Expansion of Nth order polynomials from roots

Tom Purnell

October 19th, 2022

1 Getting any polynomial from its roots

Any polynomial P where the first coefficient is 1 can be expressed as a product of $(x - r_n)$, where r_n is the nth root of P, note that N is being used to represent the order of the polynomial. using mathematical notation, we can express that as follows:

$$P = \prod_{n=1}^{N} (x - r_n) \tag{1}$$

If we want to account for the any other value of the first coefficient, we can just multiply the right side by it's value, a. In a sense we are just dividing the polynomial by the initial coefficient to force it to fit into this format.

$$P = a \prod_{n=1}^{N} (x - r_n)$$

$$\frac{P}{a} = \prod_{n=1}^{N} (x - r_n)$$

2 Expanding capital pi

If we were to expand this product for ascending values of N up to 4, we can quickly spot a pattern.

$$\frac{P_1}{a} = x^1 - x^0(r_1) + 0 - 0 + 0$$

$$\frac{P_2}{a} = x^2 - x^1(r_1 + r_2) + x^0(r_1 r_2) - 0 + 0$$

$$\frac{P_3}{a} = x^3 - x^2(r_1 + r_2 + r_3) + x^1(r_1 r_2 + r_1 r_3 + r_2 r_3) - x^0(r_1 r_2 r_3) + 0$$

$$\frac{P_4}{a} = x^4 - x^3(r_1 + r_2 + r_3 + r_4) + x^2(r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4)$$
$$- x(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4) + r_1 r_2 r_3 r_4$$

we can see that each coefficient is a sum of permutations of a certain length of the list of roots. the first term will always be the highest power of x, so naturally it is x^N. We can define a function $K(n)$ such that $K(n) = \{$the set of permuatations of length n of the roots$\}$, for a length of 0, this function returns 1. Using this we can redefine the polynomial P:

$$\frac{P}{a} = \sum_{n=0}^{N} \left[x^n \cdot K(N-n) \cdot (-1)^{N-n} \right] \tag{2}$$

Now all we need is a rigorous definition for summing permutations, purely mathematically.

3 Permutation function $K(n)$

We can factorise permutations in a much more neat manner as follows:

treating a, b, c and d as our list of roots to find the permutations of

$$\begin{aligned} K(2) &= ab + ac + ad + bc + bd + cd \\ &= a(b+c+d) + b(c+d) + cd \end{aligned}$$

replacing a .. d for r_1 .. r_4 so that we can use iteration
N is still the length of our list, which is 4 for now

$$\begin{aligned} &= r_1(r_2 + r_3 + r_4) + r_2(r_3 + r_4) + r_3 r_4 \\ &= r_1 \left(\sum_{n=2}^{N} r_n \right) + r_2 \left(\sum_{n=3}^{N} r_n \right) + r_3 \left(\sum_{n=4}^{N} r_n \right) \end{aligned}$$

if we expand our case for any value of N, and hence any order polynomial

$$= r_1 \left(\sum_{n=2}^{N} r_n \right) + r_2 \left(\sum_{n=3}^{N} r_n \right) + ... + r_{N-1} \left(\sum_{n=N}^{N} r_n \right)$$

and therefore we can generalise 2 length permutations for a list of N roots

$$= \sum_{i=1}^{N} r_i \left(\sum_{n=i+1}^{N} r_n \right)$$

note that the top of the outside sum should be $N-1$ but it makes little difference as the last term will always be 0. having it as N is helpful for furthur generalisation. We can then realise that the inner sum is the same as the result of the function $K(1)$ excluding the initial roots, up to and including i. therefore the only thing we would need to parse for the function to become recursive is the lower bound of the summation operator

$$K(2,i)=\sum_{n=i}^{N} r_n \cdot K(1,n+1)$$

Thankfully, this furthur generalises to a neat recursive function for any values of N or lengths of permutations, r is still the column matrix of our "roots", and N is it's length.

$$K(a,i)=\sum_{n=i}^{N} r_n \cdot K(a-1,n+1) \qquad \{a \neq 0\} \tag{3}$$

$$K(0,i)=1 \qquad \{a=0\} \tag{4}$$

To reiterate the definitions of each variable:
a is the length of the permutations we are summating;
i is the inital value for the lower bound of our summation operator, which will always start at 1 for our purposes, but ascend with recursions;
r is the column vector of our "roots", or any values we are finding the permutations of;
N is the length of the column vector r;
Now we can collate this all together to redefine the expansion without the capital pi

$$\prod_{n=1}^{N}(x-r_n)=\sum_{n=1}^{N}\left[x_n\left(K(N-n,1)\right)\cdot(-1)^{(N-n)}\right] \tag{5}$$

4 Conclusion

Obviously, this function can be used for more specific tasks instead of the more intensive idea of expanding an entire polynomial. if you want to transform roots of any polynomials to find a new polynomial, you could find and solve equations to connect the coefficients to the roots for each, using the permutation function.