Algorithmic Aspects of Game Theory

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5 Multi-reachability games

In a reachability games there is a set of vertices which the first player wants to reach. In multi-reachability games (MRG) there is a family of sets of vertices and the first player wins if every set in the family has been visited at least once.
1. Show that MRG game is PSpace-complete.
1. $\in$ PSPACE
HMMMMM
2. PSPACE-hard
I will show a reduction of QBF problem to an MRG game.
An input to the QBF problem is a formula, about which I will make two assumptions:

- It is in prenex normal form (i.e. all quantifiers preceed the portion containing an unquantified Boolean formula). Moreover, let's assume that the existental and universal quantifiers alternate – if it is not the case in the original input formula, we can introduce quantifiers with dummy variables, not used anywhere in the formula. For instance, $\exists_{x_1}\exists_{x_2}\phi(x_1, x_2) \mapsto \exists_{x_1}\forall_{y_1}\exists_{x_2}\phi(x_1, x_2)$ (y_1 is a "dummy" variable).
- The "body" of the formula is in conjunctive normal form.

Note that the above assumptions do not reduce the expressive power of input formulas. Every possible formula can be represented in the described format. QBF problem for such normalized formulas is still PSPACE-complete.
Let $\forall_{x_1}\exists_{x_2}\forall_{x_3}...\exists_{x_n}(y_{1,1}\vee...\vee y_{1,k_1})\wedge(y_{2,1}\vee...\vee y_{2,k_2})\wedge...(y_{m,1}\vee...\vee y_{m,k_m})$ be the input QBF formula, where $y_{...} \in \{x_1, ..., x_n, \neg x_1, ...\neg x_n\}$.
The created multi-reachability game is $G = \langle V, E, v_I, S\rangle$, where:

- V is the set of vertices. Vertices are indexed by all variables bound by quantifiers and their negations, plus there is the initial vertex. $V = \bigcup_{1\leqslant i\leqslant n}\{v_x, v_{\neg x}\} \cup \{v_I\}$
- E is the set of edges. $E = \{(v_I, v_{x_1}), (v_I, v_{\neg x_1})\} \cup \bigcup_{2\leqslant i\leqslant n}\{(v_{x_{i-1}}, v_{x_i}), (v_{x_{i-1}}, v_{\neg x_i}), (v_{\neg x_{i-1}}, v_{x_i}), (v_{\neg x_{i-1}}, v_{\neg x_i})\}$
- v_I is a starting vertex.
- S is the family of sets of vertices that the first player wants to reach. It is created directly from the CNF formula, i.e. $S = \bigcup_{1\leqslant i\leqslant m}\{\{v_{y_{i,j}} \mid 1 \leqslant j \leqslant k_i\}\}$

The first player is the existential player, their opponent is the universal player (and they start the game). The QBF formula is satisfiable iff the first player has a winning strategy.