Intro to Topology - Overview

1 Key Definitions

- Given maps $f, g : X \times I \to Y$, a homotopy from f to g is a continuous map $F : X \times I \to Y$ such that $f_0 = f$ and $f_1 = g$. If such a map exists we write $f \simeq g$.
- Given paths $f, g : I \to X$. We say f is homotopic to g relative to $\{x, y\}$ and write $f \overset{\partial}{\simeq} g$ if
  - (i) $f(0) = g(0) = x$
  - (ii) $f(1) = g(1) = y$
  - (iii) There is a homotopy $F : I \times I \to X$ such that $f_0 = f$, $f_1 = g$ and for all $t \in I$, $f_t(0) = x$ and $f_t(1) = y$.
- A pair of spaces (X, A) is a topological space together with a subspace $A \subseteq X$ using the subspace topology.

Assume we have a pair (X, A).

- A is a retract if there is a continuous map $r : X \to A$ such that $r\big|_A = id_A$.
- X deformation retracts to A if there exists a homotopy $F : X \times I \to X$ such that $f_0 = id_X$, $f_1(X) = A$ and $f_t\big|_A = id_A$ for all $t \in I$.

Assume we have topological space X and Y then,

- X is homotopy equivalent to Y there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq id_X \quad , \quad f \circ g \simeq id_Y$$

- X is contractible if it is homotopy equivalent to $\{pt\}$.

Note:

$$X \text{ deformation retracts to } A \implies X \text{ contractible}$$
$$\not\Longleftarrow$$

The reverse does not hold because the contraction does not necessarily restrict to id_A.

2 Fundamental Group

Given a pointed space (X, x_0), a loop is a path $f : I \to X$ such that $f(0) = f(1) = x_0$. We can then define an equivalence class for every loop f:

$$[f] := \left\{ g \;\middle|\; g(0) = g(1) = x_0, \quad f \overset{\partial}{\simeq} g \right\}$$

We then get the fundamental group defined to be

$$\pi_1(X, x_0) := \{[f] \mid f \text{ a loop based at } x_0\}$$

This forms a group with the operation $[f] \cdot [g] = [f * g]$ and the identity element being the constant loop.

Theorem 2.1. *If X is path connected and $x_0, x_1 \in X$ then*

$$\pi_1(X, x_0) = \pi_1(X, x_1)$$

Proof. There exists a path $h : I \to X$ from x_0 to x_1. Define $\overline{h}(s) := h(1 - s)$. We can then define a base point change homomorphism which we claim is in fact an isomorphism.

$$\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [f] \mapsto [\overline{h} * f * h]$$

We can see this is in fact an isomorphism because $\beta_{\overline{h}}$ is a left and right inverse. □

A map $p : \widetilde{X} \to X$ is a covering map if there is an open cover $\{U_\alpha\}$ of X such that

$$p^{-1}(U_\alpha) = \bigsqcup_\beta V_\alpha^\beta$$

with each V_α^β open and such that $p\big|_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ is a homomorphism.
The covering map is called n-fold if each $p^{-1}(x_0$ has n elements for every x_0.
Let $p : Y \to X$ and $q : Z \to X$ be coverings. These called isomorphic if there is a homeomorphism $h : Y \to Z$ such that

$$q \circ h = p$$

Let $p : \widetilde{X} \to X$ be a cover then a deck transformation is an isomorphism $\tau : \widetilde{X} \to \widetilde{X}$ such that $p \circ \tau = p$.

$$\mathrm{Deck}(p) := \left\{ \tau : \widetilde{X} \to \widetilde{X} \;\middle|\; \tau \text{ is a deck transformation } \right\}$$

Given a covering $p : \widetilde{X} \to X$ and a map $f : Y \to X$, a lift of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p \circ \widetilde{f}$.

Here are some useful properties of lifts:

(i) $\widetilde{f} : Y \to \widetilde{X}$ then $\widetilde{f}$ is a lift of $p \circ \widetilde{f}$.

(ii) $\widetilde{f}, \widetilde{g} : Y \to \widetilde{X}$ and $f \simeq g \implies p \circ \widetilde{f} \simeq p \circ \widetilde{g}$ (homotopies descends).

(iii) $\alpha, \beta : I \to \widetilde{X}$ such that $\alpha(1) = \beta(0)$ then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ (concatenation descends).

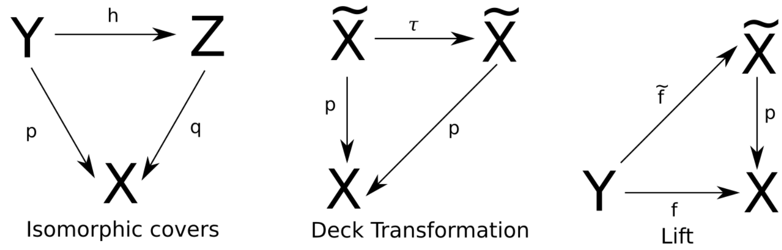

Figure 1: Commutative diagrams defining various concepts

3 Homotopy Lifting Property

A map $p : Z \to X$ has the homotopy lifting property if given any homotopy $F : Y \times I \to X$ and any lift $g : Y \times \{0\} \to Z$ there exists a unique homotopy $\widetilde{F} : Y \times I \to Z$ satisfying

(i) $\widetilde{f}_0 = g$

(ii) $p \circ \widetilde{F} = F$

i.e. given a homotopy and a lift of one endpoint, there exists a unique lift of that homotopy. As a special case where $Y = \{pt\}$, a map $p : Z \to X$ has the path lifting property if for any path $f : I \to X$, $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$ there exists a unique path $\widetilde{f} : I \to Z$ such that

(i) $\widetilde{f}(0) = \widetilde{x}_0$

(ii) $p \circ \widetilde{f} = f$

Lemma 3.1 (Local Homotopy Lifting Property)**.** *Let $p : \widetilde{X} \to X$ be a covering map and $F : Y \times I \to X$ a homotopy. Suppose we have $g : Y \times \{0\} \to \widetilde{X}$. Then for every $y \in Y$*

(a) There exists an open neighbourhood $N \subseteq Y$ and a unique homotopy $\widetilde{F}_N : N \times I \to \widetilde{X}$ such that

(i) $(\widetilde{f}_N)_0 = g$.

(ii) $p \circ \widetilde{F}_N = F\big|_{N\times I}$.

(b) If $M \subseteq Y$ with $y \in M$ is another open neighbourhood for which (a) holds then (a) also holds for $M \cap N$ and

$$\widetilde{F}_N\Big|_{(M\cap N)\times I)} = \widetilde{F}_M\Big|_{(M\cap N)\times I} = \widetilde{F}_{M\cap N}$$

Proof. This has a very long proof. □

Proposition 3.2. *Covering maps $p : \widetilde{X} \to X$ have the homotopy lifting property.*

Proof. Let $P; \widetilde{X} \to X$ be a covering map. Let $F : Y \times I \to I$ be a homotopy and choose soma arbitrary starting $g : Y \times \{0\} \to \widetilde{X}$.

We can cover $Y = \cup_\alpha N_\alpha$ such that *(a)* and *(b)* hold from the lemma.

We can then define a new homotopy by stitching these together:

$$\widetilde{F} : Y \times I \to \widetilde{X}, \qquad \widetilde{F}(y,t) := \widetilde{F}_{N_\alpha}(y,t) \text{ if } y \in N_\alpha$$

We do not get any ambiguity here thanks to property *(b)* from the lemma. The continuity of this construction follows from the pasting lemma. □

Theorem 3.3. *Let $\omega_N : I \to S^1$ be defined by $\omega_n(s) = e^{2\pi i n s}$. Then*

$$\pi_1(S^1, 1) = \{[\omega_n] \mid n \in \mathbb{Z}\}$$

Proof. Define $\Phi : \mathbb{Z} \to \pi_1(S^1, 1)$ by $n \mapsto [\omega_n]$. We claim this is an isomorphism. For this define the following useful maps

$$\begin{aligned} p(t) &= e^{2\pi i t} \\ \omega_n(t) &= e^{2\pi i n t} \\ \widetilde{\omega}_n(t) &= nt \\ \tau_m(t) &= t + m \end{aligned}$$

- <u>Φ is a group homomorphism.</u>

  Then we can see that indeed $\widetilde{\omega_n} : \mathbb{R} \to \mathbb{R}$ is a lift of ω_n. One can also see through the linear homotopy that

  $$\widetilde{\omega}_{m+n} \overset{\partial}{\simeq} \widetilde{\omega}_m * (\tau_m \circ \widetilde{\omega}_n)$$

  Now given any $m, n \in \mathbb{Z}$ we have

  $$\begin{aligned} \Phi(m+n) &= [\omega_{n+m}] && \textit{lift} \\ &= [p \circ \widetilde{\omega}_{n+m}] && \textit{homotopies descend} \\ &= [p \circ (\widetilde{\omega}_m * (\tau_m \circ \widetilde{\omega}_n))] \\ &= [p \circ \widetilde{\omega_m}] \cdot [p \circ \tau_m \circ \widetilde{\omega}_n] && \textit{deck transformation} \\ &= [p \circ \widetilde{\omega_m}] \cdot [p \circ \widetilde{\omega}_n] && \textit{lift} \\ &= [\omega_m] \cdot [\omega_n] \\ &= \Phi(m) \cdot \Phi(n) \end{aligned}$$

- <u>Φ is surjective.</u>

  Choose any $[\alpha] \in \pi_1(S^1, 1)$, we aim to find $n \in \mathbb{N}$ such that $\alpha \overset{\partial}{\simeq} \omega_n$. We certainly know that $\alpha(0) = \alpha(1) = 1$ and hence $p^{-1}(1) = \mathbb{Z}$ and in particular $0 \in p^{-1}(1) = p^{-1}(\alpha(0))$.

  So by the path lifting property there exists a unique lift $\widetilde{\alpha} : I \to R$ such that

  (i) $\widetilde{\alpha}(0) = 0$.

  (ii) $p \circ \widetilde{\alpha} = \alpha$.

  Now, $\alpha(1) = 1 \implies p(\widetilde{\alpha}(1)) = 1 \implies \widetilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$. Suppose $\widetilde{\alpha}(1) = n \in \mathbb{Z}$. So $\widetilde{\alpha}$ is a path from 0 to n in $\mathbb{R}$. By the linear homotopy we can see that $\widetilde{\alpha} \overset{\partial}{\simeq} \widetilde{\omega}_n$. But homotopies descend and hence

  $$\alpha = p \circ \widetilde{\alpha} \overset{\partial}{\simeq} p \circ \widetilde{\omega}_n = \omega_n$$

- <u>Φ is injective.</u>

  Assume that $\Phi[\omega_n] = [e]$. We aim to show that in fact $n = 0$.

  To start, $\omega_n \overset{\partial}{\simeq} e$ and hence we have a homotopy $F : I \times I \to S^1,\ (s,t) \mapsto F(s,t)$ such that $f_0 = \omega_n,\ f_1 = e$ and $f_t(0) = f_t(1) = 1$.

  Now by the HLP we see that there is a unique homotopy $F : I \times I \to R$ satisfying

(i) $\widetilde{f}_0 = \widetilde{\omega}_n$.

(ii) $p \circ \widetilde{F} = F$.

Now since the left, top and bottom edges were identically 1 in F we must have that the same edges lie in $\mathbb{Z}$ in the lifted homotopy. But consider the bottom edge $\widetilde{\omega}_n$. On the left side it is 0 but on the right it is n. By continuity along the left, top and bottom edges of $\widetilde{F}$ we must have that $n = 0$. This can be seen in Figure 3.

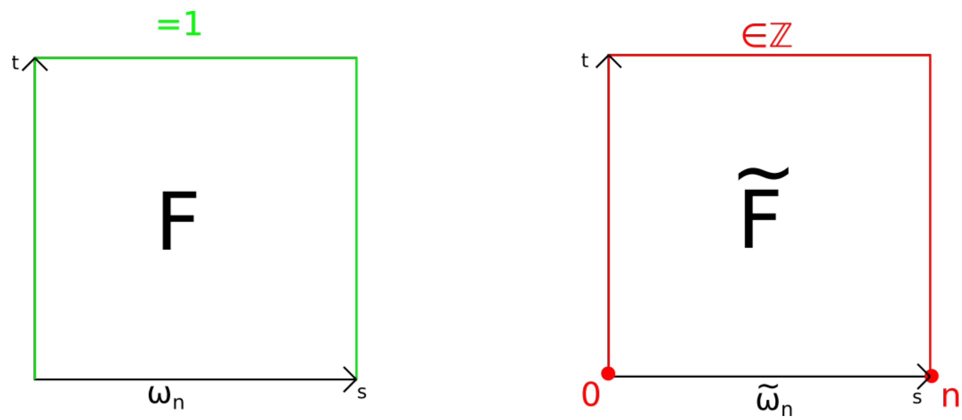

Figure 2: Diagrammatic explanation of continuity argument

□

4 Applications

A map of pairs $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subseteq B$.
The induced homomorphism of $f : (X, x_0) \to (Y, y_0)$ is the map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\alpha] &\mapsto [f \circ \alpha] \end{aligned}$$

We need to show that this is well-defined and is in fact a group homomorphism.

Lemma 4.1 (Functoriality)**.** $(g \circ f)_* = g_* \circ f_*$

Corollary 4.2. *If f is a homeomorphism then f_* is a group isomorphism.*

Theorem 4.3. *let $f : X \to Y$ be a homotopy equivalence and $x_0 \in X$. Then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. Let $y_0 := f(x_0)$ and $g : Y \to X$ be the homotopy inverse. Denote $x_1 := g(y_0)$. We get a homotopy a $K : X \times I \to X$ such that $k_0 = id_X$ and $k_1 = g \circ f$.

We define path in X by following that path of x_0 under this homotopy, i.e.

$$h : I \to X \quad t \mapsto K(x_0, t)$$

Claim: $\beta_h[\gamma] = (g \circ f)_*[\gamma]$ for all loops γ based at x_0.

We have already seen that base point change homomorphisms are in fact isomorphisms and hence we have that $(g \circ f)_* = g_* \circ f_*$ is an isomorphism. This implies that g_* is surjective and f_* is injective. Repeating this argument with the other homotopy then yields the result.

Proof of claim:

Define a new homotopy by

$$H(s,t) := \begin{cases} h(t) & \text{for } s \leq t \\ h(s) & \text{for } s \geq t \end{cases}$$

which has the properties that $h_0 = h$ and $h_1 = h(1) = x_1$.

Also define $K(\gamma(s), t)$ so that $\gamma_0 = \gamma$ and $\gamma_1 = g \circ f \circ \gamma$.

Finally, one can check that $\alpha_t := \overline{h_t} * \gamma_t * h_t$ gives a well-defined path for every t. This gives a homotopy between

$$\begin{aligned} \gamma_0 &= \overline{h} * \gamma * h \qquad\qquad \text{and} \\ \gamma_1 &= e_{x_1} * (g \circ f \circ \gamma) * e_{x_1} \end{aligned}$$

□

Proposition 4.4. *Consider the sequence of maps induced by a retract $r : X \to A$ and the 'inverse' inclusion for a point $x_0 \in A$*

$$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0)$$

1. *i_* is injective.*
2. *r_* is surjective.*
3. *If $r \overset{A}{\simeq} id_X$ then the induced maps are in fact isomorphisms.*

Proof. *(1)* and *(2)* follow immediately from the fact that $r \circ i = id_A$ and functoriality.

For *(3)* it remains to show that r_* is injective and i_* is surjective. For now we just show that r_* is injective. Suppose $r_*[\gamma] = [e]$ then we wish to show that in fact $[\gamma] = [e]$.

We know $r \circ \gamma \overset{\partial}{\simeq} e$ and $r \overset{A}{\simeq} id_X$. We have a homotopy $F : X \times I \to X$ where $f_t\big|_A = id_A$, $f_0 = r$ and $f_1 = id_X$. We define a new homotopy $G : I \times I \to X$ by $G(x,t) = F(\gamma(x), t)$ which satisfies.

(i) $g_t(0) = g_t(1) = f_t(\gamma(0)) = f_t(x_0) = x_0$ since $x_0 \in A$.

(ii) $g_0(x) = f_0(\gamma(x)) = (r \circ \gamma)(x)$.

(iii) $g_1(x) = f_1(\gamma(x)) = \gamma(x)$.

Hence we have $r \circ \gamma \overset{\partial}{\simeq} \gamma$ and hence $\gamma \overset{\partial}{\simeq} e$. The proof that i_* is surjective is very similar. □

Theorem 4.5 (No Retract Theorem)**.** *There is no retract $R : D^2 \to S^1$.*

Proof. Assume that such a retract exists then there is a surjective homomorphism r_*:

$$0 = \pi_1(D^2, 1) \to \pi_1(S^1, 1) = \mathbb{Z}$$

which is obviously nonsense. □

Theorem 4.6 (Brouwer Fixed Point Theorem)**.** *Any map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume that f has no fixed point, then we construct the following map for every x in S^1:

$$L_x(t) := tx + (1-t)f(x) \quad \forall t \geq 0$$

Then we can define a map $\phi : D^2 \to S^1$ by

$$\phi(x) = L_x(\mathbb{R}_{\geq 0}) \cap S^1$$

which we claim is a retraction. Certainly, $\phi\big|_{S^1} = id_{S_1}$ but what about continuity?

Well $\phi(x) = L_x(t)$ for the unique t which solves $|L_x(t)|^2 = 1$ and is bigger than 0. The equation for t is quadratic and so t can be shown to be a continuous function of x. Then clearly L_x is continuous so *phi* is continuous.

The no retract theorem yields a contradiction. □

Note: The Brouwer Fixed Point Theorem also holds for sets $S \cong D^2$ and their boundary.

4.1 Involutions and Borsuk-Ulam Theorem

Given a topological space X, an involution is a map $h : X \to X$ such that for every $x \in X$

$$h(h(x)) = x.$$

We often write h(x)=-x for convenience.
Given spaces X and Y each with involutions and a map $f : X \to Y$, we say f is

$$\begin{array}{rl} \text{odd} & \text{if} \quad f(-x) = -f(x) \\ \text{even} & \text{if} \quad f(-x) = f(x) \end{array}$$

for all x in X.
A map $f : (X, x_0) \to (Y, y_0$ is null-homotopic if f is homotopic to a constant map.
A map $f : (X, x_0) \to (Y, y_0)$ is null-homotopic relative to base points if there is a homotopy between $F : X \times I \to Y$ such that $f_0 = f$, $f_1 = e_{y_0}$ and $f_t(x_0) = y_0$ for all t. We then write $f \overset{x_o}{\simeq} e_{y_0}$.

Proposition 4.7. *If $f : S^1 to S^1$ is odd then f is not null-homotopic.*

Proof. Very long. □

Corollary 4.8. *If $f : S^2 \to \mathbb{R}^2$ is odd then there is a point $x \in S^2$ such that $f(x) = 0$.*

Proof. Worth doing. □

Theorem 4.9 (Borsuk-Ulam)**.** *Given a map $f : S^2 \to \mathbb{R}^2$ there is a point $x \in S^2$ such that $f(-x) = f(x)$.*

Proof. The map $g(x) := f(x) - f(-x)$ is by construction odd and hence has a vanishing point. □

5 Product Spaces

Given pointed spaces (X, x_0) and $(Y, y_0$ then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Also homotopies of paths in $X \times Y$ correspond to pairs of homotopies in X and Y.

5.1 Fundamental group of S^n for $n \geq 2$

Theorem 5.1. *For all $n \geq 2$ the fundamental group of S^n is trivial.*

Idea: Use stereographic projection into $\mathbb{R}^n$ and then use the linear homotopy.

The main issue with this is that the loop may go around or through the north pole, at which point our projection breaks down. However, in these big spheres we should be able to find a point that isn't inside the loop and project from there. Note that in S^1 we cannot do this because every non-trivial curve uses every point.

Proof. Cover the sphere with two open sets such that U_1, U_2 such that $x_0 \in U_1 \cap U_2$ and $U_1 \cap U_2$ is path connected. Now take any loop $\gamma : I \to S^n$ and subdivide the interval as

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that for all i there is a j such that $\gamma[t_{i-1}, t_i] \subseteq U_j$. So we can now view γ is the concatenation

$$\gamma = \mathop{\Huge *}_{i-1}^{m} \gamma\big|_{[t_{i-1}, t_i]}$$

Now since x_0 and $\gamma(t_i)$ are both in $U_1 \cap U_2$, we can let α_i be a path in $U_1 \cap U_2$ from $\gamma(t_i)$ to x_0. Now we can put these paths in between the subdivisions of γ:

$$\beta := \left[\gamma\big|_{[t_0,t_1]} * \alpha_1\right] * \left[\mathop{\Huge *}_{i=2}^{m-1} \overline{\alpha}_{i-1} * \gamma\big|_{[t_{i-1},t_i]} * \alpha_i\right] * \left[\overline{\alpha}_{m-1} * \gamma\big|_{[t_{m-1},t_m]}\right]$$

Now each $\overline{\alpha}_{i-1} * \gamma\big|_{[t_{i-1},t_i]} * \alpha_i$ lies in one U_j and so is homotopic to a constant loop by using the linear homotopy in $\mathbb{R}^n$. So $\beta \simeq e_{x_0}$ and $\beta \simeq \gamma$ and hence $\gamma \simeq e_{x_0}$. $\square$

6 Some Algebra

Proposition 6.1. *Let $p : \widetilde{X} \to X$ be a covering , $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$.*

1. *The induced map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*
2. *If $[\alpha] \in \pi_1(X, x_0)$ and $\widetilde{\alpha}$ is a lift of α such that $\widetilde{\alpha}(0) = \widetilde{x}_0$, then*

$$[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \iff \widetilde{\alpha} \text{ is a loop}$$

Proof. 1. We wish to show that $p_*([\widetilde{\alpha}]) = [p \circ \widetilde{\alpha}] = [e_{x_0}] \implies [\widetilde{\alpha}] = [e_{\widetilde{x}_0}]$.

Given $[\widetilde{\alpha}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ such that $[p \circ \widetilde{\alpha}] = [e_{x_0}]$, define $\alpha := p \circ \widetilde{\alpha}$ and then $\widetilde{\alpha}$ is the unique lift of α with the property $\widetilde{\alpha}(0) = \widetilde{x}_0$. Now, by assumption, $\alpha \overset{\partial}{\simeq} e_{x_0}$ and hence there is a homotopy $F : I \times I \to X$ with $f_0 = \alpha$, $f_1 = e_{x_0}$.

By the HLP, there exists a unique lift $\widetilde{F} : I \times I \to X$ such that $\widetilde{f}_0 = \widetilde{\alpha}$. Now $F(0,t)$, $F(1,t)$ and $F(s,1)$ are all constant paths at x_0 and hence lift to constant paths at $\widetilde{x}_0$. So $\widetilde{F}$ tells us that $\widetilde{\alpha} \overset{\partial}{\simeq} e_{\widetilde{x}_0}$.

2. We really only have the forward direction to prove.

   Suppose $[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ then $[\alpha] = [p \circ \widetilde{\gamma}]$ for some $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. Suppose $\widetilde{\alpha}$ is a lift of α with $\widetilde{\alpha}(0) = \widetilde{x}_0$. We have
$$\alpha = p \circ \widetilde{\alpha} \overset{\partial}{\simeq} p \circ \widetilde{\gamma} =: \gamma$$
   We know $\widetilde{\gamma}$ is a loop at $\widetilde{x}_0$. So we can lift the homotopy between α and γ to one between $\widetilde{\alpha}$ and $\widetilde{\gamma}$.

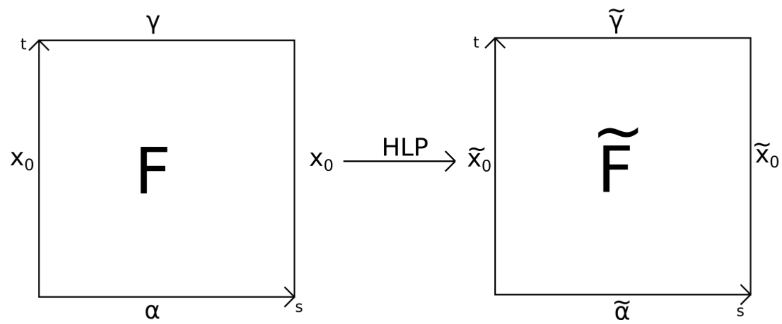

Figure 3: Diagrammatic explanation that $\widetilde{\alpha}$ is a loop at $\widetilde{x}_0$.

□

Given a group G and a subgroup $H \leq G$ we have the quotient space
$$\frac{G}{H} := \{Hg \mid g \in G\} = \{\text{right cosets}\}$$
and then the index is $[G : H] = |\frac{G}{H}| = \#$ of cosets.
Let $p : \widetilde{X} \to X$ be a cover and assume that both spaces are path connected. Then we define the degree
$$\deg(p) = |p^{-1}(x)| \quad \text{for any} \quad x \in X$$

Note: Definition of cover $\implies$ $\deg(p)$ is at least locally constant.
Path connected $\implies$ globally constant.

Proposition 6.2. *Let $p : \widetilde{X} \to X$ be a cover with $\widetilde{X}, X$ path connected, $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. Then*
$$\deg(p) := \left[\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))\right].$$

Proof. Let $G := \pi_1(X, x_0)$ and $H := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Given any $[\alpha] \in G$ we get a unique lift $\widetilde{\alpha}$ with $\widetilde{\alpha}(0) = \widetilde{x}_0$ and then $\widetilde{\alpha}(1) = \widetilde{x}_1 \in p^{-1}(x_0)$.

Now let $[\gamma] \in H$ then we have

$$\begin{aligned}[\gamma] \in H &\implies [\gamma]\cdot[\alpha] = [\gamma * \alpha] \in H[\alpha] \\ &\implies \widetilde{\gamma} \text{ (a lift of } \gamma\text{) is a loop at } \widetilde{x}_0 \\ &\implies (\gamma * \alpha) \text{ lifts to } (\widetilde{\gamma} * \widetilde{\alpha}) \text{ from } \widetilde{x}_0 \text{ to } \widetilde{x}_1 \end{aligned} \quad \textit{unique lift}$$

And hence we can construct a map

$$\Phi : \frac{G}{H} \to p^{-1}(x_0), \quad H[\alpha] \mapsto \widetilde{\alpha}(1)$$

which we just showed is independent of the representative of $H[\alpha]$. We wish to show that Φ is bijective.

Injective: Assume that $\Phi(H[\alpha]) = \Phi(H[\beta])$ then $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Write $\gamma := \alpha * \overline{\beta}$ then $\widetilde{\gamma} = \widetilde{\alpha} * \widetilde{\overline{\beta}}$ is a loop because $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{\overline{\beta}}(1) = \widetilde{x}_0$ and the assumption means that $\widetilde{\alpha}$ and $\widetilde{\beta}$ join up in the middle.

Hence $[\gamma] = [\alpha]\cdot[\beta]^{-1} \in H$ so $H[\alpha] = H[\beta]$.

Surjective: This follow from path connectivity. □

Let $\{X_\alpha, x_\alpha\}$ be a collection of pointed topological spaces. The wedge product is then defined to be

$$\bigvee_\alpha X_\alpha = \bigsqcup_\alpha \frac{X_\alpha}{x_\alpha \sim x_\beta}$$

Then $\{G_\alpha\}_\alpha$ be some groups. Then a word on $\{G_\alpha\}_\alpha$ is a finite sequence

$$g = g_1 \dots g_m$$

for some $g_i \in G_{\alpha_i}$. Then m is the length of the word g.
We can concatenate words $g = g_1, \dots, g_m$ and $h = h_1, \dots, h_n$ by

$$g * h = g_1 \dots g_m h_1 \dots h_n$$

A word is said to be reduced if

(a) $g_i \neq e_{\alpha_i} \quad \forall i$

(b) $\alpha_i \neq \alpha_{i+1} \quad \forall i$

We can define the set

$$\mathop{\huge *}_\alpha G_\alpha := \{\text{reduced words on } \{G_\alpha\}_\alpha\}$$

which we can given a product in the following way. Given $g, h \in *_\alpha G_\alpha$

1. Take the concatenation $g * h = g_1 \dots g_m h_1 \dots h_n$.
2. If g_m and h_1 lie in different G_α then $g \cdot h := g * h$.
3. Else replace $g_m h_1$ by the product.
4. If $g_m h_1 = e$ then remove it and repeat from step 1 with the word $g_1 \dots g_{m-1} h_2 \dots h_n$.

Theorem 6.3. *So defined, $(*_\alpha G_\alpha, \cdot)$ forms a group, called the free product of $\{G_\alpha\}_\alpha$.*

Proof. The only difficult part is associativity, worth reading up on. $\square$

Lemma 6.4. *Let $\{\phi_\alpha : G_\alpha \to G\}_\alpha$ be a collection of group homomorphisms. Then $\exists!$ group homomorphism $*_\alpha\phi_\alpha : G_\alpha \to G$ such that $(*_\alpha\phi_\alpha) \circ i_\alpha = \phi_\alpha \quad \forall\alpha$*

Proof. Really the only thing to prove is that this definition doesn't depend on how the word is reduced. $\square$

$$\begin{tikzcd}
& \pi_1(A_\alpha, x_0) \arrow[dr] \arrow[drr, bend left, "(i_\alpha)_*"] & & \\
\pi_1(A_\alpha \cap A_\beta, x_0) \arrow[ur, "(i_{\alpha\beta})_*"] \arrow[dr, "(i_{\beta\alpha})_*"'] & & *_\alpha\pi_1(A_\alpha, x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X, x_0) \\
& \pi_1(A_\beta, x_0) \arrow[ur] \arrow[urr, bend right, "(i_\beta)_*"'] & &
\end{tikzcd}$$

Figure 4: The holy commutative diagram

Theorem 6.5 (Seifert-van Kampen). *Let $X = \cup_\alpha A_\alpha$ be an open cover with $x_0 \in \cap - \alpha A_\alpha$, then*

(i) If $A_\alpha \cap A_\beta$ is path connected for all α, β then the induced map

$$\Phi := \mathop{\ast}_{\alpha} (i_\alpha)_* : \mathop{\ast}_{\alpha} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective

(ii) If $A_\alpha cap A_\beta \cap A_\gamma$ is path connected for all α, β, γ then

$$\ker(\Phi) = N = \langle\langle \left\{ (i_{\alpha\beta})_*(\omega) \cdot (i_{\beta\alpha})_*(\omega)^{-1} \;\middle|\; \omega \in \pi_1(A_\alpha \cap A_\beta, x_0) \text{ for some } \alpha, \beta \right\} \rangle\rangle$$

and hence

$$\pi_1(X, x_0) \cong \frac{\ast_\alpha \pi_1(A_\alpha, x_0)}{N}$$

Lemma 6.6. *If $X = \cup_\alpha A_\alpha$ is an open cover with $A_\alpha \cap A_\beta$ path connected for every pair α, β and $x_0 \in \cap_\alpha A_\alpha$, then every loop γ based at x_0 in X factors as*

$$[\gamma] = [\gamma_1] \cdot [\gamma_2] \cdot \dots \cdot [\gamma_m]$$

with each γ_i being a loop in A_{α_i}.

Proof. Note than we can cover I by the $\gamma^{-1}(A_\alpha)$ and since I is compact we can actually reduce this to a finite sub cover. Then the Lebesgue covering lemma means that we can find

$$0 = t_0 < t_1 < \dots < t_m = 1$$

such that $\gamma[t_i, t_{i+1}] \subseteq A_{\alpha_{i+1}}$ for each $i \in [0, m)$. Then for each t_i notice that $\gamma(t_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$ which is path connected and hence there is a path g_i from $\gamma(t_i)$ to x_0 in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$.

Now we can define a new path $\hat{\gamma}$ by going back to x_0 along these paths at each t_i and then returning to continue the loop. Since $g * \overline{g}$ is homotopic to $e_{\gamma(t_i)}$ we have that $\hat{\gamma} \overset{\partial}{\simeq} \gamma$ and hence we can decompose γ as required. $\square$

These essentially proves part *(i)* of the Seifert-van Kampen theorem.

A factorisation of $[f] \in \pi_1(X, x_0$ is a sequence

$$[f_1], \dots, [f_m] \quad \text{with} \quad [f_i] \in \pi_1(A_{\alpha_i}, x_0)$$

such that $f \simeq f_1 * \dots f_m$ (although this is not necessarily in reduced form).
If $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$ then a reduction is

$$[f_1] \dots [f_m] \mapsto [f+1] \dots [f_i * f_{i+1}] \dots [f_m]$$

(or an expansion if you go in the opposite direction).
If $[f_i] = (i_{\alpha\beta})_*(\omega)$ and $[g_i] = (i_{\beta\alpha})_*(\omega)$ for some $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ then an exchange is

$$[f_1] \dots [f_i] \dots [f_m] \mapsto [f_1] \dots [g_i] \dots [f_m]$$

Two factorisations of f are said to be equivalent if tone can be transformed to the other by a sequence of the above operations.

Lemma 6.7. *Under the conditions of Seifert-van Kampen (ii) any two factorisations of f are equivalent.*

Proof. In lecture notes. □

This can then be used to proved Siefert-can Kampen *(ii)*.

7 CW Complexes

Given a space X we can build up the CW complex recursively as follows:

1. X_0 is a discrete set called the 0-skeleton.
2. Given X^{n-1} and a collection of n-cells $\{D^n_\alpha\}_\alpha$ and attaching maps $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$ we subsequently define X_n by

$$X^n := \frac{\left[X^{n-1} \sqcup (\bigsqcup_\alpha D^n_\alpha)\right]}{x \sim \phi_\alpha(x)}$$

3. Then $X = \cup_n X^n$, bestowed with the weak topology is a CW complex.

Then X is called finite dimensional if $X = X^n$ for some $n \in \mathbb{N}$.
Lastly, X is finite if $\{D^n_\alpha\}_{\alpha,n}$ is finite (i.e. it is composed of finitely many cells).
A subcomplex of X is the closure in X of a collection of open cells in X.

Note: $U \subseteq X$ is open in the weak topology $\iff \forall n \quad U \cap X^n$ is open in X^n

7.1 Examples

7.1.1 Möbius Strip

The Möbius strip M can be defined as $I \times I$ after identifying $(0, x)$ with $(1, 1-x)$ for all $x \in I$.

One can give the Möbius strip a CW structure by taking two 0 cells for either side of the identified line, three 1 cells (one for the identified line and two for the remaining boundaries) and then gluing a rectangle along the inside in the natural way. Note that the Möbius strip has a circle at its centre and the strip deformation retracts to it by taking the obvious retract on $I \times I$ and then passing to the quotient. Hence $\pi_1(M) \cong \mathbb{Z}$. Notes, the Möbius strip has exactly one circle at its boundary.

7.1.2 Real projective space

Recall that we define real projective space $\mathbb{RP}^n$ by

$$\mathbb{RP}^n := \frac{S^n}{x \sim -x}$$

identifying antipodal points on the n-sphere. Let q be the quotient map from this definition. We can define a CW structure on $\mathbb{RP}^n$ recursively as follows.

Consider the open set $U_0 := \{[(x_0, x_1, \ldots, x_n] \mid x_0 \neq 0\}$. Then $\mathbb{RP}^n \setminus U_0 = \{[0, x_1, \ldots, x_n]\}$ is naturally homeomorphic to $\mathbb{RP}^{n-1}$. Moreover, since q is a two-sheeted covering we see that $q^{-1}(U_0)$ consists of the disjoint union of the sets $\{x_0 < 0\}$ and $\{x_0 > 0\}$. Each of these sets is the interior of an n-balls and maps homeomorphically to U_0 and hence $U_0 \cong e^n$.

Set $D^n = \{x_0 \geq 0\}$ and $S^{n-1} = \partial D^n = \{(0, x_1, \ldots, x_n)\}$ so we get the two-fold covering

$$\phi : S^{n-1} \to \mathbb{RP}^{n-1}$$

which we can use as the attacking map to glue D^n onto RP^{n-1} where we removed U_0 and hence

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \frac{D^n}{x \sim \phi(x)}$$

Once can continue this process on iteratively and each stage we add one extra n-cell and hence

$$\mathbb{RP}^n = \{pt\} \cup e^1 \cup e^2 \cup \cdots \cup e^n$$

n=0: $\mathbb{RP}^0 = \{pt\}$ so there is not much to say here.

n=1: $\mathbb{RP}^1 = \mathbb{RP}^0 \sqcup \frac{D^1}{0 \sim 1}$ which show that $\mathbb{RP}^1$ is just a circle.

n=2: We attach a 2-cell by gluing the boundary of a disk D^2 to $\mathbb{RP}^1$ by the usual two fold covering $S^1 \to \mathbb{RP}^1$. Perhaps a better way of thinking about $\mathbb{RP}^2$ is to start with the closed upper hemisphere. Each point corresponds to a unique point in $\mathbb{RP}^2$ except at the boundary where we identify antipodal points. So the boundary becomes a copy of $\mathbb{RP}^1$.

7.1.3 From $\mathbb{RP}^2$ to the Möbius strip

Consider Figure 5 and the space acquired by removing a circle from the interior of the disk, call this space X. Formally we can describe X as

$$X = \frac{S^1 \times I}{(x, 1) \sim (-x, 1)}$$

We claim that $X \cong M$. Visually we can see this by splitting the annulus in half and then re-glueing the segments back together along the boundary curve γ which then becomes the middle circle of the Möbius strip.

Denoting $a = a_1 * a_2$ we recover the usual characterisation of the Möbius strip. Reversing this, we can in fact obtain the projective pain by glueing a 2-cell along the boundary of the Möbius strip.

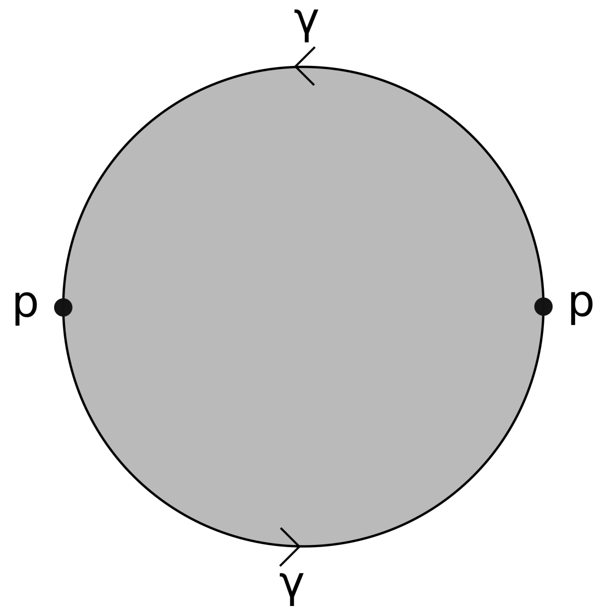

Figure 5: Cell decomposition of $\mathbb{RP}^2$, the boundary circle is a copy of $\mathbb{RP}^1$ but the interior is just a cell

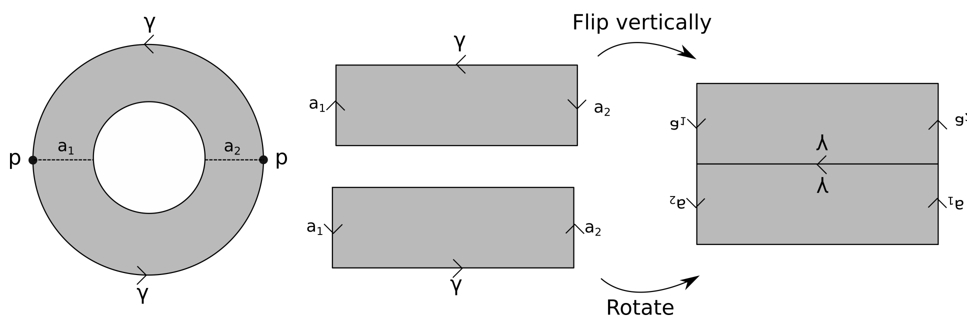

Figure 6: Showing X is homeomorphic to M

7.1.4 Computing $\pi_1(\mathbb{RP}^2)$ from the Möbius strip and S-vk

We can use our previous construction of $\mathbb{RP}^2$ to create an open cover for $\mathbb{RP}^2$ by sets whose fundamental groups we understand. Let C be some small closed disc C in the interior of the 2-cell and then take an open disc $B \supseteq C$. Then define $A = \mathbb{RP}^2 \setminus C$.

Now A and B form a cover for $\mathbb{RP}^2$, where B is a disc and A is the interior of a Möbius strip. Pick some $x_0 \in A \cap B$. Their intersection $A \cap B$ is an annulus which deformation retracts to a circle and hence $\pi_1(A \cap B, x_0)$ can be generated by a loop ω as shown in Figure 7.

A is the Möbius strip and hence $\pi_1(A, x_0) = \langle \gamma \rangle \cong \mathbb{Z}$ the fundamental group is generated by the middle circle. Finally B is contractible and hence has trivial fundamental group. The free group $\pi_1(A, x_0) * \pi_1(B, x_0)$ is generated by $[\gamma]$. Since B is trivial we only have to factor out multiplies of

$$(i_{A \cap B})_*([\omega])$$

where $i_{A \cap B}$ is the inclusion of $A \cap B$ in A. Note ω is homotopic to the boundary circle of the Möbius strip where as γ is the inner circle. Hence going round ω once is like going round γ twice.

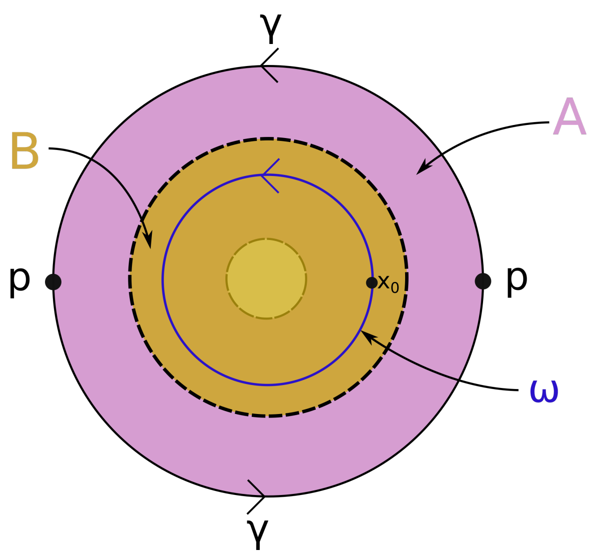

Figure 7: Covering $\mathbb{RP}^2$ in order to compute π_1 by S-vK

Consequently we have to factor out multiples of $[\gamma]^2$. In conclusion,

$$\pi(\mathbb{RP}^2, x_0) \cong \frac{\langle[\gamma]\rangle}{\langle[\gamma]^2\rangle} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

7.2 Properties of CW complexes

Given any n-cell D^n_α we construct the characteristic map as follows

$$\begin{tikzcd}
D^n_\alpha \arrow[r, hook, "i"] \arrow[rrr, bend left, "\Phi^n_\alpha"] & X^{n-1} \sqcup \coprod_\alpha D^n_\alpha \arrow[r, "q"] & X^n \arrow[r, hook] & X
\end{tikzcd}$$

Recall that a subcomplex is a space A that is a union of cells e^n_α in X such that for every cell it also contains its closure in X.

Proposition 7.1. *A compact topological subspace of a CW complex X is contained in a finite subcomplex.*

Proof. Might be worth reading this. $\square$

The C is in CW complex stands for closure finiteness meaning that the closure of every open cells meets only finitely many other cells. The W stands for weak topology.
A topological space is called

- normal if any two disjoint closed subsets have disjoint open neighbourhoods.
- Hausdorff if any two distinct point have disjoint open neighbourhoods.

- locally contractible if around every point x and neighbourhood $x \in U \subseteq X$ there is an open V with $x \in V \subseteq U$ such that V is contractible.

Example: The Warsaw circle is not locally contractible

$$W = \{(c, \sin(1/x) \mid x \in (0,1]\} \cup (\{0\} \times [-1,1]) \cup L$$

where L is a curve joining the first set to the second set, with the subspace topology.

Proposition 7.2. *CW complexes are locally contractible.*

Proposition 7.3. *If $A \subseteq X$ is a subcomplex of a CW complex X then there exists an open set $U \subseteq X$ with $A \subseteq U$ such that U deformation retracts to A.*

An important application is that we can decompose CW complexes A and B such that $A \cap B$ is a subcomplex and then use Seifert-can Kampen. One can use the above proposition to show that the fundamental group of a CW complex depends only on the 2-skeleton. However we will give a proof using the Seifert-van Kampen Theorem.

We begin with the following observation. Suppose X is a topological space and $\phi_\alpha : S^1_\alpha \to X$ is a map that attaches a 2-cell D^2_α to X then ϕ_α defines a loop $f_\alpha : I \to X$ in X which is based at $\phi_\alpha(1)$. This loop may not be null-homotopic in X but it certainly is after we attach D^2_α, namely in the space

$$Y := X \sqcup \frac{D^2_\alpha}{x \sim \phi_\alpha(x)}$$

If X is path-connected then we can choose a base point $x_0 \in X$ and a path $h_\alpha : I \to X$ from x_0 to $\phi_\alpha(1)$ and then we get a loop

$$\gamma_\alpha := h_\alpha * f_\alpha * \overline{h_\alpha}$$

In this way, every attaching map gives us a loop in Y. Note that the inclusion map $X \hookrightarrow Y$ induces a map of fundamental groups $\pi_1(X, X_0) \to \pi_1(Y, y_0)$ under which the class of every such loop, $[\gamma_\alpha]$ is sent to 0 because these loops are contractible in Y.

Proposition 7.4. *Let X be a path-connected topological space and for some fixed N let $\phi^N_\alpha : S^{n-1}_\alpha \to X$ be some collection of attaching maps. Set*

$$Y := X \sqcup \coprod_\alpha \frac{D^n_\alpha}{x \sim \phi^n_\alpha(X)}$$

Let $x_0 \in X$ be a point. Then

- *If $n = 2$, then*
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / N$$
where N is the normal subgroup generated by the element $[\gamma_\alpha]$ as defined previously.

- *If $n > 2$, then*
$$\pi(Y, x_0) \cong \pi_1(X, x_0)$$

Proof. There is a sketch proof in the notes that may be worth reading. □

Theorem 7.5. *For a path-connected CW-complex X with $x_0 \in X^2$, the inclusion map $X^2 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.*

Proof. This follows automatically from the previous proposition of X is finite dimensional because it tells us that the fundamental group does not change when we add an n-cell where $n > 2$.

If X is not finite-dimensional then note that a loop γ in X is a compact subset and therefore contained in a finite subcomplex in some X^n. Since $\pi_1(X^2, x_0 \cong \pi_1(X^n, x_0)$ every such loop is homotopic to a loop in X^2 and therefore the map $\pi_1(X^2, x_0) \to \pi(X, x_0)$ is surjective.

To see that it is injective, let γ be a loop in X^2 such that it is homotopic, in X, to the constant loop via some homotopy $F : I \times I \to X$. The image of F in X is compact and hence contained in a finite subcomplex X^n and we can safely assume that $n > 2$. It follows therefore that $[\gamma] = 0$ in $\pi_1(X^n, x_0)$. Then $\pi_1(X^2, x_0) \to \pi_1(X^n, x_0)$ is injective and hence γ is null-homotopic in X^2. $\square$

We can also prove this with Proposition 7.3. For a proof of this consult Lecture 26.