# Dynamical Systems - Proofs to Remember

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## 1 Sharkovskii's Theorem

**Theorem 1.1** (Sharkovskii's Theorem). *If $f : I \to I$ is continuous and there is a point of prime period 3. Then for each $n \in \mathbb{N}$ there is a periodic point of prime period $n$.*

The proof proceeds by a number of lemmata.

**Lemma 1.2.** *Given $I \subseteq [0, 1]$ a closed interval ,if $f(I) \supseteq I$ or $f(I) \subseteq I$ then $I$ contains a fixed point for $f$.*

*Proof.* Use the ITV on $g(x) = f(x) - x$ and consider the endpoints. □

**Lemma 1.3** (Whittling down intervals). *If $I, I' \subseteq [0, 1]$ are closed intervals and $f(I) = I'$, then $\exists$ a closed interval $I_0 \subseteq I$ such that $f(I_0) = I'$.*

*Proof.* Suppose $I' = [a, b]$ then let

$$\begin{aligned} A &:= f^{-1}(a) \cap I \\ B &:= f^{-1}(b) \cap I \end{aligned}$$

then take $x_0 = \sup(A)$ and $y_0 = \inf(B)$. Then $I_0 := [x_0, y_0]$ will do the job. □

**Lemma 1.4.** *Assume that we have closed intervals $I_1, \ldots, I_n \subseteq [0, 1]$ such that*

- *$f(I_n) \supseteq I_1$,*
- *$f(I_j) \supseteq I_{j+1}$ for all appropriate $j$,*

*then there is a fixed point $x$ for $f^n$ such that*

$$x \in I_1, f(x) \in I_2, \ldots, f^{n-1} \in I_n$$

*Proof.* We can just apply the whittling lemma to the intervals in reverse order so

$$\begin{array}{lll} \exists I'_n \subseteq I_n & s.t. & f(I'_n) = I_1 \\ \exists I'_{n-1} \subseteq I_{n-1} & s.t. & f(I'_{n-1}) = I'_n \\ & \vdots & \\ \exists I'_1 \subseteq I_1 & s.t. & f(I_1)' = I'_2 \end{array}$$

In particular we have that $f^n(I'_1) = I_1 \supseteq I'_1$ and hence the first lemma gives us the desired fixed point. □

*Proof. of Theorem 1.1.*
Let $f^3(x) = x$ be our point of prime period 3. For now we will assume that

$$\left\{x, f(x), f^2(x)\right\} = \{x_1, x_2, x_3\}$$

where $0 \leq x_1 < x_2 < x_3 \leq 1$. We also assume $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$. Other cases are similar. Let $I_0 := [x_1, x_2]$ and $I_1 := [x_2, x_3]$.
Observe that

(a) $f(I_0) \supseteq I_1$, and

(b) $f(I_1) \supseteq I_0 \cup I_1$.

We now split the proof into a number of cases:
**Case 1:** $(n = 3)$ This follows from the assumption.
**Case 2:** $(n = 1)$ This follows from the first lemma thanks to *(b)*.
**Case 3:** $(n = 2$ or $n \geq 4)$
Note that we can make a chain of intervals as in the last Lemma.

$$\begin{tikzcd}
I_0 \arrow[r, squiggly, "f"] \arrow[rrrrr, dashed, bend right, "n-1 \text{ times}"'] & I_1 \arrow[r, squiggly, "f"] & I_1 \arrow[r, squiggly, "f"] & \dots \arrow[r, squiggly, "f"] & I_1 \arrow[r, squiggly, "f"] & I_0
\end{tikzcd}$$

where $A \rightsquigarrow B$ means $f(A) \supseteq B$. Hence there is a fixed point for $f^n$ which starts in $I_0$ spends $n-1$ in $I_1$ and then returns to $I_0$. Because the earliest return is at time $n$ we can be sure that this is our prime period. $\square$

# 2 Independence of Lifts

# 3 Dense Irrational Orbits

**Theorem 3.1.** *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then for any $z \in \mathcal{K}$ we have*

$$\{R_\alpha^n(x) \mid n \in \mathbb{N}\}$$

*is a dense set in the circle $\mathcal{K}$.*

*Proof.* Get yourself some pigeons. If you put a pigeon on the circle for every point in the orbit up to time $\frac{1}{\epsilon} + 1$ then two pigeons, Kenny $k$ and Lenny $l$, must be $\epsilon$ close.

$$d(R_\alpha^l(p), R_\alpha^k(p)) < \epsilon$$

Without loss of generality, assume that Kenny is further along the orbit then Lenny so that

$$m := k - l > 0.$$

Then for any $x \in \mathcal{K}$ we have $d(R_\alpha^m(x), x < \epsilon)$. Hence the orbit $\left\{x, R_\alpha^m(x), R_\alpha^{2m}(x), R_\alpha^{3m}(x), \dots\right\}$ is $\epsilon$ dense in the circle. $\square$

# 4 Rational Points and Periodic Points

**Theorem 4.1.** *If $f : \mathcal{K} \to \mathcal{K}$ has a periodic point $x_0$ of period $m$ then $\alpha(f) \in \mathbb{Q}$.*

*Proof.* Let $F : \mathbb{R} \to \mathbb{R}$ be some lift then we must have

$$F^m(x) - x = k \in \mathbb{Z}$$

where $\rho(x) = x_0$. Then we can write any integer as $n = pm + r$ where $p \geq 0$ and $r \in [0, m)$. Hence

$$F^n(x) = F^{pm+r}(x) = F^r(x) + pk$$

Then we can conclude

$$\lim_{n\to\infty} \frac{1}{n} F^n(x) = \lim_{p\to\infty} \frac{1}{pm+r} (F^r(x) + pk) = \frac{k}{m} \in \mathbb{Q}$$

□

**Theorem 4.2.** *If $f : \mathcal{K} \to \mathcal{K}$ has 0 rotation number then $f$ has a fixed point.*

*Proof.*

- Take a lift $\widetilde{F}$ that gives $\lim_{n\to\infty} \frac{\widetilde{F}^n(x)}{n} = m$.
- Create a nicer lift $F := \widetilde{F} - m$ so that the limit becomes 0.
- Assume $d$ has no fixed point then by the IVT we can assume WLOG $F(y) > y$ for all $y \in \mathbb{R}$.
- Hence $(F^n(0))_{n\in\mathbb{N}}$ is increasing so we just need to show boundedness.
- Suppose unbounded then $|F^{n_0}(0)| > 1$ and hence for all $m$ we have $|F^{mn_0}(0)| > m$.

  $$\frac{|F^{mn_0}(0)|}{mn_0} > \frac{m}{mn_0} = \frac{1}{n_0}$$

  which cause a contradiction with the earlier 0 limit.
- It can be seen that the limit of this sequence is a fixed point.

□

**Note:** As a corollary if the rotation number is $\frac{a}{b} \in \mathbb{Q}$ then $f^b$ has 0 rotation number and hence fixed point. Therefore, $f$ has a periodic point.

## 4.1 Tending to periodic orbits

**Theorem 4.3.** *Let $f$ be a circle homeomorphism. Prove that if its rotation number $\rho \in \mathbb{Q}$ is rational then any point $x \in \mathcal{K}$ is either period or converges to some periodic orbit. More succinctly there is a point $p \in \mathcal{K}$ such that*

$$\lim_{n\to\infty} d(f^n(x), f^n(p)) = 0$$

*Proof.* We have seen that circle homeomorphisms of rational degree certainly have periodic orbits. We can therefore partition the circle using the periodic points of some period $m$. For simplicity let's assume $m = 1$.
By slicing the circle into arcs between fixed points we can assume that on each arc $f(z) > z$ or $f(z) < z$. Without loss of generality lets assume the former.
**Claim:** The iterates $(f^n(z))_{n=0}^{\infty}$ form a bounded, increasing sequence.
This follows because $f$ is a circle homeomorphism. This is thanks to injectivity which prevents iterated from "jumping over" a fixed point into the next arc where we might have $f(z) < z$.
So the sequence must have a limit $x_*$. Moreover, this limit can easily be shown to be a fixed point and therefore (thanks to our previous division of the circle) must be the fixed point at the end of the arc.
**What about $m > 1$?**
We can certainly get the iterated $f^{nm}(x)$ and $f^{nm}(p)$ to tend to one another as $n \to \infty$, but what about the points in between? Since $\mathcal{K}$ is compact there is a fixed $\delta$ such that points $\delta$ close will stay $\epsilon$ close over the next $m-1$ iterates. This gives convergence of the entire sequence. □

# 5 Poincaré's Theorem and Minimality

A homeomorphism is called minimal if every orbit is dense.

**Example:** Any irrational rotation $R_\alpha$ is minimal.

**Theorem 5.1** (Poincaré's Theorem)**.** *Any minimal circle homeomorphism is topologically conjugate to an irrational rotation.*

Given a circle homeomorphism $f : \mathcal{K} \to \mathcal{K}$ and some lift $F$ we define the following countable sets

$$\Lambda_{x_0} := \{F^n(x_0) + m \mid m, n \in \mathbb{Z}\}$$
$$\Omega := \{n\rho + m \mid m, n \in \mathbb{Z}\}$$

for some fixed $x_0 \in \mathbb{R}$ and where $\rho = \rho(f)$ is the rotation number. Note that $\Lambda_{x_0} = \pi^{-1}\{f^n(\pi x_0)\}$ and $\Omega = \pi^{-1}\left\{R_\rho^n(0)\right\}$ where $\pi$ is the usual projection.

**Lemma 5.2.** *Let $f$ be a circle homeomorphism and $x_0 \in \mathcal{K}$. If the rotation number $\rho$ is irrational then the map $T : \Lambda_{x_0} \to \Omega$ given be*

$$T(F^n(x_0) + m) = n\rho + m$$

*is a bijection. Moreover,*

1. *$T$ is strictly increasing*
2. *$T(x + 1) = T(x) + 1$*
3. *$T(F(x)) = T(x) + \rho$ for all $x \in \Lambda_{x_0}$.*

*Proof.* This is omitted but might be worth glancing over. □

*Proof. of Poincaré's Theorem* Since $f$ is minimal, it has no periodic points because their orbits would be finite and hence not dense. So the rotation number $\rho$ is irrational.
Take a lift $F$ of $f$ and $x_0 \in \mathbb{R}$ and write $\Lambda = \Lambda_{x_0}$. The sets $\Omega$ and $\Lambda$ are dense in $\mathbb{R}$ due to the minimality of $R_\rho$ and $f$ respectively.
Thus $\pi(\Omega)$ and $\pi(\Lambda)$ must be dense in $\mathcal{K}$. Moreover, the Lemma tells us that $T : \Lambda \to \Omega$ is strictly increasing. Consequently, we can extend to a unique continuous function $H : \mathbb{R} \to \mathbb{R}$ (which restricts to $T$ on $\Lambda$). Moreover, $H$ is strictly increasing, $H$ is continuous and so is its inverse.

**Note:** This is non-trivial. It is an exercise to show that given dense sets $X, Y \subseteq \mathbb{R}$ and $f : X \to Y$ a bijection, there exists a unique homeomorphism extension to $\mathbb{R}$.

By continuity $H$ inherits the properties *(2)* and *(3)* in the previous Lemma. The first says that $H$ is a lift of circle homeomorphism $h$. The second say that $h \circ f = R_\rho \circ h$. □

So we now know that if $f$ is a circle homeomorphism then there is a unique homeomorphism $h$ satisfying

$$h(f(x)) = h(x) + \rho \quad (mod 1) \quad \forall x \in \mathcal{K}$$

Note that this is a linear equation on $h$. We can conclude that a solution to this equation is unique up to adding a constant corresponding to choosing with point in $\mathcal{K}$ is sent to zero. For a hand-wavey explanation of this, see the lecture notes.

# 6 Expanding Maps

## 6.1 Fixed Points

**Theorem 6.1.** *If $f : \mathcal{K} \to \mathcal{K}$ is an expanding, orientation preserving map and $d = \deg(f)$, then there are exactly $d^n - 1$ points $p \in \mathcal{K}$ such that $f^n(p) = p$.*

*Proof.* We'll do $n = 1$ then $\deg(f^n) = \deg(f)^n$ implies the rest. Take a lift $F : \mathbb{R} \to \mathbb{R}$ and recall that $F(1) = F(0) + d$.

$$\begin{aligned}\#\text{fixed points for } f &= \#\{x \in [0,1) \mid x = F(x) \mod 1\} \\ &= \#\text{integer values assumed by } g(x) := F(x) - x \text{ in the range } [0,1)\end{aligned}$$

But $g$ is monotone increasing (take derivative) and $g(1) - g(0) = F(1) - F(0) - 1 = d - 1$. Therefore $g$ assumes $d - 1$ integer values on the range $[0, 1)$. □

**Theorem 6.2.** *Let $f : \frac{\mathbb{R}}{\mathbb{Z}} \to \frac{\mathbb{R}}{\mathbb{Z}}$ be an expanding map, than there exists a dense $G_\delta$ set of points whose orbits are dense. (Recall that a dense $G_\delta$ set is a countable intersection of open dense sets).*

*Proof.* Choose a countable dense set of points $\{x_n\}$ and for each natural $m \geq 1$ consider the ball $B\left(x_n, \frac{1}{m}\right)$. A point $x$ has a dense orbit if and only if it intersects every one of these balls. That is for all $n$ and $m$ there is a $k$ such that $f^k(x) \in B\left(x_n, \frac{1}{m}\right)$ or more precisely

$$x \in \bigcap_n \bigcap_m \bigcup_k f^{-k} B\left(x_n, \frac{1}{m}\right)$$

Note that the $\cup_k f^{-k} B\left(x_n, \frac{1}{m}\right)$ are open and dense since any expanding map is mixing and so at least transitive. □

## 6.2 Conjugacy to shift maps

**Theorem 6.3.** *If $f : \mathcal{K} \to \mathcal{K}$ is an expanding map, preserves orientation and has degree 2 then there is a semi-conjugacy $h : \Sigma \to \mathcal{K}$ to the full shift on two symbols.*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ h\downarrow & & \downarrow h \\ \mathcal{K} & \xrightarrow{f} & \mathcal{K} \end{array}$$

*Proof.* Take any $n \in \mathbb{N}$. Then $\deg f^n = (\deg f)^n$ so there are $w^n$ pre-images of $p$ under $f^n$. These are numbered $p_j$ starting with $p_0 = p$ and number consecutively anticlockwise. These points define intervals which we denote $A_{\omega_0 \ldots \omega_{n-1}}$ where the sequence of $\omega_i$ is just the binary representation of the position in the circle.
Let $K$ denote the uniform bound away from 1 of the derivative. We have a number of results:

1. $f^n(A^\circ_{\omega_0 \ldots \omega_{n-1}}) = \mathcal{K} \setminus \{p\}$
2. $A_{\omega_0 \ldots \omega_{n-1}}$ is a closed interval of length $< K^{-n}$.
3. $A_{\omega_0 \ldots \omega_{n-1}\omega_n} \subseteq A_{\omega_0 \ldots \omega_{n-1}}$.
4. $f^n(A_{\omega_0 \ldots \omega_n}) = A_{\omega_n}$.
5. $f(A_{\omega_0 \ldots \omega_n}) = A_{\omega_1 \ldots \omega_n}$.

Now we can define our conjugacy $h : \Sigma \to \mathcal{K}$. Given $\omega = (\omega_k)_{k=0}^{\infty} \in \Sigma$ let $B_n(\omega) = A_{\omega_0 \ldots \omega_{n-1}}$. These are the points in the circle that start in the $\omega_0$ interval then go to $\omega_1$, then to $\omega_2$ and after $f^{n-1}$ are in the $\omega_{n-1}$ interval. The properties implies that $B_{n+1}(\omega) \subseteq B_n(\omega)$. The sets are also closed and their diameters go to 0. Hence their infinite intersection is a single points which we define to be $h(\omega)$. The proof of their desired properties is discussed below in vague detail but is written in the lecture notes with more rigour. □

# 7 Finding semi-conjugacies/conjugacies

If you can partition your space $X$ into $n$ subsets $I_1, \ldots, I_n$ where one could conceivably go from any partition element $I_a$ to any other $I_b$, then you might be able to find a semi-conjugacy to the full shift on $n$ symbols.
The trick is to define a map $\pi : \Sigma \to X$ by

$$\pi(\mathbf{x}) = \bigcap_{n=1}^{\infty} T^{-n} I_{x_n}$$

If the sets $I(x_0, \ldots, x_n) := \cap_{k=0}^{n} T^{-k} I_{x_k}$ are closed and nested and their diameter tends to zero as $n \to \infty$ then this map is well-defined because the infinite intersection contains one point. Moreover, it is continuous because if $\mathbf{x}$ and $\mathbf{y}$ agree up to $N$ places then they both lie in $I(x_0, \ldots x_{N-1})$ whose diameters goes to 0 as $N \to \infty$.
The commutative relationship $T \circ \pi = \pi \circ \sigma$ then follows rather quickly. To get surjectivity, it suffices to show that the image of $\Sigma$ is dense. This usually involves taking in point $x \in X$ such that no $T^n x$ lies on the boundary between any $I_j$ for some $n \geq 0$ and then this points orbit will describe its pre-image in $\Sigma$.

**Note:** Shift spaces are totally disconnected, i.e. the connected components are one-point sets. In particular, they are disconnected and so this can often be used to rule out the existence of conjugacies to more familiar sets.

# 8 Transitivity and Mixing

**Note:** A compact metric space has a countable dense set of points!

**Theorem 8.1** (Baire's Theorem)**.** *Given a compact metric space $X$, the intersection of countably many open, dense subsets of $X$ is itself dense in $X$.*

**Theorem 8.2.** *If a map $T : X \to X$ on a compact metric space $X$ is topologically transitive then there exists a dense orbit.*

*Proof.* There is a countable dense set of points $\{x_k\}$ so if we can find an orbit the gets $\epsilon$ close to every $x_k$ for arbitrary $\epsilon$ then we are done. So we want $x$ such that for every $x_k$ and $m \geq 1$ there is an $n \in \mathbb{Z}$ such that

$$x \in T^{-n}\mathbb{B}\left(x_k, \frac{1}{m}\right)$$

or equivalently we want to find

$$x \in \bigcap_{k,m} \bigcup_{n \in \mathbb{Z}} T^{-n}\mathbb{B}\left(x_k, \frac{1}{m}\right)$$

which is a countable intersection (over $m$) of open dense sets. By Baire's Theorem our desired point exists. $\square$

*Proof. (Alternate).* Since we're in a compact space, there is a countable dense set. Then we can choose a sequence of open discs around these dense set $(U_n)_{n=1}^{\infty}$. Choose $N_1$ such that $T^{-N_1}(U_2) \cap U_1 \neq \emptyset$. Then choose an open disc $V_1$ of radius less than a half such that

$$V_1 \subseteq \overline{V_1} \subseteq U_1 \cap T^{-N_1}(U_2)$$

Then choose $N_2$ such that $T^{-N_2}(U_3) \cap V_1 \neq \emptyset$ and subsequently choose an open disc $V_2$ of radius less that $\frac{1}{4}$ such that

$$V_2 \subseteq \overline{V_2} \subseteq V_1 \cap f^{-N_2}(U_3)$$

By induction we get a sequence of discs

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$$

such that $diam(V_n) \leq \frac{1}{2^n}$. So choose the point $x$ in $\cap_n V_n$ then $T^{N_{n-1}}(x) \in U_n$ for each $n \geq 1$. Therefore $(T^n(x))$ forms a dense orbit. $\square$

## 8.1 Shift Spaces

**Theorem 8.3.** *The shift map $\sigma : \Sigma_A \to \Sigma_A$ is mixing if and only if the matrix $A$ is aperiodic.*

*Proof.* Suppose the matrix $A$ is aperiodic. Then it suffices to show that any two cylinder sets $U, V$ of the same length have the mixing property, i.e. there is an $N$ such that for all $n \geq N$ we have $T^{-n}U \cap V \neq \emptyset$. Write $U := [u_0, \dots u_n]$ and $V := [v_0, \dots, v_n]$. Then there is an $N$ such that we can go from any symbol to any other symbol in $N$ or more steps. So for all $m \geq N$ we can find a point in $U$ that looks like

$$u_0, \dots, u_n, \underbrace{\dots, \dots, \dots}_{\text{length } m}, v_0, \dots, v_n, \dots$$

Hence for all $m \geq N$ we have that $\sigma^{m+n+1}(U) \cap V \neq \emptyset$ and hence $\sigma$ is mixing.
Conversely, suppose that $\sigma$ is mixing then the cylinder sets are all open so there is a common $m \geq 1$ such that

$$\sigma^m[i] \cap [j] \neq \emptyset$$

So then given any pair $(i, j)$ there is a sequence $\omega$ such that $\omega_0 = i$ and $\omega_m = j$, and hence $(A^m)_{i,j} \geq 1$ because there is at least one path of length $m$ from $i$ to $j$. □

**Theorem 8.4.** *The shift map $\sigma : \Sigma_A \to \Sigma_A$ is transitive if and only if the matrix $A$ is irreducible.*

*Proof.* Suppose $A$ is irreducible then given any two open sets we can find cylinders $U$ and $V$ inside them. Write

$$U = [u_0, \dots, u_n] \quad V = [v_0, \dots, v_n]$$

then $A$ is transitive so there exists an admissible path $p_0, \dots, p_k$ of some length from $u_n$ to $v_0$. Then $(u_0, \dots, u_n, p_1, \dots, p_{k-1}, v_0, \dots, v_n, \dots) \in U \cap \sigma^{-(n+k)}V$ where we just fill the rest of the sequence out with random junk.
Now suppose that $\sigma$ is transitive. Then the cylinder sets $U := [i]$ and $V := [j]$ are open and hence there is an $n$ such that $\sigma^{-n}U \cap V \neq \emptyset$, i.e. there is a sequence which starts at $j$ and after $n$ arrives at $i$. Therefore $(A^n)_{i,j} \geq 1$ and hence $A$ is transitive.

□

# 9 Arithmetic Progressions

We say a subset $C \subseteq \mathbb{Z}$ contains arithmetic progressions of arbitrary length if

$$\forall k \geq 1 \quad \exists c \in \mathbb{Z} \text{ and } d \in \mathbb{N} \quad \text{such that}$$

$$c, c+d, c+2d, \dots, c+(k-1)d \in C$$

Similarly we say a map $T : X \to X$ is multiple mixing if for any non-empty open set $U \subseteq X$ and $k \geq 1$ there exists $d \geq 1$ such that

$$U \cap T^{-d}U \cap T^{-2d}U \cap \dots \cap T^{-(k-1)d}U \neq \emptyset$$

**Theorem 9.1** (van der Waerden's Theorem)**.** *Given any finite integer partition $\mathbb{Z} = \cup_{i=1}^M C_i$ there is an $i$ such that $C_i$ contains arithmetic progressions of arbitrary length.*

To prove this via a dynamical approach we must create a dynamical formulation. To a partition of $\mathbb{Z}$ we associate a single infinite sequence $\mathbf{x} = (x_n) \in \{1, \dots, M\}^{\mathbb{Z}}$ defined by

$$x_n = i \quad \text{if} \quad n \in C_i$$

Let $X = \overline{\cup_{n \in \mathbb{Z}} \sigma^n \mathbf{x}}$ be the closure of the orbit of $\mathbf{x}$ where $\sigma$ is the shift map.

**Lemma 9.2** (Dynamic Formulation)**.** *Assume that for some* $[i]$ *(cylinder set) we have that*

$$X \cap [i] \cap \sigma^{-d}[i] \cap \sigma^{-2d}[i] \cap \dots \sigma^{-(k-1)d}[i] \neq \emptyset$$

*for some* $k, d \geq 1$ *then* $C_i$ *contains an arithmetic progression of length* $k$.

*Proof.* The space is the closure of the orbit of $\mathbf{x}$ and this set is non-empty and open. The orbit itself is dense in $X$ and hence intersects our open set. So there is $n \in \mathbb{Z}$ such that $\sigma^n x$ is in our set. This means that $x_{n+jd} = i$ for $j = 0, \dots, k-1$ and hence $n + jd \in C_i$ for these $j$. □

**Proposition 9.3** (Multiple Recurrence)**.** *The shift map is multiple mixing when restricted to a minimal subset* $Y \subseteq X$.

*Proof. of van der Waerden's Theorem* Take a minimal subset $Y \subseteq X$. Taking $U = [i]$ where $i$ is chosen such that $[i] \cap Y \neq \emptyset$, we see the set from the dynamical formulation is open and hence non-empty by multiple recurrence so we have arbitrary arithmetic progressions. □

## 10 Hyperbolic Toral Automorphisms

Given a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$ we can associate an automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f(x, y) := (ax + by, cx + dy) \mod 1$$

We say this is hyperbolic if no eigenvalue of $A$ lives on the unit circle.

**Theorem 10.1.** *The fixed points of a hyperbolic toral automorphism are precisely those* $(x_1, x_2) \in T^2$ *such that* $x_1, x_2 \in \mathbb{Q}$.

*Proof.* Take $(x_1, x_2) \in \mathbb{T}^2$ such that $x_1, x_2 \in \mathbb{Q}$. We can therefore write $x_1 = \frac{m_1}{d}$ and $x_2 = \frac{m_2}{d}$ for some integers $m_1, m_2 \in \mathbb{Z}$. Write $m := (m_1, m_2)$ then

$$A^k x^T = \frac{1}{d} A^k m^T \quad \forall k \in \mathbb{Z}$$

But by the pigeonhole principle, since we are only looking for a fixed point $\mod 1$, there are only $l^2$ distinct pairs of values that $A^k m^T$ can assume. Hence there is $k_1 < k_2$ such that

$$A^{k_1} m^T = A^{k_2} m^t \mod 1$$

Set $n := k_2 - k_1 > 0$. Then $A^n x^t = x^t \mod 1$.
Conversely, suppose that $x$ is a periodic point of $f$. Then $A^n x = x \mod 1$ or, more succinctly, there exists integers $k_1, k_2$ such that

$$(A^n - I)x = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

But then $A^n - I$ is an integer matrix and hence its inverse has rational entries, so $x$ must have rational entries. □

**Theorem 10.2.** *The number of fixed points for* $f^n$ *where* $f$ *is a hyperbolic toral automorphism is* $|tr(A^n) - 2|$.

*Proof.* Note that the number of fixed points for $f^n$ precisely the number of $x \in \Delta := [0,1) \times [0,1)$ such that $(A^n - I)x \in \mathbb{Z}^2$. But the number of lattice points in $(A^n - I)(\Delta)$ is equal to the area of the parallelogram $(A^n - I)(\Delta)$. But $\Delta$ has unit areas so the parallelogram has area $|\det(A^n - I)|$. Then

$$|\det(A^n - I)| = |(1 - \lambda_+^n)(1 - \lambda_-^n)| = |2 - (\lambda_+^n + \lambda_-^n)| = |tr(A^n) - 2|$$

□

**Theorem 10.3.** *Hyperbolic toral automorphisms are topologically mixing.*

*Proof. (Sketch).*

- Take $U, V \subseteq \mathbb{T}^2$ open and non-empty.
- Let $l_\pm$ be the lines spanned by the eigenvectors.
- These lines have irrational slope and hence their projection to the torus is dense. (Why!?)
- Take small parallelograms $U' \subseteq U$ and $V' \subseteq V$ with sides parallel to $l_\pm$.
- Density implies that the projected lines intersect $U'$ and $V'$.
- Then as we take $f^n$ on $U'$ for larger and larger $n$ we stretch along $W_+$ and shrink along $W_-$.
- Eventually $f^n(U')$ will reach the part of $W_+$ which intersects $V'$ and continue to intersect for all future $n$.

□

### 10.1 Markov Partitions

We wish to divide the torus up into a partition $\mathcal{P} := \{P_0, \dots, P_{k-1}\}$ with the properties

- $\cup_i P_i = \mathbb{T}^2$
- $\text{int}(P_i) \cap \text{int}(P_j) = \emptyset$.
- The Markov property if there are points $x, y \in \mathbb{T}^2$ and a sequence $(\omega_n)_{n \in \mathbb{Z}}$ such that

$$T_A^n(x) \in \text{int}(P_{\omega_n}) \quad \forall n \geq 0$$
$$T_A^n(y) \in \text{int}(P_{\omega_n}) \quad \forall n \leq 0$$

  then there is a $z \in \mathbb{T}^2$ such that $T_A^n(z) \in \text{int}(P_{\omega_n}) \quad \forall n \in \mathbb{Z}$.

Such a partition is called a Markov partition.

**Theorem 10.4.** *We can divide the torus up into a Markov partition for any linear hyperbolic toral automorphisms $f : \mathbb{T}^2 \to \mathbb{T}^2$.*
*Once we have this partition we can create a semi conjugacy to a subshift of finite type $\pi : \Sigma_b \to \mathbb{T}^2$.*

*Proof. (Sketch).* We divide the torus up by extending the eigenvectors sufficiently far and making sure that when we finish we don't leave any dangling ends. Having obtained the Markov partition $\mathcal{P} = \{P_1, \dots, P_k\}$ we define a matrix $B$ by

$$B(i,j) := \begin{cases} 1 & \text{if } f(P_i^\circ) \cap P_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Thanks to the Markov property we have the following property. If $i_{-n}, \dots, i_n$ satisfy

$$\cap_{k=-n}^{n} f^{-k}(P_{i_k}^\circ) \neq \emptyset$$

and $i_{-(n+1)}, i_n$ and $i_n, i_{n+1}$ are admissible, then

$$\cap_{k=-(n+1)}^{n+1} f^{-k}(P_{i_k}^\circ) \neq \emptyset$$

That is, along as we take admissible steps, we can be sure that an admissible sequence is non-empty. Hence given a sequence $\omega = (\omega_n)_{n=-\infty}^{\infty}$ we may conclude that

$$\cap_{k=-\infty}^{\infty} f^{-k}(P_{i_k}^\circ) \neq \emptyset$$

Moreover, since the diameters of the finite intersections are decreasing, it is a single point which we denote by $\pi(\omega)$. This is a semi-conjugacy, similar to the proof for circle maps we've seen before. □

# 11 Entropy

**Theorem 11.1.** *We can calculate entropy through minimal spanning sets $S(n, \epsilon)$ and maximal separated sets $N(n, \epsilon)$.*

*Proof.*

$$S(n,\epsilon) \leq N(n,\epsilon) \leq S\left(n, \frac{\epsilon}{2}\right)$$

For the first inequality, show that an $(n, \epsilon)$ separated set is an $(n, \epsilon)$ spanning set. For the second, take an $(n, \frac{\epsilon}{2})$ spanning set and then any $(n, \epsilon)$ separated set would contain at most one point from each $(n, \frac{\epsilon}{2})$ ball. Moreover, every element of a separated set would fall in at least one ball. Hence $N(n, \epsilon) \leq S\left(n, \frac{\epsilon}{2}\right)$. □

## 11.1 of Shift Maps

**Theorem 11.2** (Gelfand's theorem). *Let $||A||$ be a norm of $A$ and $\lambda_1$ a maximal, positive, real eigenvalue. Then*

$$\lambda_1 = \lim_{k \to \infty} ||A^k||^{1/k}$$

**Theorem 11.3.** *If the transition matrix $A$ is aperiodic, then the topological entropy is*

$$h(\sigma_A) = \log \rho(A)$$

*where $\rho(A)$ is the spectral radius of the matrix $A$.*

*Proof.* 1. **Get your head around the balls**

Take a point $\alpha \in \Sigma_A$ then

$$B(\alpha, n, 2^{-k}) = [a_0, \dots, \alpha_{n+k}]$$

2. **Show that admissible cylinders are non-empty**

   Let $W_m(A)$ denote the set of admissible strings of length $m$. Since $A$ is aperiodic (although irreducible will do), given any admissible $\alpha_0 \dots \alpha_m$ the cylinder $[\alpha_0 \dots \alpha_m]$ is non-empty. This is because we can keep adding rubbish on the end.

3. **Relate admissible cylinders to separated and spanning sets**

   Note that admissible cylinders of length $m$ are pairwise disjoint and union to $\Sigma_A$. Hence

   $$S(n, 2^{-k}) \leq \#W_{n+k+1} \leq N(n, 2^{-k})$$

4. **Compute the entropy**

   When taking the limit we can get rid of the $k$.

   $$h(\sigma_A) = \limsup_{n \to \infty} \frac{\log(\#W_{n+1}(A))}{n}$$

5. **Relate #admissible cylinders to the spectral radius**

   We can choose the norm $||A^n|| = \sum_{i,j=1}^{N} |A^n_{i,j}|$. The $(i,j)$th entry of $A^n$ tells us how many admissible words of length $n$ start at $i$ and end at $j$. Hence $||A^n|| = \#W_{n+1}(A)$. Then, using Gelfand's Theorem we have that

   $$h(\sigma_A) = \lim_{n \to \infty} \frac{\log(\#W_{n+1}(A))}{n} = \lim_{n \to \infty} \frac{\log ||A^n||}{n} = \log \lambda_1$$

□

### 11.2 of Toral Automorphisms

**Theorem 11.4.** *Given a hyperbolic toral automorphism with eigenvalues $\lambda_+ > 1 > \lambda_- > 0$ then*

$$h(f) = \log \lambda_+$$

## 12 Preserved Quantities

### 12.1 Semi-Conjugacies

Given continuous maps $T : X \to X$ and $S : Y \to Y$, a semi-conjugacy from $T$ to $S$ is a continuous surjective map $\pi : Y \to X$ such that

$$T \circ \pi = \pi \circ S$$

$$\begin{tikzcd} X \arrow[r, "T"] & X \\ Y \arrow[u, "\pi"] \arrow[r, "S"'] & Y \arrow[u, "\pi"'] \end{tikzcd}$$

**Theorem 12.1.** *Let $T : X \to X$ and $S : Y \to Y$ be continuous maps on compact metric spaces and $\pi : Y \to X$ a semi-conjugacy then $h(S) \geq h(T)$.*

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\ Y & \xrightarrow[S]{} & Y \end{array}$$

*This makes sense because the dynamics of $T$ are contained in the dynamics of $S$ and hence $S$ must be at least as "complex" as $T$.*

### 12.2 Conjugacies

- Rotation number of circle homeomorphisms.
- Transitivity and mixing.
- Topological entropy.

## 13 Known Conjugacies

### 13.1 Semi-Conjugacies

- $\pi : \Sigma \to \mathcal{K}$ from full one-sided shift on two symbols to the doubling map.

### 13.2 Conjugacies

- Expanding maps of the same degree are conjugate (through the linear map of the same degree).
- Smale Horseshoe and full two-sided shift on two symbols.

**Theorem 13.1** (Poincarés Theorem)**.** *A minimal circle homeomorphism with irrational ration number is conjugate to $R_\alpha$.*

**Theorem 13.2** (Denjoy's Theorem)**.** *If $f : \mathcal{K} \to \mathcal{K}$ is a homeomorphism with irrational rotation, $f \in \mathcal{C}^1$ and $w := \log |f'|$ has bounded variation then $f$ is conjugate to $R_\alpha$.*