Radon-Nikodym - Statement and Proof

1 Statement

Theorem 1.1 (Radon-Nikodym Theorem). *Suppose μ and ν are σ-finite measures on $(X, \mathcal{A})$ such that $\nu << \mu$. Then $\exists$ a measurable function $f : X \to [0, +\infty)$ such that*

$$\forall A \in \mathcal{A}, \quad \nu(A) = \int_A f \, d\mu$$

Moreover, for all such functions h, we have that $h = f$ μ-almost everywhere. We call such function the Radon-Nikodym derivative which we denote $\frac{d\nu}{d\mu}$.

2 Existence

Proof. Suppose $\mu(x), \nu(X) < \infty$ then let

$$\mathcal{F} := \left\{ f : X \to [0, +\infty) \text{ measurable} \,\middle|\, \forall A \in \mathcal{A} \quad \int_A f \, d\mu \leq \nu(A) \right\}$$

Note $\mathcal{F} \neq \emptyset$ because it contains the 0-function.
Claim: Given $f_1, f_2 \in \mathcal{F}$ then $f_1 \vee f_2 \in \mathcal{F}$.
Choose a countable sequence $f_1, f_2, \cdots \in \mathcal{F}$ such that

$$\lim_{n \to \infty} \int f_n \, d\mu = \sup \left\{ \int f \,\middle|\, f \in \mathcal{F} \right\}$$

By replacing f_i with $f_1 \vee \cdots \vee f_i$ we can assume that this sequence is increasing. Let $g := \lim_{n\to\infty} f_n$. Then for all measurable $A \in \mathcal{A}$ we have

$$\int_A g \, d\mu = \lim_{n \to \infty} \int_A f_n \, d\mu \leq \nu(A)$$

by the monotone convergence theorem. Therefore we also have $g \in \mathcal{F}$.
We're going to define a new measure ρ by

$$\rho(A) := \nu(A) - \int_A g \, d\mu \geq 0$$

We aim to show that this is the 0 measure. Suppose for contradiction that $\rho(X) > 0$ then we can find an $\epsilon > 0$ such that $\rho(X) > \epsilon\mu(X)$ because $\mu(X)$ is finite. Now we take a Hahn decomposition $X = P \sqcup N$ of the signed measure $\rho - \epsilon\mu$.

$$\begin{aligned} \forall A \in \mathcal{A} \qquad \nu(A) &= \textstyle\int_A g \, d\mu + \rho(A) && \text{\textit{smaller set}} \\ &\geq \textstyle\int_A g \, d\mu + \rho(A \cap P) && \text{\textit{since } } P \text{ \textit{positive set for} } \rho - \epsilon\mu \\ &\geq \textstyle\int_A g \, d\mu + \epsilon\mu(A \cap P) \\ &\geq \textstyle\int_A (g + \epsilon\chi_P) \, d\mu \end{aligned}$$

This is clearly a problem because we've found an integral larger than the largest integral however $g + \epsilon\chi_P$ is the same as g up to measure 0 if $\mu(P) = 0$. Therefore we split into two cases:
Case 1: If $\mu(P) = 0$ then $\rho(P) = \nu(P) - \int_P g\, d\mu = 0 - 0$ since $\nu << \mu$. Hence, $\rho(X) - \epsilon\mu(X) = \rho(N) - \epsilon\mu(N)$ since both measure vanish on P. But then N is a negative set so $\rho(X) - \epsilon\mu(X) \leq 0$ which contradicts our definition of ϵ.
Case 2: If $\mu(P) \neq 0$ then $g + \epsilon\chi_P \in \mathcal{F}$ but if we integrate over X we get

$$\int (g + \epsilon\chi_P)\, d\mu > \int g\, d\mu$$

which is a contradiction because g was chosen to maximise this integral subject to $\int_A g\, d\mu \leq \nu(A)$. This proves that $\rho(X) = 0$ and so $\rho \equiv 0$, i.e.

$$\forall A \in \mathcal{A} \quad \nu(A) = \int_A g\, d\mu$$

Problem: What if $g(x) = \infty$ for some $x \in X$?
Well we certainly have $\int g\, d\mu \leq \nu(X) < +\infty$ and hence $\mu(\{g = +\infty\}) = 0$. We can just redefine g to be 0 on the set where it was previously infinity and this won't change any integrals. $\square$

2.1 General Case

Previously we assumed that μ and ν were finite, but actually we can do the proof when they are only σ-finite.

Proof. Decompose the space $X = \sqcup_n B_n$ where the B_n are some measurable sets such that $\mu(B_n), \nu(B_n) < \infty$. Then for each n we can define new finite measures by

$$\mu_n(A) = \mu(A \cap B_n) \quad \text{and} \quad \nu_n(A) = \nu(A \cap B_n)$$

Note that we still maintain absolute continuity $\nu_n << \mu_n$ for each n. We can now apply the first existence proof to each μ_n and ν_n to find Radon-Nikodym derivatives $g_n : X \to [0, +\infty)$. Moreover, we can choose these such that $g_n \equiv 0$ on B_n^{c} because this won't change anything with respect to μ_n and ν_n.
We now define our global derivative as

$$g = \sum_n g_n$$

which is measurable and non-negative because it is the sum of measurable and non-negative functions. Moreover, g is finite because the g_n are finite and only assume non-zero values on their respective B_n. Finally given any $A \in \mathcal{A}$ we have

$$\int_A g\, d\mu \underbrace{=}_{\text{Levi}} \sum_n \int g_n\, d\mu = \sum_n \nu_n(A) = \sum_n \nu(A \cap B_n) = \nu(A)$$

which is what we wanted. Woo hoo! 😎 $\square$

3 Uniqueness

Proof. Suppose $h, g : X \to [0, \infty)$ satisfy $\forall A \in \mathcal{A} \nu(A) = \int_A g \, d\mu = \int_A h \, d\mu$. Suppose that $\nu(X) < +\infty$. Then $g - h$ is integrable with respect to μ and $\int_{\{g>f\}} g - h \, d\mu = 0 = \int_X (g-h)^+ \, d\mu$. This implies that $(g-h)^+ = 0$ μ a.e. We have a similar argument for the negative part. Thus $g = h$ a.e.

In the σ- finite case, write $X = \sqcup_n B_n$ with $\nu(B_n) < \infty$. So $g = h$ on B_n $\mu-$ a.e. Thus $g = h$ $\mu-$ a.e. $\square$