Algebraic Topology Notes

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1 Simplicial Homology

The standard k-simplex is

$$\Delta^k := \left\{ (x_0, \dots x_k) \in \mathbb{R}^{k+1} \;\middle|\; \sum_{i=0}^{k} x_i = 1, \quad x_i \geq 0 \right\}$$

Given $v_0, \dots, v_k \in \mathbb{R}^N$ we define the simplex by

$$[v_0, \dots v_k] := \left\{ \sum_{i=0}^{k} x_i v_i \;\middle|\; \sum_{i=0}^{k} x_i = 1, \quad x_i \geq 0 \right\}$$

so than $\Delta^k = [e_0, \dots, e_k]$.
This yields an obvious map $\sigma : \Delta^k \to [v_0, \dots v_k]$

$$\sigma(x_0, \dots x_k) = \sum_{x_i v_i}$$

We will often, confusingly, denote this map $[v_0, \dots v_k]$.
We say $v_0, \dots, v_k$ are in general position if they do not lie on any $(k-1)$-dimensional affine subspace.

Proposition 1.1. *$v_0, \dots, v_k$ are in general position $\iff$ σ is a homeomorphism.*

We can then define the map $i_j : \Delta^{k-1} \to \Delta^k$ to be the map $[e_0, \dots, e_{j-1}, e_{j+1}, \dots, e_k]$. This map then parametrises the face opposite the vector e_j. The union of the $k-1$ dimensional faces of $[v_0, \dots, v_k]$ is its boundary and its interior is $[v_0, \dots, v_k]$.

A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^k \to X$ for varying k such that

1. $\sigma_\alpha : \left(\Delta^k\right)^{\mathrm{o}} \to X$ is injective and each point of X lies in the image of exactly one interior.
2. If $\sigma_\alpha : \Delta^k \to X$ is in the collection then $\sigma_\alpha \circ i_j : \Delta^{k-1} \to X$ is also for $j = 0, \dots, k$.
3. $U \subseteq X$ is open $\iff$ $\sigma_\alpha^{-1}(U)$ is open in Δ^k for all α.

We then define the spaces of formal sum of simplicies

$$\Delta_n(X) := \left\{ \sum m_\alpha \sigma_\alpha^n \;\middle|\; \text{formal sums with integer coeffs and } \sigma_\alpha^n \text{ are n-simplicies} \right\}$$

and then we can define the boundary operator $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by

$$\partial_n(\sigma_\alpha^n) = \sum_{j=0}^{n}(-1)^j(\sigma_\alpha^n \circ i_j) = \sum_{j=0}^{n}(-1)^j \sigma_\alpha^n\big|_{[e_0,\dots,\hat{e_j},\dots,e_n]}$$

where $\hat{e_j}$ means that we omit the j'th vertex.
Then the loops are elements in $\ker \partial_n$ but we don't care about loops that are themselves the boundaries of higher order simplicies since these loops can be contracted through the higher-order simplicies.

We define the n'th homology of the Δ-complex structure to be

$$H_n^\Delta := \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

2 Singular Homology

Simplicial homology is very computable but has a number of problems that prevent from having far reaching consequences to topology.

1. $\Delta_n(X)$ depends on the choice of Δ complex structure so perhaps H_n^Δ does too?
2. We would like functoriality. If X, Y have Δ-complex structures and $f : X \to Y$ is continuous, how do we then define a homomorphism $H_n^\Delta(X) \to H_n^\Delta(Y)$?

These problems can be remedied by studying the more complicated and seemingly incomputable singular homology. Instead of considering only simplicies $\Delta^n \to X$ in the Δ complex structure we allow all continuous maps $\Delta^n \to X$ which we call singular n-simplicies.

$$C_n(X) := \{\text{finite formal sums of singular n-simplicies}\} = \text{Free AbGr on singular n-simplicies}$$

We can then define $\partial_n(X) \to C_{n-1}(X)$ in exactly the same way as before and then define singular homology

$$H_n(x) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

Note:

$$\partial_n \circ \partial_{n+1} = 0$$

2.1 Functoriality

If we have a continuous map $f : X \to Y$ then we can define

$$f_\# : C_n(X) \to C_n(Y), \quad \sigma \mapsto f \circ \sigma$$

on the n-simplicies and then extend it to all of $C_n(X)$ linearly so that

$$f_\#\left(\sum_\alpha m_\alpha \sigma_\alpha\right) = \sum_\alpha m_\alpha (f \circ \sigma_\alpha)$$

Lemma 2.1.

$$\partial(f_\# \sigma) = f_\#(\partial\sigma)$$

Proof.

$$\partial(f_\#\sigma) = \partial((f \circ \sigma)) = \sum_{j=0}^{n}(-1)^j (f \circ \sigma) \circ i_j$$

$$f_\#(\partial\sigma) = f_\#\left(\sum_{j=0}^{n}(-1)^j(\sigma \circ i_j)\right) = \sum_{j=0}^{n}(-1)^j f \circ (\sigma \circ i_j)$$

□

We can also extend this result to any formal sum of n-simplicies in $C_n(X)$. Hence $f_\#$ induces a morphism of chain complexes. The morphism arises due to this commutativity and it is of chain complexes because we have $\partial^2 = 0$.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\
\dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \dots
\end{array}$$

Corollary 2.2. *From this diagram we can see that*

1. $f_\#(\underbrace{\ker\partial_n}_{\subseteq C_n(X)}) \subseteq \underbrace{\ker\partial_n}_{\subseteq C_n(Y)}$.
2. $f_\#(\operatorname{im}\partial_n) \subseteq \operatorname{im}\partial_n$.

and hence $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ because if you differ be an element of $\operatorname{im}\partial_{n+1}$ in $H_n(x)$ then you will still differ by an image element in $H_n(Y)$.

Elements of $\ker\partial$ are called cycles and elements of $\operatorname{im}\partial$ are called boundaries.

$$B_n(X) := \operatorname{im}\partial_{n+1} \subseteq C_n(X)$$
$$Z_n(X) := \ker\partial_n \subseteq C_n(X)$$

so the $H_n(X) = \frac{Z_n}{B_n}$ and the induced map f_* is given be

$$f_*(c + B_n(X)) := f_\#(c) + B_n(Y)$$

Theorem 2.3.

$$(f \circ g)_* = f_* \circ f_* \quad \textit{and} \quad (id_X)_* = id_{H_n(X)} \ \forall n$$

and hence singular homology is a functor.

This implies the important result that

$$X \underbrace{\cong}_{\text{homeo}} Y \implies H_n(X) \underbrace{\cong}_{\text{iso}} H_n(Y) \ \forall n$$

2.2 Basic computation of singular homology

Here are some important results.

Theorem 2.4.

$$H_n(\{pt\}) = \begin{cases} \mathbb{Z} & \textit{if } n = 0 \\ 0 & \textit{if } n > 0 \end{cases}$$

Note: Δ^n is path connected so every continuous map $\sigma : \Delta^n \to X$ falls in one path component. Hence

$$C_n(X) = \bigoplus_{\text{path components } X_i} C_n(X_i)$$

Moreover $\partial_n(C_n(X_i) \subseteq C_{n-1}(X_i)$ so the chain complexes of the path components are independent of one another and hence we have

$$H_n(X) = \bigoplus_{X_i} H_n(X_i)$$

Theorem 2.5. *X path connected* $\iff$ $H_0(X) \cong X$.

2.3 Reduced Homology

It often makes sense to add one extra space to the chain complex under to state results more succinctly, so we modify the chain complex as so

$$\ldots \longrightarrow C_n(X) \longrightarrow \ldots \longrightarrow C_1(x) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where we introduce anew map $\epsilon : C_0(X) \to \mathbb{Z}$ which just sums coefficients:

$$\epsilon\left(\sum m_\alpha[p_\alpha]\right) \mapsto \sum m_\alpha$$

then we define the reduced homologies to be the same $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$ but then

$$\widetilde{H}_0(X) := \frac{\ker \epsilon}{\operatorname{im} \partial_1} \neq H_0(X)$$

We can realise the relation of H_0 and $\widetilde{H}_0$ by the following commutative diagram:

$$\begin{array}{ccc} \ker \epsilon & \xrightarrow{i} & C_0(X) \\ \Big\downarrow \overline{\operatorname{im} \partial_1} & & \Big\downarrow \overline{\operatorname{im} \partial_1} \\ \widetilde{H}_0(X) & \xrightarrow{i} & H_0(X) \end{array}$$

This diagram commutes and hence the map ϵ passes to the quotient to define

$$\bar{\epsilon} : H_0(X) \to \mathbb{Z}$$

and then $\widetilde{H}_0(X) = \ker \bar{\epsilon}$.

3 Exact sequences

A sequence of abelian groups A_n and homomorphisms ϕ_n is a complex if $\operatorname{im}\phi_{n+1} \subseteq \ker\phi_n$ for all n.

$$\ldots \xrightarrow{\phi_{n+2}} A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \xrightarrow{\phi_{n-1}} \ldots$$

The complex is exact if $\operatorname{im}\phi_{n+1} = \ker\phi_n$ for all n.

Note: Whenever we have such a chain complex we can assign the n-th homology to be

$$\frac{\ker\phi_n}{\operatorname{im}\phi_{n+1}}$$

So assuming that X is non-empty we get the following exact sequence

$$0 \longrightarrow \widetilde{H}_0(X) \hookrightarrow H_0(X) \xrightarrow{\bar{\epsilon}} \mathbb{Z} \longrightarrow 0$$

Given $A \subseteq X$ both topological spaces then we say (X, A) is a pair. Further, we say they are a good pair if A is a deformation retract of some neighbourhood in X.

Theorem 3.1. *If (X, A) is a good pair then there exists a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "q_*"] & \widetilde{H}_n(\frac{X}{A}) \arrow[dll, "\partial"'] & \\
& \widetilde{H}_{n-1}(A) \arrow[r, "i_*"] & \widetilde{H}_{n-1}(X) \arrow[r, "q_*"] & \widetilde{H}_{n-1}(\frac{X}{A}) \arrow[dll, "\cdots"'] & \\
& \widetilde{H}_0(A) \arrow[r, "i_*"] & \widetilde{H}_0(X) \arrow[r, "q_*"] & \widetilde{H}_0(\frac{X}{A}) \arrow[r] & 0
\end{tikzcd}$$

We can then use this long exact sequence with the pair (S^{n-1}, D^n).

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_n(S^{n-1}) \arrow[r, "i_*"] & \cancelto{0}{\widetilde{H}_n(D^n)} \arrow[r, "q_*"] & \widetilde{H}_n\left(\frac{D^n}{S^{n-1}}\right) \arrow[dll, "\partial"'] & \\
& \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "i_*"] & \cancelto{0}{\widetilde{H}_{n-1}(D^n)} \arrow[r, "q_*"] & \widetilde{H}_{n-1}\left(\frac{D^n}{S^{n-1}}\right) \arrow[dll, "\cdots"'] & \\
& \widetilde{H}_0(S^{n-1}) \arrow[r, "i_*"] & \cancelto{0}{\widetilde{H}_0(D^n)} \arrow[r, "q_*"] & \widetilde{H}_0\left(\frac{D^n}{S^{n-1}}\right) \arrow[r] & 0
\end{tikzcd}$$

Thanks to the exactness of this sequence we see that

$$\widetilde{H}_n(S^{n-1}) \cong \widetilde{H}_n\left(\frac{D^n}{S^{n-1}}\right) \cong \widetilde{H}_n(S^n)$$

and hence

$$\widetilde{H}_k(S^n) = \begin{cases} 0 & \text{if } k < n \\ \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

We will often find short exact sequences with are sequences that look like

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

in which case the first isomorphism theorem tells us that $C \cong \frac{B}{A}$.

4 Relative homologies

Given a pair (X, A) we can define the relative chain group

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)}$$

i.e. we consider two continuous maps into X to be the same if the agree on $X \setminus A$.
Now our maps ∂_n pass to the quotient by

$$\overline{\partial}_n(c + C_n(A)) := \partial_n(c) + C_{n-1}(A)$$

which is well defined because the boundary of a chain in A is still a chain in A. We still get $\overline{\partial}_n \circ \overline{\partial}_{n+1} = 0$ and hence we can define the relative homology groups

$$H_n(X, A) := \frac{\ker \overline{\partial}_n}{\operatorname{im} \overline{\partial}_{n+1}}$$

Notice that given a pair of spaces (X, A), we get a short exact sequence of chain complexes

$$0 \longrightarrow C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j_\#} C_n(X, A) \longrightarrow 0$$

Theorem 4.1. *Given any short exact sequence of chain complexes*

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$$

we get a long exact sequence of homology:

$$\begin{tikzcd} 0 \arrow[r] & H_n(A_\bullet) \arrow[r, "i_*"] & H_n(B_\bullet) \arrow[r, "q_*"] & H_n(C_\bullet) \arrow[dll, "\partial"] \\ & H_{n-1}(A_\bullet) \arrow[r, "i_*"] & H_{n-1}(B_\bullet) \arrow[r, "q_*"] & H_{n-1}(C_\bullet) \arrow[dll, "\cdots"] \\ & H_0(A_\bullet) \arrow[r, "i_*"] & H_0(B_\bullet) \arrow[r, "q_*"] & H_0(C_\bullet) \arrow[r] & 0 \end{tikzcd}$$

where the connecting homomorphism is defined by diagram chasing.

Proof. We construct the connecting homomorphism $\delta : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ as per the following diagram. Remember that

- Columns are exact sequences.
- Rows are complexes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We need to prove this is well defined, that is:

1. Independent of our choice of representative $c_n \in [c_n]$
2. Independent of our choice of b_n such that $j(b_n) = c_n$.

Then we need to prove exactness at each point in the sequence. ☐

This gives us a long exact sequence of relative homology but unfortunately we cannot yet use it to prove anything about $\widetilde{H_n}\left(\frac{X}{A}\right)$ because the sequence only includes terms like $\widetilde{H_n}(X, A)$. We would like to show that for a good pair

$$\widetilde{H}_n\left(\frac{X}{A}\right) \cong \widetilde{H}_n(X, A)$$

First homotopy invariance of homology and a relationship between π_1 and H_1.

5 Homotopy Invariance of Homology

Theorem 5.1. *Homotopy maps $f, g : X \to Y$ induce the same homomorphism of homology, i.e. $f_* = g_*$.*

Proof. This is the proof where we construct the prism operator. It is very long and complicated but the main ideas should be remembered. ☐

Corollary 5.2. *If $h : X \to Y$ is a homotopy equivalence then $h_* : H_n(X) \to H_n(Y)$ is an isomorphism for every $n \in \mathbb{N}$.*

Proof. This is now a simple application of functoriality. ☐

Note: In particular, a contractible space has the same homology as a point.

Suppose p, q are morphisms of chain complexes $A_\bullet \to B_\bullet$.

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, bend right, "q"'] \arrow[d, bend left, "p"] \arrow[dl, dashed, "P"'] & A_n \arrow[r] \arrow[d, bend right, "q"'] \arrow[d, bend left, "p"] \arrow[dl, dashed, "P"'] & A_{n-1} \arrow[r] \arrow[d, bend right, "q"'] \arrow[d, bend left, "p"] \arrow[dl, dashed, "P"'] & \cdots \arrow[dl, dashed, "P"'] \\
\cdots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We say they are chain homotopic if there is a $P : A_n \to B_{n+1}$ for each n such that

$$p - q = \partial P + P\partial$$

6 Relation between π_1 and H_1

For this section assume that X is path connected and $x_0 \in X$ Consider the map which realises a loop in X based at x_0 as a cycle in $Z_1(X)$. We claim that his passes to the quotient to define

$$h : \pi_1(X, x_0) \to Z_1(X)$$

Moreover, we claim

(i) h is a homomorphism.

(ii) h is surjective.

(iii) $\ker h = [\pi_1(X, x_0), \pi_1(X, x_0)]$ is the commutator subgroup.

Hence by the first isomorphism theorem $H_1(X)$ can be seen as the abelianisation of $\pi_1(X, x_0)$

$$H_1(X) \cong \frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]} =: \pi_1(X, x_0)_{ab}$$

Proof. Again this is very long but an understanding of the main argument would be useful. Consult Hatcher for more details. $\square$

7 Homology: Civil War

Theorem 7.1. *The inclusion of $\Delta_\bullet(X) \to C_\bullet(X)$ induces an isomorphism*

$$H_n^\Delta(X) \to H_n(X)$$

when X is a Δ-complex.

This is quite a long road so we start with the following very important technical result. The idea is that if we remove chains deep inside A then we do not effect the relative homology because they have already been quotiented out.

Theorem 7.2 (Excision). *Given spaces $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^o$. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism of relative homology*

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A) \quad \forall n$$

Version 2: *If $A, B \subseteq X$ such that $A^o \cup B^o \subseteq X$ then the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism $H_n(B, A \cap B) \to H_n(X, A)$ for every n.*

We prove venison 2. Notice that $\{A, B\}$ forms a cover of X.
Suppose we have a cover $\mathcal{U} = \{U_\alpha\}_\alpha$ then we can define

$$C_n^{\mathcal{U}}(X) := \{n\text{-chains in } X \text{ where each simplex lies in some } \alpha\}$$

and we clearly get an inclusion $i : C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$.

Proposition 7.3. *As defined above i is a chain homotopy equivalent. That is there is some $S : C_\bullet(X) \to C_\bullet(X)^{\mathcal{U}}$ such that*

- *$i \circ S : C_\bullet(X) \to C_\bullet(X)$ is chain homotopic to the identity.*
- *$S \circ i : C_\bullet^{\mathcal{U}}(X) \to C_\bullet^{\mathcal{U}}(X)$ is chain homotopic to the identity.*

Proof. This was done by barycentric subdivision. By iterated barycentric subdivision, given any singular n-simplex $\sigma : \Delta^n \to X$ we subdivide Δ^n to define a chain $S(\sigma)$ which lies in $C_n^{\mathcal{U}}(X)$. This works because $\partial(\textit{subdivision}) = \textit{subsdivison}(\partial)$. $\square$

Out first application of excision gives us a relationship between relative homology and homology of quotient spaces.

Proposition 7.4. *For a **good pair** (X, A), the quotient map $q : (X, A) \to \left(\frac{X}{A}, \frac{A}{A}\right)$ induces an isomorphism*

$$H_n(X, A) \to H_n\left(\frac{X}{A}, \frac{A}{A}\right) \cong \widetilde{H}_n\left(\frac{X}{A}\right)$$

8 Useful Results

Lemma 8.1. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proposition 8.2. *Given an equivalent relation $\sim$ and a continuous map $f : X \to Y$ that respects the relation, $\frac{f}{\sim} : \frac{X}{\sim} \to Y$ is also continuous.*