary digits in the representation of xor, thus it is no longer 0.

[4] (h^b means binary representation)

Thus, $W_E = \{x \in \mathbb{N}^R : \otimes x \neq \vec{0}\}$.

2. Represent NIM as a game on graph.
The set of positions is the set of all stack configurations. In general, the space of positions is infinite, but for a fixed game we have finite set of positions.
However, this is not enough (in a game on graph, we want Pos to be disjoint set $Pos = Pos_E \cup Pos_A, Pos_E \cap Pos_A = \emptyset$). Thus we also add information who moves next.
Graph: $G = \langle V_E \dot{\cup} V_A, E\rangle$.
Strategy is a function $\sigma : V^*V \to V$, $\sigma(v_0v_1v_2...v_nv_{n+1}) \to v$.
Let $w = v_1...v_n...$ be a winning position. We say that first player wins if their strategy $L \in V^w$.
In NIM, the winning condition does not depend on history, so we can collapse the states with the same stacks configuration in the last move (and of course the same currrent player).

3. Chocolate game.
We have a grid $m \times n$. A player chooses a field and everything on the right and up is erased. One restriction: you cannot choose position $(1, 1)$. The player who makes last move wins. [5] Is the game determined? Who has a winning strategy?

$1 \times n$: Eve wins (obvious)

$2 \times n$: Eve wins: she eats the position $(n, 2)$ and always maintains the bottom row has one piece more than the top one.

$\omega \times \omega$: [6] Eve wins: eat the position $(2, 2)$ and then maintain $(1, n); (n, 1)$

$m \times n$: [7] Assume the 2nd player wins. That means that for any move made in the first move, there exists such move from a second player, that the second player has a winning strategy after the second move.
We can use a strategy stealing argument: First player removes a rectangle larger than 1×1. Then by assumption the second player has a winning strategy. But instead we can start by removing only one piece of chocolate. Then second player must make a move such that only a single rectangular area is removed (which could be made in one move by the first player). Then first player can copy second player's winning strategy. Thus second player does not have a winning strategy.

Above we have shown that the second player has no w.s.
$\neg\exists_\pi\forall_\sigma\pi$ wins with $\sigma \Leftrightarrow \forall_\pi\exists_\sigma\sigma$ wins with π. But this doesn't imply $\exists_\sigma\forall_\pi\sigma > \pi$ (although there is an implication the other way).
We will just show the game is determined, without showing the winning strategy:
We can build the graph of positions. All paths from $m \times n$ (starting position) are finite, the size of graph is finite as well. We can thus infer the winning position by searching the graph bottom-up (from node $0, 0$ to m, n). We have thus shown the game is determined for a finite size. Since Adam does not have a winning strategy, thus Eve must have it.

[5] Other definition: players eat chocolate, position $(1, 1)$ is poisoned.
[6] $\omega = \{0, 1, ...\}$
[7] $m, n \in \mathbb{N}$

For $\omega \times \omega$ we use an argument that after the first move the game graph must have a finite height.

At home: think about chess determinacy, also Armageddon version (black wins if he doesn't lose, also for draw).