Intro to Topology - Overview

1 Key Definitions

- Given maps $f, g : X \times I \to Y$, a homotopy from f to g is a continuous map $F : X \times I \to Y$ such that $f_0 = f$ and $f_1 = g$. If such a map exists we write $f \simeq g$.
- Given paths $f, g : I \to X$. We say f is homotopic to g relative to $\{x, y\}$ and write $f \overset{\partial}{\simeq} g$ if
  - (i) $f(0) = g(0) = x$
  - (ii) $f(1) = g(1) = y$
  - (iii) There is a homotopy $F : I \times I \to X$ such that $f_0 = f$, $f_1 = g$ and for all $t \in I$, $f_t(0) = x$ and $f_t(1) = y$.
- A pair of spaces (X, A) is a topological space together with a subspace $A \subseteq X$ using the subspace topology.

Assume we have a pair (X, A).

- A is a retract if there is a continuous map $r : X \to A$ such that $r\big|_A = id_A$.
- X deformation retracts to A if there exists a homotopy $F : X \times I \to X$ such that $f_0 = id_X$, $f_1(X) = A$ and $f_t\big|_A = id_A$ for all $t \in I$.

Assume we have topological space X and Y then,

- X is homotopy equivalent to Y there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq id_X \quad , \quad f \circ g \simeq id_Y$$

- X is contractible if it is homotopy equivalent to $\{pt\}$.

Note:

$$X \text{ deformation retracts to } A \implies X \text{ contractible}$$
$$\not\Longleftarrow$$

The reverse does not hold because the contraction does not necessarily restrict to id_A.

2 Fundamental Group

Given a pointed space (X, x_0), a loop is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.
We can then define an equivalence class for every loop f:

$$[f] := \left\{ g \;\middle|\; g(0) = g(1) = x_0, \quad f \overset{\partial}{\simeq} g \right\}$$

We then get the fundamental group defined to be

$$\pi_1(X, x_0) := \{[f] \mid f \text{ a loop based at } x_0\}$$

This forms a group with the operation $[f] \cdot [g] = [f * g]$ and the identity element being the constant loop.

Theorem 2.1. *If X is path connected and $x_0, x_1 \in X$ then*

$$\pi_1(X, x_0) = \pi_1(X, x_1)$$

Proof. There exists a path $h : I \to X$ from x_0 to x_1. Define $\overline{h}(s) := h(1 - s)$. We can then define a base point change homomorphism which we claim is in fact an isomorphism.

$$\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [f] \mapsto [\overline{h} * f * h]$$

We can see this is in fact an isomorphism because $\beta_{\overline{h}}$ is a left and right inverse. □

A map $p : \widetilde{X} \to X$ is a covering map if there is an open cover $\{U_\alpha\}$ of X such that

$$p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta$$

with each V_α^β open and such that $p\big|_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ is a homomorphism.
The covering map is called n-fold if each $p^{-1}(x_0$ has n elements for every x_0.
Let $p : Y \to X$ and $q : Z \to X$ be coverings. These called isomorphic if there is a homeomorphism $h : Y \to Z$ such that

$$q \circ h = p$$

Let $p : \widetilde{X} \to X$ be a cover then a deck transformation is an isomorphism $\tau : \widetilde{X} \to \widetilde{X}$ such that $p \circ \tau = p$.

$$\mathrm{Deck}(p) := \left\{ \tau : \widetilde{X} \to \widetilde{X} \;\middle|\; \tau \text{ is a deck transformation } \right\}$$

Given a covering $p : \widetilde{X} \to X$ and a map $f : Y \to X$, a lift of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p \circ \widetilde{f}$.

Here are some useful properties of lifts:

(i) $\widetilde{f} : Y \to \widetilde{X}$ then $\widetilde{f}$ is a lift of $p \circ \widetilde{f}$.

(ii) $\widetilde{f}, \widetilde{g} : Y \to \widetilde{X}$ and $f \simeq g \implies p \circ \widetilde{f} \simeq p \circ \widetilde{g}$ (homotopies descends).

(iii) $\alpha, \beta : I \to \widetilde{X}$ such that $\alpha(1) = \beta(0)$ then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ (concatenation descends).

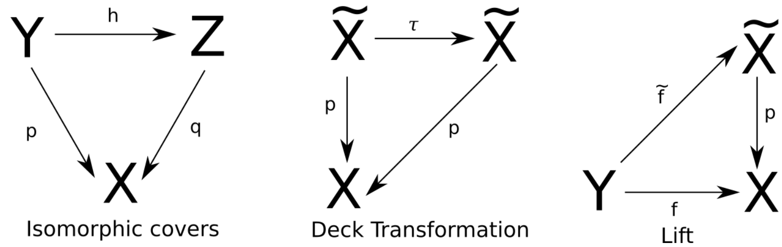

Figure 1: Commutative diagrams defining various concepts

3 Homotopy Lifting Property

A map $p : Z \to X$ has the homotopy lifting property if given any homotopy $F : Y \times I \to X$ and any lift $g : Y \times \{0\} \to Z$ there exists a unique homotopy $\widetilde{F} Y \times I \to Z$ satisfying

(i) $\widetilde{f}_0 = g$

(ii) $p \circ \widetilde{F} = F$

i.e. given a homotopy and a lift of one endpoint, there exists a unique lift of that homotopy. As a special case where $Y = \{pt\}$, a map $p : Z \to X$ has the path lifting property if for any path $f : I \to X$, $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$ there exists a unique path $\widetilde{f} : I \to Z$ such that

(i) $\widetilde{f}(0) = \widetilde{x}_0$

(ii) $p \circ \widetilde{f} = f$

Lemma 3.1 (Local Homotopy Lifting Property). *Let $p : \widetilde{X} \to X$ be a covering map and $F : Y \times I \to X$ a homotopy. Suppose we have $g : Y \times \{0\} \to \widetilde{X}$. Then for every $y \in Y$*

(a) There exists an open neighbourhood $N \subseteq Y$ and a unique homotopy $\widetilde{F}_N : N \times I \to \widetilde{X}$ such that

(i) $(\widetilde{f}_N)_0 = g$.

(ii) $p \circ \widetilde{F}_N = F\big|_{N \times I}$.

(b) If $M \subseteq Y$ with $y \in M$ is another open neighbourhood for which (a) holds then (a) also holds for $M \cap N$ and

$$\widetilde{F}_N\Big|_{(M \cap N) \times I)} = \widetilde{F}_M\Big|_{(M \cap N) \times I} = \widetilde{F}_{M \cap N}$$

Proof. This has a very long proof. □

Proposition 3.2. *Covering maps $p : \widetilde{X} \to X$ have the homotopy lifting property.*

Proof. Let $P; \widetilde{X} \to X$ be a covering map. Let $F : Y \times I \to I$ be a homotopy and choose soma arbitrary starting $g : Y \times \{0\} \to \widetilde{X}$.
We can cover $Y = \cup_\alpha N_\alpha$ such that *(a)* and *(b)* hold from the lemma.
We can then define a new homotopy by stitching these together:

$$\widetilde{F} : Y \times I \to \widetilde{X}, \qquad \widetilde{F}(y, t) := \widetilde{F}_{N_\alpha}(y, t) \text{ if } y \in N_\alpha$$

We do not get any ambiguity here thanks to property *(b)* from the lemma. The continuity of this construction follows from the pasting lemma. □

Theorem 3.3. *Let $\omega_N : I \to S^1$ be defined by $\omega_n(s) = e^{2\pi i n s}$. Then*

$$\pi_1(S^1, 1) = \{[\omega_n] \mid n \in \mathbb{Z}\}$$

Proof. Define $\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ by $n \mapsto [\omega_n]$. We claim this is an isomorphism. For this define the following useful maps

$$\begin{aligned} p(t) &= e^{2\pi i t} \\ \omega_n(t) &= e^{2\pi i n t} \\ \widetilde{\omega}_n(t) &= nt \\ \tau_m(t) &= t + m \end{aligned}$$

- Φ is a group homomorphism.

  Then we can see that indeed $\widetilde{\omega_n} : \mathbb{R} \to \mathbb{R}$ is a lift of ω_n. One can also see through the linear homotopy that

  $$\widetilde{\omega}_{m+n} \overset{\partial}{\simeq} \widetilde{\omega}_m * (\tau_m \circ \widetilde{\omega}_n)$$

  Now given any $m, n \in \mathbb{Z}$ we have

  $$\begin{aligned} \Phi(m+n) &= [\omega_{n+m}] && \\ &= [p \circ \widetilde{\omega}_{n+m}] && \textit{lift} \\ &= [p \circ (\widetilde{\omega}_m * (\tau_m \circ \widetilde{\omega}_n))] && \textit{homotopies descend} \\ &= [p \circ \widetilde{\omega_m}] \cdot [p \circ \tau_m \circ \widetilde{\omega}_n] && \\ &= [p \circ \widetilde{\omega_m}] \cdot [p \circ \widetilde{\omega}_n] && \textit{deck transformation} \\ &= [\omega_m] \cdot [\omega_n] && \textit{lift} \\ &= \Phi(m) \cdot \Phi(n) && \end{aligned}$$

- Φ is surjective.

  Choose any $[\alpha] \in \pi_1(S^1, 1)$, we aim to find $n \in \mathbb{N}$ such that $\alpha \overset{\partial}{\simeq} \omega_n$. We certainly know that $\alpha(0) = \alpha(1) = 1$ and hence $p^{-1}(1) = \mathbb{Z}$ and in particular $0 \in p^{-1}(1) = p^{-1}(\alpha(0))$.

  So by the path lifting property there exists a unique lift $\widetilde{\alpha} : I \to R$ such that

  (i) $\widetilde{\alpha}(0) = 0$.

  (ii) $p \circ \widetilde{\alpha} = \alpha$.

  Now, $\alpha(1) = 1 \implies p(\widetilde{\alpha}(1)) = 1 \implies \widetilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$. Suppose $\widetilde{\alpha}(1) = n \in \mathbb{Z}$. So $\widetilde{\alpha}$ is a path from 0 to n in $\mathbb{R}$. By the linear homotopy we can see that $\widetilde{\alpha} \overset{\partial}{\simeq} \widetilde{\omega}_n$. But homotopies descend and hence

  $$\alpha = p \circ \widetilde{\alpha} \overset{\partial}{\simeq} p \circ \widetilde{\omega}_n = \omega_n$$

- Φ is injective.

  Assume that $\Phi[\omega_n] = [e]$. We aim to show that in fact $n = 0$.

  To start, $\omega_n \overset{\partial}{\simeq} e$ and hence we have a homotopy $F : I \times I \to S^1,\ (s,t) \mapsto F(s,t)$ such that $f_0 = \omega_n,\ f_1 = e$ and $f_t(0) = f_t(1) = 1$.

  Now by the HLP we see that there is a unique homotopy $F : I \times I \to R$ satisfying

(i) $\widetilde{f}_0 = \widetilde{\omega}_n$.

(ii) $p \circ \widetilde{F} = F$.

Now since the left, top and bottom edges were identically 1 in F we must have that the same edges lie in $\mathbb{Z}$ in the lifted homotopy. But consider the bottom edge $\widetilde{\omega}_n$. On the left side it is 0 but on the right it is n. By continuity along the left, top and bottom edges of $\widetilde{F}$ we must have that $n = 0$. This can be seen in Figure 3.

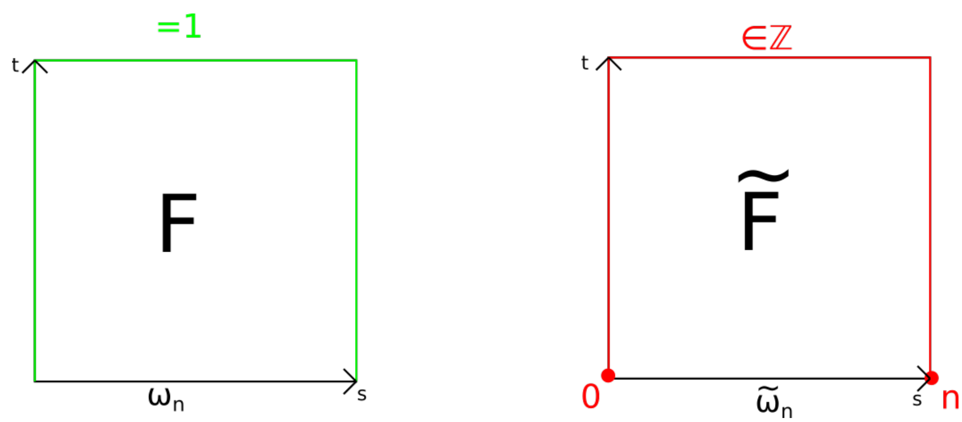

Figure 2: Diagrammatic explanation of continuity argument

□

4 Applications

A map of pairs $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subseteq B$.
The induced homomorphism of $f : (X, x_0) \to (Y, y_0)$ is the map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\alpha] &\mapsto [f \circ \alpha] \end{aligned}$$

We need to show that this is well-defined and is in fact a group homomorphism.

Lemma 4.1 (Functoriality)**.** $(g \circ f)_* = g_* \circ f_*$

Corollary 4.2. *If f is a homeomorphism then f_* is a group isomorphism.*

Theorem 4.3. *let $\phi : X \to Y$ be a homotopy equivalence and $x_0 \in X$. Then*

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism.

Proposition 4.4.