Ergodic Theory Notes

1 Basic Definitions

For this section we fix a probability space $(X, \mathcal{B}, \mu)$and we have a transformation $T : X \to X$ which is measurable in our probability space.

We say T is a measure preserving transformation (m.p.t.) or μ is a T-invariant measure if

$$\mu(T^{-1}B) = \mu(B) \quad \forall B \in \mathcal{B}$$

The push forward of μ by T is defiend to be

$$T_*\mu(B) = \mu(T^{-1}B) \quad \forall B \in \mathcal{B}$$

We say a measure μ is regular if $\forall B \in \mathcal{B}$ we have $\forall \epsilon > 0 \exists U \subseteq X$ open such that

$$B \subseteq U \quad \text{and} \quad \mu(U) < \mu(B) + \epsilon$$

An m.p.t T is said to be ergodic if

$$\forall B \in \mathcal{B},\ T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1$$

2 Facts on Fourier Series

Suppose $f \in L_1(\mathbb{T}^k)$ then we can define the Fourier coefficients by

$$\hat{f}(n) = \int_{\mathbb{T}^k} f(x)e^{-2\pi i n\cdot x}dx \quad \forall n \in \mathbb{Z}^k$$

Theorem 2.1 (Fejér's Theorem)**.** *The average of the partial Fourier sums converges uniformly to f, i.e.*

$$\frac{1}{N}\sum_{k=0}^{N-1} S_k f \to f \quad \textit{uniformly}$$

Theorem 2.2 (Riemann-Lebesgue Lemma)**.** *For all $f \in L_1(\mathbb{T}^k)$,*

$$\lim_{|n|\to\infty} \hat{f}(n) = 0$$

Theorem 2.3 (Reisz-Fisher Theorem)**.** *Define $S_N f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{2\pi i(n\cdot x)}$ then $S_n f \to f$ in L^2 for all $f \in L^2(\mathbb{T}^k)$.*

Corollary 2.4. *If $f \in L^2(\mathbb{T}^k)$ and $\hat{f}(n) = 0\ \forall n \in \mathbb{Z}^k \setminus \{0\}$, then f is constant.*

3 Criteria for measure preserving

Theorem 3.1. *Given $T : X \to X$ on a probability space (X, μ), the following are equivalent:*

1. *T is m.p.t*
2. $\int f \circ T d\mu = \int f d\mu \qquad \forall f \in L_1(X)$.

Recall the space $L_1(X) = \{f : x \to \mathbb{R} : \text{measurable} \quad ||f||_1 := \int |f| d\mu < \infty\}$

Theorem 3.2. *Given $T : X \to X$ on a probability space (X, μ), the following are equivalent:*

1. *T is m.p.t*
2. $\int f \circ T d\mu = \int f d\mu \qquad \forall f \in C(X)$.

So we see that in fact it suffices to check that T does not affect the integral of any continuous function f. However, we can extend this further using the density of trigonometric polynomials in the space of continuous functions. First, we need to define a trigonometric polynomial in arbitrary dimension on the k-torus $X = \mathbb{T}^k$ with $\mu = leb$ and $\mathcal{B} = Borel$.

$P : \mathbb{T}^k \to \mathbb{T}^k$ is a trigonometric polynomial if for some $N \geq 1$ and $c_n \in \mathbb{C}$ we can write

$$P(x) = \sum_{|n| \leq N} c_n e^{2\pi i n \cdot x}$$

where $n = (n_1, \dots, n_k) \in \mathbb{Z}^k, x = (x_1, \dots, x_k), |n| = |n_1| + \cdots + |n_k|$.

Note:

$$\int_{\mathbb{T}^k} e^{2\pi n \cdot x} dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

and hence

$$\int_{\mathbb{T}^k} P = c_0$$

Theorem 3.3. *Given $T : \mathbb{T}^k \to \mathbb{T}^k$ continuous and denoting by μ the Lebesgue measure.*

1. *T is m.p.t*
2. $\int P \circ T d\mu = \int P d\mu \qquad$ *$\forall$ trigonometric polynomials P.*

4 Criteria for Ergodicity

First another few definitions.

Given $A, B \subseteq X$, their symmetric difference is

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

A function f is T-invariant if $f \circ T = f$ a.e.
A function f is constant if $\exists c \in \mathbb{R}$ such that $f(x) = c$ almost everywhere.

Theorem 4.1. *Given a measure preserving transformation $T : X \to X$ and some $1 \leq p \leq \infty$. TFAE:*

1. *T is ergodic.*
2. *For all f measurable f invariant $\iff$ f constant.*
3. *For all $f \in L^p(X)$, f invariant $\iff$ f constant.*

Note: To check that T is ergodic it suffices to show that all invariant L^2 functions have zero Fourier coefficients away from zero.

To this end we present the following formula for computing the Fourier coefficients of invariant L^2 functions.

Theorem 4.2. *Given $f \in L^2$ which is invariant*

$$\hat{f}(n) = \lim_{N\to\infty} \int (S_N f)(Tx) e^{-2\pi i n \cdot x} dx$$

5 Theorems using Measure Preserving

Theorem 5.1 (Poincaré Recurrence Theorem)**.** *Given a probability space $(X, \mathcal{B}, \mu)$and $T : X \to X$ measure preserving. Then*

$$\mu\{x \in B : T^n x \in B \textit{ infinitely often}\} = \mu(B) \quad \forall B \in \mathcal{B}$$

6 Theorems using Ergodicity

Theorem 6.1 (Pointwise Ergoic Theorem - Birkhoff 1931)**.** *Given a measure space $(X, \mathcal{B}, \mu)$and a measure preserving transformation $T : X \to X$ and $f \in L^1(X)$. Then $\exists f^* \in L^1(X)$ invariant such that*

$$\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j \to f^* \textit{ a.e.} \quad \textit{and} \quad \int f^* = \int f$$

Note this does not actually need ergodicity. However, if we additionally assume ergodicity we can prove the following stronger result.

Corollary 6.2. *Given a probability space $(X, \mathcal{B}, \mu)$, T measure preserving and ergodic, $f \in L^1(x)$, then*

$$\underbrace{\frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j}_{\textit{Time average}} \to \underbrace{\int f d\mu}_{\textit{Space average}} \textit{ a.e.}$$