

Radon-Nikodym - Statement and Proof

1 Statement

Theorem 1.1 (Radon-Nikodym Theorem). Suppose μ and ν are σ -finite measures on (X, \mathcal{A}) such that $\nu \ll \mu$. Then \exists a measurable function $f : X \rightarrow [0, +\infty)$ such that

$$\forall A \in \mathcal{A}, \quad \nu(A) = \int_A f d\mu$$

Moreover, for all such functions h , we have that $h = f$ μ -almost everywhere. We call such function the *Radon-Nikodym derivative* which we denote $\frac{d\nu}{d\mu}$.

2 Existence

Proof. Suppose $\mu(X), \nu(X) < \infty$ then let

$$\mathcal{F} := \left\{ f : X \rightarrow [0, +\infty) \text{ measurable} \mid \forall A \in \mathcal{A} \quad \int_A f d\mu \leq \nu(A) \right\}$$

Note $\mathcal{F} \neq \emptyset$ because it contains the 0-function.

Claim: Given $f_1, f_2 \in \mathcal{F}$ then $f_1 \vee f_2 \in \mathcal{F}$.

Choose a countable sequence $f_1, f_2, \dots \in \mathcal{F}$ such that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup \left\{ \int f d\mu \mid f \in \mathcal{F} \right\}$$

By replacing f_i with $f_1 \vee \dots \vee f_i$ we can assume that this sequence is increasing. Let $g := \lim_{n \rightarrow \infty} f_n$. Then for all measurable $A \in \mathcal{A}$ we have

$$\int_A g d\mu = \lim_{n \rightarrow \infty} \int_A f_n d\mu \leq \nu(A)$$

by the monotone convergence theorem. Therefore we also have $g \in \mathcal{F}$.

We're going to define a new measure ρ by

$$\rho(A) := \nu(A) - \int_A g d\mu \geq 0$$

We aim to show that this is the 0 measure. Suppose for contradiction that $\rho(X) > 0$ then we can find an $\epsilon > 0$ such that $\rho(X) > \epsilon\mu(X)$ because $\mu(X)$ is finite. Now we take a Hahn decomposition $X = P \sqcup N$ of the signed measure $\rho - \epsilon\mu$.

$$\begin{aligned} \forall A \in \mathcal{A} \quad \nu(A) &= \int_A g d\mu + \rho(A) \\ &\geq \int_A g d\mu + \rho(A \cap P) && \left. \begin{array}{l} \downarrow \text{smaller set} \\ \downarrow \text{since } P \text{ positive set for } \rho - \epsilon\mu \end{array} \right\} \\ &\geq \int_A g d\mu + \epsilon\mu(A \cap P) \\ &\geq \int_A (g + \epsilon\chi_P) d\mu \end{aligned}$$

This is clearly a problem because we've found an integral larger than the largest integral however $g + \epsilon\chi_P$ is the same as g up to measure 0 if $\mu(P) = 0$. Therefore we split into two cases:

Case 1: If $\mu(P) = 0$ then $\rho(P) = \nu(P) - \int_P g d\mu = 0 - 0$ since $\nu \ll \mu$. Hence, $\rho(X) - \epsilon\mu(X) = \rho(N) - \epsilon\mu(N)$ since both measure vanish on P . But then N is a negative set so $\rho(X) - \epsilon\mu(X) \leq 0$ which contradicts our definition of ϵ .

Case 2: If $\mu(P) \neq 0$ then $g + \epsilon\chi_P \in \mathcal{F}$ but if we integrate over X we get

$$\int (g + \epsilon\chi_P) d\mu > \int g d\mu$$

which is a contradiction because g was chosen to maximise this integral subject to $\int_A g d\mu \leq \nu(A)$. This proves that $\rho(X) = 0$ and so $\rho \equiv 0$, i.e.

$$\forall A \in \mathcal{A} \quad \nu(A) = \int_A g d\mu$$

Problem: What if $\mu(X) = \infty$?

Well we certainly have $\int g d\mu \leq \nu(X) < +\infty$ and hence $\mu(\{g = +\infty\}) = 0$. We can just redefine g to be 0 on the set where it was previously infinity and this won't change any integrals. Not sure why this fixes the problem???

□

3 Uniqueness

Proof.

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