Fluid Dynamics Notes

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1 Mathematical Modelling of Fluid Flow

1.1 Validity of continuum mechanics

Continuum mechanics is valid when

$$\frac{l}{L} << 1$$

where l and L are length scales characterising molecular motion and flow dimension respectively.

Typical examples:

- $l_{gas} \sim 100nm = 10^{-7}m$ is the mean free path of a typical gas
- $l_{liquid} \sim 1nm = 10^{-9}m$ is the typical distance between molecules.

1.2 Lagrangian v.s. Eulerian Description

A fluid will consist of a number of material points/fluid elements/fluid particles/fluid volumes. In order to uniquely identify each fluid element we can tag it according to its Eulerian coordinates at some initial condition

$$x_j(t - t_0) = X_j = (X_1, X_2, X_3)$$

Alternatively we allow the coordinates to move and deform with time to get the X_j Largrangian coordinates which follow the fluid. This allows us to derive useful equation but these equations are usually simpler when transformed back to Eulerian coordinates.

1.3 Flow Visuation

1.3.1 Particle Paths

Particle paths are intuitively the path a particle will follow if dropped into a flow and depends only on initial conditions:

$$\frac{\partial t}{\partial t} = u_i \qquad \text{for fixed} \quad x_i^p(t = 0) = X_i$$

We integrate these equations and eliminate t to get particle paths.

To calculate particle paths for the flow $\mathbf{u}(t) = (u_0, kt, 0)$, we integrate to get

$$\mathbf{x}(t) = (u_0 t + a, \frac{kt^2}{2} + b, c) \implies y = \frac{k}{2}\left(\frac{x - a}{u_0}\right)^2 + b \implies \text{Parabola}$$

1.3.2 Stream lines

Intuitively, we pause time and then draw lines in the vector field. We parameterise these lines in terms of arc length s

$$\frac{\partial s}{\partial s} = u_i \qquad \text{for fixed} \quad t$$

We integrate these equations and eliminate s to get stream lines. Note, for a steady flow (i.e $\frac{\partial t}{\partial t} = 0$), this is the same as the particle paths.

For the same flow $\mathbf{u}(t) = (u_0, kt, 0)$, suppose at $s = 0$ we have $(x, y, z) = (a, b, c)$ then

$$\mathbf{x}(t) = (u_0 s + a, kts + b, c) \implies y = kt\left(\frac{x-a}{u_0}\right) + b \implies \text{Straight lines}$$

fsubsectionMaterial Derivative The material derivative is the rate of change of something which follows the flow. To figure out what this is we consider the derivative of a function dependant on time and the particle paths $f(\mathbf{x}^p(t), t)$. Then

$$\begin{aligned} \frac{\partial t}{\partial t} &= \frac{\partial t}{\partial t} + \sum_i \frac{\partial x_i}{\partial x_i}\frac{\partial t}{\partial t} \\ &= \frac{\partial t}{\partial t} + \sum_i u_i \frac{\partial x_i}{\partial x_i} \\ &= \left(\frac{\partial t}{\partial t} + \sum_i u_i \frac{\partial x_i}{\partial x_i}\right) f \end{aligned}$$

We then define the material derivative to be

$$\frac{D}{Dt} := \frac{\partial t}{\partial t} + \sum_i u_i \frac{\partial x_i}{\partial x_i} = \frac{\partial t}{\partial t} + \mathbf{u} \cdot \bigtriangledown$$

The first time is the local rate of change at a fixed Eulerian position whereas the second term represents the convective rate of change caused by driving fluid elements through gradients of f.

For a concentration of pollutant c=c(x) in a river with steady flow $\mathbf{u} = (u_0, 0, 0)$, how does the concetration change in a fluid element that follows the fluid?

$$\begin{aligned} \frac{Dc}{Dt} &= \frac{\partial t}{\partial t} + (\mathbf{u} \cdot \bigtriangledown)c \\ &= u_0 \frac{\partial x}{\partial x} + 0\frac{\partial y}{\partial y} + 0\frac{\partial z}{\partial z} \\ &= u_0 \frac{\partial x}{\partial x} \end{aligned}$$

Note:

- $\frac{Df}{Dt} \equiv 0 \implies f$ is constant in fluid elements but, in general, has different values in different elements.

- Parameterising streamlines by s, let $\mathbf{e}_s$ be the unit tangent vector at s. Then

$$(\mathbf{u}\cdot\triangledown)f = |u|\mathbf{e}_s\cdot\triangledown f = |u|\frac{\partial s}{\partial s}$$

  Hence $(u\cdot\triangledown)f = 0 \implies f$ constant on streamlines

We can now define acceleation as

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial t}{\partial t} + (\mathbf{u}\cdot\triangledown)\mathbf{u}.$$

We also say a flow is steady if $\frac{\partial t}{\partial t} = 0$ (normal partial derivaive).

Consider the 1D flow $\mathbf{u} = (u_1(x), 0, 0)$, noticing we have steady flow but

$$\begin{aligned} a_1 &= \frac{\partial t}{\partial t} + \sum_j u_j\frac{\partial x_j}{\partial x_j} \\ &= 0 + u_1\frac{\partial x}{\partial x} = u_1\frac{\partial x}{\partial x} \end{aligned}$$

individual fluid elements still accelerate. To visualise this consider a flow in a pipe which constricts.

1.4 Vorticity and Rate of Strain

We want to understand how flow deforms fluid elements which will show us that rate of deformations is what generates stress in a fluid.
To do this we Taylor expand velocity to first order, at two points differing by some δ_{x_j} :

$$\begin{aligned} u_i(x_j + \delta_{x_j}, t) &= u_i(x_j, t) + \sum_j \frac{\partial x_j}{\partial x_j}\delta x_j \\ &= \underbrace{u_i(x_j, t) + \sum_j r_{ij}\delta x_j}_{\text{rigid body}} + \underbrace{\sum_j e_{ij}\delta x_j}_{\text{Shearing + Extension}} \\ &=: u_i^T + u_i^R + u_i^S \end{aligned}$$

$$\begin{aligned} \text{where} \quad r_{ij} &= -r_{ji} := \frac{1}{2}\left(\frac{\partial x_j}{\partial x_j} - \frac{\partial x_i}{\partial x_i}\right) && \text{is the rate of rotation tensor} \\ e_{ij} &= e_{ji} := \frac{1}{2}\left(\frac{\partial x_j}{\partial x_j} + \frac{\partial x_i}{\partial x_i}\right) && \text{is the rate of strain tensor} \end{aligned}$$

This term has three components of interest which we will now explore.

1.4.1 Translation

The first term is certainly translation because

$$u_i(x_j + \delta_{x_j}, t) = u_i(x_j, t) \implies \text{velocity is locally constant}$$

This induces no internal stress.

1.4.2 Rotation

Because the rate of rotation tensor is skew-symmetric we only get 3 non-zero terms:

$$(r_{ij}) = \begin{bmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix}$$

so we can rewrite the rotation term as

$$u_i^R = \sum_j r_{ij}\delta_{x_j} = \sum_j \sum_k \epsilon_{ijk}\Omega j \delta_{x_k}$$

where $\Omega = (r_{32}, r_{13}, r_{21}) = (a, b, c)$. We then define vertocity to be $\omega := \bigtriangledown \times \mathbf{u} = 2\Omega$. We can view this as the rotation term where Ω is the local rate of rotation. Rotation induces no internal stress.

1.4.3 Shearing + Extension

The strain term u_i^S involves the relative motion of fluid particles. They can be split into two types:

$$\text{Diagonal terms} \longleftrightarrow \text{Extensional terms}$$
$$\text{Off-diagonal terms} \longleftrightarrow \text{Sharing terms}$$

Read written notes for diagrammatic explanation of these terms. **Note:**

Denote by V the volume of a fluid element then

$$\boxed{\frac{1}{V}\frac{DV}{Dt} = \bigtriangledown \cdot \mathbf{u}}$$

Therefore $\frac{DV}{Dt} = 0 \implies \underbrace{\bigtriangledown \cdot \mathbf{u} = 0}_{Incompressible flow} \iff \sum_k e_{kk} = 0$

1.5 Conservation of Mass

We want an equation which encompasses conservation of mass in a fluid. Consider a fluid element cube with (infinetesimal) side length d and centred at (x_0, y_0, x_0).

$$\text{Rate of increase of fluid mass} = \text{Volume} \cdot \text{Rate of change of density}$$
$$= d^3 \cdot \frac{\partial t}{\partial t}$$

Summing over all sides we see

$$\begin{aligned} d^3\frac{\partial t}{\partial t} = d^2[&-\rho(x_0+d, y, z)u(x_0+d, y, z) + \rho(x_0, y, z)u(x_0, y, z) \\ &-\rho(x, y,_0+d, z)u(x, y_0+d, z) + \rho(x, y_0, z)u(x, y_0, z) \\ &-\rho(x, y, z_0+d)u(x, y, z_0+d) + \rho(x, y, z_0)u(x, y, z_0)] \end{aligned}$$

Dividing by d^3 and taking the limit $d \to 0$ we see

$$\frac{\partial t}{\partial t} = -\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} - \frac{\partial z}{\partial z} = -\bigtriangledown \cdot (\rho \mathbf{u})$$

So we have our equation

$$\frac{\partial t}{\partial t} + \bigtriangledown \cdot (\rho\, \mathbf{u}) = 0$$

Note:

We have derived this equation in Cartesian coordinates but obtained a vector equation, so we may apply it to any coordinate system.

1.6 Conservation of Momentum

We would like to apply Newton's 2^{nd} law to fluids. So far we can write

$$m\mathbf{a} = \underbrace{\mathbf{F}_{body}}_{internal} + \underbrace{\mathbf{F}_{stress}}_{external}$$

We define stress to be the force per unit area and then the stress tensor T_{ij} is the i^{th} component of stress on the surface with normal $\mathbf{n}_j$. We normally the symmetric class of tensors where $T_{ij} = Tji$. We now consider the total force acting on our infinitesimal cube and by a similar calculation as the one in conservation of momentum we find

$$\delta F_i = d^3 \sum_j \frac{\partial x_j}{\partial x_j}$$

1.6.1 Cauchy's Momentum Equation

We can now substitute all of the formulae into Newton's second law to get:

$$\underbrace{(\rho d^3)}_{mass} \underbrace{\frac{Du_i}{Dt}}_{acceleration} = \underbrace{d^3 \sum_j \frac{\partial x_j}{\partial x_j}}_{internal} - \underbrace{(\rho d^3)}_{mass} \underbrace{g\delta_{i3}}_{gravity}$$

Then we can divide by d^3 on both sides to yield the Cauchy Momentum Equation:

Cauchy's Momentum Equation:

$$\rho \frac{Du_i}{Dt} = \frac{\partial x_j}{\partial x_j} - \rho g \delta_{i3}$$

or in vector form

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho} \bigtriangledown \cdot \underline{\underline{\mathbf{T}}} - g\mathbf{e}_z$$

This is a very important formula but it requires us to know the stress tensor. We would like to relate $\underline{\underline{\mathbf{T}}}$ to velocity $\mathbf{u}$ and pressure $\mathbf{P}$.

1.6.2 Inviscid Flow

In inviscid flows, there is no internal friction and so there are no sheering terms to the stress tensor, i.e. all off-diagonal terms are 0. Therefore, the only stress is the inward pressure acting perpendicular to the sides of the fluid element. We can write this as the following constitutive relation:

$$T_{ij} = -p\delta i_j$$

Substituting $\frac{\partial x_j}{\partial x_j} = -\frac{\partial x_i}{\partial x_i}$ into Cauchy's Momentum Equation we see

Euler's Equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \bigtriangledown p - g\mathbf{e}_z$$

Note:

- In the absence of gravitational forces we see that fluid elements are accelerated by pressure gradients from high to low pressure.
- If we assume fluid elements are not being accelerated then we recover a hydrostatic balance:

$$0 = -\frac{1}{\rho}\frac{\partial z}{\partial z} - g \implies p = -\rho g z + p_0$$

1.6.3 Stress Tensor for a Viscous Fluid

This time we have the normal inward pressure as with inviscid flows but we also have an additional viscous stress tensor σ_{ij}.

$$T_{ij} = -p\delta_{ij} + \sigma_{ij}$$

We need another constitutive relation between the rate of strain tensor e_{ij} and the viscous stress tensor σ_{ij}. When this relationship is linear, we call this flow Newtonian. In the case that the flow is incompressible we get $\sum_k e_{kk} = 0$ and hence

$$\sigma_{ij} = 2\mu e_{ij}$$

where μ is the dynamic viscosity given in $kgm^{-1}s^{-1}$. The dynamic viscosity is the coefficient of proportionality between the rate of strain tensor and the stress tensor. This describes how easily a fluid moves under a shear force. Then we can write

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + 2\mu e_{ij} \\ &= -p\delta_{ij} + \mu\left(\frac{\partial x_j}{\partial x_j} - \frac{\partial x_j}{\partial x_j}\right) \end{aligned}$$

Now $\sum_i T_{ii} = -3p$ and hence $p = -\frac{1}{3}(\sum_i T_{ii})$ is the mechanical (NOT thermodynamic) pressure.

Note:

The flow of incompressible Newtonian fluids covers a huge range of phenomena. However, there are many cases in which non-Newtonian behaviour is encountered and hence we need more complex constitutive relations than those above.

We can now substitute these relations into Cauchy's momentum equation to get:

The incompressible Navier-Stokes equations:

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial x_i}{\partial x_i} + \nu\frac{\partial^2 x_k}{\partial {x_k}^2} - g\delta_{i3}$$

or in vector form

$$\frac{D\mathbf{u}}{Dt} = \underbrace{-\frac{1}{\rho} \bigtriangledown p}_{Pressure\ gradients} + \underbrace{\nu \bigtriangledown^2 \mathbf{u}}_{Viscous\ forces} \underbrace{-g\mathbf{e}_z}_{Gravity}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient.

1.7 Controlling Flow Parameters

We have two key questions to answer before we can start using these formulae:

- When is the inviscid assumption valid?
- When is the incompressible assumption valid?

To this end, we shall define two dimensionless parameters, namely the Reynold's Number and Mach Number.

1.7.1 Reynold's Number

Our aim is to determine when it is safe to ignore the viscous forces in NS equations. To do this we compare the approximate sizes of typical terms within the viscous and acceleration term. Suppose we have a flow with characteristic length L and characteristic speed u with a kinematic viscosity ν.

$$\text{A typical viscosity term is } \left|\nu\frac{\partial^2 x}{\partial x^2}\right| \sim \frac{\nu u}{L^2}$$
$$\text{A typical acceleration term is } \left|u\frac{\partial x}{\partial x}\right| \sim \frac{u^2}{L}$$

Dividing these two terms we should get (we can check a dimensional parameter. We call this the Reynold's number.

$$\frac{|(u\cdot\triangledown)u|}{|\nu\triangledown^2 u|} \sim \frac{u^2}{L}\cdot\frac{L^2}{\nu u} = \underbrace{\frac{uL}{\nu}}_{Reynold's\ Number}$$

We now have two cases:

- $Re >> 1 \implies \frac{Inertia}{Viscosity} >> 1 \implies$ Viscosity is negligible so we can use the Euler equations.
- $Re << 1 \implies \frac{Inertia}{Viscosity} << 1 \implies$ Inertia is negligible so we can use the Stoke's equation (where we remove inertia term).
- $Re \approx 1 \implies$ We have to use the NS equations

Stokes's Equation:

$$0 = -\frac{1}{\rho}\triangledown p + \nu\triangledown^2\mathbf{u} - g\mathbf{e}_z$$