MEET TREES STUFF

1. SOME OBSERVATIONS ABOUT TREES

$\mathcal{O} = \{x_0, x_1, \ldots\}$ is an orbit of a finite partial automorphism p of a $\wedge$-tree (so $x_j = p^j(x_0)$).

Remark 1.1. If for some i, j we have $x_i \geqslant x_j$, then whenever (for some $k \in \mathbf{Z}$) x_{i+k}, x_{j+k} exist, then $x_{i+k} \geqslant x_{j+k}$.

In particular, if x_{2j-i} exists, then $x_j \geqslant x_{2j-i}$, so $x_i \geqslant x_{2j-i}$, and likewise, if $x_{j+k(j-i)}$ exists, then $x_i \geqslant x_{j+k(j-i)}$. $\Diamond$

Proposition 1.2. *Suppose (in a $\wedge$-tree) $x_1 \leqslant x_2$ and $y \not\geqslant x_1$. Then $x_1 \wedge y = x_2 \wedge y$.*

Proof. Trivially, $x_1 \wedge y \leqslant x_2 \wedge y$.

On the other hand, $x_2 \wedge y < x_2$, so $x_2 \wedge y \not\leqslant x_1$. Furthermore, since $y \geqslant x_2 \wedge y$ and $y \not\geqslant x_1$, we have $x_2 \wedge y \not\geqslant x_1$. Thus $x_2 \wedge y \leqslant x_1$, whence $x_2 \wedge y \leqslant x_1 \wedge y$. $\square$

The following proposition describes the "spiral" behaviour of orbits.

Proposition 1.3. *Suppose $n > 0$ is minimal such that x_0 is comparable to x_n. Then:*

(1) x_i is comparable to x_j if and only if $i \equiv j \pmod n$,
(2) if $i_1 \equiv i_2 \not\equiv j \pmod n$, then $x_{i_1} \wedge x_j = x_{i_2} \wedge x_j$.

Proof. (1): $\Leftarrow$ is an easy consequence of Remark 1.1. For $\Rightarrow$, suppose this is not true. Then we can choose $i \not\equiv j \pmod n$ such that x_i is comparable to x_j. Choose i, j such that $|i - j|$ is minimal.

Note that we may assume without loss of generality that $i < j$, and then furthermore, that $i = 0$. Thus j is the smallest number such that $n \nmid j$ and x_j is comparable to x_0. Necessarily (by choice of n), $j > n$.

Notice also that x_n and x_j cannot be comparable: otherwise, x_0 and x_{j-n} are comparable, which contradicts minimality of j.

Now, we have one of the three:

- $x_n \geqslant x_0$, in which case $x_{n+j} \geqslant x_n, x_j$,
- $x_n < x_0$ and $x_0 \leqslant x_j$, or
- $x_n, x_j < x_0$.

In all three cases, x_n and x_j are comparable, a contradiction.

(2): Since x_0 and x_n are comparable, it follows that x_{i_1} and x_{i_2} are also comparable. Suppose without loss of generality that $x_{i_1} \geqslant x_{i_2}$.

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Since of course $x_{i_1} \wedge x_j \leqslant x_{i_1}$, we have that x_{i_2} and $x_{i_1} \wedge x_j$ are comparable.

If $x_{i_2} \geqslant x_{i_1} \wedge x_j$, then $x_{i_2} \wedge x_j \geqslant x_{i_1} \wedge x_j$, and since $x_{i_1} \geqslant x_{i_2}$, we also have $x_{i_1} \wedge x_j \geqslant x_{i_2} \wedge x_j$, so we are done.

Otherwise, $x_{i_2} < x_{i_2} \wedge x_j$, so in particular, $x_{i_2} < x_j$, which contradicts (1), so we are done. □

Proposition 1.4. *Suppose that neither $x_0 \wedge x_1 < x_1 \wedge x_2$ nor $x_0 \wedge x_1 > x_1 \wedge x_2$.*

Then $x_0 \wedge x_1$ is a fixed point, and whenever x_{i+1} exists, $x_0 \wedge x_1 = x_i \wedge x_{i+1}$. Moreover, $x_0 \wedge x_1 = \bigwedge \mathcal{O}$, i.e. it is the smallest element of the semilattice generated by $\mathcal{O}$.

More generally, if $x_0 \wedge x_k$ is not strictly greater nor strictly smaller than $x_k \wedge x_{2k}$, then, then $x_k \wedge x_{2k}$ is a fixed point of p^k and it is minimal in the $\wedge$-lattice generated by $\{x_{nk} \mid n \in \mathbf{N}\}$.

Proof. Note that $x_0 \wedge x_1, x_1 \wedge x_2 \leqslant x_1$, so they are comparable. By the assumption, $x_0 \wedge x_1 = x_1 \wedge x_2$. It follows immediately that $x_0 \wedge x_1$ is a fixed point, so for every i, $x_i \wedge x_{i+1} = x_0 \wedge x_1$ (as long as x_{i+1} exists).

But then for $i > 1$ we have $x_i \wedge x_0 \wedge x_1 = x_i \wedge x_i \wedge x_{i-1} = x_i \wedge x_{i-1} = x_0 \wedge x_1$. Since $\bigwedge \mathcal{O} = \bigwedge_{i=0}^m x_i$, the conclusion follows.

The "more generally" part is a special case with the partial automorphism p^k in place of p. □

For each $j, k \in \mathbf{N}$, put $y_j^k := x_j \wedge x_{j+k}$. Notice that $p^l(y_j^k) = y_{j+l}^k$.

Proposition 1.5. *Suppose $n > 0$ is minimal such that y_0^k is comparable to y_n^k (in particular, suppose that such n exists). Then n divides k.*

Proof. We have the following claim.

Claim. y_0^k and y_k^k are comparable.

Using Claim, by Proposition 1.3(1) applied to y_i^k with $i = 0$ and $j = k$, we conclude that $0 \equiv k \pmod{n}$, i.e. n divides k. Thus, we only need to prove the claim.

Proof of claim. Suppose that $y_0^k \geqslant y_n^k$ (the other case is analogous). Then we have:

$$y_0^k \geqslant y_n^k \geqslant y_{2n}^k \geqslant \ldots \geqslant y_{kn}^k. \tag{1}$$

Note that immediately by definition of y_j^k, $x_k \geqslant y_0^k, y_k^k$. Thus y_0^k and y_k^k are comparable. As in the preceding paragraph, we conclude that if $y_0^k < y_k^k$, then $y_0^k < y_{nk}^k = y_{kn}^k$. This contradicts (1), so we must have $y_0^k \geqslant y_k^k$. □(claim)

□

Proposition 1.6. *Suppose $k \in \mathbf{N}$ is minimal such that for some $n > 0$, y_0^k is comparable but not equal to y_n^k. Suppose in addition that $k > 0$.*

Then k divides every such n, and the minimal n is equal to k.

Proof. We have the following claim.

Claim. If $n = 1$, then $k = 1$.

Proof. The proof is by contraposition. Suppose that $k > 1$. Then for every n, $y_0^1 = x_0 \wedge x_1$ is either equal or incomparable to $y_n^1 = x_n \wedge x_{n+1}$.

In particular, y_0^1 is neither greater nor smaller than y_1^1. By Proposition 1.4, it follows that $y_0^1 \wedge y_1^1 = y_1^1 \wedge y_2^1$. In particular, $n > 1$. □

Suppose for now that n is minimal. Then by Proposition 1.5, n divides k. Put $k' := k/n$, consider the partial automorphism p^n, and put $x'_j = x_{nj}$, and for $y'_j = x_{nj} \wedge x_{nj+k} = x'_j \wedge x'_{j+k'}$.

Then we have that $y'_0 = x'_0 \wedge x'_{k'} = x_0 \wedge x_k$ is comparable but not equal to $y'_1 = x'_1 \wedge \wedge x'_{1+k'} = x_n \wedge x_{n+k}$. Moreover, by minimality of k, it follows that k' is minimal such that for some n' the meets $x'_0 \wedge x'_{k'}$ and $x'_{n'} \wedge x'_{n'+k'}$ are comparable but not equal. By Claim it follows that $k' = 1$, i.e. the minimal n is k.

The case of arbitrary n follows from Proposition 1.3(1) (applied to y_i^k). □

Definition 1.7. Let $\mathcal{O} = \{x_0, \ldots, x_m\}$ be the orbit or a finite partial automorphism p (such that $p(x_i) = x_{i+1}$).

Then:

- if for some positive integer $n \leqslant m$ we have that $x_0 = x_n$, and n is minimal, then we say that $\mathcal{O}$ is a *loop of length n*;
- if for some positive integer n we have that x_0 is comparable (but not equal) to x_n, and n is minimal, then we say that $\mathcal{O}$ is a *spiral of length n*;
- if for some positive integer $k \leqslant m/2$, we have that x_0 and x_k are incomparable, but $x_0 \wedge x_k$ is not equal to $x_k \wedge x_{2k}$, and k is minimal, then we say that $\mathcal{O}$ is a *comb spiral of lenth k*;
- otherwise (if $\mathcal{O}$ is not a loop, a spiral, nor a comb loop), we say that $\mathcal{O}$ is a *pseudoloop.*

Remark 1.8. Note that if $x_0, x_1, \ldots, x_m$ is of comb type k, then $y_0^k, y_1^k, \ldots, y_{m-k}^k$ is of spiral type k.

Conversely, it is not hard to see that if $y_0, \ldots$ is and orbit of p of spiral type k, then for some $\bar{p} \supseteq p$, there is a $\bar{p}$-orbit $x_0, \ldots$ of comb type k, such that $y_j = x_j \wedge x_{j+k}$. ◊

Proposition 1.9. *If $\mathcal{O}$ is of finite or spiral type, then for every k, either $x_0 \wedge x_k = x_k \wedge x_{2k}$ or x_0 and x_k are comparable.*

Proof. Suppose that $\mathcal{O}$ is of finite or spiral type n. If n divides k, then trivially x_0 and x_{nk} are comparable. Otherwise, by Proposition 1.3, we have that $x_0 \wedge x_k = x_{kn} \wedge x_{kn+k}$.

But if $x_0 \wedge x_k < x_k \wedge x_{2k}$, then $x_0 \wedge x_k < x_{kn} \wedge x_{kn+k}$, and likewise if $x_0 \wedge x_k > x_k \wedge x_{2k}$, then $x_0 \wedge x_k > x_{kn} \wedge x_{kn+k}$. In both cases, we have a contradiction. But since $x_0 \wedge x_k, x_k \wedge x_{2k} \leqslant x_k$, they are compatible, so they must be equal. □

Note that Proposition 1.9 implies that orbits of finite or spiral type are not of comb type.

Remark 1.10. Proposition 1.6 implies that $\mathcal{O}$ is of comb type k if and only if k is minimal such that for some n, the meets $x_0 \wedge x_k$ and $x_n \wedge x_{n+k}$ are comparable. $\lozenge$

Conjecture 1.11. *If $\mathcal{O}$ is of open type, then it can be extended to an orbit of any other type.*

If $\mathcal{O}$ is not of open type, then it is determined.

2. Psuedoloops analysis

In this section, we have a blanket assumption that $\mathcal{O}$ is a loop or a pseudoloop, i.e. $\mathcal{O}$ is an antichain and for every k, we have $x_0 \wedge x_k = x_k \wedge x_{2k}$.

Remark 2.1. If $x_0 \wedge x_k = x_k \wedge x_{2k}$, then for every integer i we have $x_0 \wedge x_k = x_{ik} \wedge x_{(i+1)k}$. $\lozenge$

Proposition 2.2. *Suppose $\mathcal{O}$ is a pseudoloop and k, n are positive integers.*

Then $x_0 \wedge x_k \leqslant x_0 \wedge x_{kn}$.

Proof. This is immediate from Proposition 1.4. $\square$

Proposition 2.3. *Let $d = (k_1, k_2)$ be the greatest common divisor of k_1 and k_2. Then either $x_0 \wedge x_{k_1} = x_0 \wedge x_d$ or $x_0 \wedge x_{k_2} = x_0 \wedge x_{k_2}$.*

Proof. Let a, b be positive integers such that $ak_1 = bk_2 + d$. Then (by Remark 2.1)

$$(x_0 \wedge x_{ak_1}) \wedge (x_0 \wedge x_{bk_2}) \leqslant x_{ak_1} \wedge x_{bk_2} = x_0 \wedge x_d.$$

On the other hand, by Proposition 1.4, $x_0 \wedge x_d \leqslant x_0 \wedge x_{ak_1}$ and $x_0 \wedge x_d \leqslant x_0 \wedge x_{bk_2}$, thus

$$(x_0 \wedge x_{ak_1}) \wedge (x_0 \wedge x_{bk_2}) = x_0 \wedge x_d$$

Since $x_0 \geqslant x_0 \wedge x_{ak_1}, x_0 \wedge x_{bk_2}$, the latter two are comparable, so

$$x_0 \wedge x_d = (x_0 \wedge x_{ak_1}) \wedge (x_0 \wedge x_{bk_2}) = \min(x_0 \wedge x_{ak_1}, x_0 \wedge x_{bk_2}),$$

which completes the proof. $\square$

Define a sequence $(n_i)_i$ in the following way: $n_0 = 1$, while n_{i+1} is the smallest positive integer n such that $x_0 \wedge x_n > x_0 \wedge x_{n_i}$ (if it exists).

Remark 2.4. Immediately by the definition, for every k, there is some i such that $x_0 \wedge x_k = x_0 \wedge x_{n_i}$. $\lozenge$

Proposition 2.5. *For each i, $x_0 \wedge x_k \geqslant x_0 \wedge x_{n_i}$ if and only if n_i divides k.*

It follows that $(n_i)_i$ is strictly $|$-increasing and (by Remark 2.4) that for $n(k)$ being the largest n_i such that n_i divides k, we have $x_0 \wedge x_k = x_0 \wedge x_{n(k)}$.

Proof. $\Leftarrow$ is immediate by Proposition 1.4.

For $\Rightarrow$, note that if $d = (k, n_i)$, then by Proposition 2.3, we have $x_0 \wedge x_d = x_0 \wedge x_{n_i}$. By minimality of n_i, it follows that $d = n_i$, so n_i divides k. $\square$

Proposition 2.6. *For each i and positive $m < n_i$, we have $x_0 \wedge x_{n_i} \neq x_m \wedge x_{m+n_i}$.*

Proof. Suppose $x_0 \wedge x_{n_i} = x_m \wedge x_{m+n_i}$. Then $x_0 \wedge x_{n_i} \leqslant x_m$, so $x_0 \wedge x_{n_i} \leqslant x_0 \wedge x_m$. But then by Proposition 2.5, n_i divides m, so $m \geqslant n_i$. □

Corollary 2.7. *If y is in the $\wedge$-lattice generated by $\mathcal{O}$, $k \leqslant 2n$ and $y < x_0$, then $y \geqslant x_0 \wedge x_k$ if and only if $p^k(y) = y$.*

Proposition 2.8. *Consider a partial orbit $x_0, \ldots, x_{n+1}$.*

Let i_0 be such that n_{i_0} is maximal not greater than n.

If n_{i_0} divides $n+1$, then there are two ways to amalgamate $x_0 \ldots, x_n$ and $x_1, \ldots, x_{n+1}$: either $x_0 \wedge x_{n+1} > x_0 \wedge x_{n_{i_0}}$ (so $n_{i_0+1} = n+1$), or $x_0 \wedge x_{n+1} = x_0 \wedge x_{n_{i_0}}$.

Given any $k < n$, we have $x_{n+1} \wedge x_{n+1-k} = x_{n+1} \wedge x_{n+1-n(k)}$. If $n(k) < \sqrt{n}$, then $x_{n+1} \wedge x_{n+1-n(k)}$

Otherwise, if n_i does not divide $n+1$, then