Complex Analysis Notes

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1 Complex Algebra

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i} \quad \cosh(z) := \frac{e^z + e^{-z}}{2} \quad \sinh(z) := \frac{e^z - e^{-z}}{2}$$

Note:

$$\cosh(z) = \cos(iz) \quad \sinh(z) = -i \sinh(iz)$$

- The principal value of the argument is the unique $\theta \in (-\pi, \pi]$. This is a continuous function on $\mathbb{C}$ without any half-line (including 0).
- $\xi + i\eta$ is the logarithm of $re^{i\theta}$ if

$$\xi = \log(r) \qquad \eta = \theta + 2\pi n \;\; n \in \mathbb{Z}$$

- The principal value of the logarithm corresponds to $n = 0$.
- We say that $\xi + i\eta$ is an element of $z_0^{z_1}$ if

$$\xi + i\eta \in e^{z_1 \log(z_0)}$$

- The extended complex plane is $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.
- We can extended inversion to the $\hat{\mathbb{C}}$ by setting

$$\frac{1}{0} := \infty \qquad \frac{1}{\infty} := 0$$

1.1 Riemann Sphere

To represent the complex plane, we use stereographic projection of $S^2 \setminus \{\text{north pole}\}$ into $\mathbb{C}$ and then send the north pole to ∞.

$$\begin{aligned} \pi : S^2 \setminus \{(0,0,1)\} &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3} \end{aligned}$$

Lemma 1.1. *A circle on S^2 is the intersection of S^2 with some plane. The image of every non-vanishing circle on S^2, under π is a line or circle in $\mathbb{C}$.*

In this proof we notice that circles through the north pole go to lines and circles not through the north pole go to circles. So we can define $\pi(\text{north pole}) := \infty$ and see that $\pi(S^2) = \hat{\mathbb{C}}$.
We can use this to define a metric on $\hat{\mathbb{C}}$.

$$\forall z, w \in \mathbb{C} \qquad d(z, w) := \left|\left|\pi^{-1}(z) - \pi^{-1}w\right|\right|$$

where $||\cdot||$ is the Euclidean norm on S^2.

Note: We can compute everything in this definition in terms of complex algebra to find

$$d(z,w) = \frac{2\,|z-w|}{\sqrt{1+|z|^2}+\sqrt{1+|w|^2}}$$
$$d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}$$

When doing complex algebra we stick to the following conventions

- $\infty + z = z + \infty = \infty \qquad \forall z \in \mathbb{C}$
- $\infty \cdot z = z \cdot \infty = \infty \qquad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$
- $\frac{z}{\infty} = 0 \qquad \forall z \in \mathbb{C}$
- $\frac{z}{0} = \infty \qquad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$

2 Mobius Transformations

Given $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ we can define a Mobius transformation

$$f(z) := \frac{az+b}{cz+d} \qquad \forall z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$$

We can extend this to $\hat{\mathbb{C}}$ by defining $\hat{f}(-\frac{d}{c}) = \infty$ and $\hat{f}(\infty) = \frac{a}{c}$.
Notice we can multiply a, b, c, d by any non-zero complex number and recover the same function.
We say that f is normalised if $ad - bc = 1$.
It can be noticed that composing two Mobius transformations yields another Mobius transformation.
We can calculate the coefficients of the transformation by multiplying the corresponding matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lemma 2.1. *Extended Mobius transforms are invertible and their inverse is another Mobius transform.*

2.1 Decomposing Mobius transformations

Let inv be the inversion map $z \mapsto \frac{1}{z}$.

Lemma 2.2. *Let $\mathcal{C}$ be a circle or a line then inv$(\mathcal{C})$ is a circle or a line.*

Proof. Worth going over. □

The elementary Mobius transformations are

(a)	Inversion:	$\mathrm{inv}(z) = \frac{1}{z}$
(b)	Translation:	$z \mapsto z + b$
(c)	Rotation:	$z \mapsto az$ for $a = e^{i\theta}$
(d)	Expansion/Contraction:	$z \mapsto rz$ for $z \in \mathbb{R}, z > 0$

Lemma 2.3. *Every Mobius transformation can be written as a composition of elementary Mobius transformations.*

Proof. **Case 1:** $c \neq 0$
We can write

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

Case 2: $c = 0$
$c = 0$ and $ad - bc \neq 0 \implies d \neq 0$ and hence we can write

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}$$

In both cases these transformations can be easily decomposed. □

Theorem 2.4. *The image of a circle or line in $\hat{\mathbb{C}}$ under a Mobius transformation is another circle or line.*

Theorem 2.5. *Given 3 distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and three other distinct points $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ there exists a unique Mobius transform f with $f(z_i) = w_i$ for all i.*

Proof. **Existence:** We define two helper functions, assuming that none of the points are ∞

$$S(z) := \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

and if any z_i is ∞ then we simply remove any term containing that z_i. Notice

$$S(z_1) = 1 \quad S(z_2) = 0 \quad S(z_3) = \infty$$

We define T in the same way but replacing each z_i with w_i. Then we can notice that defining $f := T^{-1}S$ yields a function with the desired properties.
Uniqueness: It suffices to check the cases when $w_1 = 1$, $w_2 = 0$ and $w_3 = \infty$ because we can always compose with T. Then we can just pick two suitable Mobius transformations f_1 and f_2, then show that $g := f_1 \circ f_2^{-1}$ is the identity Mobius transformation. □

Note: Look up the cross ratio.

- A non-identity Mobius transform has at most two fixed points because

$$z = \frac{az+b}{cz+d} \iff 0 = cz^2 + (d-a)z - b$$

3 Complex Differentiability

Given $D \subseteq \mathbb{C}$ open, a function $f : D \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

Note: This definition of f' can be restated as

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \ s.t. \ \ |z - z_0| < \delta \implies \left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \leq \epsilon\,|z - z_0|$$

Prop 3.1. *$f : D \to \mathbb{C}$ complex differentiable at $z_0 \in D$ implies f is continuous at z_0.*

The complex derivative also satisfies all of the usual algebra of derivative functions from real analysis, including the chain rule

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Theorem 3.2 (Cauchy-Riemann Equations)**.** *The following are equivalent, given $f : D \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in D$*

(a) f is $\mathbb{C}$-differentiable at z_0.

(b) f is $\mathbb{R}$-differentiable at (x_0, y_0) and $df(z_0)$ is complex linear

(c) f is $\mathbb{R}$-differentiable at (x_0, y_0) and the CR equations hold:

$$u_x = v_y \qquad u_y = -v_x$$

Proof. *(i)* $\iff$ *(ii)* is somewhat immediate. Consider the alternative definition given in the notes. We see that being $\mathbb{C}$-differentiable is equivalent to the existence of a complex number ξ such that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - \xi \cdot h}{h} = 0$$

We can view thus view the derivative as a $\mathbb{C}$-linear function $h \mapsto \xi \cdot h$. This is equivalent to the definition of $\mathbb{R}$-differentiability with the additional requirement that the map is $\mathbb{C}$-linear. In practice this means that the Jacobian matrix is some real number multiplied by a rotation matrix.
This explains *(ii)* $\iff$ *(iii)* as well because the Jacobian must be given by

$$r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Alternatively, writing out the Jacobian we see that the derivative as a $\mathbb{C}$-linear map

$$M(h) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

and then the condition $M(ih) = iM(h) \; \forall h \in \mathbb{C}$ is equivalent to the Cauchy-Riemann equations. $\square$

Theorem 3.3 (Power Series Expansion)**.** *Given a sequence $(a_k)_{k \in \mathbb{N}_0}$ with $a_k \in \mathbb{C}$, consider the power series*

$$\sum_{k=0}^{\infty} a_k z^k \tag{1}$$

(a) There exists a radius of convergence $r \in [0, \infty]$ such that for all z with $|z| < r$ the series (1) converges, and for all z with $|z| = r' > r$ the series (1) does not converge.

(b) The series

$$\sum_{k=1}^{\infty} k a_k z^{k-1} \tag{2}$$

head the same radius of convergence.

(c) $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $\mathcal{B}_r(0) = \{|z| < r\}$.

Proof. (a) If we convergence for some z_0 then $(a_k z_0^k) \to 0$ is bounded by $C > 0$ say and hence for all z with $|z| < |z_0|$ we have

$$\sum_{k=0}^{\infty} \left| a_k z^k \right| = \sum_{k=0}^{\infty} \left| a_k z_0^k \right| \frac{|z|^k}{|z_0|^k} \leq C \frac{1}{|z_0| - |z|}$$

and hence we get convergence.

The radius of convergence is therefore $\sup \{\eta \geq 0 \mid \exists z \text{ with } |z| = \eta \; s.t. \; (1) \text{ converges}\}$

(b) We now consider (2). Suppose $|z| < \hat{r} < r$, then we have

$$\sum_{k=1}^{\infty} \left| k a_k z^{k-1} \right| \leq \frac{1}{\hat{r}} \sum_{k=1}^{\infty} \underbrace{k \left(\frac{|z|^{k-1}}{\hat{r}^{k-1}} \right)}_{\to 0} \underbrace{\left| a_k \hat{r}^k \right|}_{\text{convergent}}$$

and hence the sum converges. Likewise the sum diverges wherever the other one does.

(c) Confusing proof.

□

4 Cauchy's Collection of Complex Corollaries

4.1 Complex Integration

Let $f : D \to \mathbb{C}$ be continuous and γ a smooth curve with $\Gamma = \gamma[a, b] \subseteq D$

$$\int_{\Gamma} f(z) \, dz = \int_{\gamma} f(z) \, dz := \int_a^b f(\gamma(t)) \dot{\gamma}(t) \, dt$$

The length of a curve is defined to be

$$L(\gamma) := \int_a^b |\dot{\gamma}| \, dt$$

Two curves $\gamma : [a, b] \to \mathbb{C}$ and $\lambda : [c, d] \to \mathbb{C}$ are smoothly equivalent parametrisations for Γ if there is a smooth function $\rho : [a, b] \to [c, d]$ such that

(i) $\dot{\rho}(t) \neq 0 \quad \forall t$.

(ii) $\rho^{-1} \in \mathcal{C}^1$ and is never zero.

(iii) $\gamma = \lambda \circ \rho$.

(iv) $\rho(a) = c$ and $\rho(b) = d$.

Lemma 4.1. *The complex line integral is invariant under change of parametrisation.*

Lemma 4.2. *If γ and λ are smoothly equivalent then $L(\gamma) = L(\lambda)$.*

Lemma 4.3. *$f : D \to \mathbb{C}$ holomorphic and $\gamma \in \mathcal{C}^1([a,b])$ such that $\Gamma \subseteq D$ then*

$$\begin{aligned} \int_\gamma f'(z)\,dz &= \int_a^b f'(\gamma(t))\dot{\gamma}(t)\,dt \\ &= \int_a^b \frac{d}{dt} f(\gamma(t))\,dt \\ &= f(\gamma(b)) = f(\gamma(a)) \end{aligned}$$

4.2 My First Cauchy's Theorem

Theorem 4.4 (Goursat's Theorem)**.** *Take $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic. Take a rectangle $Q \subseteq D$ such that $Q \cup \partial Q = \overline{Q} \subseteq D$. Take a $\mathcal{C}^1$ parametrisation $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma[a,b] = \partial Q$ and γ circles around Q exactly once in the positive direction. Then*

$$\int_\gamma f(z)\,dz = 0$$

Proof. We split the proof into a number of steps:

1. $f \equiv 1$.

   This proof follows easily from the FTC.

2. $f(z) = z$.

   This proof also follows easily from the FTC because

   $$\int \gamma(t)\dot{\gamma}(t)\,dt = \frac{1}{2}\int_a^b \frac{d}{dt}\left(\gamma(t)\right)^2\,dt = \frac{1}{2}\left[\gamma(b)^2 - \gamma(a)^2\right]$$

3. f holomorphic in D.

   Divide into rectangles, this is a very long proof in Lecture 9.

□

Corollary 4.5 (Cauchy's Theorem for images of rectangles)**.** *Given $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic such that $\overline{Q} \subseteq D$. Suppose $\phi : \overline{Q} \to D$ is $\mathcal{C}^1$. Let γ be a $\mathcal{C}^1$ parametrisation of ∂Q then*

$$\int_{\phi\circ\gamma} f(z)\,dz = 0$$

We say $D \subseteq \mathbb{C}$ is

- a region if it is non-empty and connected.
- polygonally connected if between every two points are joined by a path consisting of a finite collection of straight lines all contained within D.

A contour is a simple closed curve.

Theorem 4.6. *Given a non-empty open set $D \subseteq C$*

$$D \text{ is a region} \quad \iff \quad D \text{ is polygonally connected}$$

Theorem 4.7 (Jordan Curve Theorem)**.** *Let γ be a contour and $\Gamma = \gamma[a,b]$ then ${}^{c}\gamma$ consists of*

$$I(\gamma) \cup O(\gamma)$$

where $I(\gamma)$ is bounded and $O(\gamma)$ is unbounded and the two regions are disjoint.

Note: Jordan Curve Theorem $\implies$ Cauchy's theorem for contours

4.3 Cauchy's Integral Formula

Theorem 4.8 (Cauchy's Integral Formula)**.** *Given $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic, suppose that $\overline{\mathcal{B}_r(a)} \subseteq D$ for some $a \in D$ and $r > 0$. Then for all $z_0 \in \mathcal{B}_r(a)$*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(a)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Not too difficult, worth going over (Lecture 11) $\square$

4.4 Applications

4.4.1 Taylor's Theorem

Theorem 4.9 (Taylor's Theorem)**.** *Given $D \subseteq C$ open and polygonally connected and $f : D \to C$ holomorphic. Assume $\exists R > 0$ and $z_0 \in D$ such that $\overline{\mathcal{B}_R(z_0)} \subseteq D$ then for all $z \in \mathcal{B}_R(z_0)$ we have*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Proof. We want to use the Cauchy's Integral formula so we need to rewrite the integrand. Note

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \frac{\xi - z_0}{\xi - z} = \frac{1}{\xi - z_0} \frac{1}{\left(\frac{\xi - z}{\xi - z_0}\right)} = \frac{1}{\xi - z_0} \frac{1}{\left(1 - \frac{z - z_0}{\xi - z_0}\right)}$$

Now by assumption we have that $|z - z_0| < |\xi - z_0| = R$. So we can write this as a geometric series

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^k$$

which we can plug back into Cauchy's Integral formula to see

$$f(z) = \frac{1}{2\pi i} \int_{\partial_R(z_0)} \frac{f(\xi)}{\xi - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^k \, d\xi$$

But the sum converges uniformly in the integration variable ξ so we can switch the sum and the integral

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \, d\xi \right) (z - z_0)^k$$

$\square$

Corollary 4.10. *Every holomorphic function on D is in fact $C^\infty(D)$.*

Corollary 4.11. *$D \subseteq \mathbb{C}$ is open and polygonally collected and $f : D \to \mathbb{C}$ then the following are equivalent:*

(i) f is holomorphic in D.

(ii) f is real differentiable on D and the CR equations hold.

(iii) f can be expressed in a power series.

Corollary 4.12. *Suppose $f(z) = \sum_{k\in\mathbb{N}} a_k z^k$ is holomorphic on $\mathcal{B}_R(0)$ for some $R > 0$ and suppose f is bounded in that ball, say by M. Then for all $k \in \mathbb{N}$*

$$|a_k| \le \frac{M}{R^k}$$

where R is the radius of convergence.

Note: Here are some standard power series expansions:

$$\cos(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} \qquad \sin(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \sinh(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

These all have infinite radius of convergence.

We often need to calculate the radius of convergence for given power series. Here are some tools for doing so, they are more less the same as real analysis.

(i) **d'Alembert's Ratio Test**

Assume that a sequence (a_n) is such that

$$l := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

Case 1: If $l < 1$ then $\sum_n |a_n|$ convergences.

Case 2: If $l > 1$ then $\sum_n |a_n|$ diverges.

Case 3: If $l = 1$ then we have no information.

(ii) **Cauchy's n'th root test**

Assume that a sequence (a_n) is such that

$$l := \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists. The cases and their consequences are the same as before. The radius of convergence is then seen to be R where

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

4.4.2 Other Applications

Theorem 4.13 (Liouville's Theorem)**.** *Any entire, bounded function is constant.*

Proof. Pick $z_0 \in \mathbb{C}$ and $M > 0$ such that $|f(z)| \leq M \; \forall z \in \mathbb{C}$. Define

$$m(f, R, z_0) := \max_{z \in \partial \mathcal{B}_R(z_0)} |f(z)|$$

Then by Taylor's theorem we see that

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} m(f, R, z_0) \leq \frac{n!}{R^n} M$$

and in particular $|f'(z_0)| \leq \frac{M}{R} \to 0$ as $R \to \infty$. Hence $f'(z_0) = 0$. □

Corollary 4.14 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial has at least one zero in* $\mathbb{C}$*.*

Proof. Take some polynomial $P(z) := a_n z^n + \cdots + a_1 z + a_0$ such that $a_n \neq 0$. Then for any $\epsilon > 0$ there is a radius R such that $\forall |z| > R$ we have

$$(1 - \epsilon) |a_n| |z|^n \leq |P(z)| \leq (1 + \epsilon) |a_n| |z_n|^n$$

Suppose $P(z)$ has no zeros in $\mathbb{C}$ then $\frac{1}{P(z)}$ is complex differentiable in $\mathbb{C}$ and there is an $R >)$ such that for all $|z| > R$

$$\frac{1}{2} |a_n| z_n \leq |P(z)| \implies \left| \frac{1}{P(z)} \right| \leq \frac{2}{|a_n| |z^n|} \leq \frac{2}{|a_n| R^n}$$

and hence $\frac{1}{P}$ is bounded on $\{|z| > R\}$ and it is obviously bounded inside by compactness. Hence by Liouvilles's Theorem P is constant. This is a contradiction. □

Theorem 4.15 (Morea's Theorem)**.** *Given a region $D \subseteq \mathbb{C}$ and a $f : D \to \mathbb{C}$ continuous, suppose given any triangle T with $T \cup \partial T \subseteq D$ we have $\int_{\partial T} f(z)\, dz = 0$. Then f is holomorphic in D.*

Theorem 4.16 (Schwarz Reflection Principle)**.** *Suppose D is open in $\overline{H^+} := \{z \in \mathbb{C} \mid \mathcal{I}(z) \geq 0\}$ and $f : D \to \mathbb{C}$ is continuous on D and holomorphic on D°. Then*

$$\widetilde{f}(z) := \begin{cases} f(z) & z \in D \\ \overline{f(\overline{z})} & z \in \widetilde{D} \end{cases}$$

where $\widetilde{D}$ is the complex conjugate of D, is well-defined and holomorphic on $D \cup \widetilde{D}$.

Proof. By composition of reflections we can easily sow that $\widetilde{f}$ is holomorphic on $\widetilde{D}^\circ$ so that only the lines remain. We show that the integral of f over any triangle with one edge on the line is 0 by a continuity argument, approaching from both sides. □

5 Zeros of Holomorphic Functions

Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic, the order of any zero $z_0 \in D$ is

$$\text{ord}(f, z_0) := \inf \left\{ k \in \mathbb{N} \;\middle|\; f^{(k)}(z_0) \neq 0 \right\} \in \mathbb{N} \cup \{\infty\}$$

We say that $f : D \to \mathbb{C}$ is a conformal mapping if f is holomorphic in D and it's derivative is non-vanishing on D.
We say that f is biholomorphic if f is a conformal mapping such that f^{-1} exists and is also conformal.

Prop 5.1. *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic, suppose we have a zero $z_0 \in D$ of order $k \in \mathbb{N}$. Then there is a neighbourhood U_0 of z_0 and a holomorphic function $h : U_0 \to V_0$ such that $h(z_0) = 0$, ord$(f, z_0) = 1$ and*

$$f(z) = (h(z))^k \quad \forall z \in U_0$$

Proof. WLOG we may assume that $z_0 = 0$, then we apply Taylor's theorem to write f as

$$f(z) = \sum_{n=k}^{\infty} c_n z^n$$

because $\text{ord}(f, z_0) = k$ and hence the first k terms vanish. For simplicity we can also assume that $c_k = 1$. Hence we can write

$$f(z) = z^k \left(\underbrace{1 + \sum_{n=k+1}^{\infty} c_n z^{n-k}}_{=:g(z)} \right) = \left(\underbrace{z\sqrt[k]{1 + g(z)}}_{=:\text{h}(z)} \right)^k$$

Note that g is holomorphic and $g(0) = 0$, and $h(0) = 0$. Moreover, $h'(0) = \sqrt[k]{1 + g(0)} + 0\left(\sqrt[k]{1 + g(z)}\right)' = 1 \neq 0$. Hence $\text{ord}(h, 0) = 1$. Read up on making it holomorpic (Lecture 14). ∎

Note: This implies that all zeros of finite order are isolated.

Theorem 5.2. *If ord$(f, z_0) = k \in \mathbb{N}$ for $f : D \to \mathbb{C}$ holomophic then $\forall \epsilon > 0$ the exists a $U_\epsilon \subseteq D$ with $z_0 \in U_\epsilon$ such that $f(U_\epsilon) = \mathcal{B}_\epsilon(0)$ and $f\big|_{U_\epsilon}$ takes every w with $0 < |w| < \epsilon$ exactly k times and 0 for z_0.*

Proof. Without loss of generality we may assume that $z_0 = 0$.
If $f(z) = z^k$ then any $w = re^{i\theta}$ has exactly k roots.
In the general case we can write $f(z) = (h(z))^k$ for $h : U \to V$ holomorphic such that $h(0) = 0$ and $h'(0) \neq 0$. Moreover, h is locally biholomorphic around a neighbourhood of 0. Choose $\epsilon > 0$ sufficiently small that

$$A := \left\{ \xi \in \mathbb{C} \;\middle|\; |\xi| \leq \sqrt[k]{\epsilon} \right\} \subseteq V$$

Then define $U_\epsilon := h^{-1}(A)$. This set has the desired properties because the original roots of $z \mapsto z^k$ lie in A. ∎

Note: Every bijective holomorphic function is biholomorphic.

Theorem 5.3 (Identity Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and connected with $f_1, f_2 : D \to \mathbb{C}$ holomorphic, assume that $\{f_1 = f_2\}$ has at least one accumulation point in D. Then $f_1 = f_2$ on D.*

Proof. Define $g := f_1 - f_2$ and let z_0 be one of the accumulation points. Then z_0 is a zero of infinite order for g. Apparently this is a proof? □

Theorem 5.4 (Open Mapping Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic and non-constant, $f(D)$ is open and connected.*

Theorem 5.5 (Maximum Modulus Principle)**.** *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic and non-constant, $|f|$ does not have any maxima.*

Lemma 5.6 (Schwarz Lemma)**.** *Suppose $f : \Delta \to \Delta$ is holomorphic such that $f(0) = 0$ then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \Delta$.

(ii) $|f'(0)| \leq 1$.

(iii) *If for some $z \in \Delta \setminus \{0\}$ we have $|f(z)| = |z|$ or $|f'(z)| = 1$ then $\exists \theta \in \mathbb{R}$ such that $f(\tilde{z}) = e^{i\theta}\tilde{z}$ for all $\tilde{z} \in \Delta$.*

6 Singularities

Given $D \subseteq \mathbb{C}$ open and connected and $f \in \mathcal{H}(D)$,

- f has an isolated singularity at $z_0 \notin D$ if there is an $\epsilon > 0$ such that f is defined on $\mathcal{B}_\epsilon(z_0) \setminus \{z_0\}$.
- $z_0 \in D$ is a regular point if f is complex differentiable at $z_0 \in D$.

Given an isolated singularity z_0 then it has order

$$\operatorname{ord}(f, z_0) := -\inf\left\{ n \in \mathbb{Z} \;\middle|\; \lim_{z \to z_0} (z - z_0)^n f(z) \text{ exists and } < \infty \right\}$$

then we say z_0 is a

- removable singularity if $\operatorname{ord}(f, z_0) \geq 0$.
- pole of order $n \in \mathbb{N}$ if $\operatorname{ord}(f, z_0) = -n \in (-\infty, -1]$.
- essential singularity if $\operatorname{ord}(f, z_0) = -\infty$.

Let $S \subseteq D$ be a discrete set, then a holomorphic function $f : D \setminus S \to \mathbb{C}$ is called meromorphic on D if none of the isolated singularities in S are essential.

Prop 6.1. *Let $\mathcal{Z}_f$ and $\mathcal{P}_f$ be the set of zeros and poles respectively of $f : D \to \mathbb{C}$ meromorphic, $f \neq 0$. Then neither set has an accumulation point in D.*

Proof. Certainly any pole of f is an isolated singularity and hence cannot be an accumulation point of $\mathcal{P}_f$. Any other $z \in D$ where f is holomorphic cannot be an accumulation point of poles either. Suppose now that $\mathcal{Z}_f$ has an accumulation point at $z_0 \in D$ then z_0 cannot be a pole otherwise we'd be able to write

$$f(z) = \frac{g(z)}{(z-z_0)^m} \quad \text{with } m \in \mathbb{N}, g(z_0) \neq 0$$

and hence for some $\epsilon > 0$ we have that $f(z) \neq 0$ for all $0 \leq |z - z_0| < \epsilon$ which means that z_0 is not an accumulation point.
So any accumulation point must be a complex differentiable point. We are left to show that $D \setminus \mathcal{P}_f$ see open and connected because then the identity theorem tells us that $f \equiv 0$ because $\{f = 0\}$ has an accumulation point in $D \setminus \mathcal{P}_f$. ☐

Lemma 6.2. *Suppose $D \subseteq \mathbb{C}$ is open and connected. Suppose $M \subseteq D$ has no accumulation point in D. Then $D \setminus M$ is open and connected.*

Proof. Openness is immediate, for connectedness just draw a picture. ☐

6.1 Laurent Series

A Laurent series is a series of the form

$$\sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

such that the positive terms converge inside some ball around z_0 and the negative terms converge outside sum larger ball around z_0. Hence the Laurent series converges in an annulus around the point z_0.

Theorem 6.3 (Cauchy's Theorem for annuli)**.** *Given $0 \leq R_1 < R_2 < \infty$ and $D \subseteq \mathbb{C}$ open and connected such that $\overline{A} := \overline{A(R_1, R_2, z_0)} \subseteq D$, for any $z \in A$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_{R_2}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi - \frac{1}{2\pi i} \int_{\partial \mathcal{B}_{R_1}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Do normal Cauchy on a small ball contained in the annulus then do some appropriate contour integration. ☐

Theorem 6.4 (Laurent's Theorem)**.** *Given f holomorphic on a neighbourhood of an annulus $A = A(R_1, R_2, z_0)$ and any $z \in A$,*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

where for all $\rho \in [R_1, R_2]$ we can write

$$a_k = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_\rho(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Corollary 6.5. *Under the same assumption, if f is bounded on $\{|z - z_0| = \rho\}$ for sum $\rho \in [R_1, R_2]$ then*

$$|a_k| \leq \frac{M}{\rho^k} \quad \text{for all } k \in \mathbb{Z}$$

6.2 Classification of Singularities

Theorem 6.6 (Riemann's removable singularity theorem)**.** *Gain an isolated singularity z_0 for a function $f \in \mathcal{H}(D \setminus \{z_0\})$, assume that $|f|$ is bounded in a neighbourhood of z_0. Then there is a holomorphic function $\widetilde{f} \in \mathcal{H}(D)$ which extend to f. Moreover, z_0 was a removable singularity.*

Proof. In the neighbourhood we can expand in a Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

and then for all sufficiently small $\rho > 0$ we have $|a_k| \leq \frac{M}{\rho^k}$ for all $k \in \mathbb{Z}$. Taking $\rho \to 0$ we see that $a_k = 0$ for all $k < 0$. So we can extend f by taking $\widetilde{f}(z_0) = a_0$. □

Corollary 6.7. *Given $f : D \to \mathbb{C}$ which is holomorphic except for an isolated singularity at z_0, the following are equivalent:*

(i) *f has a pole at z_0.*

(ii) *At least coefficient of negative order in the Laurent series around z_0 in non-zero, but at most finitely many.*

(iii) $\lim_{z \to z_0} |f(z)| = +\infty$.

Theorem 6.8 (Casorati-Weirstrass)**.** *Given $f \in \mathcal{H}(D \setminus \{z_0\})$ with an isolated, essential singularity at z_0, for all $\epsilon > 0$ the set $f(\mathcal{B}_\epsilon(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Suppose it's not dense. Then $\exists \delta > 0$ and $w \in \mathbb{C}$ such that $|f(z) - w| > \delta$ for all $z \in \mathcal{B}_\epsilon(z_0) \setminus \{z_0\}$. So we can define a function

$$g(z) := \frac{1}{f(z) - w}$$

and it may be extended to a function which is holomorphic on $\mathcal{B}_\epsilon(z_0)$ because the bottom is always bigger than δ hence g is bounded. So we can write $f(z) = \frac{1}{g(z)} + w$ which clearly doesn't have an essential singularity $\Rightarrow\Leftarrow$. □

Note: Suppose z_0 is an essential singularity for f, then certainly $\lim_{z \to z_0} f(z)$ cannot have any finite value $c \in \mathbb{C}$. Suppose $\lim_{z \to z_0} f(z) = \infty$ then there would be some ϵ ball around z_0 such that $|z - z_0| < \epsilon \implies |f(z)| > 1$. Hence the image of this ball could not be dense. This shows that essential singularities exhibit no limiting behaviour.

7 Residual Theory

7.1 Winding Numbers

Given $z_0, z_1 \in \mathbb{C} \setminus \{0\}$ such that $\frac{z_0}{|z_0|} \neq \frac{-z_1}{|z_1|}$ then there is a unique $\theta \in (-\pi, pi)$ such that

$$\frac{z_0}{|z_0|} e^{i\theta} = \frac{z_1}{|z_1|}$$

then we define $\measuredangle(z_0, z_1) := \theta$.
We say $\gamma[t_0, t_1]$ is a half-plane curve if the image of γ is contained entirely in one half plane.

Then we define $\measuredangle\gamma := \measuredangle(\gamma(t_0), \gamma(t_1))$.
Given any other piecewise $\mathcal{C}^1$ curve we can break it up into a sequence of half-plane curves and then sum the angles to define $\measuredangle\gamma$.

Lemma 7.1. *Given a closed $\mathcal{C}^1$ curve $\gamma; [t_0, t_1] \to \mathbb{C} \setminus \{0\}$, there exists a unique integer $\inf(\gamma, 0)$ called the index or winding number of γ around 0 such that $\measuredangle\gamma = 2\pi\, ind(\gamma, 0)$.*

Prop 7.2. *Given a closed $\mathcal{C}^1$ curve γ with $a \notin \gamma[t_0, t_1]$ we have*

$$ind(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

Proof. Assume $a = 0$, $t_0 = 0$ and $t_1 = 1$ then we can split up into the half-plane decomposition:

$$0 = \tau_0 \leq \cdots \leq \tau_n = 1$$

Let α_i be the straight line connected $\gamma(\tau_i)$ with $\frac{\gamma(\tau_i)}{|\gamma(\tau_i)|}$ and let β_i be the counter-clockwise path along the unit circle connecting $\frac{\gamma(\tau_{i-1}}{|\gamma(\tau_{i-1}|}$ with $\frac{\gamma(\tau_i)}{|\gamma(\tau_i)|}$. Then we can see

$$\int_{\gamma[\tau_{i-1},\tau_i]} \frac{dz}{z} = \int_{\alpha_{i-1}} \frac{dz}{z} + \int_{\beta_i} \frac{dz}{z} - \int_{\alpha_i} \frac{dz}{z} \quad \forall i$$

Then summing over i and noting $\alpha_0 = \alpha_n$ obtains the result. □

A cycle is a formal linear combination of closed curves $\gamma = \sum_{i=1}^n \alpha_i\gamma_i$ with $\alpha_i \in \mathbb{Z}$ then we define

$$\operatorname{ind}(\gamma, a) = \sum_{i=1}^n \alpha_i \operatorname{ind}(\gamma_i, a)$$

Then γ is called homologous to 0 in D if for every $a \in \mathbb{C} \setminus D$ we have

$$\operatorname{ind}(\gamma, a) = 0$$

Theorem 7.3 (Cauchy's Theorem (homotopy version))**.** *Let $D \subseteq C$ be open and connected and γ a $\mathcal{C}^1$ cycle that is homologous to 0 in D. Then $\forall f \in \mathcal{H}(D)$*

$$\int_\gamma f(z)\, dz = 0$$

7.2 Residual Theorem

Given f holomorphic with an isolated singularity at z_0, the residue of f at z_0 is defined to be

$$\operatorname{res}(f, z_0) := \frac{1}{2\pi i}\int_{\partial\mathcal{B}_\epsilon(z_0)} f(\xi)\, d\xi$$

There are a number of convenient ways of calculating the residue.

1. For a Laurent series
$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z-z_0)^k$$
we have $\operatorname{res}(f, z_0) = a_{-1}$.

2. Suppose that f has a pole of order n at z_0. Define $g(z) := (z - z_0)^n f(z)$ then g is holomorphic at z_0 and in fact

$$\text{res}(f, z_0) = a_{-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!} = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right) \right]$$

3. If $f(z) = \frac{h(z)}{k(z)}$ and $k(z)$ has a simple zero at z_0 then

$$\text{res}(f, z_0) = \lim_{z \to z_0} \frac{h(z)}{\left(\frac{k(z) - k(z_0)}{z - z_0} \right)} = \frac{h(z_0)}{k'(z_0)}$$

Given $\gamma; [t_0, t_1] \to \mathbb{C}$ a closed curve with image Γ

$$\text{Int}(\gamma) := \{ z \in \mathbb{C} \setminus \Gamma \mid \text{ind}(\gamma, z) \neq 0 \}$$
$$\text{Ext}(\gamma) := \{ z \in \mathbb{C} \setminus \Gamma \mid \text{ind}(\gamma, z) = 0 \}$$

Lemma 7.4. *(a) $[a \mapsto \text{ind}(\gamma, a)]$ is locally constant in $\mathbb{C} \setminus \Gamma$.*

(b) $\text{Int}(\gamma)$ is bounded.

(c) $\text{Ext}(\gamma)$ is non-empty and unbounded.

Theorem 7.5 (The Residue Theorem). *Given $D \subseteq \mathbb{C}$ open and connected and $f \in \mathcal{H}(D \setminus S)$ for some discrete set S of isolated singularities, suppose γ is a closed $\mathcal{C}^1$ curve homologous to 0 in D such that $\Gamma \cap S = \emptyset$ and $\Gamma \subseteq D$. Then γ winds around at most a finite number of singularities in S and*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{a \in S} \text{ind}(\gamma, a)\, \text{res}(f, a)$$

Proof. First things first, $A := \{ a \in S \mid \text{ind}(\gamma, a) \neq 0 \}$ is bounded. Assume that γ winds around infinitely many points in S then A is infinite. Hence there is a sequence of points (a_n) in A such that $a_n \to a$.
Case 1: $a \in \Gamma$
Case 2: $\text{ind}(\gamma, a) \neq 0$
Both lead to a contradiction apparently.
Now suppose $a_1, \ldots, a_N$ are the points around with γ winds and define $\alpha_i := \text{ind}(\gamma, a_i)$.
Choose $\epsilon > 0$ small such that $\overline{\mathcal{B}_\epsilon(a_i)} \cap \Gamma = \emptyset$ for all i. Then define $\gamma_i(t) := a_i + \epsilon e^{i2\pi t}$ for $t \in [0, 1]$. Let β_i be some little paths joining γ to γ_i and then concatenate all the γ_is and β_is with γ in some appropriate way to form $\widetilde{\gamma}$ Then

$$\int_{\widetilde{\gamma}} = 0 \implies \int_\gamma f(z) dz = 2\pi i \sum_{a \in S} \text{ind}(\gamma, a)\, \text{res}(f, a)$$

$\square$

Theorem 7.6 (Argument Principle). *Given $D \subseteq \mathbb{C}$ open and connected and f meromorphic on D, let $A \subseteq D$ be open with boundary ∂A being a closed $\mathcal{C}^1$ curve γ. Assuming $\partial A \subseteq D$ and $\Gamma \cap \mathcal{P}(f) = \Gamma \cap \mathcal{Z}(f) = \emptyset$, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \mathcal{Z}_A(f) - \mathcal{P}_A(f)$$

where

$$\mathcal{Z}_A(f) := \sum_{z \in f^{-1}(0) \cap A} ord(f,z) \quad and \quad \mathcal{P}_A(f) := \sum_{z \in \mathcal{P}(f) \cap A} |ord(f,z)|$$

Proof. The proof uses the residual theorem to calculate the integral on the left hand side. The isolated singularities of $\frac{f'}{f}$ are the poles and zeros of f. We need to understand the residue of $\frac{f'}{f}$ (the logarithmic derivative) at these points.
To this end, let z_0 be a zero or a pole of f. Then for some $n \in \mathbb{Z}$ and function g which is holomorphic in neighbourhood of z_0 with $g(z_0) \neq 0$ we can write

$$f(z) = (z - z_0)^n g(z)$$

Then we can calculate the logarithmic derivative as follows

$$\frac{f'(z)}{f(z)} = \frac{1}{f(z)} \left(n(z-z_0)^{n-1} g(z) + (z - z_0)^n g'(z) \right) = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Since $g(z_0) \neq 0$ we see that $\frac{f'}{f}$ has a simple pole at z_0 and hence $\operatorname{res}\left(\frac{f'}{f}, z_0\right) = n$.
So when we integrate $\frac{f'}{f}$ we sum $\operatorname{ord}(f, z_0)$ for all zeros and poles of f with a factor of $2\pi i$ out front. This counts zeros positively and poles negatively, each with multiplicity respecting the order. □

8 Rouches Theorem

Theorem 8.1 (Rouches Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and connected and γ a closed $\mathcal{C}^1$ curve with $\Gamma = im(\gamma) \subseteq D$. Suppose $f, g \in \mathcal{H}(D)$ satisfy*

$$|f(\xi) - g(\xi)| < |g(\xi)| \quad \forall \xi \in \Gamma \tag{3}$$

Then f and g have the same number of zeros in $Int(\gamma)$.

Proof. Thanks to the strict inequality in *(3)*, there is a neighbourhood $U \supseteq \Gamma$ such that $h := \frac{f}{g}$ is holomorphic in U. Then again by *(3)*, we see

$$|h(z) - 1| = \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \forall z \in U$$

and hence $h(U) \subseteq \mathcal{B}_1(1) \subseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathcal{R}(z) \leq 0,\ \mathcal{I}(z) = 0\}$ and so $\log(h)$ is well-defined on U and we can write

$$(\log(h))' = \frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g}$$

then since h is holomorphic on U we can integrate its derivative over γ and get

$$0 = \frac{1}{2\pi i} \int_\gamma \frac{f'(\xi)}{f(\xi)}\, d\xi - \frac{1}{2\pi i} \int_\gamma \frac{g'(\xi)}{g(\xi)}\, d\xi$$

□

Note: This can alternatively be restated as follows:
Given $h, w \in \mathcal{H}(D)$, such that we can write $h = f + g$ and $w = f$, suppose that

$$|g(\xi)| < |f(\xi)| \quad \forall \xi \in \Gamma$$

then $f+g$ and f have the same number of zeros in $\mathrm{Int}(\gamma)$.

Proof. Define the meromorphic function

$$F(z) := \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$$

Note that we want to show that $Z_A(F) = P_A(F)$. By the argument principle we have that $Z_A(F) - P_A(F)$ is the winding number of $F \circ \gamma$ about 0 since

$$\int_\gamma \frac{F'(z)}{F(z)} dz = \int_{F\circ\gamma} \frac{1}{z} dz$$

But by assumption we have that for all z in the image of γ $|F(z) - 1| < 1$. So the image of γ under F is contained □

9 Functional Convergence

Suppose $f_n : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$ is open. We say that f_n converges locally uniformly to f as $n \to \infty$ if

$$\forall \text{compact } K \subseteq D, \quad f_n\big|_K \to f\big|_K \text{ uniformly}$$

Theorem 9.1 (Weirstrass Convergence Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and connected and a sequence (f_n) of holomorphic functions on D such that f_n converges locally uniformly to f, then $f \in \mathcal{H}(D)$.*

Proof. Pick $z_0 \in D$ and $\delta > 0$ sufficiently small that $\overline{\mathcal{B}_\delta(z_0)} \subseteq D$. We will prove that f is holomorphic on all of this ball by use of Morea's Theorem. So take γ closed curve in $\mathcal{B}_\delta(z_0)$. We can write

$$\int_\gamma f = \int_\gamma f_n + \int_\gamma (f - f_n)$$

But $f(z) - f_n(z) \to 0$ uniformly on γ and hence $\int_\gamma (f - f_n) \to 0$ as $n \to \infty$. Moreover, f_n is holomorphic on D and hence the integral over γ is 0 and thus $\int_\gamma f_n = 0$. We may conclude $\int_\gamma f = 0$ and so by Morea's theorem f is holomorphic in the ball. □

Theorem 9.2. *Given $D \subseteq \mathbb{C}$ open and connected and a sequence of functions $f_n \in \mathcal{H}(D)$, if f_n converges locally uniformly to f then the derivatives also converge locally uniformly.*

Theorem 9.3 (Hurwitz Theorem)**.** *$D \subseteq \mathbb{C}$ open and connected, $f_n : D \to \mathbb{C}$ and $f_n \in \mathcal{H}(D)$. Suppose that f_n converges locally uniformly to f and that none of the f_n have more than some $k \in \mathbb{N}$ zeros. Then either f is constant or has at most k zeros.*

Proof. Suppose that f is not constant. Suppose that f has K zeros of multiplicity $m_1, \ldots, m_k$ at distinct $z_1, \ldots, z_K \in D$. Fix $\delta > 0$ such that for all no other z_j lies in the δ ball around z_i. Define

$$\epsilon := \inf_{i=1,\ldots,k} \inf_{\xi \in \partial\mathcal{B}_\delta(z_i)} |f(\xi)| > 0$$

By compactness we have that $\epsilon > 0$. Then by locally uniform convergence we know there is a n_0 such that

$$\sup_{i=1,\ldots,k} \sup_{\xi \in \partial\mathcal{B}_\delta(z_i)} |f_n(\xi) - f(\xi)| < \frac{\epsilon}{2} \quad \forall n > n_0$$

Rouche's theorem tells us that f has exactly m_i zeros in $\mathcal{B}_\delta(x_i)$ since

$$|f_n(\xi) - f(\xi)| < |f(\xi)| \quad \forall \xi \in \mathcal{B}_\delta(z_i)$$

$\square$

In words ϵ was the 'minimum' value of f on the boundary of these balls. Then thanks to locally uniform convergence we were able to ensure that the difference between f_n and f was always less than this minimum value and thus we could use Rouche's theorem.
We say a sequence (f_n) is locally bounded if for every compact set $K \subseteq D$ there is a constant $C > 0$ such that for all $n \geq 1$ and $z \in K$ we have $|f_n(z)| \leq C$.

Lemma 9.4. *If we have a sequence of holomorphic functions (f_n) on a region D that is locally bounded and that converges pointwise on a dense subset of D, then the whole sequence converges locally uniformly to a holomorphic function.*

Theorem 9.5 (Montel's Theorem)**.** *Every locally bounded sequence of holomorphic functions on an open and connected set has a locally uniform convergent subsequence.*

Proof. Take a countable dense subset and then do lion hunting to get a subsequence that converges on this dense set. Then apply the Lemma. $\square$

10 Special Functions

10.1 The Gamma Function

To begin we define the function for $\mathcal{R}(z) > 0$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$$

Theorem 10.1. *The Gamma-integral defines a holomorphic function and the k'th derivative is*

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1} (\log t)^k e^{-t} dt$$

One can easily show directly from the integral that Γ satisfies the functional relationship

$$\Gamma(z+1) = z\Gamma(z)$$

which we can use to extend Γ to the rest of $\mathbb{C}$. Given any $z \in \mathbb{C}$ we just choose $n \in \mathbb{N}$ such that $\mathcal{R}(z+n) > 0$ and then define

$$\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1)\dots(z+n)}$$

which is well-defined and does not depend on n. Of course this gives us ∞ at the negative integers and so

Theorem 10.2. *The gamma function extends via. the functional relationship to a meromorphic function with poles at $z = 0, -1, -2\dots,$ each of order 1 and satisfies*

$$res(\Gamma, -n) = \frac{(-1)^n}{n!}$$

10.2 Infinite Products

We say $\prod_{k=1}^{\infty} w_k$ converges if

- only finitely many $w_n = 0$
- The sequence of partial products converge with non-zero limit

Note that if the first few terms are 0 then we just ignore them.
Then $\prod_{k=1}^{\infty} w_K$ converges absolutely if $\exists n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \log w_k$$

converges.

Note:

$$w_n = \frac{\prod_{k=1}^{n} w_n}{\prod_{k=1}^{n-1} w_n} \to 1 \quad \text{as} \quad n \to \infty$$

Prop 10.3. *Defining $C^- := \mathbb{C} \setminus \{y = 0, x \leq 0\}$*

(a) $\prod_{n=1}^{\infty} w_n$ with $w_n \in \mathbb{C}^- \ \forall n \in \mathbb{N}$ converges $\iff$ $\sum_{n=1}^{\infty} \log w_n$ converges.

(b) $\prod_{n=0}^{\infty}(1 + a_n)$ converges absolutely $\iff$ $\sum_{k=1}^{\infty} |a_k|$ converges

We can use infinite products to further characterise the Gamma function.

Lemma 10.4. *The infinite product*

$$H(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

converges and absolutely and defines an entire function.

Corollary 10.5.

$$\begin{aligned} G_N(z) &:= z e^{-z \log(N)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) \\ &= z e^{-z\left(\log N - \sum_{n=1}^{N} \frac{1}{n}\right)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \end{aligned}$$

Theorem 10.6. *For any $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ we can write*

$$\frac{1}{\Gamma(z)} = G(z) = \lim_{N \to \infty} G_N(z)$$

10.3 The Zeta Function

Given any $z \in \mathbb{C}$ with $\mathcal{R}(z) > 1$ we can define the Riemann zeta function by

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Theorem 10.7. *We can express this as an infinite product over the primes* $\mathbb{P} = \{2, 3, 5, 7, \dots\}$ *by*

$$\frac{1}{\zeta(z)} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^z}\right)$$

Note: The ζ-function does not have any zeros in $\{\mathcal{R}(z) > 1\}$.

Lemma 10.8. *We can also relate the* ζ*-function to the* Γ*-function by*

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}e^{-t}}{1 - e^{-t}} dt$$

Note: The ζ-function can also be characterised by the Hankel contour but we really didn't do much on this. It might be worth reading over.

11 Riemann Mapping Theorem

Two open set $U, V \subseteq \mathbb{C}$ are said to be conformally equivalent if there exists $\phi : U \to V$ such that

- ϕ is holomorphic.
- ϕ is a bijection.
- The inverse map ϕ^{-1} is holomorphic.

If ϕ can be given by $z \mapsto az + b$ for some complex a, b then we say U and V are congruent.
Given $D \subseteq \mathbb{C}$ and loops $\gamma_1, \gamma_2 : [t_0, t_1] \to D$ then they are homotopic if there exists a continuous mapping $h : [0, 1] \times [t_0, t_1] \to D$ such that for all $t \in [t_0, t_1]$ and $s \in [0, 1]$ we have

- $h(0, t) = \gamma_1(t)$
- $h(1, t) = \gamma_2(t)$
- $h(s, t_0) = \gamma_1(t_0) = \gamma_2(t_0)$
- $h(s, t_1) = \gamma_1(t_1) = \gamma_2(t_1)$

Then we define $h_\tau(t) := (\tau, t)$ so that $h_0 = \gamma_1$ and $h_1 = \gamma_2$.
Finally, a connected set $D \subseteq \mathbb{C}$ is called simply connected if any two continuous curves with the same base point are homo topic to one another.

Note: Congruent $\implies$ Conformally equivalent $\implies$ Homeomorphic

Theorem 11.1. *Given γ_1, γ_2 homotopic and $f : D \to \mathbb{C}$ holomorphic*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Theorem 11.2 (Cauchy's theorem for simply connected set)**.** *Given a closed curve γ in D an open and simply connected domain along with any $f \in \mathcal{H}(D)$,*

$$\int_{\gamma} f(z)dz = 0$$

Proof. Using the previous theorem, denote by e the constant path at any point along γ then $\int_\gamma f(z)dz = \int_e f(z)dz = 0$ $\square$

Lemma 11.3. *Given $D \subseteq \mathbb{C}$ open and simply connected, $f : D \to \mathbb{C} \setminus \{0\}$ holomorphic there exists a holomorphic function $g : D \to \mathbb{C}$ such that*

$$f(z) = e^{g(z)} \quad \forall z \in D$$

moreover, that g is unique up to an additive constant $2\pi n$ for $n \in \mathbb{Z}$.

Proof. Choose $z_0 \in D$ then $f(z_0) \neq 0$ so there exists $w_o \in \mathbb{C}$ such that $f(z_0) = e^{w_0}$. Now given any other $z \in D$ choose a path γ in D which connects z_0 to z. Now define

$$g(z) := w_0 + \int_{f \circ \gamma} \frac{1}{z} dz$$

This is well-defined because our set is simply connected. Now we can see that in fact

$$g(z) = w_0 + \int_\gamma \frac{f'(z)}{f(z)} dz$$

and so g is holomorphic with derivative $g'(z)\frac{f'(z)}{f(z)}$. Hence we have that

$$\left(f(z)e^{-g(z)}\right)' = f'(z)e^{-g(z)} - f(z)e^{-g(z)}\frac{f'(z)}{f(z)} = 0$$

and so $f(z)e^{-g(z)}$ is a constant say α and moreover

$$\alpha = f(z_0)e^{-g(z_0)} = 1$$

$\square$

Theorem 11.4 (Riemann Mapping Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and simply connected such that $D \neq \emptyset$ and $D \neq \mathbb{C}$, then D is conformally equivalent to $\Delta = \{|z| < 1\}$.*

Proof. Bit of a mess. $\square$

THE END