Proposition 0.1. *Suppose $(L,<)$ is a an infinite linear order. Then at least one of the following holds:*

(1) there are arbitrarily long finite intervals in L,
(2) there is a dense interval in L,
(3) there is a uniform bound on the size of finite intervals in L, and the set of left endpoints of the maximal closed intervals is dense.

Proof. Let us call a point $a \in L$ *semi-isolated* if it has a successor or a predecessor in L, and denote by $S(L)$ the set of all semi-isolated points in L.

Let us say that $I \subseteq S(L)$ is a *component* of $S(L)$ if it consists of all points in finite distance to a given $a \in S(L)$, i.e. for some (equivalently, every) $a \in I$, we have that $b \in I$ if and only if (a,b) and (b,a) are finite.

Now, if $S(L)$ has arbitrarily large components, then it clearly has arbitrarily long finite intervals, so 1 is satisfied, and we are done.

Otherwise, the sizes of components of $S(L)$ are uniformly bounded. In particular, they are all finite. Denote by $S_-(L)$ the set of left endpoints of components in $S(L)$. If $S_-(L)$ is dense, we are done (3 is satisfied).

Otherwise, there are some $a,b \in S_-(L)$ such that $a<b$ and $(a,b)\cap S_-(L)=\emptyset$. But then if we take a^+ to be the largest element in the component of a in $S(L)$ — such element exists, because the component is finite — then $(a^+,b)\cap S(L)=\emptyset$, so no point in the interval (a^+,b) has a successor or a predecessor. On the other hand, the interval (a^+,b) cannot be empty: otherwise, b would be the successor of a^+, which would contradict the choice of a^+. Thus, (a^+,b) is a dense interval, so 2 is satisfied, and we are done. □

Corollary 0.2. *Suppose $(L,<)$ is an infinite $\aleph_0$-saturated linear order (or even just one which realises all 2-types over $\emptyset$). Then $(L,<)$ interprets (without parameters) a linear order $(P,<)$ which has no endpoints and is discrete or dense.*

Proof. Apply Proposition 0.1. If 2 or 3 occurs, we have a definable subset $(P',<)$ of L which is dense. If it has a left endpoint a, then $(a,+\infty)\cap P'$ is definable, has no left endpoint and is still dense. Likewise with the right endpoint.

The only possibility to consider is when $(L,<)$ has arbitrarily long finite intervals. Note that then it follows by compactness that there is some interval (a,b) in L which is infinite and discrete. Thus we may assume without loss of generality that L is discrete.

Suppose also that L has both a minimal and maximal elements $a_{-\infty}, a^{+\infty}$. The other cases are similar. Then we m □

1. Hereditary G-compactness, hereditary Borel cardinality

Definition 1.1. We say that a theory T is *G-compact* if $\equiv_{\mathrm{L}}$ is type-definable, or equivalently, if the Galois group $\mathrm{Gal}(T)$ is a Hausdorff group.

Definition 1.2. We say that a structure is *G-compact* if its theory is G-compact.

Definition 1.3. If M and $N = (S_i, R_j, f_k, c_l)_{i,j,k,l}$ (where S_i are sorts, R_j are predicates, f_k are functions and c_l are constants) are structures (possibly in different languages), then we say that M *interprets* N if we can in M^{eq} definable sets S'_i, definable relations R'_j, definable functions f'_k and definable points c'_l, such that $(S'_i, R'_j, f'_k, c'_l)_{i,j,k,l} \cong N$.

Definition 1.4. A theory T is said to be *hereditarily G-compact* if for every model $M \models T$, and every structure N interpreted by M (with parameters), N is G-compact.

Remark 1.5. It is enough to consider any single $|T|^+$-saturated $M \models T$ (the only way the choice of the model matters is in the realised types of the parameters used in the interpretation). Furthermore, one can check that T is hereditarily G-compact if and only if its restriction to every countable sublanguage is hereditarily G-compact. (?)

Fact 1.6. *Every simple theory is G-compact, and every theory interpreted by a simple theory is simple. Therefore, every simple theory is hereditarily G-compact.*

Example 1.7. Let T be any non-G-compact theory. Then T^{Sk}, the Skolemization of T, is G-compact but not hereditarily G-compact (because it interprets T).

Example 1.8. Any o-minimal expansion of a group (with at least two definable points) is G-compact (because it has definable Skolem functions). On the other hand, we will see below that no theory which interprets dense linear orderings is hereditarily G-compact (this includes many o-minimal theories, such as all expansions of the real field).

Question 1.9. *Is simplicity equivalent to hereditary G-compactness?*

Question 1.10. *Is NSOP equivalent to hereditary G-compactness?*

Definition 1.11. Given two posets $P = (P, <_P), (Q, <_Q)$, the *linear sum* $P \oplus Q$ is defined as $(P \sqcup Q, <)$ where $a < b$ if:

- $a \in P$ and $b \in Q$, or
- $a, b \in P$ and $a <_P b$,
- or $a, b \in Q$ and $a <_Q b$.

Informally, $P \oplus Q$ is the disjoint union of P and Q with Q put after P.

Remark 1.12. The linear sum $\oplus$ is clearly associative. $\diamondsuit$

Theorem 1.13. *Suppose $(P, \leq)$ is a partially ordered set such that the three natural embeddings of P into $P \oplus P \oplus P$ are elementary. Then P is not hereditarily G-compact (even without parameters, and without imaginary sorts if one has sorts with arbitrarily many definable constants).*

Proof. Let c_n^i, $n \in \mathbf{N}^+$, $i = 0, \dots, n-1$ be any (possibly imaginary) pairwise distinct, definable constants. Let $P_n = \bigcup_{i=0}^{n-1} P \times \{c_n^i\}$. Furthermore, define a cyclic order S_n on P_n by saying that $S_n((p_1, c_n^i), (p_2, c_n^j), (p_3, c_n^k))$ whenever $S(i,j,k)$ (where S is the standard cyclic ordering on $\{1, \dots, n\}$) or $i = j$ and $p_1 < p_2$ or $j = k$ and $p_2 < p_3$ or $k = i$ and $p_3 < p_1$. Finally, put $R_n(p, a_n^i) := (p, a_n^{i+1})$ (with addition modulo n in the upper index).

Put $\mathcal{P} = (P_n, S_n, R_n)_{n \in \mathbf{N}^+}$. Clearly P interprets $\mathcal{P}$ (without parameters). We will show that $\mathcal{P}$ is not G-compact.

Denote by $\mathcal{P}^*$ a model constructed analogously to $\mathcal{P}$, only with P replaced by its monster model extension P^*. It is easy to see that then $\mathcal{P} \preceq \mathcal{P}^*$ (because they are interpreted in the same way from P and P^*, respectively).

Choose arbitrary $p_0 \in P$. We will show that for $n > 1$,

$$n/2 - 1 < d_L((p_0, c_n^0), (p_0, c_n^{\lfloor n/2 \rfloor}) < \infty.$$

This will complete the proof, as the diameter of $[(p_0, c_n^0)_{n \in \mathbf{N}^+}]_{\equiv_{\mathrm{L}}}$ will be unbounded.

For the first inequality, note that if $\bar{p}_1, \bar{p}_2 \in P_n^*$ have the same type over some $\mathcal{M} \preceq \mathcal{P}^*$, we must have $S_n(\bar{p}_1, \bar{p}_2, R_n(\bar{p}_1))$ or $S_n(\bar{p}_1, \bar{p}_2, R_n^{-1}(\bar{p}_1))$ (because there are some points in $\mathcal{M}$ between $\bar{p}_1$ and each of $R_n(\bar{p}_1)$ and $R_n^{-1}(\bar{p}_2)$). In particular, the second coordinate can change by at most one step between $\bar{p}_1$ and $\bar{p}_2$. Therefore, since c_n^0 and $c_n^{\lfloor n/2 \rfloor}$ differ by $\lfloor n/2 \rfloor$, we obtain the first inequality.

For the second, more substantial step, by the assumption and saturation of P^*, we can find some $P^-, P^+ \preceq P^*$ such that $P^- < P < P^+$ and $P^- \cup P \cup P^+ \preceq P^*$, and such that there exists an $f\colon P^- \cup P \cup P^+$ such that restrictions to P^-, P and P^+ are isomorphisms with P, P^+ and P^-, respectively.

Now, define partial functions $\sigma_-, \sigma, \sigma_+ \colon \mathcal{P}^* \to \mathcal{P}^*$ by the formulas:

$$\sigma_-((p, c_n^i)) := \begin{cases} (f(p), c_n^i) \text{ if } p \in P^- \\ (p, c_n^i) \text{ if } p \in P^+ \end{cases}$$

$$\sigma((p, c_n^i)) := \begin{cases} (f(p), c_n^i) \text{ if } p \in P \\ (p, c_n^i) \text{ if } p \in P^- \end{cases}$$

$$\sigma_+((p, c_n^i)) := \begin{cases} (f(p), c_n^{i+1}) \text{ if } p \in P^+ \\ (p, c_n^i) \text{ if } p \in P \end{cases}$$

It is easy to check that of these is elementary partial maps in $\mathcal{P}^*$ (because their domains and ranges are models of $\mathrm{Th}(\mathcal{P})$, and they are clearly isomorphisms between those), so they extend to automorphisms of $\mathcal{P}^*$. Moreover, it is easy to see that the extensions are all Lascar strong automorphisms, and $\sigma_- \circ \sigma_+ \circ \sigma((p, c_n^i)) = (p, c_n^{i+1})$. ☐

Proposition 1.14. *If P is a model complete poset such that for any $Q \equiv P$ we have $P \oplus Q \equiv P$, then P satisfies the hypothesis of Theorem 1.13.*

Proof. By applying assumption for $Q = P$, we have $P \equiv P \oplus P$, and by applying it again for $Q = P \oplus P$, we get $P \equiv P \oplus P \oplus P$. The result follows immediately by model completeness. $\square$

Proposition 1.15. $(\mathbf{Q}, <)$ *and* $(\mathbf{Z}, <)$ *satisfy the assumptions of Theorem 1.13, so they are not hereditarily G-compact (without parameters).*

Consequently, every infinite linear ordering without endpoints which is discrete or dense is not hereditarily G-compact (without parameters).

Proof. In both cases, the conclusion follows from a straightforward application of the Ehrenfeucht-Fraïssé games. For example, to check that the middle copy of $\mathbf{Z}$ is elementarily embedded in $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} = \mathbf{Z}^{\oplus 3}$, choose some an arbitrary finite sequence $k_1, \ldots, k_n \in \mathbf{Z}$, and enumerate $\mathbf{Z}^{\oplus 3}$ as $\mathbf{Z} \times \{1, 2, 3\}$. We need to show that $(\mathbf{Z}, <, k_1, \ldots, k_n) \equiv (\mathbf{Z}^{\oplus 3}, (k_1, 2), \ldots, (k_n, 2))$. For an Ehrenfeucht-Fraïssé game of length N, $\square$

Proposition 1.16. *If* $(L, <)$ *is an* $\aleph_1$*-saturated infinite linear order, then there is an infinite definable set* D *such that* $(D, <\restriction_D)$ *is dense or discrete.*

Proof. Notice that since $(L, <)$ is $\aleph_1$-saturated, it follows immediately that given a definable $D \subseteq L$, $(D, <\restriction_D)$ is also $\aleph_1$-saturated.

If L defines an infinite discrete linear order, we are done. So suppose it does not. We aim to show that there is a definable dense linear order.

Given a linear order $(K, <)$, denote by $S(K)$ the set of all immediate successors in K, and denote by $P(K)$ the set of all immediate predecessors. It is clear that $S(K)$ and $P(K)$ are both definable in $(K, <)$.

Claim 1: Let $(K, <)$ be an arbitrary linear order. If I is a convex component of $P(K)$ in K (i.e. a maximal subset of $P(K)$ which is convex in K), then either I is finite or I contains arbitrarily long finite intervals.

Proof. If I is finite, we are done. Suppose, then, that I is infinite.

For brevity, given $a \in S(K)$, write $a + (-1)$ or $a - 1$ for its predecessor, and given $a \in P(K)$, write $a + 1$ for its successor in K. In the same way, let us write $a + k$ for all $k \in \mathbf{Z}$ for the k-th successor (when it exists).

Take any $a \in I$, and consider the set S_a of $a + k$ for all $k \in \mathbf{Z}$ (for which this makes sense).

Note that S_a is convex and all elements of S_a, except the last one (if it exists) are contained in I.

Thus, if S_a is infinite, we are done (because we can find arbitrarily long finite intervals in $I \cap S_a$). So suppose S_a is finite. Then it has a smallest element a_- and a largest element a^+. Note that this implies that $a^+ \notin P(K)$, so $I \setminus S_a = I \setminus [a_-, a+] = I \cap (-\infty, a_-)$. Thus, because I is infinite, there is some $b \in I$ such that $b < a_-$. But then for each $n \in \mathbf{N}_{>0}$, if $b + n$ exists, then $b + n \in (b, a_-) \subseteq I$ (otherwise, for some n we would have $b + n = a_-$, which is impossible, because a_-

is not a successor). This implies that $b+n+1$ exists. Thus by induction, for all $n \in \mathbf{N}$, $b+n$ exists, and $(b, b+n+1)$ is an interval with n elements. $\square$(claim)

Now, if some convex component of $P(L)$ contains arbitrarily long finite intervals, then by compactness, there is an infinite discrete interval, which contradicts the assumption from the second paragraph.

Otherwise, all convex components of $P(L)$ are finite, so $L' = L \setminus P(L)$ is an infinite discrete linear order: indeed, if there are finitely many convex components, then $P(L)$ is finite, in which case L' is clearly infinite (because L is infinite). Otherwise, if $L \setminus P(L)$ was finite, then $P(L) = L \setminus (L \setminus P(L))$ would have only finitely many convex components, all of them finite. But then L itself would be finite, a contradiction.

Similarly, we show that all convex components of $S(L')$ are finite, so $L_1 = L' \setminus S(L')$ is an infinite linear order.

In the same way, we define L_n for each $n \in \mathbf{N}_{>0}$ recursively as $L_{n+1} := L_n \setminus P(L_n) \setminus S(L_n \setminus P(L_n))$, and they are all infinite linear orders. There are two cases: either L_n stabilises (i.e. eventually $S(L_n) = P(L_n) = \emptyset$), or it does not. In the former case, L_n is clearly dense. $\square$

Corollary 1.17. *Every o-minimal expansion of a group is G-compact but not hereditarily G-compact.*

Proposition 1.18. *By an analgous construction, the theory of (infinite) discrete linear orders is not hereditarily G-compact.*

Proposition 1.19. *If $(L, \leq)$ is a totally ordered set, L interprets a dense linear order or L interprets a discrete linear order.*

Proposition 1.20 (?). *If $P = (P, \leq)$ is a poset such that $P \equiv P \sqcup P$ ordered in such a way that the first copy goes before the second copy, and P is model complete, then an analogous construction works (probably).*

Remark 1.21. For the "circles" construction to work for a general poset P, we certainly need P to be upwards and downwards directed, without least or largest element.

Remark 1.22. If P is a poset, and b is an element such that b is not the supremum of some two elements, then $(-\infty, b)$ is upwards-directed.

Question 1.23. *Suppose P has the property that every element is the supremum of some two elements. Does this imply IP (or some other general property)? (Note: atomless Boolean algebras have this property, as do binary ordered trees (growing down). They both have IP and SOP.)*

Question 1.24. *Are atomless Boolean algebras hereditarily G-compact?*

Remark 1.25. In atomless Boolean algebras, we can always eliminate the parameters (up to a definable bijection). From now on, we will assume that there are no parameters.

Lemma 1.26. *Let T be the theory of atomless boolean algebras. Let $\varphi(x_1, \ldots, x_n)$ be any formula such that $\varphi(x^1, \ldots, x^n) \vdash \bigwedge_i p(x^i)$ where p is a complete $\emptyset$-type. Then we can assume without loss of generality (up to a $\emptyset$-definable bijection) that p is the type of a partition and $\varphi(x^1, \ldots, x^n)$ is equivalent to a formula of the form $\bigvee \bigwedge x^i_j \cap x^{i'}_{j'} [=/\neq] \emptyset$ (i.e. a set of* permissible *intersection tables).*

Proof. The first part is obvious. The second part is just calculation (and q.e.). □

Corollary 1.27. *Atomless boolean algebras don't interpret infinite linear orders.*

Proof. Suppose a formula $\varphi(x, y)$ defines a linear preorder with infinitely many equivalence classes. By ω-categoricity, one of the $|x|$-types intersects infinitely many of these classes, so we can assume without loss of generality that φ implies this type, and so, by the preceding lemma, that x is a partition.

Now, let x and y be arbitrary partitions of the appropriate size. Let z be a partition of the same size, independent of x and y. Then by linearity, either $\models \varphi(x, z)$ or $\models \varphi(z, x)$. But the type of x over z is the same as the type of z over x, so in fact $\models \varphi(x, z) \wedge \varphi(z, x)$. For the same reason, $\models \varphi(z, y) \wedge \varphi(y, z)$, and hence by transitivity $\models \varphi(x, y) \wedge \varphi(y, x)$, so x and y are equivalent and, as they were arbitrary, φ defines a total relation on its domain, a contradiction. □

Remark 1.28. Suppose atomless boolean algebras interpret a nontrivially directed poset. Let $\varphi(x, y)$ be the formula defining it. Then we can restrict φ to a single complete type in such a way that it still defines a nontrivially directed poset.

Proof. Note that a (countable) poset is nontrivially directed if and only if it has a cofinal chain. By pigeonhole principle, infinitely many members of the chain lie in a single type, and so we can apply the lemma. □

Corollary 1.29. *Suppose $\varphi(x, y)$ defines a nontrivially directed poset on partitions of a given size. Let x be any partition, and let $\bar{x}$ be the upper bound of all permutations of x. Then all row permutations of the intersection table of x and $\bar{x}$ are permitted by φ.*

Remark 1.30. Given a poset P, for some interval in P to be nontrivially directed, it is necessary that for all $p_0, p_1 \in P$ there are some p, p' such that $p_0 < p, p' < p_1$ and p_1 is the supremum of p and p'.

Example 1.31. Define a poset P as follows. Let $P_0 = \{p_{\min}, p_{\max}\}$, where $p_{\min} < p_{\max}$. We construct P_{n+1} from P_n by adding, for each pair $p_0, p_1 \in P_n$ of distinct points such that $[p_0, p_1] = \{p_0, p_1\}$ in P_n, a pair of points p, p' such that $p_0 < p, p' < p_1$ and p and p' are incomparable. Then put $P = \lim_n P_n$ with inclusions.

Remark 1.32. The poset P in the preceding example has the property that no interval is nontrivially directed. In fact, all (open) intervals are isomorphic.

Remark 1.33. P is a non-distributive lattice.

Remark 1.34. (?) The poset P has q.e. in the language of bounded lattices (i.e. $\cup, \cap, \leq, 0, 1$).

Proof. Let $\varphi(x, \bar{a})$ be quantifier-free formula with parameters $\bar{a}$, where $|x| = 1$. We can assume without loss of generality that $\bar{a}$ enumerates a sublattice of P. □

Remark 1.35. Intervals in P have NIP.

Conjecture 1.36 (Well-known?). *Every NIP unstable theory interprets an infinite linear order. (Note: every unstable weakly VC-minimal theory does.)*

Question 1.37. *Does an IP+SOP theory intepret an infinite linear order?*

Question 1.38. *For a poset $(P, \leq)$, how do the following properties relate:*
(1) P is directed (both ways?).
(2) "circles with P" "work".
(3) P has NIP.
(4) P is nice with respect to VC-codimensions.
What if we only want them up to interpretation?

Question 1.39. *Suppose P and Q are posets such that their induced circular orders are elementarily equivalent. Does the "circles" construction yield elementarily equivalent results?*

Example 1.40. $\omega \sqcup \omega^*$ yields the same thing as $\mathbf{Z}$.

2. Adding parameters an G-compactness

Example 2.1. Consider an $F = (F, \cdot, +, c)_{c \in \mathbf{Q}^{\mathrm{alg}}}$ algebraically closed field of characteristic 0 with parameters for all algebraic numbers. Then the Shelah strong types and types over the empty set coincide, but for any transcendental t, the types over $\{t\}$ do not coincide with strong types (e.g. $\sqrt{t}$ and $-\sqrt{t}$ have different strong types but the same type over t).

Question 2.2. *Suppose T is G-compact (KP and L strong types coincide). If we add to the language some parameters for elements of a model of T, is the resulting theory still G-compact (i.e. do the KP and L strong types with parameters coincide)?*

Proposition 2.3. *Suppose G is a group definable in $\mathfrak{C}$. Let I be a large set (independent of $\mathfrak{C}$). Consider a structure $\mathfrak{C}' = (\mathfrak{C}, I \times G', I, \cdot, \pi)$, where G' is a copy of G, the structure on $\mathfrak{C}$ is standard, there is no internal structure on $I \times G'$ or I, $\cdot \colon G \times (I \times G') \to (I \times G')$ is the left action $g \cdot (i, g') = (i, gg')$, and $\pi \colon I \times G' \to I$ is the standard projection.*

Then $\operatorname{Aut}(\mathfrak{C}') = G^I \rtimes (S_I \times \operatorname{Aut}(\mathfrak{C}))$*, where the action of S_I is obvious (and fixes G' coordinates and $\mathfrak{C}$),* $\operatorname{Aut}(\mathfrak{C})$ *acts on $\mathfrak{C}$ in the usual way and permutes the G' coordinates in the standard manner (fixing I), while G^I permutes the G' coordinates* *by acting on the right coordinatewise (and fixes $I, \mathfrak{C}$). Moreover,* $\mathrm{Autf}_L(\mathfrak{C}') = G^I \rtimes (S_I \times \mathrm{Autf}_L(\mathfrak{C}))$*, and similarly for Shelah and Kim-Pillay strong automorphism groups.*

Meanwhile, for any $i_0 \in I$ and $g' \in G$ we have that the orbits of (i_0, g') via the action of automorphisms, strong Shelah, Kim-Pillay and Lascar automorphisms respectively are the same as their G, G^0, G^{00} and G^{000} orbits, respectively.

Proof. The "meanwhile" part is standard, as once we add i_0 as a parameter, $\{i_0\} \times G'$ is just a principal homogenous space for G, which was analysed in detail in [**?**].

For the $\mathrm{Aut}(\mathfrak{C}')$, first note that $\mathrm{Aut}(\mathfrak{C})$ and S_I act naturally on $\mathfrak{C}'$ (by acting on $\mathfrak{C}$ part and permuting the G' coordinates in $I \times G'$, and by permuting the I coordinates in $I \times G'$, respectively), and the two actions commute. Thus it is easy to see that it is enough to show that $\mathrm{Aut}(\mathfrak{C}'/I\mathfrak{C}) = G^I$. But from the same arguments as in the preceding paragraph, the automorphisms of $\{i_0\} \times G'$ fixing $\mathfrak{C}$ are the same as G acting on the right. Clearly, the actions on $\{i\} \times G'$ and $\{j\} \times G'$ are independent, and each of them must be preserved setwise by $\mathrm{Aut}(\mathfrak{C}'/I\mathfrak{C})$, so we are done.

For the $\mathrm{Autf}_L(\mathfrak{C}')$, notice that clearly $S_I, G^I \leq \mathrm{Autf}_L(\mathfrak{C}')$ (because given a small model $M \preceq \mathfrak{C}'$ such that $M \cap I = I_0$ we have $S_{I \setminus I_0}, G^{I \setminus I_0} \leq \mathrm{Aut}(\mathfrak{C}'/M)$), and clearly $\mathrm{Autf}_L(\mathfrak{C}) \leq \mathrm{Autf}_L(\mathfrak{C}')$ (as we have no additional structure on $\mathfrak{C}$), so we have $G^I \rtimes (S_I \times \mathrm{Autf}_L(\mathfrak{C})) \leq \mathrm{Autf}_L(\mathfrak{C}')$. The converse is trivial, because $\mathrm{Autf}_L(\mathfrak{C}')$ acts on $\mathfrak{C}$ by Lascar strong automorphisms.

The argument for other two strong automorphism groups is analogous. $\square$

Example 2.4. Suppose $\mathfrak{C} = (\mathbf{R}^*, \mathbf{Z}^*)$ (a large real closed field and a monster extension of the additive group of integers, with a constant for $1 \in \mathbf{Z}$), and let G be the universal cover of $SL_2(\mathbf{R})$. Then the theory of $\mathfrak{C}$ is G-compact (and, in fact, $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Aut}(\mathfrak{C})$, because it has a pointwise definable model) and $G^{00} \neq G^{000}$. Consider $\mathfrak{C}'$ as in the last proposition.

By the proposition $\mathrm{Autf}_L(\mathfrak{C}') = \mathrm{Autf}_{KP}(\mathfrak{C}')$, but given any $i_0 \in I$ we have that $\mathrm{Autf}_L(\mathfrak{C}'/i_0) \neq \mathrm{Autf}_{KP}(\mathfrak{C}'/i_0)$.

Example 2.5. If we take for $\mathfrak{C}$ a real closed field and $G = (S^1)^*$, we will have, as before, $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Aut}(\mathfrak{C})$, whereas $G^0 \neq G^{00}$ (because G/G^0 is trivial and $G/G^{00} = G(\mathbf{R}) = S^1$). Thus by the proposition we obtain a theory which has $\equiv_{KP} = \equiv_{Sh}$, but not so after adding a single parameter.

Example 2.6. We can combine the two previous example by taking $\mathfrak{C}$ as in the first example, and $G = \widetilde{\mathrm{SL}_2(\mathbf{R})}^* \times (S^1)^*$, which yields an example with $\mathrm{Autf}_L(\mathfrak{C}') = \mathrm{Aut}(\mathfrak{C}')$, such that the automorphism group and all strong automorphism groups are pairwise distinct over any $i_0 \in I$.

Proposition 2.7. *The theory DLO of dense linear orderings (without endpoints) interprets without parameters a (structure analogous to) "circles" structure given* *by Casanovas et al, only without the "even" circles. Therefore, it is not hereditarily G-compact.*

Proof. Let us denote by N the entire model of DLO. Let $(c_{n,m})_{n,m}$ be dummy parameters (which can easily be defined in N^{eq}), where n natural numbers, and $1 \leq m \leq n$.

Then we interpret the structure as follows:

- M_n is the disjoint union $\bigcup_{m \leq n} \{c_{n,m}\} \times N$.
- g_n is defined by $g_n \cdot (c_{n,m}, x) = (c_{n,m+1}, x)$, where $c_{n,n+1} = c_{n,1}$.
- For each n, we have a natural circular order S'_n on $\{c_{n,1}, \ldots, c_{n,n}\}$. This allows us to define the (strict) circular ordering S_n on M_n.
  - If two elements a_1, a_3 have the same first coordinate, then an element a_2 is between them if and only if it is also has the same first coordinate, and is between according to the ordering on the second coordinate.
  - If a_1, a_2, a_3 have pairwise distinct first coordinates, then a_2 is between them if and only if the first coordinate is between the first coordinates of a_1 and a_3 (in the sense of S'_n).
  - If a_1, a_2 have the same first coordinate, distinct from that of a_3, then a_2 is between a_1 and a_3 if and only if the second coordinate of a_2 is larger or smaller than the second coordinate of a_1, depending in the obvious way on the relationship between the first coordinates of a_1, a_3.
  - The interpretation in case when a_2, a_3 share the first coordinate, distinct from that of a_1, is analogous.

This $(M_n, g_n, S_n)_{n \in 2\mathbf{N}+1}$ is clearly a model of the theory described in Casanovas et al, restricted to odd circles. Therefore, it is not G-compact, as was proven there, and DLO is not hereditarily G-compact. □

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