

# Measure Theory - Overview

## 1 Starting definitions

Start with a set  $X$ .  $\mathcal{A} \subseteq \mathcal{P}(X)$  is called an **algebra** if

- $\mathcal{A}$  is non-empty
- $X \in \mathcal{A}$
- $\mathcal{A}$  is closed and under complementation
- $\mathcal{A}$  is closed under finite unions and intersections

We obtain a  **$\sigma$ -algebra** if we also have closure under countable unions and intersections. We say a set  $A \in \mathcal{A}$  is  **$\mathcal{A}$ -measurable**.

Note that intersecting  $\sigma$ -algebras obtains a new  $\sigma$ -algebra but taking unions does not necessarily work. A very important  $\sigma$ -algebra is the **Borel  $\sigma$ -algebra**,

$$\mathcal{B}(\mathbb{R}^d) := \sigma\left(\{\text{open sets in } \mathbb{R}^d\}\right)$$

Note that it can also be formed by all closed sets, closed half-rays or half-open intervals.

Given a  $\sigma$ -algebra  $\mathcal{A}$  on a set  $X$ , a **measure** on  $(X, \mathcal{A})$  is a function  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that

- $\mu(\emptyset) = 0$
- Given disjoint  $A_1, A_2, \dots \in \mathcal{A}$ ,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

This gives us a **measure space**  $(X, \mathcal{A}, \mu)$ .

We call this measure **finite** if  $\mu(X) < \infty$  and  **$\sigma$ -finite** if we can write  $X$  as a union of finite measure sets.

Note measures are always increasingly monotonous and countably sub-additive. We also have the following very important property:

**Proposition 1.1** (Continuity of measure). *Given a measure space  $(X, \mathcal{A}, \mu)$ .*

- $A_1 \subseteq A_2 \subseteq \dots$  in  $\mathcal{A}$  then

$$\mu\left(\bigcup_i A_i\right) = \lim_i \mu(A_i)$$

- $A_1 \supseteq A_2 \supseteq \dots$  in  $\mathcal{A}$  and  $\mu(A_n) < +\infty$  for some  $n$ , then

$$\mu \left( \bigcap_k A_k \right) = \lim_k \mu(A_k)$$

**Note:** When taking the limit of an infinite intersection of nested sets, we only get continuity if our measure is finite or one of the nested sets has finite measure.

**Example:** Take  $\mu$  to be the counting measure on  $\mathbb{N}$  then consider the nested sets

$$A_k := \{k, k+1, k+2, \dots\}$$

then  $\bigcap_k A_k = \emptyset$  but  $\lim_{k \rightarrow \infty} \mu(A_k) = +\infty$ .

A very important measure is the Lebesgue measure since it coincides with our natural intuition for the measure of subsets of  $\mathbb{R}^d$ . First we need to define an outer measure.

An **outer measure** is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  such that

- $\mu^*(\emptyset) = 0$
- $A \subseteq B \subseteq X \implies \mu^*(A) \leq \mu^*(B)$
- Given a countable collection of subsets  $A_i \subseteq X$ , we have countable sub-additivity

$$\mu^* \left( \bigcup_i A_i \right) \leq \sum_i \mu^*(A_i)$$

Notably we require monotonicity as an axiom and also require that the outer measure is defined on every subset. This is a weaker notion than a measure.

Given  $A \subseteq \mathbb{R}$ ,  $\mathcal{C}_A := \{\text{Collections } \{(a_i, b_i)\}_{i=1}^\infty \mid -\infty < a_i < b_i < \infty, \bigcup_{i=1}^\infty (a_i, b_i) \supseteq A\}$ . This is the set of collections of finite open intervals which cover the set  $A$ . We can then define the **Lebesgue outer measure** on  $A$  to be

$$\lambda^*(A) := \inf \left\{ \sum_i (b_i - a_i) \mid \{(a_i, b_i)\}_{i=1}^\infty \in \mathcal{C}_A \right\}$$

**Proposition 1.2.**  $\lambda^*$  is an outer measure on  $\mathbb{R}$  and  $\lambda^*([a, b]) = b - a$ ,  $\forall a, b \in \mathbb{R}$  such that  $a \leq b$ .

*Proof.* The only difficult things to prove are countable sub-additivity and the desired value for intervals.

- (i) Given  $A_1, A_2, \dots \subseteq \mathbb{R}$  we may assume that  $\lambda^*(A_i) < \infty$  for all  $i$  else countable sub-additivity holds trivially. Given  $\epsilon > 0$  we can pick  $\{(a_{i_n}, b_{i_n})\}_{i=1}^\infty \in \mathcal{C}_{A_i}$  such that

$$\sum_{n=1}^\infty (b_{i_n} - a_{i_n}) < \lambda^*(A_i) + \frac{\epsilon}{2^i}$$

We can union these countably many countable collections to get another countable collection covering  $\cup_i A_i$ .

$$\begin{aligned}
\lambda^* \left( \bigcup_i A_i \right) &\leq \sum_j (b_j - a_j) \\
&= \sum_i \left( \sum_n (b_{i_n} - a_{i_n}) \right) \\
&\leq \sum_i \left( \lambda^*(A_i) + \frac{\epsilon}{2^i} \right) \\
&\leq \left( \sum_i \lambda^*(A_i) \right) + \epsilon
\end{aligned}$$

Taking  $\epsilon \rightarrow 0$  yields the result.

- (ii) Just think of a nice cover than does the job either exactly or to within  $\epsilon$ , depending on your philosophy surrounding the set  $\mathcal{A}$ .

□

By taking  $d$ -dimensional ‘rectangular’ intervals we can use the same procedure to define a Lebesgue measure on  $\mathbb{R}^d$  which similarly assigns expected ‘volumes’ to these rectangles. The proof of this is somewhat more involved. Now for a weird definition.

Given an outer measure  $\mu^*$  on  $X$ ,  $B \subseteq X$  is  $\mu^*$ -measurable if

$$\forall A \in \mathcal{P}(X) \quad \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

Intuitively,  $B$  is ‘nice’ if when we want to measure any other set we just measure the part inside and the part outside  $B$  and then add the measures together.

It’s easy to show that any set with zero outer measure or whose complement has zero outer measure is outer measurable. Define

$$M_{\mu^*} := \{\mu^*\text{-measurable sets}\}$$

**Theorem 1.3.** *Given an outer measure  $\mu^*$ ,  $M = M_{\mu^*}$  is a  $\sigma$ -algebra and  $\mu^*$  yields a measure when restricted to  $M_{\mu^*}$ .*

*Proof.* We certainly have  $\sigma, X \in M$  and closure under complementation. First let’s prove closure under finite union. Take  $B_1, B_2 \in M$  and choose  $A \subseteq X$  arbitrary.

$$\begin{aligned}
&\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) \\
&= \mu^*[(A \cap (B_1 \cup B_2)) \cap B_1] + \mu^*[(A \cap (B_1 \cup B_2)) \cap B_1^c] \\
&\quad + \mu^*[A \cap (B_1 \cap B_2)^c] \quad \left. \vphantom{\mu^*} \right\} B_1 \text{ measurable} \\
&= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) \quad \left. \vphantom{\mu^*} \right\} \text{simplify sets} \\
&= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \quad \left. \vphantom{\mu^*} \right\} B_2 \text{ measurable} \\
&= \mu^*(A) \quad \left. \vphantom{\mu^*} \right\} B_1 \text{ measurable}
\end{aligned}$$

So  $M$  is certainly an algebra. To obtain countable unions we note the following can be proved by induction. Given  $B_1, B_2, \dots \in M$  and any  $A \subseteq X$ .

$$\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^* \left( A \cap \left( \bigcup_{i=1}^n B_i \right)^c \right) \quad \forall n \in \mathbb{N}$$

Letting  $n \rightarrow \infty$ , by monotonicity of the outer measure on right term we get

$$\begin{aligned} \mu^*(A) &\geq \underbrace{\sum_{i=1}^{\infty} \mu^*(A \cap B_i)}_{\text{converges since all terms +ve}} + \mu^* \left( A \cap \left( \bigcup_{i=1}^{\infty} B_i \right)^c \right) \\ &\geq \mu^* \left( A \cap \left( \bigcup_{i=1}^{\infty} B_i \right) \right) + \mu^* \left( A \cap \left( \bigcup_{i=1}^{\infty} B_i \right)^c \right) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{sub-additivity}$$

and hence  $\cup_i B_i \in M$  because the other inequality is an axiomatic assumption. For arbitrary sets we can just take appropriate complementation to express their union as a union of pairwise disjoint sets.

It remains to show that we get a measure. Again the only thing to really show is the remaining inequality to get countable additivity. Given disjoint  $B_1, B_2, \dots$  in  $M$  just take  $A = \cup_i B_i$  in the above inequality to get

$$\mu^* \left( \bigcup_{i=1}^{\infty} B_i \right) \geq \sum_{i=1}^{\infty} \mu^*(B_i) + \mu^*(\emptyset)$$

□

## 1.1 Working with Lebesgue Measures

We are now able to form a measure from the Lebesgue outer measure.

The **Lebesgue measurable sets** are exactly the  $\lambda^*$ -measurable sets. The resulting  $\sigma$ -algebra is denoted  $\mathcal{L}^d$ . Restricting  $\lambda^*$  to  $\mathcal{L}^d$  yields the **Lebesgue measure**  $\lambda_d$ .

### Proposition 1.4.

$$\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$$

*Proof.* Take  $b \in \mathbb{R}$ , we will show that  $(-\infty, b] \in \mathcal{L}$  so that we can take  $\sigma$  on either side to obtain the result. Pick any  $A \subseteq \mathbb{R}$  such that  $\lambda^*(A) < \infty$  and take arbitrary  $\epsilon > 0$ .

Choose  $\{(a_i, b_i)\} \in \mathcal{C}_A$  such that  $\sum_i (b_i - a_i) < \lambda^*(A) + \epsilon$ . Notice that  $(a_i, b_i) \cap B$  and  $(a_i, b_i) \cap B^c$  are disjoint intervals whose lengths sum to  $b_i - a_i$ .

$$\begin{aligned} &\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \\ &\leq \lambda^*([\cup_i (a_i, b_i)] \cap B) + \lambda^*([\cup_i (a_i, b_i)] \cap B^c) && \left. \begin{array}{l} \\ \end{array} \right\} \text{monotonicity} \\ &\leq \sum_i \lambda^*((a_i, b_i) \cap B) + \sum_i \lambda^*((a_i, b_i) \cap B^c) && \left. \begin{array}{l} \\ \end{array} \right\} \text{countable sub-additivity} \\ &\leq \sum_i [\text{length}((a_i, b_i) \cap B) + \text{length}((a_i, b_i) \cap B^c)] && \left. \begin{array}{l} \\ \end{array} \right\} \text{rearrange +ve terms} \\ &= \sum_i (b_i - a_i) < \lambda^*(A) + \epsilon \end{aligned}$$

Now taking  $\epsilon \rightarrow 0$  we obtain the troublesome inequality. □

It is often useful to be able to approximate the Lebesgue measure from above and from below.

**Proposition 1.5** (Regularity of Measure). *Let  $A \in \mathcal{L}(\mathbb{R}^d)$  then*

- (a)  $\lambda(A) = \inf \{ \lambda(U) \mid U \text{ open}, U \supseteq A \}$
- (b)  $\lambda(A) = \sup \{ \lambda(K) \mid K \text{ compact}, K \subseteq A \}$
- (c)  $RHS(a) = RHS(b) \implies A \in \mathcal{L}(\mathbb{R}^d)$

*Proof.* Every measure is monotonous so we only have one inequality to prove in each case.

- (a) Assume  $\lambda(A) < \infty$  else we are already done. Given any  $\epsilon > 0$ , pick  $\{R_i\} \in \mathcal{C}_A$  such that  $\sum_i \text{vol}(R_i) \leq \lambda^*(A) + \epsilon < \lambda(A) + \epsilon$ . Define  $U := \cup_i R_i$  which is then an open set such that  $A \subseteq U$ . Now we have

$$\lambda(U) \leq \sum_i \lambda(R_i) = \sum_i \lambda^*(R_i) = \sum_i \text{vol}(R_i) \leq \lambda(A) + \epsilon$$

Taking  $\epsilon \rightarrow 0$  yields the result

- (b) Again take  $\epsilon > 0$  arbitrarily, we split into cases.

**Case 1:**  $A$  is a bounded set.

Take  $C \supseteq A$  which is compact. Now by (a) there is  $U$  open with  $U \supseteq C \setminus A$  such that

$$\lambda(U) \leq \lambda(C \setminus A) + \epsilon$$

Now define  $K := C \setminus U$ . Then  $C$  is closed and  $U$  is open so  $K$  is closed and  $K$  lives within  $C$  so is bounded. Hence  $K$  is bounded. Also note  $K \subseteq A$ .

$$\begin{aligned} \lambda(C) &\leq \lambda(K) + \lambda(U) \\ &\leq \lambda(K) + \lambda(C \setminus A) + \epsilon \end{aligned}$$

Hence

$$\lambda(K) \geq \lambda(C) - \lambda(C \setminus A) - \epsilon = \lambda(A) - \epsilon$$

Taking  $\epsilon \rightarrow 0$  yields  $\sup \geq \lambda(A)$ .

**Case 2:**  $A$  is an unbounded set.

The issue this time is we can't really choose that  $C$ . This time define  $A_i := A \cap [-i, i]^d$  and set  $A := \cup_i A_i$  to see

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

Continuity of measure tells us that  $\lim_{n \rightarrow \infty} \lambda(A_i) = \lambda(A)$ . So given  $\epsilon > 0$  there is  $n \in \mathbb{N}$  such that  $\lambda(A_n) \geq \lambda(A) - \frac{\epsilon}{2}$ . Case 1 tells us that there is compact  $K$  such that  $K \subseteq A_n \subseteq A$  with the property that

$$\lambda(K) \geq \lambda(A_n) - \frac{\epsilon}{2} \geq \lambda(A) - \epsilon$$

Taking  $\epsilon \rightarrow 0$  yields our result.

□

One nice property of the Lebesgue measure is translation invariance.

**Proposition 1.6** (Translation invariance). *Fix  $x \in \mathbb{R}^d$  then*

$$(a) \quad \forall A \in \mathcal{P}(\mathbb{R}^d) \quad \lambda^*(A) = \lambda^*(A + x)$$

$$(b) \quad A \in \mathcal{L} \implies A + x \in \mathcal{L}, \quad \lambda(A + x) = \lambda(A)$$

*Proof.* (a) Given a covering collection  $\{(a_i, b_i)\} \in \mathcal{C}_A$  we can just translate these intervals and the volume is preserved.

(b) We first show that  $A + x$  is  $\lambda^*$ -measurable.

$$\begin{aligned} & \lambda^*(B \cap (A + x)) + \lambda^*(B \cap (A + x)^c) \\ &= \lambda^*((B - x) \cap A) + \lambda^*((B - x) \cap A^c) \quad \left. \begin{array}{l} \text{using (a)} \\ A \in \mathcal{L} \end{array} \right\} \\ &= \lambda^*(B - x) \\ &= \lambda^*(B) \quad \left. \begin{array}{l} \text{using (a)} \end{array} \right\} \end{aligned}$$

and hence  $A + x \in \mathcal{L}$ . Also  $\lambda(A + x) = \lambda^*(A + x) = \lambda^*(A) = \lambda(A)$ . □

The question arises whether there exists a set which cannot be measured by the omnipotent Lebesgue. This depends on your view of the Axiom of Choice.

**Theorem 1.7** (Vitali Set). *Assuming the axiom of choice,  $\exists E \subseteq (0, 1)$  such that  $E \notin \mathcal{L}(\mathbb{R})$ .*

*Proof.* We start by defining an equivalence relation  $x \sim y$  if  $x - y \in \mathbb{Q}$ . This gives us equivalence classes  $\mathbb{Q} + x$  for  $x \in \mathbb{R}$ . By the axiom of choice, we can choose  $E \subseteq (0, 1)$  such that we pick exactly one element from each class.

Label the rationals in  $(-1, 1)$  by  $r_1, r_2, \dots$  and define  $E_n := E + r_n$  for each rational. Then the  $E_n$  are pairwise disjoint as follows. Suppose  $E_n \ni e + r_n = e' + r_{n'} \in E_{n'}$  for  $n \neq n'$ . Then we have two elements of the Vitali set such that  $e - e' = r_n - r_{n'} \in \mathbb{Q}$ . These two are therefore in the same equivalence class  $\mathbb{Q} + x$  so we must have  $e = e'$  and hence  $n = n'$ .

Note also that every real number is a rational distance away from a unique member of the Vitali set and hence

$$\bigcup_{r \in \mathbb{Q}} (E + r) = \mathbb{R} \quad \text{and} \quad (0, 1) \subseteq \bigcup_n E_n \quad \text{and} \quad \bigcup_n E_n \subseteq (-1, 2)$$

Now we wish to show that  $E$  is not Lebesgue measurable. Suppose for contradiction that  $E \in \mathcal{L}$  then by translation invariance so too is  $E_n$  for all  $n$ . Then

$$3 = \lambda((-1, 2)) \geq \lambda(\cup_n E_n) = \sum_n \lambda(E_n) = \sum_n \lambda(E)$$

To avoid the right hand side shooting off to infinity we must have  $\lambda(E) = 0$ . But then

$$\lambda(0, 1) = 1 \leq \sum_n \lambda(E_n) = \sum_n \lambda(E) = 0$$

This is a clear contradiction. □

## 2 Extended Real Line

We can extend the real line by adding two points

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$$

and abiding by the following conventions

- $+\infty + x = +\infty \quad \forall x \in (-\infty, +\infty]$

- $-\infty + x = -\infty \quad \forall x \in [-\infty, +\infty)$

- 

$$x \cdot (+\infty) = (+\infty) \cdot x = \begin{cases} -\infty & x \in [-\infty, 0) \\ 0 & x = 0 \\ +\infty & x \in (0, +\infty] \end{cases}$$

- 

$$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} +\infty & x \in [-\infty, 0) \\ 0 & x = 0 \\ -\infty & x \in (0, +\infty] \end{cases}$$

We need a topology for this by giving a base of open sets

$$\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{[-\infty, a) \mid a \in \mathbb{R}\} \cup \{(a, \infty] \mid a \in \mathbb{R}\}$$

Then a set is closed if and only if all sequences contain their limits (including limits at infinity). Under this topology  $\overline{\mathbb{R}}$  is compact.

### 3 Measurable Functions

Before we can define measurable functions we need to note a few equivalences.

**Proposition 3.1.** *Given a measurable space  $(X, \mathcal{A})$ . If  $Y = \mathbb{R}$  or  $\overline{\mathbb{R}}$ ,  $A \in \mathcal{A}$  and  $f : A \rightarrow Y$ . The following are equivalent:*

(a)  $\forall t \in \mathbb{R} \quad f^{-1}([-\infty, t]) \in \mathcal{A}$

(b)  $\forall t \in \mathbb{R} \quad f^{-1}((t, +\infty]) \in \mathcal{A}$

(c)  $\forall t \in \mathbb{R} \quad f^{-1}([-\infty, t)) \in \mathcal{A}$

(d)  $\forall t \in \mathbb{R} \quad f^{-1}([t, +\infty]) \in \mathcal{A}$

(e)  $\forall \text{ open } U \subseteq Y \quad f^{-1}(U) \in \mathcal{A}$

(f)  $\forall \text{ closed } B \subseteq Y \quad f^{-1}(B) \in \mathcal{A}$

(g)  $\forall B \in \mathcal{B}(Y) \quad f^{-1}(B) \in \mathcal{A}$

*Proof.* There's an awful lot to prove here. □

A function  $f : A \rightarrow \overline{\mathbb{R}}$  or  $\mathbb{R}$  is **A-measurable** if  $\forall t \in \mathbb{R}$ ,

$$\{f < t\} := \{x \in A \mid f(x) < t\} \in \mathcal{A}$$

A function  $f$  is **simple** if  $f(A)$  is finite.

One consequence is that all Borel-measurable functions are Lebesgue-measurable since  $\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}(\mathbb{R}^d)$ .

**Proposition 3.2.** *If  $f, g : A \rightarrow \overline{\mathbb{R}}$  are measurable then  $\{f < g\}$ ,  $\{f \leq g\}$  and  $\{f = g\}$  are in  $\mathcal{A}$ .*

*Proof.* Notice that we only really need to show the first is in  $\mathcal{A}$ . We express this as a countable combination of measurable sets:

$$B = \bigcup_{r \in \mathbb{Q}} \{f < r \text{ and } g > r\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g > r\})$$

□

We can define maximum and minimum functions which by this last proposition are measurable functions themselves.

$$(f \vee g)(x) = \max \{f(x), g(x)\}$$

$$(f \wedge g)(x) = \min \{f(x), g(x)\}$$

Pointwise sup, inf, lim sup, lim inf and lim of sequences of measurable functions also define measurable functions.

**Note:** Given  $f_n : A \rightarrow \overline{\mathbb{R}}$  measurable we can show that  $B := \{x \in A \mid \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$  is measurable and is the domain we use to define the pointwise limit function  $\lim_n f_n$ .

Also, given functions

$$(X, \mathcal{A}) \xrightarrow[\text{measurable}]{f} (\mathbb{R}, \mathcal{B}) \xrightarrow[\text{Borel measurable}]{g} (\mathbb{R}, \mathcal{B})$$

their composition  $f \circ g$  is also measurable.

We can also see that the set of measurable functions forms a vector space under appropriate pointwise operations. We can see that  $f^2$  is measurable because

$$\{f^2 < t\} = \{f < \sqrt{t}\} \cap \{f > -\sqrt{t}\}$$

Define the following two very important functions:

$$f^+ := f \vee 0$$

$$f^- := -(f \wedge 0)$$

We will come to use the following technical proposition very often:

**Proposition 3.3.** *Given  $f : A \rightarrow [0, +\infty]$  measurable, there exist measurable simple functions  $f_n : [0, +\infty)$  such that  $f_1 \leq f_2 \leq f_3 \leq \dots$  and  $f = \lim_n f_n$ .*

*Proof.* Given  $n \in \mathbb{N}$ , for every  $k \in (1, \dots, n \cdot 2^n)$  define the set

$$A_{n,k} := \left\{ \frac{k-1}{2^n} \leq f < \frac{k}{2^n} \right\} \in \mathcal{A}$$

Then define

$$f_n(x) := \begin{cases} \frac{k-1}{2^n} & \text{if } \exists k \in \{1, \dots, n \cdot 2^n\} \text{ such that } x \in A_{n,k} \\ n & \text{otherwise} \end{cases}$$

Where  $f$  has a finite value, the maximum error is  $\frac{1}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ . Where  $f$  has infinite value  $f_n(x) = n \rightarrow \infty$  as  $n \rightarrow \infty$ . Certainly  $f_1 \leq f_2 \leq f_3 \leq \dots$  □

By applying this proposition to  $f^+$  and  $f^-$  separately and combining the results we can see that any measurable  $f$  is the limit of measurable simple functions.



**Note:** It is possible to construct a set this is Lebesgue measurable but not Borel measurable. Its rather long winded but worth a read.

## 4 Limits of measurable functions

Should show that the liminf and what not of measurable functions are measurable and the sets where they are defined.

### 4.1 Some Generalisations

Given spaces  $(X, \mathcal{A})$  and  $(Y, \mathcal{C})$  we can say that  $f : X \rightarrow Y$  is  $(\mathcal{A}, \mathcal{C})$ -measurable if

$$\forall C \in \mathcal{C} \quad f^{-1}(C) \in \mathcal{A}$$

We can see very clearly that composition of measurable functions yields another measurable function. Checking something is measurable can be quite challenging because we have a lot of sets to check. The following allows us to check a basis of sets rather than there  $\sigma$ -algebra.

**Proposition 4.1.** *Suppose  $\mathcal{C} = \sigma(C_0)$  for some  $C_0 \subseteq \mathcal{P}(Y)$  then*

$$f \text{ is measurable} \iff \forall C \in C_0 \quad f^{-1}(C) \in \mathcal{A}$$

## 5 Integration

The aim of this section is to define the integral on a measure space  $(X, \mathcal{A}, \mu)$ . We define this function iteratively on an increasingly large subset of functions.

### 5.1 Simple Functions

Define

$$S_+ := \{f : X \rightarrow [0, +\infty) \mid f \text{ simple and } \mathcal{A}\text{-measurable}\}$$

So given  $f \in S_+$  we can write  $f = \sum_i a_i \chi_{A_i}$  for some  $a_i \in [0, +\infty)$  and  $A_1, \dots, A_m$  disjoint and measurable. The  $a_i$  are not distinct and so this is not a unique presentation.

We can now define the **integral** to be

$$\int f \, d\mu := \sum_{i=1}^m a_i \mu(A_i) = \sum_{a \in f(X)} a \mu(f^{-1}(a))$$

It can be shown with some ease that this is a linear, increasing function. We also get the desirable property that we can swap limit and integral in certain circumstances.

**Proposition 5.1.** *Let  $f$  and  $f_1 \leq f_2 \leq f_3 \leq \dots$  in  $S_+$  with  $f = \lim_n f_n \in S_+$ , then*

$$\int f \, d\mu = \lim_n \int f_n \, d\mu$$

*Proof.* By monotonicity we certainly have

$$\lim_n \int f_n d\mu \leq \int f d\mu$$

For the opposite inequality, write  $f = \sum_i a_i \chi_{A_i}$ . Take some arbitrary  $\epsilon > 0$ . Define the following sets

$$A_{n,i} := \{x \in A_i \mid f_n(x) \geq (1 - \epsilon)a_i\} \in \mathcal{A}$$

and notice these are nested sets satisfy

$$A_{1,i} \subseteq A_{2,i} \subseteq A_{3,i} \subseteq \dots \quad \text{such that} \quad \cup_n A_{n,i} = A_i$$

Define  $g_n := \sum_{i=1}^k (1 - \epsilon)a_i \chi_{A_{n,i}} \leq f_n$  which also satisfies  $g_1 \leq g_2 \leq g_3 \leq \dots$

$$\begin{aligned} \lim_n \int f_n d\mu &\geq \lim_n \int g_n d\mu \\ &= \sum_{i=1}^k (1 - \epsilon)a_i \mu(A_{n,i}) \\ &= (1 - \epsilon) \sum_{i=1}^k a_i \lim_n \mu(A_{n,i}) \\ &= (1 - \epsilon) \sum_{i=1}^k a_i \mu(A_i) \\ &= (1 - \epsilon) \int f d\mu \end{aligned} \quad \left. \vphantom{\sum_{i=1}^k} \right\} \text{measure cty}$$

Taking  $\epsilon \rightarrow 0$  yields the remaining inequality. □

## 5.2 Non-negative measurable functions

Define

$$\overline{S_+} := \{\text{measurable } f : X \rightarrow [0, +\infty]\}$$

Given  $f \in \overline{S_+}$  we can define the **integral** by

$$\int f d\mu := \sup \left\{ \int g d\mu \mid g \in S_+, g \leq f \right\}$$

Note that this is certainly consistent with our original definition for  $S_+$

**Proposition 5.2.** *Given  $f_1 \leq f_2 \leq \dots$  in  $S_+$ , and  $d := \lim_n f_n$  then  $f \in \overline{S_+}$ . Moreover,  $\int f d\mu = \lim_n \int f_n d\mu$ .*

*Proof.* We have already seen that  $f \in \overline{S_+}$  because it is the limit of a sequence of measurable functions. By our new definition of the integral we have

$$\int f_1 d\mu \leq \int f_2 d\mu \leq \dots \leq \int f d\mu$$

and hence certainly  $\lim_n \int f_n d\mu \leq \int f d\mu$ . So if the limit is an upper bound, it is certainly the least such upper bound.

So for the converse inequality it suffices to show that given  $g \in S_+$  such that  $g \leq f$  we have  $\int g d\mu \leq \lim_n \int f_n d\mu$ . Well consider

$$g \wedge f_1 \leq g \wedge f_2 \leq \dots \in S_+$$

We have that  $f_n \rightarrow f \geq g$  and hence  $\lim_{n \rightarrow \infty} (g \wedge f_n) = g$ . So the previous proposition tells us that

$$\int g d\mu = \lim_{n \rightarrow \infty} \int (g \wedge f_n) d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu$$

□

Again we can show that this new integral is still a linear, increasing operator on  $\overline{S_+}$ .

### 5.3 Arbitrary Measurable Functions

Finally given any  $f : X \rightarrow \overline{\mathbb{R}}$  define the **integral** to be

$$\int f d\mu := \begin{cases} \text{UNDEFINED} & \text{if } \int f^+ d\mu = \int f^- d\mu = +\infty \\ \int f^+ d\mu - \int f^- d\mu & \text{otherwise} \end{cases}$$

$f$  is called  **$\mu$ -integrable** if  $\int f^+ d\mu < +\infty$  and  $\int f^- d\mu < +\infty$ .

In the case  $f \in \overline{S_+}$ , then  $f^- = 0$  and hence the definitions coincide.

**Note:** It might be worth going over the proof that the integral is linear in the arbitrary case.

### 5.4 Playing with the Integral

One property we will often use to estimate integrals.

**Proposition 5.3.** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be measurable then*

$$f \text{ integrable} \iff |f| \text{ integrable}$$

Moreover,  $|\int f d\mu| \leq \int |f| d\mu$ .

We say that a measure space  $(X, \mathcal{A}, \mu)$  is **complete** if

$$\forall A \in \mathcal{A} \text{ such that } \mu(A) = 0 \quad \forall B \subseteq A \quad B \in \mathcal{A}$$

i.e. every subset of a 0-measure set is measurable.

The **completion** of  $(X, \mathcal{A}, \mu)$  is  $(X, \mathcal{A}_\mu, \bar{\mu})$  where

$$\begin{aligned} \mathcal{A}_\mu &:= \{A \subseteq X \mid \exists E, F \in \mathcal{A} \text{ such that } E \subseteq A \subseteq F, \mu(F \setminus E) = 0\} \supseteq \mathcal{A} \\ \bar{\mu}(A) &:= \mu(F) = \mu(E) \end{aligned}$$

The proof that the completion of a measure space is in fact a complete measure space is omitted and non-examinable. A property  $P : X \rightarrow \{\text{true}, \text{false}\}$  holds **almost everywhere** if

$$\exists N \in \mathcal{A} \text{ such that } \mu(N) = 0, N \supseteq P^{-1}(\text{false})$$

**Proposition 5.4.** *Suppose  $(X, \mathcal{A}, \mu)$  is complete and  $f, g : X \rightarrow \overline{\mathbb{R}}$  such that  $f(x) = g(x)$  for almost every  $x$ . Then  $f$  is measurable  $\iff g$  is measurable.*

*Proof.* Suppose that  $f$  is measurable and  $\exists N \in \mathcal{A}$  such that  $\mu(N) = 0$  and  $\{f \neq g\} \subseteq N$ .

$$\{g \leq t\} = (\{f \leq t\} \cap N^c) \cup (\{g \leq t\} \cap N)$$

Note  $\{f \leq t\} \in \mathcal{A}$  since  $f$  is measurable and certainly  $N^c \in \mathcal{A}$ . The second set is a subset of  $N$  and  $N$  has 0 measure and hence the second set is measurable by completeness. So  $\{g \leq t\} \in \mathcal{A}$  and so  $g$  is measurable.  $\square$

**Proposition 5.5.** *Suppose  $f, g : X \rightarrow \overline{\mathbb{R}}$  are measurable such that  $f = g$  almost everywhere. If  $f$  is integrable then  $g$  is integrable. Moreover  $\int f d\mu = \int g d\mu$ .*

*Proof.* Pick  $N \in \mathcal{A}$  with  $\mu(N) = 0$  such that  $\{f \neq g\} \subseteq N$ . Define

$$h(x) := \begin{cases} +\infty & x \in N \\ 0 & x \notin N \end{cases}$$

Consider the following sequence of simple measurable, non-negative functions.

$$\chi_N \leq 2\chi_N \leq 3\chi_N \leq \cdots \leq \lim_n (n\chi_N) = h$$

Hence

$$\int h d\mu = \lim_{n \rightarrow \infty} \int n\chi_N d\mu = \lim_{n \rightarrow \infty} n\mu(N) = \lim_{n \rightarrow \infty} 0 = 0$$

Certainly  $g^+ \leq f^+ + h$  and hence  $\int g^+ d\mu \leq \int f^+ d\mu + \int h d\mu \leq \int f^+ d\mu < +\infty$ . Similarly we can show that  $\int g^- d\mu \leq \int f^- d\mu < +\infty$  and so  $g$  is integrable. We can repeat this whole proof in the opposite direction to get the opposite inequalities and hence  $\int f d\mu = \int g d\mu$ .  $\square$

## 5.5 Application to Probability Theory

Suppose we have a **random variable**  $Y$ . We need a measure space with the following structure.

- $X = \{\text{elementary outcomes}\}$
- $\mathcal{A} = \{\text{events}\}$
- $\mu(A) = \mathbb{P}(A)$
- $\mu(X) = 1$  so that this is a probability space.

Then  $Y : X \rightarrow \overline{\mathbb{R}}$  is a measurable function. We define the **expectation** of  $Y$  to be

$$\mathbb{E}(Y) := \int Y d\mu$$

**Proposition 5.6** (Markov's Inequality). *Given  $f : X \rightarrow [0, +\infty]$  measurable and  $t \in (0, +\infty)$ . Let  $A := \{f \geq t\}$ . Then*

$$\mu(A) \leq \frac{1}{t} \int_A f d\mu \leq \frac{1}{t} \int f d\mu$$

*Proof.*

$$t\chi_A \leq f\chi_A \leq f \xRightarrow[\text{integrate}]{} t\mu(A) \leq \int_A f d\mu \leq \int f d\mu$$

$\square$

Phrasing this in terms of random variables we see that given a random variable  $Y \geq 0$  then

$$\mathbb{P}(Y \geq t) \leq \frac{1}{t} \mathbb{E}(Y) \quad \forall t \in (0, +\infty)$$

**Corollary 5.7.** *Suppose  $f : X \rightarrow \overline{\mathbb{R}}$  is a measurable function. Then*

$$\int |f| d\mu = 0 \implies f = 0 \text{ almost everywhere}$$

*Proof.* Given any  $n \in \mathbb{N}$

$$\mu \left\{ |f| \geq \frac{1}{n} \right\} \leq n \int |f| d\mu = 0$$

Now  $\{f \neq 0\} = \cup_{n \in \mathbb{N}} \{|f| \geq \frac{1}{n}\}$  and  $\mu(\cup_{n \in \mathbb{N}} \{|f| \geq \frac{1}{n}\}) = 0$ . □

**Corollary 5.8.**

$$f : X \rightarrow \overline{\mathbb{R}} \text{ integrable} \implies |f| < +\infty \text{ a.e.}$$

*Proof.* The proof is very similar to the previous corollary. □

The following space will be of vital importance

$$\mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R}) := \{f : X \rightarrow \mathbb{R} \mid \text{integrable}\}$$

We will often just refer to this as  $\mathcal{L}^1$ .

**Corollary 5.9.** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a measurable function. Then*

$$f \text{ integrable} \iff \exists g \in \mathcal{L}^1 \text{ s.t. } g = f \text{ a.e.}$$

*Proof.* Just set  $g$  to be the same as  $f$  except on a set of 0-measure where  $f$  is  $\infty$  where we define  $g$  to be 0. □

## 5.6 Limit Theorems

**Theorem 5.10** (Monotone Convergence Theorem). *Let  $f$  and  $f_1, f_2, \dots$  be measurable functions  $X \rightarrow [0, +\infty]$  such that for almost every  $x$*

$$f_1(x) \leq f_2(x) \leq \dots \quad \text{and} \quad f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

*then  $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$ .*

*Proof.* We will suppose that the inequalities hold for every  $x \in X$ . This leaves us with one inequality left to prove. We approximate each  $f_n$  by an increasing sequence of  $S_+$  functions and then select a subsequence of these.

So for each  $n \in \mathbb{N}$  we can pick  $g_{n,1} \leq g_{n,2} \leq g_{n,3} \leq \dots$  in  $S_+$  such that  $f_n = \lim_{k \rightarrow \infty} g_{n,k}$ . Then for each  $k \in \mathbb{N}$  we define

$$h_k := \max \{g_{1,k}, g_{2,k}, \dots, g_{k,k}\} \in S_+$$



**Theorem 5.13** (Dominated Convergence Theorem). *Suppose:*

(i)  $g : X \rightarrow [0, +\infty]$  is integrable

(ii)  $f, f_1, f_2, \dots : X \rightarrow \mathbb{R}$  are measurable such that for almost every  $x \in X$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{and} \quad \forall n \in \mathbb{N} \quad |f_n(x)| \leq g(x)$$

Then:

1.  $f$  and each  $f_i$  are integrable

2.  $\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$

*Proof.* We may assume that (ii) holds for every  $x \in X$  since this won't change any integrals. Likewise we may assume that  $g(x) \neq +\infty$  for all  $x \in X$ .

1. Given  $n \in \mathbb{N}$ ,  $|f_n| \leq g \implies \int |f_n| < \int g < +\infty \implies f_n$  integrable.

Then  $|f| = \lim_n |f_n| \leq \lim_n g = g \implies f$  integrable.

2. **Claim:**  $\int (g + f) \, d\mu \leq \liminf_n \int (g + f_n) \, d\mu$

This follows by Fatou's Lemma because  $g + f_n \geq 0$  is measurable and  $g + f = \lim_n (g + f_n)$ .

Now,

$$\begin{aligned} \int g \, d\mu + \int f \, d\mu &= \int (g + f) \, d\mu \\ &\leq \liminf_n \left( \int (g + f_n) \, d\mu \right) \\ &= \int g \, d\mu + \liminf_n \int f_n \, d\mu \\ \text{and hence} \quad \int f \, d\mu &\leq \liminf_n \int f_n \, d\mu \end{aligned}$$

Now applying the same argument to  $-f$  and  $\{-f_n\}$  yields

$$\int (-f) \, d\mu \leq \liminf_n \int (-f_n) \, d\mu \implies \int f \, d\mu \geq \limsup_n \int f_n \, d\mu$$

And hence we have  $\int f \, d\mu = \lim_n \int f_n \, d\mu$ .

□

It's also worth knowing that

**Theorem 5.14.** *Given bounded function  $f : [a, b] \rightarrow \mathbb{R}$*

(a)  $f$  is Riemann integrable  $\iff$  for almost every  $x$ ,  $f$  is continuous at  $x$

(b) In this case Riemann Integral = Lebesgue Integral

## 5.7 The Riemann Integral

Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$ , a **partition** is  $P = \{a_i\}_{i=0}^k$  where

$$a = a_0 < a_1 < \cdots < a_{k-1} < a_k = b$$

We say  $P'$  **refines**  $P$  if  $P \subseteq P'$  and  $P'$  is a partition.

We define the **lower sum**

$$l(f, P) := \sum_{i=1}^k (a_i - a_{i-1}) \inf(f[a_{i-1}, a_i])$$

and the **upper sum**

$$u(f, P) := \sum_{i=1}^k (a_i - a_{i-1}) \sup(f[a_{i-1}, a_i])$$

$f$  is **Riemann integrable** if

$$\sup_P l(f, P) = \inf_P u(f, P)$$

Then this common value is the **Riemann integral (RI)**  $\int_a^b f(x)dx$ .

**Theorem 5.15.** *Let  $a < b$  and  $f : [a, b] \rightarrow \mathbb{R}$  be bounded*

*(a)  $f$  is Riemann integrable if and only if*

$$\lambda(\{x \in [a, b] \mid f \text{ not continuous at } x\}) = 0$$

*i.e. the set is  $\lambda$ -measurable and has measure 0.*

*(b) If one of (a) holds then the  $f$  is Lebesgue integrable and*

$$(RI) \int_a^b f(x)dx = \int_a^b f d\lambda$$

*Proof.* Very long, should definitely be read. □

## 6 Theorems on Measures

$D \subseteq \mathcal{P}(X)$  is a **d-system** or **Dynkin class** if

- (a)  $X \in D$ .
- (b)  $\forall A, B \in D$  such that  $B \subseteq A$  we have  $A \setminus B \in D$ .
- (c)  $D$  is closed under countable union.

Given any collection of sets  $\mathcal{C}$ ,  $d(\mathcal{C})$  is the smallest d-system containing  $\mathcal{C}$ .

$\mathcal{C} \subseteq \mathcal{P}(X)$  is a  **$\pi$ -system** if it is closed under finite intersections.

**Lemma 6.1.** *Let  $\mathcal{C}$  be a  $\pi$ -system then  $\sigma(\mathcal{C}) = d(\mathcal{C})$ .*



*Proof.* A  $\sigma$ -algebra is a d-system and  $d(\mathcal{C})$  is the smallest d-system containing  $\mathcal{C}$  and hence we easily see  $\sigma(\mathcal{C}) \supseteq d(\mathcal{C})$ .

For the opposite direction we want to show that  $\mathcal{D} := d(\mathcal{C})$  is a  $\sigma$ -algebra. First we show that it is closed under finite intersections. To this end define

$$D_1 := \{A \in \mathcal{D} \mid \forall C \in \mathcal{C} \quad A \cap C \in \mathcal{D}\}$$

**Claim:**  $D_1$  is a d-system.

Once this has been shown we can see that  $D_1 \supseteq \mathcal{C}$  because  $\mathcal{C}$  is closed under intersections. Hence

$$D_1 \supseteq d(\mathcal{C}) \implies d(\mathcal{C}) = \mathcal{D} \supseteq D_1 \supseteq d(\mathcal{C}) \implies \mathcal{D} = D_1$$

Next define

$$D_2 := \{A \in \mathcal{D} \mid \forall C \in \mathcal{D} \quad A \cap C \in \mathcal{D}\}$$

**Claim:**  $D_2$  is also a d-system.

Then again one can easily see that  $D_2 \supseteq \mathcal{C}$  and hence

$$\mathcal{D} \supseteq D_2 \supseteq d(\mathcal{C}) = \mathcal{D} \implies D_2 = \mathcal{D}$$

which shows that  $\mathcal{D}$  is closed under finite intersections.

So we have that  $\mathcal{D}$  is a  $(\pi + d)$ -system which means that in fact  $\mathcal{D}$  is a  $\sigma$ -algebra and thus yields the opposite inequality because  $\sigma(\mathcal{C})$  is the smallest  $\sigma$ -algebra containing  $\mathcal{C}$ .  $\square$

**Corollary 6.2.** *Given a measurable space  $(X, \mathcal{A})$  and a  $\pi$ -system  $\mathcal{C} \subseteq \mathcal{P}(X)$  such that two measures  $\mu$  and  $\nu$  coincide on  $\mathcal{C}$ . If there exists an increasing sequence of subsets*

$$C_1 \subseteq C_2 \subseteq \dots \quad \text{in } \mathcal{C}$$

*such that  $\cup C_n = X$  and  $\mu(C_n) < \infty$  then  $\mu = \nu$ .*

Given  $f : \mathbb{R} \rightarrow \mathbb{R}$  non-decreasing and right continuous at every  $x \in \mathbb{R}$ , we define the **Lebesgue-Stieltjes measure** by

$$\lambda_f^* := \inf \left\{ \sum_{i=1}^{\infty} (f(b_i) - f(a_i)) \mid A \subseteq \cup_i (a_i, b_i] \right\}$$

**Proposition 6.3.** *Given a measure  $\mu$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that  $\mu([a, b]) < \infty$  for all such  $a < b$  define*

$$F_\mu(x) := \begin{cases} \mu((0, x]) & \text{for } x \geq 0 \\ -\mu((x, 0]) & \text{for } x < 0 \end{cases}$$

*Then  $F_\mu$  is non-decreasing and right continuous and  $f(0) = 0$ .*

This gives us a nice bijection

$$\{f : \mathbb{R} \rightarrow \mathbb{R} \mid \text{non-decreasing, right continuous, } f(0) = 0\} \leftrightarrow \{\text{measure } \mu \mid \mu((a, b]) < \infty \forall a < b\}$$

## 6.1 Product Measures

Given measurable spaces  $(X, \mathcal{A})$  and  $(Y, \mathcal{C})$  a **rectangle** is any set  $A \times C$  with  $A \in \mathcal{A}$  and  $C \in \mathcal{C}$ . Define  $\mathcal{R} := \{\text{rectangles}\}$  then the **product  $\sigma$ -algebra** is

$$\mathcal{A} \times \mathcal{C} := \sigma(\mathcal{R})$$

Given any subset  $E \subseteq X \times Y$  and  $f : X \times Y \rightarrow Z$ , for  $x \in X$  we define the **section**

$$E_x := \{y \in Y \mid (x, y) \in E\}$$

and then

$$f_x : Y \rightarrow Z \quad \text{by} \quad y \mapsto f(x, y)$$

i.e. we restrict  $f$  to the vertical line  $E_x$ . We likewise define  $E^y$  and  $f^y : X \rightarrow Z$  by restricting  $f$  to the horizontal line  $E^y$ .

**Example:**

$$\mathcal{B}(\mathbb{R}^2) = \sigma(\text{2D intervals}) \subseteq \sigma(\text{rectangles}) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$$

Given a rectangle  $A \times C \in \mathcal{R}$  we can write  $A \times C = A \times \mathbb{R} \cap \mathbb{R} \times C$ . If we define projection to the first coordinate  $\pi_1$  then we see

$$A \times \mathbb{R} = \pi_1^{-1}(A) \in \mathcal{B}(\mathbb{R}^2)$$

since  $A \in \mathcal{B}(\mathbb{R})$  and projection is a continuous function. Likewise  $\mathbb{R} \times C \in \mathcal{B}(\mathbb{R}^2)$  and hence  $A \times C \in \mathcal{B}(\mathbb{R}^2)$ . We may conclude that

$$\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$$

**Lemma 6.4.** (a)  $E \in \mathcal{A} \times \mathcal{C} \implies \forall x \ E_x \in \mathcal{C} \text{ and } \forall y \ E^y \in \mathcal{A}$ .

(b) If  $f : X \times Y \rightarrow \mathbb{R}$  is  $(\mathcal{A} \times \mathcal{C}, \mathcal{B}(\mathbb{R}))$ -measurable then  $f_x$  is  $\mathcal{C}$ -measurable and  $f^y$  is  $\mathcal{A}$ -measurable  $\forall x, y$ .

*Proof.* This is done by the standard procedure:

(i) Prove that  $\{E \subseteq X \times Y \mid E_x \in \mathcal{C}\}$  is a  $\sigma$ -algebra.

(ii) Prove that all rectangles belong to this set.

□

**Proposition 6.5.** Given  $\sigma$ -finite measure spaces  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{C}, \nu)$  and  $E \in \mathcal{A} \times \mathcal{C}$ , the function

$$I_E : X \rightarrow [0, +\infty], \quad x \mapsto \nu(E_x)$$

is  $\mathcal{A}$ -measurable for all  $x \in X$ .

*Proof.* Show that  $I_E$  is measurable for all rectangles and then that the set

$$\mathcal{F} := \{E \in \mathcal{A} \times \mathcal{C} \mid I_E \text{ is measurable}\}$$

is a d-system. Then since the rectangles form a  $\pi$ -system we get

$$\mathcal{F} \supseteq d(\mathcal{R}) = \sigma(\mathcal{R}) = \mathcal{A} \times \mathcal{C}$$

□

**Theorem 6.6.** *Given  $\sigma$ -finite measure spaces  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{C}, \nu)$  and  $E \in \mathcal{A} \times \mathcal{C}$ , there is a unique measure  $\mu \times \nu$  on  $(X \times Y, \mathcal{A} \times \mathcal{C})$  such that for all  $A \times C \in \mathcal{R}$*

$$(\mu \times \nu)(A \times C) = \mu(A) \cdot \nu(C)$$

and moreover given any  $E \in \mathcal{A} \times \mathcal{C}$

$$(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

*Proof.* We get uniqueness from the rectangle equality and our previous result about measure uniqueness. We then show that the last formula defines a measure with the desired properties.  $\square$

## 6.2 Fubini's Theorem

**Proposition 6.7** (Tonelli's Theorem). *Given  $\sigma$ -finite measure spaces  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{C}, \nu)$  and some function  $f : X \times Y \rightarrow [0, +\infty]$  which is  $(\mathcal{A} \times \mathcal{C})$ -measurable, the following holds*

- (a)  $x \mapsto \int_Y f_x d\nu$  is measurable.
- (b)  $\int_{X \times Y} f d(\mu \times \nu) = \int_X \left( \int_Y f_x d\nu \right) d\mu$ .

*Proof.* This proof follows the standard format of proving the result for simple functions and then extending it to measurable function by the monotone convergence theorem.  $\square$

**Theorem 6.8** (Fubini's Theorem). *Given  $\sigma$ -finite measure spaces  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{C}, \nu)$  and  $E \in \mathcal{A} \times \mathcal{C}$  and some function  $f : X \times Y \rightarrow \overline{\mathbb{R}}$  which is  $(\mu \times \nu)$ -integrable then*

- (a) For almost every  $x \in X$ ,  $f_x$  is  $\nu$ -integrable and for almost every  $y \in Y$ ,  $f^y$  is  $\mu$ -integrable.
- (b) The function

$$I_f(x) := \begin{cases} \int_Y f_x d\nu & \text{if } f_x \text{ is integrable} \\ 0 & \text{otherwise} \end{cases}$$

is  $\mu$ -integrable and likewise  $I^f(y)$  is  $\nu$ -integrable.

- (c)  $\int_{X \times Y} f d(\mu \times \nu) = \int_X I_f d\mu = \int_Y I^f d\nu$ .

### Example: Algorithm for Fubini's Theorem

Given some measurable  $f : X \times Y \rightarrow \overline{\mathbb{R}}$

1. Write  $f = f^+ - f^-$  which are both measurable.
2. Apply Tonelli's tells us  $x \mapsto \int f_x^+ d\nu$  and  $x \mapsto \int f_x^- d\nu$  are both measurable.
3. Compute

$$A^+ := \int_X \left( \int_Y f_x^+ d\nu \right) d\mu$$

$$A^- := \int_X \left( \int_Y f_x^- d\nu \right) d\mu$$

4. If both  $A^+, A^- < \infty$  then Tonelli tells us that

$$\int_{X \times Y} f^+ d(\mu \times \nu) = A^+ < +\infty$$

$$\int_{X \times Y} f^- d(\mu \times \nu) = A^- < +\infty$$

5. Hence  $f$  is  $(\mu \times \nu)$ -integrable and Fubini tells us

$$\int_{X \times Y} f d(\mu \times \nu) = A^+ - A^-$$

### 6.3 Signed measures

For a measurable space  $(X, \mathcal{A})$  and a function  $\mu : \mathcal{A} \rightarrow [-\infty, +\infty]$  is called a **signed measure** if

- (a)  $\mu(\emptyset) = 0$
- (b) Given any measurable disjoint sets  $A_1, A_2, \dots$  we have countable additivity

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

#### Note:

- Since the left hand side of (b) is defined so to the right hand side must be defined. Hence there is no disjoint  $A, B \in \mathcal{A}$  such that  $\mu(A) = \infty$  and  $\mu(B) = -\infty$  otherwise their union would not have a well-defined measure.
- Even more strongly, if  $\mu(A) = \infty$  and  $\mu(B) = -\infty$  for some  $A, B \in \mathcal{A}$  then one of the following occurs:
  - $\mu(A \cap B) \neq \mu(B) \implies \mu(A \cap B) = \mu(B) - \mu(A^c \cap B) \implies \mu(B \setminus A) = -\infty$
  - $\mu(A \cap B) \neq \mu(A) \implies \mu(A \setminus B) = +\infty$

These are both contradictions and so we can assume that one of  $\pm\infty$  never occurs.

For a signed measure  $\mu$  on  $(X, \mathcal{A})$ , a set  $A \subseteq X$  is called a **positive set** (resp. **negative set**) if:

- (i)  $A \in \mathcal{A}$ .
- (ii)  $\forall B \subseteq A$  such that  $B$  is measurable we have  $\mu(B) \geq 0$  (resp.  $\mu(B) \leq 0$ ).

**Lemma 6.9.** *Given a signed measure  $\mu$  and  $A \in \mathcal{A}$ ,*

$$-\infty < \mu(A) < 0 \implies \exists \text{ negative set } B \subseteq A \text{ such that } \mu(B) \leq \mu(A)$$

*Proof.* We proceed by induction on  $n$ , contracting a measurable set  $A_n$  each time. For each  $n \in \mathbb{N}$  define  $\delta_n := \sup \left\{ \mu(E) \mid E \text{ measurable, } E \subseteq A \setminus \left( \bigcup_{i=1}^{n-1} A_i \right) \right\}$ . Note that  $\delta_n \geq 0$  since we may always take the empty set.

Now pick any measurable  $A_n \subseteq A \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$  such that  $\mu(A_n) \geq \min(\frac{\delta_n}{2}, 1)$ .

Having done this process let  $A_\infty = \bigcup_n A_n$  and then let  $B := A \setminus A_\infty$ .

Then  $\mu(A_\infty) = \sum_n \mu(A_n) \geq 0$ . So by finite additivity  $\mu(A) = \mu(A_\infty) + \mu(B) \geq \mu(B)$ .

We claim that  $B$  is a negative set. Since  $\mu(A) > -\infty$  and  $\sum_n \mu(A_n) = \mu(A_\infty) < +\infty$ . Then since the sum converges we must have  $\mu(A_n) \rightarrow 0$  and hence  $\delta_n \rightarrow 0$ .

Now take any measurable  $E \subseteq B$ , we must have that  $\mu(E) \leq \delta_n$  for all  $n$  and hence  $\mu(E) \leq 0$ .  $\square$

**Theorem 6.10** (Kahn Decomposition Theorem). *Given any signed measure  $\mu$  on  $(X, \mathcal{A})$  there is a partition  $X = P \sqcup N$  such that  $P$  is a positive set and  $N$  is a negative set.*

*Proof.* WLOG we can assume that  $\mu : \mathcal{A} \rightarrow (-\infty, +\infty]$  then we define

$$L := \inf \{ \mu(A) \mid A \text{ is a negative set} \}$$

then choose any negative sets  $A_n$  such that  $\mu(A_n) \rightarrow L$  (note we don't know yet whether they are disjoint).

Define  $N := \bigcup_n A_n$  which is negative since for all measurable  $B \subseteq N$

$$\mu(B) = \underbrace{\mu(B \cap A_1)}_{\subseteq A_1} + \underbrace{\mu(B \cap (A_2 \setminus A_1))}_{\subseteq A_2} + \cdots \leq 0$$

and hence  $L \leq \mu(N)$ . Now for every  $n$  we have that

$$\mu(N) = \mu(A_n) + \underbrace{\mu(N \setminus A_n)}_{\leq 0} \leq \mu(A_n) \implies \mu(N) \leq L$$

by taking the limit. Hence we have  $\mu(N) = L > -\infty$ . Now let  $P = X \setminus N$ , this is a positive set.  $\square$

**Theorem 6.11** (Jordan Decomposition Theorem). *For every Hahn decomposition theorem of  $X = P \sqcup N$  of a signed measure  $\mu$  on  $(X, \mathcal{A})$  then there are measures  $\mu^+, \mu^-$  such that at least one is finite such that  $\mu = \mu^+ - \mu^-$  and  $\mu^+(N) = 0$  and  $\mu^-(P) = 0$ .*

*Moreover, such measures are unique and do not depend on the choice of  $N$  and  $P$ .*

*Proof. Existence:* Define  $\mu^+(A) = \mu(A \cap P)$  and  $\mu^-(A) = -\mu(A \cap N)$  for all  $A \in \mathcal{A}$ . Then since  $A$  is measurable we have that  $\mu(A) = \mu(A \cap P) + \mu(A \cap N) = \mu^+(A) - \mu^-(A)$  and  $\mu(N \cap P) = \mu(\emptyset) = 0$ . At least one of the holds  $\mu(N) = -\infty$ ,  $\mu(P) = -\infty$  since they are disjoint sets. Hence one of the new measures is finite.

**Independence on decomposition:** Given any  $A \in \mathcal{A}$  we would like to show that

$$\begin{aligned} \mu^+(A) &= \sup \{ \mu(B) \mid B \subseteq A \text{ measurable} \} \\ \mu^-(A) &= \sup \{ -\mu(B) \mid B \subseteq A \text{ measurable} \} \end{aligned}$$

These do not depend on  $N$  or  $P$  so we get our uniqueness, we will just prove the first identity. Given any  $B \subseteq A$  we can notice

$$\mu(B) = \mu^+(B) - \mu^-(B) \leq \mu^+(B) \leq \mu^+(A)$$

and hence we have  $\geq$ . Then we need to find a measurable  $B$  such that  $\mu(B) \geq \mu^+(A)$ . Well notice

$$\mu^+(A) = \mu^+(A \cap P) + \mu^+(A \cap N) = \mu^+(A \cap P) = \mu^+(A \cap P) - \mu^-(A \cap P) = \mu(A \cap P)$$

and so we can just take  $B = A \cap P$ .  $\square$

## 6.4 Absolute continuity

Given two measures  $\mu, \nu$  on a  $(X, \mathcal{A})$  we say that  $\nu$  is **absolutely continuous** with respect to  $\mu$  if

$$\forall A \in \mathcal{A} \quad \mu(A) = 0 \implies \nu(A) = 0$$

and we write  $\nu \ll \mu$ .

**Lemma 6.12.** Suppose that  $\mu$  and  $\nu$  are measures on  $(X, \mathcal{A})$  and that  $\nu(X) < +\infty$ , Then

$$\nu \ll \mu \iff \forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \mu(A) < \delta \implies \nu(A) < \epsilon$$

*Proof.* " $\Leftarrow$ ": Let  $\mu(A) = 0$  then given any  $\epsilon > 0$  there is a  $\delta > 0$  such that  $\mu(A) < \delta \implies \nu(A) < \epsilon$ . But  $\mu(A) < \delta$  for any such delta and hence  $\nu(A) < \epsilon$  for any such  $\epsilon$  and hence  $\nu(A) = 0$ .

" $\Rightarrow$ ": Suppose not then there exists an  $\epsilon > 0$  and a sequence of sets  $A_k \in \mathcal{A}$  such that

$$\mu(A_k) < \frac{1}{2^k} \quad \text{but} \quad \nu(A_k) \geq \epsilon$$

Now let  $B_n := \bigcup_{k=n}^{\infty} A_k$ . Notice that  $\mu(B_n) \leq \sum_{k=n}^{\infty} \mu(A_k) < \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$ . Moreover,  $\nu(B_n) \geq \nu(A_n) \geq \epsilon$ . Let  $B := \bigcap_{n=1}^{\infty} B_n$  so that

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \supseteq B$$

Since we assumed that  $\nu$  was finite, Borel-Cantelli tells us that

$$\nu(B) = \lim_{n \rightarrow \infty} \nu(B_n) \geq \epsilon$$

But by assumption  $\mu(B) = 0$ . This contradicts absolute continuity.  $\square$

**Theorem 6.13** (Radon-Nikodym Theorem). Suppose  $\mu$  and  $\nu$  are  $\sigma$ -finite measures on  $(X, \mathcal{A})$  such that  $\nu \ll \mu$ . Then  $\exists$  a measurable function  $f : x \rightarrow [0, +\infty)$  such that

$$\forall A \in \mathcal{A}, \quad \nu(A) = \int_A f \, d\mu$$

Moreover, for all such functions  $h$ , we have that  $h = f$   $\mu$ -almost everywhere. We call such function the **Radon-Nikodym derivative** which we denote  $\frac{d\nu}{d\mu}$ .

*Proof.* Look over this!  $\square$

**Note: We do need the  $\sigma$ -finite assumption**

Let  $\mu$  be the counting measure and  $\lambda$  the Lebesgue measure on  $([0, 1], \mathcal{B})$ . Then  $\lambda \ll \mu$  since  $\mu$  is only 0 on the empty set.

Can we have  $\lambda(A) = \int_A g \, d\mu$ ? No:

If  $g$  is non-zero on at least one point then look at  $A = \{x\}$  then  $\lambda(A) = 0$  but  $\int_A g \, d\mu = g(x) \neq 0$ .

So  $g$  must be identically zero which easily leads to a contradiction.

## 6.5 $\mathcal{L}^p$ spaces

Fix some  $(X, \mathcal{A})$  measure space and real number  $p \in [1, +\infty)$  then

$$\mathcal{L}^p := \{\text{measurable } f : X \rightarrow \mathbb{R} \mid |f|^p \text{ is integrable}\}$$

We can also define

$$\mathcal{L}^\infty := \{\text{bounded measurable } f : X \rightarrow \mathbb{R}\}$$

We can give this space a norm by

$$\|f\|_\infty := \inf \{M \geq 0 \mid \{|f| > M\} \text{ is locally } \mu\text{-null}\} \in [0, +\infty)$$

$A \subseteq X$  is called  **$\mu$ -null** if  $\exists N \in \mathcal{A}$  such that  $\mu(N) = 0$  and  $N \supseteq A$ .

$A$  is called **locally  $\mu$ -null** if  $\forall B \in \mathcal{A}$  with  $\mu(B) < +\infty$ ,  $A \cap B$  is null.

$p, q \in (1, +\infty)$  are **conjugate exponents** if  $\frac{1}{p} + \frac{1}{q} = 1$  or  $\{p, q\} = \{1, \infty\}$ .

**Lemma 6.14** (Young's Inequality). *Given conjugate exponents  $p, q \in (1, \infty)$  and  $x, y \geq 0$*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

**Proposition 6.15** (Holder's Inequality). *Given conjugate exponents  $p, q \in [1, +\infty]$ . If  $f \in \mathcal{L}^p$  and  $g \in \mathcal{L}^q$  then  $fg \in \mathcal{L}^1$  and*

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q$$

**Proposition 6.16** (Minkowski's Inequality). *Given any  $p \in [1, +\infty]$ ,*

$$f, g \in \mathcal{L}^p \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

**Corollary 6.17.** *Given any  $p \in [1, +\infty]$ , then  $\mathcal{L}^p(X, \mathcal{A}, \mu)$  is a vector space and  $\|\cdot\|_p$  is a semi-norm.*

Let  $\mathcal{N}^p = \mathcal{N}^p(X, \mathcal{A}, \mu) := \{f \in \mathcal{L}^p \mid \|f\|_p = 0\}$

Then we can define  $L^p := \frac{\mathcal{L}^p}{\mathcal{N}^p}$ . This can be seen to be a normed space.

**Theorem 6.18.** *Given any  $p \in [1, +\infty]$ ,  $(L^p, \|\cdot\|_p)$  is complete.*

*Proof.* It is enough to show that for all  $\{f_n\}$  in  $\mathcal{L}^p$

$$\sum_{k=1}^{\infty} \|f_k\|_p < +\infty \implies \exists f \in \mathcal{L}^p \quad \text{s.t.} \quad \left\| \sum_{k=1}^n f_k - f \right\|_p \rightarrow 0$$

Define  $g_n : X \rightarrow [0, +\infty]$  by  $g_n(x) = \sum_{k=1}^n |f_k(x)|$  and then  $g(x) := \lim_{n \rightarrow \infty} g_n(x)$ .

By the monotone convergence theorem we can see that  $\int |g| d\mu < +\infty$ .

Then define

$$f(x) := \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } g(x) < +\infty \\ 0 & \text{otherwise} \end{cases}$$

Then by the dominated convergence theorem we see that

$$\int |f|^p d\mu \leq \lim_{n \rightarrow \infty} \int \sum_{k=1}^n |f_k|^p d\mu < +\infty$$

so on and so forth...

□

## 7 Modes of Convergence

Given a measure space  $(X, \mathcal{A}, \mu)$  and measurable functions  $f, f_1, f_2, f_3, \dots : X \rightarrow \mathbb{R}$  we say

- $(f_n)$  **converges to  $f$  almost everywhere** if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for almost every } x \in X$$

- $(f_n)$  **converges to  $f$  in measure** if

$$\forall \epsilon > 0 \quad \lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| > \epsilon\}) = 0$$

**Proposition 7.1.** *Given a finite measure space, almost everywhere convergence  $\implies$  convergence in measure.*

*Proof.* Given  $\epsilon > 0$  define  $A_n := \{|f_n - f| > \epsilon\}$ . Let  $B_n := \cup_{i \geq n} A_i$  be the set of points where  $f_n$  and  $f$  differ by more than  $\epsilon$  at some point in the future. Notice that  $B_1 \supseteq B_2 \supseteq \dots$  and all the  $B_i$  are measurable. Now define  $B = \cap_n B_n$  and note that  $B \subseteq \{x \in X \mid f_n(x) \not\rightarrow f(x)\}$  since  $B$  is the set of points where no matter how far in the sequence you go there will always be a time in the future where  $f_n$  and  $f$  disagree significantly. Therefore

$$0 = \mu(B) = \mu(\cap_n B_n) = \lim_{n \rightarrow \infty} \mu(B_n)$$

where the last inequality is thanks to  $\mu$  being a finite measure. But note that  $A_n \subseteq B_n$  for all  $n$  and hence  $\mu(A_n) \rightarrow 0$  as required.  $\square$

**Lemma 7.2** (Borel-Cantelli). *Given a measure space  $(X, \mathcal{A}, \mu)$  and  $A_1, A_2, \dots \in \mathcal{A}$  such that  $\sum_n \mu(A_n) < \infty$ , let*

$$A := \{x \in X \mid x \in A_n \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k \right)$$

*then  $\mu(A) = 0$ .*

**Corollary 7.3.** *Given a measure space  $(X, \mathcal{A}, \mu)$  and  $f_1, f_2, f_3, \dots : X \rightarrow \mathbb{R}$  all measurable, if  $f_n \rightarrow f$  in measure then  $\exists n_1 < n_2 < \dots$  such that  $f_{n_i} \rightarrow f$  almost everywhere.*

**Theorem 7.4** (Egorov's Theorem).  *$(X, \mathcal{A}, \mu)$  a measure space and  $f, f_1, f_2, \dots : X \rightarrow \mathbb{R}$  measurable such that  $f_n \rightarrow f$  almost everywhere. If  $\mu(X) < \infty$  then*

$$\forall \epsilon > 0 \exists B \in \mathcal{A} \text{ s.t. } \mu(B^c) < \epsilon \text{ and } f_n \rightarrow f \text{ uniformly on } B$$

*Proof.* Take arbitrary  $\epsilon > 0$ . For each  $n \in \mathbb{N}$  define

$$g_n(x) := \sup_{j \geq n} |f_j(x) - f(x)|$$

Then  $g_n : X \rightarrow [0, +\infty]$  is measurable and since  $f_n \rightarrow f$  almost everywhere, so too  $g_n \rightarrow 0$  almost everywhere. Now these  $g_n$  may have some infinite points so we define

$$g'_n(x) := \begin{cases} g_n(x) & \text{if } g_n(x) < \infty \\ 0 & \text{otherwise} \end{cases}$$



Note we still have  $g'_n \rightarrow 0$  almost everywhere and hence  $g'_n \rightarrow 0$  in measure. Therefore, for every  $k \in \mathbb{N}$  we have  $n_k$  such that

$$\mu \left\{ g'_{n_k} > \frac{1}{k} \right\} < \frac{\epsilon}{2^k}$$

Define  $B_k := \{g_{n_k} \leq \frac{1}{k}\}$  and subsequently  $B := \cap_k B_k$ . Note also that  $\mu(B_k^c) < \frac{\epsilon}{2^k}$  and hence

$$\mu(B^c) \leq \sum_k \mu(B_k^c) < \sum_k \frac{\epsilon}{2^k} < \epsilon$$

Finally, to prove uniform convergence, given any  $\delta > 0$  take  $k > \frac{1}{\delta}$  then for any  $x \in B$  and  $n \geq n_k$  we have

$$|f_n(x) - f(x)| \leq g_{n_k}(x) \underbrace{\leq}_{\text{since}} \frac{1}{k} < \delta$$

□

For  $f, f_1, f_2, \dots \in \mathcal{L}^1(X, \mathcal{A}, \mu)$  we say  $(f_n)$  converges to  $f$  in mean if

$$\int |f_n(x) - f(x)| d\mu(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

**Lemma 7.5.** *Convergence in mean  $\implies$  convergence in measure.*

*Proof.* Using Markov's inequality, given any  $\epsilon > 0$ ,

$$\mu \{ |f_n - f| > \epsilon \} \leq \frac{1}{\epsilon} \int |f_n - f| d\mu \rightarrow 0$$

□

**Proposition 7.6.**  *$(X, \mathcal{A}, \mu)$  a measure space and  $f, f_1, f_2, \dots \in \mathcal{L}^1(X, \mathcal{A}, \mu)$  If  $f_n \rightarrow f$  almost everywhere or in measure and there is an integrable  $g : X \rightarrow [0, +\infty]$  such that for almost every  $x$ ,  $|f| \leq g$  and for all  $n \in \mathbb{N}$   $|f_n| \leq g$  then  $f_n \rightarrow f$  in mean.*

*Proof.* Suppose  $f_n \rightarrow f$  almost everywhere then almost everywhere we have  $|f_n - f| \leq 2g$ . We then apply dominated convergence theorem to get

$$\lim_{n \rightarrow \infty} \int |f_n - f| d\mu = \int \lim_{n \rightarrow \infty} \underbrace{|f_n - f|}_{0 \text{ a.e.}} d\mu = 0$$

Now suppose  $f_n \rightarrow f$  in measure but, for contradiction, not in mean. So there is an  $\epsilon > 0$  and a sequence  $n_1 < n_2 < \dots$  such that for all  $k$

$$\int |f_{n_k} - f| d\mu > \epsilon \tag{1}$$

Convergence in measure implies the existence of an almost everywhere convergent subsequence so we have  $k_1 < k_2 < \dots$  such that  $f_{n_{k_i}} \rightarrow f$  almost everywhere as  $i \rightarrow \infty$ . By the first part of the proof  $f_{n_{k_i}} \rightarrow f$  in mean which contradicts (1). □

**Theorem 7.7** (Lusin's Theorem). *Let  $A \in \mathcal{L}(\mathbb{R}^d)$  with  $\lambda(A) < \infty$  and  $f : A \rightarrow \mathbb{R}$  Lebesgue measurable. Then  $\forall \epsilon > 0 \exists$  compact  $K \subseteq A$  such that  $\lambda(A \setminus K) < \epsilon$  and  $f|_K$  is continuous.*