Manifolds Notes on things I don't understand/remember

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1 Orientations

2 Abstract Manifolds

3 Vector Bundles

In defining a vector bundle over a manifold we associate a vector space to each point in the manifold and then arrange these together in a smooth way. Examples of this will include the tangent and normal bundles as defined in times gone past.

Suppose that M is an m-manifold. A family of vector spaces over M is a manifold E together with a smooth map $p : E \to M$ such that for all $x \in M$, the fibre $E_x := p^{-1}(x)$ has a vector space structure. Given $U \subseteq M$ we write $E\big|_U := p^{-1}U$ which we equip with the restriction of p.

Given another family over vector spaces $p' : E' \to M$ we define an isomorphism to be a diffeomorphism

$$\psi : E \to E'$$

with the property that $p' \circ \psi = p$ and $\psi : E_x \to E'_x$ restricts to a linear isomorphism for all $x \in M$. If such a map exists then E and E' are called isomorphic. Finally, a family E over M is called trivial if it is isomorphic to $M \times \mathbb{R}^q$ for some $q \in \mathbb{N}$.

$$\begin{tikzcd} E' \arrow[rr, "\psi"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

A vector bundle over M is a family of vector spaces $p : E \to M$ such that $\forall x \in M$ there is some open $U \subseteq M$ with $x \in U$ such that $E\big|_U$ is trivial.

Putting this another way, E is a vector bundle over M if we can find an atlas $\{\phi_alpha : U_\alpha \to V_\alpha\}$ for M such that for each α we get a diffeomorphism

$$\psi_\alpha : E\big|_{U_\alpha} \to V_\alpha \times \mathbb{R}^q \subseteq \mathbb{R}^{m+q}$$

which restricts to a linear isomorphism on each fibre. Note than then the maps $\{\psi_\alpha\}$ form an atlas for E which is called a locally trivialising atlas for E.

Note:

- Given a vector bundle, $p : E \to M$ is a surjective smooth submersion.

- Each fibre E_x is an embedded sub-manifold and the vector space operations are smooth.
- The notion of a vector bundle is preserved under isomorphism.

Example:

1. If M is a smooth manifold then the tangent bundle is a vector bundle over M. For a manifold in Euclidean space, this follows for the proof that TM was a manifold. For an abstract space, we can form a locally trivialising atlas as follows:

   Given a chart $\phi : U \to V \subseteq \mathbb{R}^m$ for M we construct a new map

   $$\psi : TU \to V \times \mathbb{R}^m \quad \text{by} \quad \psi(v) = (\phi(x), d_x\phi(v))$$

   where $x = pv$. These maps then form a locally trivialising atlas because when restricted to some x the map becomes the derivative map

   $$\psi_{p^{-1}(x)} : T_xM \to \{\phi(x)\} \times \mathbb{R}^m \quad \psi(v) = (\phi(x), d_x\phi(v))$$

   which is a linear isomorphism.

2. When can (and perhaps should) check that the atlas given for the normal bundle consists of maps which restrict to linear isomorphisms on each fibre.

3. The spaces B_n together with the projection to the circle yields a vector bundles. In particular, the Möbius band is a bundle over the circle. Note B_m is isomorphic to B_n if $|m-n|$ is even. But, B_0 cannot be isomorphic to B_1 since they are not even diffeomorphic. Hence the Möbius band is non-trivial.

Given a vector bundle $p : E \to M$ we say that section of E is a smooth map $s : E \to M$ such that $p \circ s$ is the identity on M. This generalises our previous notion of a section of the tangent space which is often referred to as a vector field on M. We say that a section is non-vanishing if $s(x) \neq 0$ for all $x \in M$. Given any smooth map $f : M \to \mathbb{R}^q$ we get a section of $M \times \mathbb{R}^q$.

Note: Any section is an embedding of M into E.

Theorem 3.1 (Hairy Ball Theorem)**.** *Every vector field on S^2 has at least one zero.*

Note: It turns out that any odd-dimensional manifold admits a non-vanishing vector field. An even-dimensional manifold may or may not. The torus does but it is the only compact connected orientable 2-manifold which does.

Lemma 3.2. *A vector bundle E is trivial if and only if there are sections $s_1, s_2, \dots, s_q$ such that $\{s_1, \dots, s_q(x)\}$ forms a basis for E_x for all $x \in M$.*

Proof. If $f : M \times \mathbb{R}^q$ is an isomorphism then set $s_i(x) = f(x, e_i)$ for some fixed basis $e_1, \dots, e_q$ for $\mathbb{R}^q$.

Conversely, if we have such sections s_i then we can define a map $f : M \times \mathbb{R}^q \to E$ by

$$f(x, (\lambda_1, \dots, \lambda_q)) = \sum_{i=1}^{q} \lambda_i s_i(x)$$

which yields an isomorphism. □

Example:

1. Using the Intermediate Value Theorem, one can show that the Möbius has no non-vanishing section and is hence a non-trivial bundle by the above Lemma.
2. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere in $\mathbb{R}^n$ then the normal bundle is trivial because the outward pointing unit normal is a nowhere vanishing section.

A manifold M is called parallelisable if the tangent bundle TM is trivial.

In view of the last lemma we can say that a manifold is parallelisable if and only if M admits a global frame field. Recall, a frame field is a family of vector fields v_i such that the $v_i(x)$ form a basis for T_xM at every $x \in M$. We know that frame fields exists locally but not necessarily globally.

Lemma 3.3. *Any parallelisable manifold is orientable.*

Proof. Exercise. □

Example:

- The circle S^1 is parallelisable.
- S^2 is not by the Hairy Ball Theorem.
- S^3 is parallelisable because given $x = (x_1, x_2, x_3, x_4) \in S^3$ we can form a frame field with

$$\begin{matrix} (-x_2, & x_1, & x_4, & -x_3) \\ (-x_3, & -x_4, & x_1, & x_2) \\ (-x_4, & x_3, & -x_2, & x_1) \end{matrix}$$

- S^5 is not strangely enough but S^7 is. It may be worth consulting your written notes from lectures to figure out how this frame was calculated.

3.1 Vector Fields

3.2 Whitney Sums

3.3 Cotangent Bundle

3.4 1-forms

4 Smooth Function Extension

5 Differential Forms