Complex Analysis Notes

Thomas Chaplin

1 Complex Algebra

- The principal value of the argument is the unique $\theta \in (-\pi, \pi]$. This is a continuous function on $\mathbb{C}$ without any half-line (including 0).
- $\xi + i\eta$ is the logarithm of $re^{i\theta}$ id

$$\xi = \log(r) \qquad \eta = \theta + 2\pi n \;\; n \in \mathbb{Z}$$

- The principal value of the logarithm corresponds to $n = 0$.
- We say that $\xi + i\eta$ is an element of $z_0^{z_1}$ if

$$\xi + i\eta \in e^{z_1 \log(z_0)}$$

- The extended complex plane is $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.
- We can extended inversion to the $\hat{\mathbb{C}}$ by setting

$$\frac{1}{0} := \infty \qquad \frac{1}{\infty} := 0$$

1.1 Riemann Sphere

To represent the complex plane, we use stereographic projection of $S^2 \setminus \{\text{north pole}\}$ into $\mathbb{C}$ and then send the north pole to ∞.

$$\begin{aligned} \pi : S^2 \setminus \{(0,0,1)\} &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3} \end{aligned}$$

Lemma 1.1. *A circle on S^2 is the intersection of S^2 with some plane. The image of every non-vanishing circle on S^2, under π is a line or circle in $\mathbb{C}$.*

In this proof we notice that circles through the north pole go to lines and circles not through the north pole go to circles. So we can define $\pi(\text{north pole}) := \infty$ and see that $\pi(S^2) = \hat{\mathbb{C}}$.
We can use this to define a metric on $\hat{\mathbb{C}}$.

$$\forall z, w \in \mathbb{C} \qquad d(z, w) := ||\pi^{-1}(z) - \pi^{-1}w||$$

where $||\cdot||$ is the Euclidean norm on S^2.

Note: We can compute everything in this definition in terms of complex algebra to find

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}+\sqrt{1+|w|^2}}$$
$$d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}$$

When doing complex algebra we stick to the following conventions

- $\infty + z = z + \infty = \infty \qquad \forall z \in \mathbb{C}$
- $\infty \cdot z = z \cdot \infty = \infty \qquad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$
- $\frac{z}{\infty} = 0 \qquad \forall z \in \mathbb{C}$
- $\frac{z}{0} = \infty \qquad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$

2 Mobius Transformations

Given $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ we can define a Mobius transformation

$$f(z) := \frac{az+b}{cz+d} \qquad \forall z \in \mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$$

We can extend this to $\hat{\mathbb{C}}$ by defining $\hat{f}(-\frac{d}{c}) = \infty$ and $\hat{f}(\infty) = \frac{a}{c}$.
Notice we can multiply a, b, c, d by any non-zero complex number and recover the same function.
We say that f is normalised if $ad - bc = 1$.
It can be noticed that composing two Mobius transformations yields another Mobius transformation.
We can calculate the coefficients of the transformation by multiplying the corresponding matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lemma 2.1. *Extended Mobius transforms are invertible and their inverse is another Mobius transform.*

2.1 Decomposing Mobius transformations

Let inv be the inversion map $z \mapsto \frac{1}{z}$.

Lemma 2.2. *Let $\mathcal{C}$ be a circle or a line then inv$(\mathcal{C})$ is a circle or a line.*

Proof. Worth going over. □

The elementary Mobius transformations are

(a)	Inversion:	$\text{inv}(z) = \frac{1}{z}$
(b)	Translation:	$z \mapsto z + b$
(c)	Rotation:	$z \mapsto az$ for $a = e^{i\theta}$
(d)	Expansion/Contraction:	$z \mapsto rz$ for $z \in \mathbb{R}, z > 0$

Lemma 2.3. *Every Mobius transformation can be written as a composition of elementary Mobius transformations.*

Proof. **Case 1:** $c \neq 0$
We can write

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

Case 2: $c = 0$
$c = 0$ and $ad - bc \neq 0 \implies d \neq 0$ and hence we can write

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}$$

In both cases these transformations can be easily decomposed. □

Theorem 2.4. *The image of a circle or line in $\hat{\mathbb{C}}$ under a Mobius transformation is another circle or line.*

Theorem 2.5. *Given 3 distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and three other distinct points $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ there exists a unique Mobius transform f with $f(z_i) = w_i$ for all i.*

Proof. **Existence:** We define two helper functions, assuming that none of the points are ∞

$$S(z) := \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

and if any z_i is ∞ then we simply remove any term containing that z_i. Notice

$$S(z_1) = 1 \quad S(z_2) = 0 \quad S(z_3) = \infty$$

We define T in the same way but replacing each z_i with w_i. Then we can notice that defining $f := T^{-1}S$ yields a function with the desired properties.
Uniqueness: It suffices to check the cases when $w_1 = 1$, $w_2 = 0$ and $w_3 = \infty$ because we can always compose with T. Then we can just pick two suitable Mobius transformations f_1 and f_2, then show that $g := f_1 \circ f_2^{-1}$ is the identity Mobius transformation. □

Note: Look up the cross ratio.

- A non-identity Mobius transform has at most two fixed points because

$$z = \frac{az+b}{cz+d} \iff 0 = cz^2 + (d-a)z - b$$

3 Complex Differentiability

Given $D \subseteq \mathbb{C}$ open, a function $f : D \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

Note: This definition of f' can be restated as

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; s.t. \;\; |z - z_0| < \delta \implies |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|$$

Prop 3.1. *$f : D \to \mathbb{C}$ complex differentiable at $z_0 \in D$ implies f is continuous at z_0.*

The complex derivative also satisfies all of the usual algebra of derivative functions from real analysis, including the chain rule

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Theorem 3.2 (Cauchy-Riemann Equations)**.** *The following are equivalent, given $f : D \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in D$*

(a) f is $\mathbb{C}$-differentiable at z_0.

(b) f is $\mathbb{R}$-differentiable at (x_0, y_0) and $df(z_0)$ is complex linear

(c) f is $\mathbb{R}$-differentiable at (x_0, y_0) and the CR equations hold:

$$u_x = v_y \qquad u_y = -v_x$$

Proof. *(i)* $\iff$ *(ii)* is somewhat immediate. Consider the alternative definition given in the notes. We see that being $\mathbb{C}$-differentiable is equivalent to the existence of a complex number ξ such that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - \xi \cdot h}{h} = 0$$

We can view thus view the derivative as a $\mathbb{C}$-linear function $h \mapsto \xi \cdot h$. This is equivalent to the definition of $\mathbb{R}$-differentiability with the additional requirement that the map is $\mathbb{C}$-linear. In practice this means that the Jacobian matrix is some real number multiplied by a rotation matrix.
This explains *(ii)* $\iff$ *(iii)* as well because the Jacobian must be given by

$$r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Alternatively, writing out the Jacobian we see that the derivative as a $\mathbb{C}$-linear map

$$M(h) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

and then the condition $M(ih) = iM(h) \; \forall h \in \mathbb{C}$ is equivalent to the Cauchy-Riemann equations. $\square$

Theorem 3.3 (Power Series Expansion)**.** *Given a sequence $(a_k)_{k \in \mathbb{N}_0}$ with $a_k \in \mathbb{C}$, consider the power series*

$$\sum_{k=0}^{\infty} a_k z^k \tag{1}$$

(a) *There exists a radius of convergence $r \in [0,\infty]$ such that for all z with $|z| < r$ the series (1) converges, and for all z with $|z| = r' > r$ the series (1) does not converge.*

(b) *The series*

$$\sum_{k=1}^{\infty} k a_k z^{k-1} \tag{2}$$

head the same radius of convergence.

(c) *$f(z) := \sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $\mathcal{B}_r(0) = \{|z| < r\}$.*

Proof. (a) If we convergence for some z_0 then $(a_k z_0^k) \to 0$ is bounded by $C > 0$ say and hence for all z with $|z| < |z_0|$ we have

$$\sum_{k=0}^{\infty} |a_k z^k| = \sum_{k=0}^{\infty} |a_k z_0^k| \frac{|z|^k}{|z_0|^k} \leq C \frac{1}{|z_0| - |z|}$$

and hence we get convergence.

The radius of convergence is therefore $\sup\{\eta \geq 0 \mid \exists z \text{ with } |z| = \eta \;\; s.t. \text{ (1) converges}\}$

(b) We now consider (2). Suppose $|z| < \hat{r} < r$, then we have

$$\sum_{k=1}^{\infty} |k a_k z^{k-1}| \leq \frac{1}{\hat{r}} \sum_{k=1}^{\infty} \underbrace{k \left(\frac{|z|^{k-1}}{\hat{r}^{k-1}} \right)}_{\to 0} \underbrace{|a_k \hat{r}^k|}_{\text{convergent}}$$

and hence the sum converges. Likewise the sum diverges wherever the other one does.

(c) Confusing proof.

□

4 Cauchy's Collection of Complex Corollaries

4.1 Complex Integration

Let $f : D \to \mathbb{C}$ be continuous and γ a smooth curve with $\Gamma = \gamma[a,b] \subseteq D$

$$\int_\Gamma f(z)\, dz = \int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\dot{\gamma}(t)\, dt$$

The length of a curve is defined to be

$$L(\gamma) := \int_a^b |\dot{\gamma}|\, dt$$

Two curves $\gamma : [a,b] \to \mathbb{C}$ and $\lambda : [c,d] \to \mathbb{C}$ are smoothly equivalent parametrisations for Γ if there is a smooth function $\rho : [a,b] \to [c,d]$ such that

(i) $\dot{\rho}(t) \neq 0 \quad \forall t$.

(ii) $\rho^{-1} \in \mathcal{C}^1$ and is never zero.

(iii) $\gamma = \lambda \circ \rho$.

(iv) $\rho(a) = c$ and $\rho(b) = d$.

Lemma 4.1. *The complex line integral is invariant under change of parametrisation.*

Lemma 4.2. *If γ and λ are smoothly equivalent then $L(\gamma) = L(\lambda)$.*

Lemma 4.3. *$f : D \to \mathbb{C}$ holomorphic and $\gamma \in \mathcal{C}^1([a,b])$ such that $\Gamma \subseteq D$ then*

$$\begin{aligned}\int_\gamma f'(z)\, dz &= \int_a^b f'(\gamma(t))\dot{\gamma}(t)\, dt \\ &= \int_a^b \frac{d}{dt} f(\gamma(t))\, dt \\ &= f(\gamma(b)) = f(\gamma(a))\end{aligned}$$

4.2 My First Sony Cauchy's Theorem

Theorem 4.4 (Goursat's Theorem)**.** *Take $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic. Take a rectangle $Q \subseteq D$ such that $Q \cup \partial Q = \overline{Q} \subseteq D$. Take a $\mathcal{C}^1$ parametrisation $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma[a,b] = \partial Q$ and γ circles around Q exactly once in the positive direction. Then*

$$\int_\gamma f(z)\, dz = 0$$

Proof. We split the proof into a number of steps:

1. $f \equiv 1$.

   This proof follows easily from the FTC.

2. $f(z) = z$.

   This proof also follows easily from the FTC because

   $$\int \gamma(t)\dot{\gamma}(t)\, dt = \frac{1}{2}\int_a^b \frac{d}{dt}\left(\gamma(t)\right)^2\, dt = \frac{1}{2}\left[\gamma(b)^2 - \gamma(a)^2\right]$$

3. f holomorphic in D.

   Divide into rectangles, this is a very long proof in Lecture 9.

□

Corollary 4.5 (Cauchy's Theorem for images of rectangles)**.** *Given $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic such that $\overline{Q} \subseteq D$. Suppose $\phi : \overline{Q} \to D$ is $\mathcal{C}^1$. Let γ be a $\mathcal{C}^1$ parametrisation of ∂Q then*

$$\int_{\phi\circ\gamma} f(z)\, dz = 0$$

We say $D \subseteq \mathbb{C}$ is

- a region if it is non-empty and connected.
- polygonally connected if between every two points are joined by a path consisting of a finite collection of straight lines all contained within D.

A contour is a simple closed curve.

Theorem 4.6. *Given a non-empty open set $D \subseteq C$*

$$D \text{ is a region} \quad \iff \quad D \text{ is polygonally connected}$$

Theorem 4.7 (Jordan Curve Theorem)**.** *Let γ be a contour and $\Gamma = \gamma[a, b]$ then ${}^{\mathsf{c}}\gamma$ consists of*

$$I(\gamma) \cup O(\gamma)$$

where $I(\gamma)$ is bounded and $O(\gamma)$ is unbounded and the two regions are disjoint.

Note: Jordan Curve Theorem $\implies$ Cauchy's theorem for contours

4.3 Cauchy's Integral Formula

Theorem 4.8 (Cauchy's Integral Formula)**.** *Given $D \subseteq \mathbb{C}$ open and $f : D \to \mathbb{C}$ holomorphic, suppose that $\overline{\mathcal{B}_r(a)} \subseteq D$ for some $a \in D$ and $r > 0$. Then for all $z_0 \in \mathcal{B}_r(a)$*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(a)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Not too difficult, worth going over (Lecture 11) □

4.4 Applications

4.4.1 Immediate Applications

Theorem 4.9 (Taylor's Theorem)**.** *Given $D \subseteq C$ open and polygonally connected and $f : D \to C$ holomorphic. Assume $\exists R > 0$ and $z_0 \in D$ such that $\overline{\mathcal{B}_R(z_0)} \subseteq D$ then for all $z \in \mathcal{B}_R(z_0)$ we have*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Corollary 4.10. *Every homolomorphic function on D is in fact $C^\infty(D)$.*

Corollary 4.11. *$D \subseteq \mathbb{C}$ is open and polygonally collected and $f : D \to \mathbb{C}$ then the following are equivalent:*

(i) f is holomorphic in D.

(ii) f is real differentiable on D and the CR equations hold.

(iii) f can be expressed in a power series.

Corollary 4.12. *Suppose $f(z) = \sum_{k \in \mathbb{N}} a_k z^k$ is holomorphic on $\mathcal{B}_R(0)$ for some $R > 0$ and suppose f is bounded in that ball, say by M. Then for all $k \in \mathbb{N}$*

$$|a_k| \leq \frac{M}{R^k}$$

where R is the radius of convergence.

Theorem 4.13 (Liouville's Theorem)**.** *Any entire, bounded function is constant.*

Proof. Pick $z_0 \in \mathbb{C}$ and $M > 0$ such that $|f(z)| \leq M \; \forall z \in \mathbb{C}$. Define

$$m(f, R, z_0) := \max_{z \in \partial \mathcal{B}_R(z_0)} |f(z)|$$

Then by Taylor's theorem we see that

$$|f^(n)(z_0)| \leq \frac{n!}{R^n} m(f, R, z_0) \leq \frac{n!}{R^n} M$$

and in particular $|f'(z_0)| \leq \frac{M}{R} \to 0$ as $R \to \infty$. Hence $f'(z_0) = 0$. $\square$

Corollary 4.14 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial has at least one zero in* $\mathbb{C}$*.*

Proof. Take some polynomial $P(z) := a_n z^n + \cdots + a_1 z + a_0$ such that $a_n \neq 0$. Then for any $\epsilon > 0$ there is a radius R such that $\forall |z| > R$ we have

$$(1 - \epsilon)|a_n||z|^n \leq |P(z)| \leq (1 + \epsilon)|a_n||z_n|^n$$

Suppose $P(z)$ has no zeros in $\mathbb{C}$ then $\frac{1}{P(z)}$ is complex differentiable in $\mathbb{C}$ and there is an $R >)$ such that for all $|z| > R$

$$\frac{1}{2}|a_n|z_n \leq |P(z)| \implies |\frac{1}{P(z)}| \leq \frac{2}{|a_n||z^n|} \leq \frac{2}{|a_n|R^n}$$

and hence $\frac{1}{P}$ is bounded on $\{|z| > R\}$ and it is obviously bounded inside by compactness. Hence by Liouvilles's Theorem P is constant. This is a contradiction. $\square$

Theorem 4.15 (Morea's Theorem)**.** *Given a region $D \subseteq \mathbb{C}$ and a $f : D \to \mathbb{C}$ continuous, suppose given any triangle T with $T \cup \partial T \subseteq D$ we have $\int_{\partial T} f(z)\, dz = 0$. Then f is holomorphic in D.*

Theorem 4.16 (Schwarz Reflection Principle)**.** *Suppose D is open in $\overline{H^+} := \{z \in \mathbb{C} \mid \mathcal{I}(z) \geq 0\}$ and $f : D \to \mathbb{C}$ is continuous on D and holomorphic on D°. Then*

$$\widetilde{f}(z) := \begin{cases} f(z) & z \in D \\ \overline{f(\overline{z})} & z \in \widetilde{D} \end{cases}$$

where $\widetilde{D}$ is the complex conjugate of D, is well-defined and holomorphic on $D \cup \widetilde{D}$.

Proof. By composition of reflections we can easily sow that $\widetilde{f}$ is holomorphic on $\widetilde{D}^\circ$ so that only the lines remain. We show that the integral of f over any triangle with one edge on the line is 0 by a continuity argument, approaching from both sides. $\square$

5 Zeros of Holomorphic Functions

Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic, the order of any zero $z_0 \in D$ is

$$\mathrm{ord}(f, z_0) := \inf \left\{ k \in \mathbb{N} \;\middle|\; f^{(k)}(z_0) \neq 0 \right\} \in \mathbb{N} \cup \{\infty\}$$

We say that $f : D \to \mathbb{C}$ is a conformal mapping if f is holomorphic in D and it's derivative is non-vanishing on D.

We say that f is biholomorphic if f is a conformal mapping such that f^{-1} exists and is also conformal.

Prop 5.1. *Given $D \subsetneq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic, suppose we have a zero $z_0 \in D$ of order $k \in \mathbb{N}$. Then there is a neighbourhood U_0 of z_0 and a holomorphic function $h : U_0 \to V_0$ such that $h(z_0) = 0$, $ord(f, z_0) = 1$ and*

$$f(z) = (h(z))^k \quad \forall z \in U_0$$

Proof. WLOG we may assume that $z_0 = 0$, then we apply Taylor's theorem to write f as

$$f(z) = \sum_{n=k}^{\infty} c_n z^n$$

because $\mathrm{ord}(f, z_0) = k$ and hence the first k terms vanish. For simplicity we can also assume that $c_k = 1$. Hence we can write

$$f(z) = z^k \left(\underbrace{1 + \sum_{n=k+1}^{\infty} c_n z^{n-k}}_{=:g(z)} \right) = \Bigg(\underbrace{z \sqrt[k]{1 + g(z)}}_{=:\text{h(z)}} \Bigg)^k$$

Note that g is holomorphic and $g(0) = 0$, and $h(0) = 0$. Moreover, $h'(0) = \sqrt[k]{1 + g(0)} + 0 \left(\sqrt[k]{1 + g(z)} \right)' = 1 \neq 0$. Hence $\mathrm{ord}(h, 0) = 1$. Read up on making it holomorpic (Lecture 14). $\square$

Note: This implies that all zeros of finite order are isolated.

Theorem 5.2. *If $ord(f, z_0) = k \in \mathbb{N}$ for $f : D \to \mathbb{C}$ holomophic then $\forall \epsilon > 0$ the exists a $U_\epsilon \subseteq D$ with $z_0 \in U_\epsilon$ such that $f(U_\epsilon) = \mathcal{B}_\epsilon(0)$ and $f\big|_{U_\epsilon}$ takes every w with $0 < |w| < \epsilon$ exactly k times and 0 for z_0.*

Proof. Without loss of generality we may assume that $z_0 = 0$.
If $f(z) = z^k$ then any $w = re^{i\theta}$ has exactly k roots.
In the general case we can write $f(z) = (h(z))^k$ for $h : U \to V$ holomorphic such that $h(0) = 0$ and $h'(0) \neq 0$. Moreover, h is locally biholomorphic around a neighbourhood of 0. Choose $\epsilon > 0$ sufficiently small that

$$A := \left\{ \xi \in \mathbb{C} \;\middle|\; |\xi| \leq \sqrt[k]{\epsilon} \right\} \subseteq V$$

Then define $U_\epsilon := h^{-1}(A)$. This set has the desired properties because the original roots of $z \mapsto z^k$ lie in A. $\square$

Note: Every bijective holomorphic function is biholomorphic.

Theorem 5.3 (Identity Theorem)**.** *Given $D \subseteq \mathbb{C}$ open and connected with $f_1, f_2 : D \to \mathbb{C}$ holomorphic, assume that $\{f_1 = f_2\}$ has at least one accumulation point in D. Then $f_1 = f_2$ on D.*

Proof. Define $g := f_1 - f_2$ and let z_0 be one of the accumulation points. Then z_0 is a zero of infinite order for g. Apparently this is a proof? $\square$

Theorem 5.4 (Open Mapping Theorem). *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic and non-constant, $f(D)$ is open and connected.*

Theorem 5.5 (Maximum Modulus Principle). *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \to \mathbb{C}$ holomorphic and non-constant, $|f|$ does not have any maxima.*

Lemma 5.6 (Schwarz Lemma). *Suppose $f : \Delta \to \Delta$ is holomorphic such that $f(0) = 0$ then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \Delta$.

(ii) $|f'(0)| \leq 1$.

(iii) If for some $z \in \Delta \setminus \{0\}$ we have $|f(z)| = |z|$ or $|f'(z)| = 1$ then $\exists \theta \in \mathbb{R}$ such that $f(\widetilde{z}) = e^{i\theta}\widetilde{z}$ for all $\widetilde{z} \in \Delta$.

6 Singularities

Given $D \subseteq \mathbb{C}$ open and connected and $f \in \mathcal{H}(D)$,

- f has an isolated singularity at $z_0 \notin D$ if there is an $\epsilon > 0$ such that f is defined on $\mathcal{B}_\epsilon(z_0) \setminus \{z_0\}$.
- $z_0 \in D$ is a regular point if f is complex differentiable at $z_0 \in D$.

Given an isolated singularity z_0 then it has order

$$\operatorname{ord}(f, z_0) := -\inf\left\{ n \in \mathbb{Z} \;\middle|\; \lim_{z \to z_0} (z - z_0)^n f(z) \text{ exists and } < \infty \right\}$$

then we say z_0 is a

- removable singularity if $\operatorname{ord}(f, z_0) \geq 0$.
- pole of order $n \in \mathbb{N}$ if $\operatorname{ord}(f, z_0) = -n \in (-\infty, -1]$.
- essential singularity if $\operatorname{ord}(f, z_0) = -\infty$.

Let $S \subseteq D$ be a discrete set, then a holomorphic function $f : D \setminus S \to \mathbb{C}$ is called meromorphic on D if none of the isolated singularities in S are essential.

Prop 6.1. *Let $\mathcal{Z}_f$ and $\mathcal{P}_f$ be the set of zeros and poles respectively of $f : D \to \mathbb{C}$ meromorphic. Then neither set has an accumulation point in D.*

Proof. Certainly any pole of f is an isolated singularity and hence cannot be an accumulation point of $\mathcal{P}_f$. Any other $z \in D$ where f is holomorphic cannot be an accumulation point either.
Suppose now that $\mathcal{Z}_f$ has an accumulation point at $z_0 \in D$ then z_0 cannot be a pole otherwise we'd be able to write

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{with } m \in \mathbb{N}, g(z_0) \neq 0$$

and hence $f(z) \neq 0$ for all $0 \leq |z - z_0| \epsilon$ which means that z_0 is not an accumulation point.

So any accumulation pole must be a complex differentiable point. We are left to show that $D \setminus \mathcal{P}_f$ see open and connected because then the identity theorem tells us $\mathcal{Z}_f$ cannot have an accumulation point in $D \setminus \mathcal{P}_f$. ☐

Lemma 6.2. *Suppose $D \subseteq \mathbb{C}$ is open and connected. Suppose $M \subseteq D$ has no accumulation point in D. Then $D \setminus M$ is open and connected.*

Proof. Openness is immediate, for connectedness just draw a picture. ☐

6.1 Laurent Series

A Laurent series is a series of the form

$$\sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

such that the positive terms converge inside some ball around z_0 and the negative terms converge outside sum larger ball around z_0. Hence the Laurent series converges in an annulus around the point z_0.

Theorem 6.3 (Cauchy's Theorem for annuli)**.** *Given $0 \leq R_1 < R_2 < \infty$ and $D \subseteq \mathbb{C}$ open and connected such that $\overline{A} := \overline{A(R_1, R_2, z_0)} \subseteq D$, for any $z \in A$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_{R_2}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi - \frac{1}{2\pi i} \int_{\partial \mathcal{B}_{R_1}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Do normal Cauchy on a small ball contained in the annulus then do some appropriate contour integration. ☐

Theorem 6.4 (Laurent's Theorem)**.** *Given f holomorphic on a neighbourhood of an annulus $A = A(R_1, R_2, z_0)$ and any $z \in A$,*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

where for all $\rho \in [R_1, R_2]$ we can write

$$a_k = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_\rho(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Corollary 6.5. *Under the same assumption, if f is bounded on $\{|z - z_0| = \rho\}$ for sum $\rho \in [R_1, R_2]$ then*

$$|a_k| \leq \frac{M}{\rho^k} \quad \text{for all } k \in \mathbb{Z}$$

6.2 Classification of Singularities

Theorem 6.6 (Riemann's removable singularity theorem)**.** *Gain an isolated singularity z_0 for a function $f \in \mathcal{H}(D \setminus \{z_0\})$, assume that $|f|$ is bounded in a neighbourhood of z_0. Then there is a holomorphic function $\widetilde{f} \in \mathcal{H}(D)$ which extend to f. Moreover, z_0 was a removable singularity.*

Proof. In the neighbourhood we can expand in a Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

and then for all sufficiently small $\rho > 0$ we have $|a_k| \leq \frac{M}{\rho^k}$ for all $k \in \mathbb{Z}$. Taking $\rho \to 0$ we see that $a_k = 0$ for all $k < 0$. So we can extend f by taking $\widetilde{f}(z_0) = a_0$. $\square$

Corollary 6.7. *Given $f : D \to \mathbb{C}$ which is holomorphic except for an isolated singularity at z_0, the following are equivalent:*

(i) *f has a pole at z_0.*

(ii) *At least coefficient of negative order in the Laurent series around z_0 in non-zero, but at most finitely many.*

(iii) $\lim_{z \to z_0} |f(z)| = +\infty$.

Theorem 6.8 (Casorati-Weirstrass)**.** *Given $f \in \mathcal{H}(D \setminus \{z_0\})$ with an isolated, essential singularity at z_0, for all $\epsilon > 0$ the set $f(\mathcal{B}_\epsilon(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

7 Residual Theory

8 Rouches Theorem

9 Montels Theorem

10 Riemann Mapping Theorem