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Statement

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The Ph.D. thesis is based on papers [KPR15], [KR18] and [Rze17]. So I will first describe contributions of the authors of these papers, and then explain briefly how it is related to the thesis itself.

[KPR15]: Already during preparation of [KP17] we (Anand Pillay and myself) had an idea of applying the results obtained there to study Borel cardinalities of quotients of definable groups by their model-theoretic components. Then, during my visit in Notre Dame in 2014, together with Anand Pillay, we started to think about possible adaptation of the results from [KP17] to the context of the group of automorphism of the monster model of a given theory (in place of a definable group), and we discussed some very general reasons why it is plausible to expect that this could be used to say something about Borel cardinalities of Lascar strong types.

Then, I adapted the proofs of the main results of [KP17] to the context of $\mathrm{Aut}(\mathfrak{C})$ (where $\mathfrak{C}$ is a monster model) acting on the appropriate space of types, yielding a presentation of the Lascar Galois group of any theory as a quotient of a compact Hausdorff group, and similarly for arbitrary strong types. All of this is contained in Section 2 of [KPR15]. Next, using Section 2, I proved one of the the main results of [KPR15] (Theorem 4.1), which roughly says that for countable theories, for any strong type defined on a single complete type, type-definability coincides with smoothness (in the sense of descriptive set theory). All the corollaries and discussions in Section 4 of [KPR15] are mine. In order to prove Theorem 4.1, some additional preparatory technical work was needed, which was also done by myself in Subsections 3.1 and 3.2 (including fundamental Lemma 3.1).

Having proved Theorem 4.1, I suggested that Tomasz should use similarly the main results of Section 2 in the case of uncountable theories. Tomasz immediately noticed Proposition 3.7 and Corollary 3.8 (a variant of Mycielski's theorem) and used it together with Section 2 to prove that for arbitrary language (possibly uncountable), for arbitrary analytic strong type E (i.e. a bounded invariant equivalence relation refining the relation of having the same type over $\emptyset$) defined on a single complete type, E is type definable or has at least $2^{\aleph_0}$-classes. Then he came up with the idea of considering the property of relative definability of E. As a result, he proved the theorem which says that either E is relatively definable (and so has finitely many classes) or it has at least $2^{\aleph_0}$-classes. All of this (in a more general context) is contained in another key result of [KPR15] (Theorem 5.1). In fact, all results in Section 5 of [KPR15] were obtained by Tomasz (including Section 5.2 about trying to embed E_0 in the case of non-countable theories). My contribution to Section 5 were only minor corrections.

Theorem 4.1 and Tomasz's Theorem 5.1 allowed us to formulate the main result of [KPR15] in the form of an elegant trichotomy (see Theorem 6.1 for a general form, and see the abstract of [KPR15] for a particular case).

The appendix is a joint work of Tomasz and myself. I originally proved directly that stability implies the existence of the desired semigroup operation, and I suggested that Tomasz try to prove the converse, which I expected to be more complicated. And Tomasz proved the converse (with my little contribution in the form of Proposition A.6 and some corrections, e.g. in Corollary A.7). It is worth emphasizing that key Proposition A.1 and the notion of piecewise definability are due to Tomasz.

[Rze17]: Tomasz came up with a very natural idea of extending the context of [KPR15] to strong types which are not necessarily defined on a single complete type. He introduced the notion of weakly-orbital equivalence relations, extending both invariant equivalence relations for transitive actions and orbital equivalence relations. Then, roughly speaking, he proved in various contexts that for weakly-orbital relations "closedness" of all classes is equivalent to "closedness" of the relation itself. Combining this with the main result from [KPR15], he extended this result to weakly-orbital equivalence relations. Paper [Rze17] is solely due to Tomasz (both the ideas of what to do and their realization are his).

[KR18]: Right after completing the work on [KPR15], I had an idea that, in a countable theory, one should be able to use topological dynamics methods (similar to those from [KP17] and Section 2 of [KPR15]) in order to present any strong type defined on a single complete type as a quotient of a compact Polish group by some subgroup and to use this to give a more direct proof of Theorem 4.1 of [KPR15]. The problem in [KPR15] was that the monster model was uncountable (so too big to produce a Polish group), but even if it was countable, the compact group produced in [KPR15] could be not metrizable (so not Polish). So I came up with the idea of replacing the monster model by a sufficiently homogeneous countable model (which we called ambitious) and, more importantly, I divided the compact group obtained analogously to [KPR15] by a certain subgroup in order to get a Polish space and a Borel reduction in one direction (the produced Polish space "divided by a certain subgroup" Borel reduces to the original strong type) which yielded conceptually a more natural proof of Theorem 4.1 from [KPR15]. (In fact, I did it first in the context of definable groups and their quotients by certain model-theoretic component (namely G^{000}), and later in the context of ambitious models and strong types.) Tomasz then improved this by showing that we can divide the compact group obtained as in [KPR15] by the normal core of the group that I produced, obtaining a Polish group (instead of a Polish space) and still yielding a Borel reduction from the quotient of this Polish group by a certain subgroup to the strong type in question. Next, using (or rather extending) a correspondence between model-theoretic NIP and topological dynamical tameness, I proved that under NIP, we get reductions in both directions, i.e. the Borel cardinality of the given strong type equals to the Borel cardinality of a quotient of a compact Polish group by some subgroup.

At the end of completing [KPR15], Tomasz noticed that for strong types coarser than Kim-Pillay strong type a presentation as a quotient of a compact Polish group can be obtained in a much simpler way, without using topological dynamics (just using the Kim-Pillay Galois group instead). In particular, he distinguished descriptive set theoretic tools (which are really basic) essentially needed in the proof. In consequence, he got a more canonical proof in this particular case, allowing to transfer various good properties of the strong type in question to the corresponding properties of the group by which we have to divide the Kim-Pillay Galois group in order to obtain our strong type, both as a topological space and also as Borel cardinality (without the NIP assumption!). This is contained in Section 3 of [KR18] which is due to Tomasz.

Sections 4.2, 4.3 and 5 contain mainly known material, but Tomasz did a great job in finding in the literature the necessary references, so that the whole presentation may serve as a future reference. The notion of tame model is new and invented by Tomasz. The concept of ambitious model also seems new, but it was invented by myself (though the name was proposed by Tomasz).

Section 6 (containing essential for this paper development in pure abstract topological dynamics, the main outcome of which is a construction of a Polish compact group associated with a given metrizable dynamical system) is a joint work of Tomasz and myself. As described above (in the first paragraph of the description of [KR18]), I obtained the results of Sections 6.1 and 6.2 for topological dynamics of some flows in model-theoretic contexts of both definable groups and strong types, but Tomasz noticed that this generalizes to general topological dynamics, even slightly simplifying some of my proofs. So one could say that the results in Sections 6.1 and 6.2 are my results extended by Tomasz to a general context. The results of Section 6.3 were invented and proved by Tomasz: the trick with dividing by the normal core, and also the observation that for tame systems the group obtained as in [KPR15] is already Polish, so there is no need to divide this group by anything.

Section 7.1 is an adaptation of Section 2 of [KPR15] to the context of ambitious models (in [KPR15] it was done for monster models). It is a rather straightforward adaptation, but of highly non-trivial arguments. I more or less checked it while proving the results described above, and later Tomasz did it carefully, simplifying slightly some of the arguments.

Section 7.2 is new and due to Tomasz.

The main result of [KR18] is Theorem 7.13 from Section 7.3. Using his clear presentation of the argument in Section 3 (for strong types coarser than Kim-Pillay strong type) and Section 6, Tomasz was able to state this theorem in a very comprehensive way, including my original results (described in the first paragraph of the description of [KR18]) as an essential part, expanded by additional information about correspondence between good properties of the strong type in question and similar properties of the group by which we have to divide the Polish group that we produced in previous sections. Another important aspect of Theorem 7.13 is that the Polish group obtained there is the same for all strong types. This was observed by Tomasz and required an additional lemma, namely Lemma 7.12 from Section 7.2 which was proved by Tomasz.

Thus, in the current form and with the current proof, Theorem 7.13 is really a joint result of Tomasz and myself (but with the most essential part done by myself), as is Theorem 8.1. Theorem 8.4 is due to Tomasz. As I said in the first paragraph concerning [KR18], I originally obtained the essential part of Theorem 8.4 but for a definable group G and its particular subgroup $K := G^{000}$; Tomasz generalized it to any type-definable group G and an arbitrary invariant, bounded index subgroup K, and, as in the case of Theorem 7.13, he obtained additional information besides Borel reductions.

Having Theorem 7.13 in such a comprehensive form allowed us to recover not only Theorem 4.1 of [KPR15], but also the full trichotomy theorem from [KPR15]. The additional part from this trichotomy (the one involving relative definability) was recovered by Tomasz in Corollary 7.18. The comprehensive form of Theorem 8.4 allowed Tomasz to extend the analogous trichotomy obtained in Corollary 6.2 of [KPR15] for some quotients of type-definable subgroups of definable groups to quotients of general type-definable groups (which is an essential generalization).

The appendix, with an application of the main result to some concrete examples, is due to Tomasz.

Ph.D. thesis: Working on adaptations of topological dynamics methods (developed by Pillay and myself in [KP17] for definable groups) to other model-theoretic contexts, Tomasz came up with an idea that there should be a general abstract (purely topological dynamical) context unifying all the contexts considered so far and possibly many others. Indeed, Tomasz realized this idea in Section 5. He introduced group-like equivalence relations and several variants of them, the most important being [weakly] uniformly properly group-like equivalence relations. Then he adapted the original ideas from [KP17] (which were also adapted in Section 2 of [KPR15] and in Section 7.1 of [KR18]) and the idea of the proof of Theorem 7.13 of [KR18] to his abstract context, obtaining very general theorems (e.g. Theorems 5.42, 5.43, 5.44) in the spirit of Theorem 7.13 of [KR18] and other important results from Section 5 of [KPR15]. This required a lot of non-trivial work, and I did not give Tomasz any suggestions concerning this abstract context.

In Section 6, Tomasz obtained all the main results from [KPR15] and [KR18] as corollaries of his abstract theorems from Section 5. For this, it was enough to check that in the contexts considered in [KPR15] and [KR18] the equivalence relations in question are [weakly] uniformly properly group-like and then to apply results from Section 5. In fact, one can expect that the abstract context developed in Section 5 will apply to virtually all equivalence relations occurring in model-theoretic situations and possibly beyond. At the end of Section 6.???, according to my suggestion, Tomasz points out that it applies to topological groups considered as first order structures as in my recent paper with Anand Pillay, as well as to local Galois groups introduced by Kim, Lee, and Dobrowolski. This is an important motivation for the results in Section 5, because checking whether the original ideas from [KP17] and further ideas from [KR18] can be applied to each new context that naturally appears is a hard work.

The last subsection of Section 6 contains the material from the appendix in [KR18] (concrete examples), and is entirely due to Tomasz (as mentioned above).

Section 3 is entirely due to Tomasz. It contains the material from Section 3 of [KR18] discussed above, extended by analogous results in purely topological settings of equivalence relations on compact spaces invariant with respect to a transitive action of a compact group and of orbital equivalence relations of actions of compact groups.

Section 4.1 contains the material from Section 6 of [KR18] (which is my joint work with Tomasz, see above), and Section 4.2 – the material from Section 5 of [KR18], extended by Tomasz e.g. by introducing the notion of NIP type-definable set and relativized version of ambitious models. Tomasz used these additional notions in Section 6 (in particular, to weaken slightly the assumption in Theorem 8.1 of [KR18]).

Section 7 is entirely due to Tomasz. It contains the material from [Rze17] with some nice improvement. Namely, Tomasz introduced a certain abstract context involving "pseudo-closed" sets and agreeable actions, and proved some general theorems on [weakly] orbital equivalence relations with respect to agreeable actions. Then he used them together with some results from Sections 3 and 6 to deduce as corollaries all the results from [Rze17].

In Section 8, Tomasz extends the main result from Section 5.2 of [KR18] (which was also obtained by him) to the context of weakly uniformly properly group-like equivalence relations, and obtains the result from Section 5.2 of [KPR15] as a corollary.

Appendix A contains a concise but complete presentation of the fundamental for this thesis knowledge from topological dynamics. I think it can serve as a good reference in the future. Appendix B.1 essentially coincides with the appendix of [KPR15], which is a joint work of Tomasz and myself (see above). Appendix B.2 is due to Tomasz.

Section 2 contains various preliminaries important for this thesis.

References

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