Manifolds Notes on things I don't understand/remember

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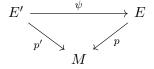
In defining a vector bundle over a manifold we associate a vector space to each point in the manifold and then arrange these together in a smooth way. Examples of this will include the tangent and normal bundles as defined in times gone past.

Suppose that M is an m-manifold. A family of vector spaces over M is a manifold E together with a smooth map $p: E \to M$ such that for all $x \in M$, the fibre $E_x := p^{-1}(x)$ has a vector space structure. Given $U \subseteq M$ we write $E|_U := p^{-1}U$ which we equip with the restriction of p.

Given another family over vector spaces $p': E' \to M$ we define an isomorphism to be a diffeomorphism

$$\psi: E \to E'$$

with the property that $p' \circ \psi = p$ and $\psi : E_x \to E'_x$ restricts to a linear isomorphism for all $x \in M$. If such a map exists then E and E' are called isomorphic. Finally, a family E over M is called trivial if it is isomorphic to $M \times \mathbb{R}^q$ for some $q \in \mathbb{N}$.



A vector bundle over M is a family of vector spaces $p: E \to M$ such that $\forall x \in M$ there is some open $U \subseteq M$ with $x \in U$ such that $E|_U$ is trivial.

Putting this another way, E is a vector bundle over M if we can find an atlas $\{\phi_a lpha : U_\alpha \to V_\alpha\}$ for M such that for each α we get a diffeomorphism

$$\psi_{\alpha}: E|_{U_{\alpha}} \to V_{\alpha} \times \mathbb{R}^q \subseteq \mathbb{R}^{m+q}$$

which restricts to a linear isomorphism on each fibre. Note than then the maps $\{\psi_{\alpha}\}$ form an atlas for E which is called a locally trivialising atlas for E.

Note:

• Given a vector bundle, $p: E \to M$ is a surjective smooth submersion.

- Each fibre E_x is an embedded sub-manifold and the vector space operations are smooth.
- The notion of a vector bundle is preserved under isomorphism.

Example:

1. If M is a smooth manifold then the tangent bundle is a vector bundle over M. For a manifold in Euclidean space, this follows for the proof that TM was a manifold. For an abstract space, we can form a locally trivialising atlas as follows:

Given a chart $\phi: U \to V \subseteq \mathbb{R}^m$ for M we construct a new map

$$\psi: TU \to V \times \mathbb{R}^m$$
 by $\psi(v) = (\phi(x), d_x \phi(v))$

where x = pv. These maps then form a locally trivialising atlas because when restricted to some x the map becomes the derivative map

$$\psi_{n^{-1}(x)}: T_xM \to \{\phi(x)\} \times \mathbb{R}^m \quad \psi(v) = (\phi(x), d_x\phi(v))$$

which is a linear isomorphism.

- 2. When can (and perhaps should) check that the atlas given for the normal bundle consists of maps which restrict to linear isomorphisms on each fibre.
- 3. The spaces B_n together with the projection to the circle yields a vector bundles. In particular, the Möbius band is a bundle over the circle. Note B_m is isomorphic to B_n if |m-n| is even. But, B_0 cannot be isomorphic to B_1 since they are not even diffeomorphic. Hence the Möbius band is non-trivial.

Given a vector bundle $p: E \to M$ we say that section of E is a smooth map $s: E \to M$ such that $p \circ s$ is the identity on M. This generalises our previous notion of a section of the tangent space which is often referred to as a vector field on M. We say that a section is non-vanishing if $s(x) \neq 0$ for all $x \in M$. Given any smooth map $f: M \to \mathbb{R}^q$ we get a section of $M \times \mathbb{R}^q$.

Note: Any section is an embedding of M into E.

Theorem 3.1 (Hairy Ball Theorem). Every vector field on S^2 has at least one zero.

Note: It turns out that any odd-dimensional manifold admits a non-vanishing vector field. An even-dimensional manifold may or may not. The torus does but it is the only compact connected orientable 2-manifold which does.

Lemma 3.2. A vector bundle E is trivial if and only if there are sections s_1, s_2, \ldots, s_q such that $\{s_1, \ldots, s_q(x)\}$ forms a basis for E_x for all $x \in M$.

Proof. If $f: M \times \mathbb{R}^q$ is an isomorphism then set $s_i(x) = f(x, e_i)$ for some fixed basis e_1, \ldots, e_q for \mathbb{R}^q

Conversely, if we have such sections s_i then we can define a map $f: M \times \mathbb{R}^q \to E$ by

$$f(x, (\lambda_1, \dots, \lambda_q)) = \sum_{i=1}^q \lambda_i s_i(x)$$

which yields an isomorphism.

Example:

- 1. Using the Intermediate Value Theorem, one can show that the Möbius has no non-vanishing section and is hence a non-trivial bundle by the above Lemma.
- 2. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere in \mathbb{R}^n then the normal bundle is trivial because the outward pointing unit normal is a nowhere vanishing section.

A manifold M is called parallelisable if the tangent bundle TM is trivial.

In view of the last lemma we can say that a manifold is parallelisable if and only if M admits a global frame field. Recall, a frame field is a family of vector fields v_i such that the $v_i(x)$ form a basis for T_xM at every $x \in M$. We know that frame fields exists locally but not necessarily globally.

Lemma 3.3. Any parallelisable manifold is orientable.

Proof. Exercise. \Box

Example:

- The circle S^1 is parallelisable.
- S^2 is not by the Hairy Ball Theorem.
- S^3 is parallelisable because given $x = (x_1, x_2, x_3, x_4) \in S^3$ we can form a frame field with

- S^5 is not strangely enough but S^7 is. It may be worth consulting your written notes from lectures to figure out how this frame was calculated.
- 3.1 Vector Fields
- 3.2 Whitney Sums
- 3.3 Cotangent Bundle
- 3.4 1-forms
- 4 Smooth Function Extension
- 5 Differential Forms