Manifolds Notes on things I don't understand/remember

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1 Orientations

Let V be a vector space of dimension $m > 0$. Let $I(V)$ be the set of linear isomorphism $V \to \mathbb{R}^m$. Given $\rho, \sigma \in I(V)$ we get a linear automorphism

$$(\sigma \circ \rho^{-1}) : \mathbb{R}^m \to \mathbb{R}^m$$

Note that that that the determinant of $(\sigma \circ \rho^{-1})$ is non-zero. We can define an equivalence relation $I(V)$ by

$$\sigma \sim \rho \iff \det(\sigma \circ \rho^{-1}) > 0$$

We write $Or(V) := I(V)/\sim$. Thus $|Or(V)| = 2$. In the special case $\dim(V) = 0$ we set $Or(V) := \{-1, +1\}$. An orientation on V is an element of $Or(V)$. When equipped with an orientation, a vector space obtains the adjective oriented.

If we let V be an oriented vector space of positive dimension. A basis $v_1, \ldots, v_m$ of V determines an element of $I(V)$ by sending the basis to a standard basis of $\mathbb{R}^m$. We then say $\{v_i\}$ is positively oriented if this map lies in the class of the orientation.

If V and W are oriented vector spaces of the same dimension, then we say a linear isomorphism $L : V \to W$ is orientation preserving if it sends some positively oriented basis of V to some positively oriented basis for W. Otherwise we say L is orientation reversing.

$\mathbb{R}^m$ is said to have the standard orientation if its orientation class contains the identity. If L is an automorphism of $\mathbb{R}^m$ with the standard orientation then L is orientation preserving if and only if $\det(L) > 0$.

Note: If V and W are vector spaces and $E = V \oplus W$ then orientations on V and W determine an orientation on E. Likewise, orientation on any two of $\{E, V, W\}$ determine an orientation on the remaining space.

Let M be an m-manifold where $m > 0$. An orientation on M is an assignment of an orientation to each tangent space T_xM such that there is an atlas of charts $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ such that

$$\forall \alpha \quad \forall x \in U_\alpha \quad d_x\phi_\alpha : T_xM \to \mathbb{R}^m \text{ is orientation preserving.}$$

When equipped with such an assignment, M is called an oriented manifold. We say M is orientable if it admits an orientation.

Note: We can obtain an 'opposite' orientation on M by post-composing every chart with a transposition.

Not every manifold is orientable however every manifold is locally orientable. It is also easily seen that orientability is invariant under diffeomorphism.

Example:

1. $S^n \subseteq \mathbb{R}^{n+1}$ is orientable. Take the atlas given by stereographic projection. Then the transition function reverses orientation so we just post-compose one of the charts with any orientation-reversing diffeomorphism of $\mathbb{R}^n$.

2. The directed product of any two orientable manifolds is orientable. For example, the Torus or cylinder.

3. The Möbius band is not orientable. In particular, it is not diffeomorphic to the cylinder.

Lemma 1.1. *A connected orientable manifold has exactly two orientations.*

1.1 Criterion for Orientability

Suppose that $M \subseteq \mathbb{R}^n$ is a smooth manifold. Given $x \in M$,

$$\mathbb{R}^n \cong T_x\mathbb{R}^n = T_xM \times T_xM^\perp$$

Since $\mathbb{R}^n$ comes with the standard orientation, an orientation on T_xM determines one on $T_xM^\perp$ (and vice versa).

Consider the case where $n = m + 1$. Then $T_xM^\perp \cong \mathbb{R}$. So an orientation on T_xM, which gives an orientation on $T_xM^\perp$, gives us a unique $\kappa(x) \in T_xM^\perp$ with

1. $||\kappa(x)|| = 1$

2. $\{\kappa(x)\}$ forms a positively oriented basis for $T_xM^\perp$.

We can think of $\kappa(x)$ as the 'outward' unit normal vector. If M is oriented then $\kappa(x)$ is defined everywhere. Also $[x \mapsto \kappa(x)]$ is smooth and so $\kappa(x)$ is a normal field.

Conversely, if κ is a global nowhere-vanishing normal field on M, then κ gives rise to an orientation on T_xM for all $x \in M$. From this we can construct an orientable atlas through great pain.

Theorem 1.2. *If $M \subseteq \mathbb{R}^{m+1}$ is an m-manifold, then the following are equivalent:*

(a) M is orientable.

(b) M admits a nowhere-vanishing normal field.

(c) M admits a unit normal field.

Suppose that M and N or oriented manifolds and $f : M \to N$ is a diffeomorphism. We say that f preserves orientation if $d_xf : T_xM \to T_{fx}N$ respects the given orientation for all $x \in M$. It reverse orientation if d_xf reverses orientation for all $x \in M$.

2 Abstract Manifolds

An open cover of a space X is a collection $\mathcal{U}$ of open subsets of X that union to the whole space.
A refinement of $\mathcal{U}$ is an open cover $\mathcal{V}$ of X such that $\forall V \in \mathcal{V}$ there is a $U\mathcal{U}$ such that $V \subseteq U$.
We say that a cover $\{U_\alpha\}$ of a Hausdorff space X is locally finite if

$$\forall x \in X \quad \exists O \subseteq X \text{ open, s.t. } x \in O \text{ and } |\{\alpha \mid O \cap U_\alpha \neq \emptyset\}| < \infty$$

that is around every point there is an open set which meets at most finitely many members of the cover.
We say X is paracompact if every open cover has a locally finite refinement.

Lemma 2.1. *Any (abstract) manifold is paracompact.*

3 Vector Bundles

In defining a vector bundle over a manifold we associate a vector space to each point in the manifold and then arrange these together in a smooth way. Examples of this will include the tangent and normal bundles as defined in times gone past.

Suppose that M is an m-manifold. A family of vector spaces over M is a manifold E together with a smooth map $p : E \to M$ such that for all $x \in M$, the fibre $E_x := p^{-1}(x)$ has a vector space structure. Given $U \subseteq M$ we write $E\big|_U := p^{-1}U$ which we equip with the restriction of p.

Given another family over vector spaces $p' : E' \to M$ we define an isomorphism to be a diffeomorphism

$$\psi : E \to E'$$

with the property that $p' \circ \psi = p$ and $\psi : E_x \to E'_x$ restricts to a linear isomorphism for all $x \in M$. If such a map exists then E and E' are called isomorphic. Finally, a family E over M is called trivial if it is isomorphic to $M \times \mathbb{R}^q$ for some $q \in \mathbb{N}$.

$$\begin{tikzcd}
E' \arrow[rr, "\psi"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\
& M &
\end{tikzcd}$$

A vector bundle over M is a family of vector spaces $p : E \to M$ such that $\forall x \in M$ there is some open $U \subseteq M$ with $x \in U$ such that $E\big|_U$ is trivial.

Putting this another way, E is a vector bundle over M if we can find an atlas $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ for M such that for each α we get a diffeomorphism

$$\psi_\alpha : E\big|_{U_\alpha} \to V_\alpha \times \mathbb{R}^q \subseteq \mathbb{R}^{m+q}$$

which restricts to a linear isomorphism on each fibre. Note than then the maps $\{\psi_\alpha\}$ form an atlas for E which is called a locally trivialising atlas for E.

Note:

- Given a vector bundle, $p : E \to M$ is a surjective smooth submersion.

- Each fibre E_x is an embedded sub-manifold and the vector space operations are smooth.
- The notion of a vector bundle is preserved under isomorphism.

Example:

1. If M is a smooth manifold then the tangent bundle is a vector bundle over M. For a manifold in Euclidean space, this follows for the proof that TM was a manifold. For an abstract space, we can form a locally trivialising atlas as follows:

   Given a chart $\phi : U \to V \subseteq \mathbb{R}^m$ for M we construct a new map

   $$\psi : TU \to V \times \mathbb{R}^m \quad \text{by} \quad \psi(v) = (\phi(x), d_x\phi(v))$$

   where $x = pv$. These maps then form a locally trivialising atlas because when restricted to some x the map becomes the derivative map

   $$\psi_{p^{-1}(x)} : T_xM \to \{\phi(x)\} \times \mathbb{R}^m \quad \psi(v) = (\phi(x), d_x\phi(v))$$

   which is a linear isomorphism.

2. One can (and perhaps should) check that the atlas given for the normal bundle consists of maps which restrict to linear isomorphisms on each fibre.

3. The spaces B_n together with the projection to the circle yields a vector bundles. In particular, the Möbius band is a bundle over the circle. Note B_m is isomorphic to B_n if $|m-n|$ is even. But, B_0 cannot be isomorphic to B_1 since they are not even diffeomorphic. Hence the Möbius band is non-trivial.

Given a vector bundle $p : E \to M$ we say that section of E is a smooth map $s : E \to M$ such that $p \circ s$ is the identity on M. This generalises our previous notion of a section of the tangent space which is often referred to as a vector field on M. We say that a section is non-vanishing if $s(x) \neq 0$ for all $x \in M$. Given any smooth map $f : M \to \mathbb{R}^q$ we get a section of $M \times \mathbb{R}^q$.

Note: Any section is an embedding of M into E.

Theorem 3.1 (Hairy Ball Theorem)**.** *Every vector field on S^2 has at least one zero.*

Note: It turns out that any odd-dimensional manifold admits a non-vanishing vector field. An even-dimensional manifold may or may not. The torus does but it is the only compact connected orientable 2-manifold which does.

Lemma 3.2. *A vector bundle E is trivial if and only if there are sections $s_1, s_2, \ldots, s_q$ such that $\{s_1, \ldots, s_q(x)\}$ forms a basis for E_x for all $x \in M$.*

Proof. If $f : M \times \mathbb{R}^q$ is an isomorphism then set $s_i(x) = f(x, e_i)$ for some fixed basis $e_1, \ldots, e_q$ for $\mathbb{R}^q$.

Conversely, if we have such sections s_i then we can define a map $f : M \times \mathbb{R}^q \to E$ by

$$f(x, (\lambda_1, \ldots, \lambda_q)) = \sum_{i=1}^{q} \lambda_i s_i(x)$$

which yields an isomorphism. □

Example:

1. Using the Intermediate Value Theorem, one can show that the Möbius has no non-vanishing section and is hence a non-trivial bundle by the above Lemma.
2. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere in $\mathbb{R}^n$ then the normal bundle is trivial because the outward pointing unit normal is a nowhere vanishing section.

A manifold M is called parallelisable if the tangent bundle TM is trivial.

In view of the last lemma we can say that a manifold is parallelisable if and only if M admits a global frame field. Recall, a frame field is a family of vector fields v_i such that the $v_i(x)$ form a basis for T_xM at every $x \in M$. We know that frame fields exists locally but not necessarily globally.

Lemma 3.3. *Any parallelisable manifold is orientable.*

Proof. Exercise. □

Example:

- The circle S^1 is parallelisable.
- S^2 is not by the Hairy Ball Theorem.
- S^3 is parallelisable because given $x = (x_1, x_2, x_3, x_4) \in S^3$ we can form a frame field with

$$\begin{array}{rrrr} (-x_2, & x_1, & x_4, & -x_3) \\ (-x_3, & -x_4, & x_1, & x_2) \\ (-x_4, & x_3, & -x_2, & x_1) \end{array}$$

- S^5 is not strangely enough but S^7 is. It may be worth consulting your written notes from lectures to figure out how this frame was calculated.

3.1 Vector Fields

Let X be a vector field on M (that is a section of TM). Suppose that $f \in C^\infty(M)$. We can define a function $Xf : M \to \mathbb{R}$ as

$$(Xf)(x) := X \cdot [f]$$

where $[f] \in \mathcal{G}_x(M)$.

Claim: Xf as defined above is a smooth function

Let x_i be local coordinates of a neighbourhood $U \subseteq M$ of $x \in M$. Then we write $X = \sum_i \lambda_i \frac{\partial}{\partial x_i}$ where $\lambda_i : U \to \mathbb{R}$ are smooth functions. So we have $Xf = \sum_i \lambda_i \frac{\partial f}{\partial x_i}$ which is smooth.

One can also see that $[f \mapsto Xf]$ is a linear map $C^\infty(M) \to C^\infty(M)$. Directly from our definition of T_xM we have that

$$X(fg) = fXg + gXf$$

for all $f, g \in C^\infty(M)$.

Proposition 3.4. *Suppose $L : C^\infty(M) \to C^\infty(M)$ is a linear map satisfying*

$$L(fg) = fL(g) + gL(f)$$

then there is a unique vector field X on M such that $L(f) = Xf$ for all $f \in C^\infty(M)$.

Proof. Probably worth going through this proof. □

Suppose X, Y are vector fields on M. One can check that

$$[f \mapsto X(Yf) - Y(Xf)]$$

satisfies the conditions of Proposition 3.4. It follows that there is a unique vector field, denoted $[X, Y]$, on M such that

$$[X, Y]f = X(Yf) - Y(Xf)$$

for all $f \in C^\infty(M)$. This vector field is called the Lie bracket. Manifolds is full of lies. □

3.2 Whitney Sums

Suppose we have two vector bundles $p : E \to M$ and $p' : E' \to M$. We'd like to sum these vector bundles. Each fibre is a vector space so we can take their sums and stitch the fibres back together

$$\hat{E} := \bigsqcup_{x \in M} (E_x \oplus E'_x) \qquad \hat{p} : \hat{E} \to M$$

where $\hat{p}$ is the obvious map. We need to give this a manifold structure.

Let $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ be an atlas for M which gives rise to locally trivialising atlases

$$\left\{\psi_\alpha : p^{-1}U_\alpha \to V_\alpha \times \mathbb{R}^q\right\}_\alpha \quad \text{and} \quad \left\{\psi'_\alpha : p'^{-1}U_\alpha \to V_\alpha \times \mathbb{R}^r\right\}_\alpha$$

for E and E' respectively. The first coordinate of these charts is always $\phi_\alpha(x)$. If we want to stitch two charts ψ_α and ψ'_α together, we only need one copy of this first coordinate.

Given $v \in E_x \oplus E'_x$ write $v = u + u'$ for $u \in E_x$ and $u' \in E'_x$. Let $w \in \mathbb{R}^q$ and $w' \in \mathbb{R}^r$ be the second coordinates of $\psi_\alpha(u)$ and $\psi'_\alpha(u')$ respectively. Note we can be sure this is the same α because it only depends on x. We then form a chart for the Whitney sum as follows

$$\theta_\alpha : (\hat{p})^{-1}U_\alpha \to V_\alpha \times \mathbb{R}^q \times \mathbb{R}^r \qquad \theta_\alpha(v) = (\phi_\alpha(x), w, w')$$

Topology:

We deem a set $O \subseteq \hat{E}$ to be open if $\theta_\alpha(O)$ is open in $V_\alpha \times \mathbb{R}^{q+r}$ for every chart α.

Note: One can check this does indeed give us a topology for $\hat{E}$, endows it with the structure of a vector bundle, and the resulting structure is independent of all the atlases we chose.

The resulting vector bundle $\hat{E}$ is called the Whitney sum of the vector bundles E and E'. We often denote this $E \oplus E'$.

3.2.1 Alternative approach

Up to isomorphism, we can construct the Whitney sum in an alternative fashion. This time we start with the full direct product $E \times E'$ and consider the submanifold

$$F := \{(v, w) \in E \times E' \mid pv = p'w\}$$

There is a map $\widetilde{p} : F \to M$ where $(v, w) \mapsto pv = p'w$. Under this map $\widetilde{p}^{-1}(x) = E_x \times E'_x$ Since this is just a direct product of vector spaces, there is a natural identification $E_x \oplus E'_x \to E_x \times E'_x$. Combining these identifications across the Whitney sum we get a bijection $E \oplus E' \to F \subseteq E \times E'$. This can be shown to be an embedding of manifolds and that these constructions are isomorphic as vector bundles.

Note: While $E_x \times E'_x$ and $E_x \oplus E'_x$ are essentially the same, $E \times E'$ and $E \oplus E'$ are vastly different!

It might be worth be looking at how we stitch together dual spaces.

3.3 Cotangent Bundle

We will often be interested in the dual to the tangent bundle $(TM)^*$ which is more usually denoted T^*M. Its fibres are of the form $(T_xM)^*$ but more commonly written T_x^*M. An element of T_x^*M is called a covector at x. A section of T^*M is called a covector field, but more frequently referred to as a 1-form.

Suppose that we have a smooth function $f \in C^\infty(M)$. Then f determines an element of T_x^*M given by the linear map $[v \mapsto v \cdot f]$. This element is denoted $df(x) \in T_x^*M$. So the map $[f \mapsto df(x)] : M \to T^*M$ is a section of T^*M, i.e. a 1-form. Note that we get the product rule

$$d(fg) = f(dg) + g(df)$$

To better understand T_x^*M we find a basis. Take a chart $\phi : U \to V$ on the underlying manifold such that $x \in U$. We get 1-forms $dx_1, \ldots, dx_m$ defined by

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

These dx_i form a basis for T_x^*M. This is, in fact, the dual basis to $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ for T_xM. Note that, if f is a smooth function then we have

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

Suppose $\{\phi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ is a chart for M. Then for each α we get 1-forms $dx_1^\alpha, \ldots, dx_m^\alpha$. On the overlap $U_\alpha \cap U_\beta$, these 1-forms transform according to the rule

$$dx_i^\alpha = \sum_{j=1}^m \frac{\partial x_i^\alpha}{\partial x_j^\beta} dx_j^\beta$$

Note that $\left(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\right)_{i,j}$ is the Jacobian of the transition function $\phi_\alpha \circ \phi_\beta^{-1}$.

Integration:

1-forms can be used to integrate along curves. Suppose that $\gamma : [a, b] \to M$ is a smooth curve and ω is a 1-form on M. Then we get a smooth function $[t \mapsto \omega(\gamma(t))(\gamma'(t))] : [a, b] \to \mathbb{R}$. Then we write

$$\int_\gamma \omega := \int_a^b \omega(\gamma(t))(\gamma'(t))\, dt$$

3.4 Pull-backs

Suppose we have a smooth function of manifolds $f : M \to N$. Given a 1-form ω on N we can define a 1-form η on M as follows: Given $x \in M$, write $\eta(x)(v) := \omega(f(x))(f_*(v))$ where $f_*(v) := d_x f(v)$. Thus $\eta(x) \in T_x^* M$.

This gives us a map $\eta : M \to T^*M$ which is smooth and hence a section to T^*M. In other words, η is a 1-form on T^*M which is called the pull-back of ω.

4 Smooth Function Extension

Lemma 4.1. *There is a smooth function $\theta_0 : \mathbb{R}^n \to [0, 1]$ such that*

1. *$\theta_0(x) = 1$ whenever $||x|| \leq 1$.*
2. *$\theta_0(x) = 0$ whenever $||x|| \geq 2$.*

Proof. This is done in a series of steps:

1. Define $\theta_1 : \mathbb{R} \to \mathbb{R}$ by
$$\theta_1(t) := \begin{cases} e^{-1/t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$
which is smooth.
2. Set $\theta_2(t) := \frac{\theta_1(t)}{\theta_1(t)+\theta_1(1-t)}$. So $\theta_2\big|_{(1,\infty)} \equiv 1$ and $\theta_2\big|_{(-\infty,0]} \equiv 0$.
3. Define $\theta_0(x) := \theta_2(2 - ||x||)$.

□

Lemma 4.2. *Let M be an m-manifold and $W \subseteq M$ open such that $x \in W$. Then there is a smooth function $\theta : M \to [0, 1]$ such that $\theta\big|_{M\setminus W} \equiv 0$ and $\theta\big|_U \equiv 1$ for some open neighbourhood U of x.*

Proof. Let $\phi : U \to V$ with $x \in U$. After pre-composing with a translation we can assume $0 \in \phi(W \cap U)$. After post-composing with a dilation we can assume $\mathbb{B}_2(0) \subseteq \phi(W \cap U)$. Now set

$$\theta(x) := \begin{cases} \theta_0(\phi(x)) & \text{for } x \in U \\ 0 & \text{otherwise} \end{cases}$$

□

Corollary 4.3. *Suppose $f : W \to \mathbb{R}$ is a smooth function on some open $W \subseteq M$ with $x \in W$. Then there is a smooth function $g : M \to \mathbb{R}$ which agree with f on some neighbourhood of x in W.*

Proof.

$$g(x) := \begin{cases} f(x) \cdot \theta(x) & \text{for } x \in W \\ 0 & \text{otherwise} \end{cases}$$

□

5 Differential Forms

Given vector spaces E, F and $p \in \mathbb{N}$ we denote

$$A(E^p, F) := \{p-\text{linear alternating maps } E^p \to F\}$$

where by alternating we mean that swapping any two coordinates negates the output. Equivalently, if two coordinates are the same then the output is 0.

Lemma 5.1. *Given E and p there is a vector space V together with a surjective map $\mu \in A(E^p, V)$ with the property that if $\theta \in A(E^p, F)$ then there is a linear map $\hat{\theta} : V \to F$ such that $\theta = \hat{\theta} \circ \mu$*

Proof. Go through this! □

$$\begin{tikzcd}
 & V \arrow[dr, dashed, "\hat{\theta}"] & \\
E^p \arrow[ur, "\mu"] \arrow[rr, "\theta"'] & & F
\end{tikzcd}$$

Note: The $\hat{\theta}$ is unique given θ and V. The V is unique up to isomorphism.

We write $V = \Lambda^p E$ and given $v_1, \ldots, v_p \in E$ we write

$$v_1 \wedge \cdots \wedge v_p := \mu(v_1, \ldots, v_p)$$

We say $\Lambda^p E$ is the p-th exterior power of E.

5.1 Basis for $\Lambda^p E$

Let $e_1, \ldots, e_m$ be a basis for E. Since μ is surjective $\Lambda^p E$ is spanned by

$$\left\{ e_{i_1} \wedge \cdots \wedge e_{i_p} \;\middle|\; i_k \in I(m) \right\}$$

where we can assume that the i_k are distinct else their image would be null. We can also assume that the indices are in order up to sign.

Lemma 5.2. *These elements are linearly independent and hence form a basis.*

Therefore we can say $\dim(\Lambda^p E) = \binom{m}{p}$.

5.2 Wedge Product

Given $p, q \in \mathbb{N}$ with $p, q \geq 1$ we can define the bilinear wedge product

$$\cdot \wedge \cdot : (\Lambda^p E \times \Lambda^q E) \to \Lambda^{p+q} E$$

First we define on it on a basis. So take a basis $e_1, \ldots, e_m$ of E and then define

$$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_q}) = e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$$

This can then be extended linearly to arbitrary elements and hence doesn't depend on our initial choice of basis.

5.3 Induced maps

Suppose we have a linear map between finite dimensional vector spaces

$$\phi : E \to F$$

then we get a multi linear map in the natural way

$$\phi^p : E^p \to F^p$$

By composing with the surjective map μ_F we get an alternating map

$$\mu_F \circ \phi^p : E^p \to \Lambda^p F$$

Hence by the defining property of $\Lambda^P E$ we get a linear map

$$\Lambda^p \phi : \Lambda^p E \to \Lambda^p F$$

with the property that the outer diamond in the below diagram commutes.

$$\begin{tikzcd}
& \Lambda^p E \arrow[dr, dashed, "\Lambda^p\phi"] & \\
E^p \arrow[ur, "\mu_E"] \arrow[rr, "\mu_f \circ \phi^p"'] \arrow[dr, "\phi^p"'] & & \Lambda^p F \\
& F^p \arrow[ur, "\mu_F"'] &
\end{tikzcd}$$

Essentially, if $e_1, \ldots, e_m$ is a basis for E, then we can describe $\Lambda^p \phi$ by

$$(\Lambda^p \phi)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (\phi e_{i_1}) \wedge \cdots \wedge (\phi e_{i_p}).$$

5.4 ☠ The dreaded p-form ☠

Let M be an m-manifold. Given $x \in M$ we can form the p-th exterior power of the cotangent space

$$\Lambda^p(T_x^* M)$$

We can assemble these together into a vector bundle $\Lambda^p(T^*M)$. Subsequently, a p-form on M is define to be a section of the bundle $\Lambda^p(T^*M)$

What the fuck does this mean???

A more natural way to think about p-forms is to take local coordinates. Let $\phi : U \to \mathbb{R}^m$ be a chart yielding local coordinates $x_1, \ldots, x_m$. We have locally defined 1-forms $dx_1, \ldots, dx_m$ which form a basis for the cotangent space

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}$$

Then given $I \in \mathcal{I}(m,p)$ we write $\mathbf{d}x_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Thus $\{\mathbf{d}x_I \mid I \in \mathcal{I}(m,p)\}$ forms a basis for $\Lambda^P(T^*M)$. It follows that any p-form ω on U can be uniquely written in the form

$$\omega = \sum_{I \in \mathcal{I}(m,p)} \lambda_I \mathbf{d}x_I$$

where each $\lambda_I : U \to \mathbb{R}$ is a locally-defined smooth function.

Note: This is all we really need from the bundle structure of $\Lambda^p(T^*M)$.

In particular, if $p = m$ then an m-form locally looks like

$$\lambda \left(dx_1 \wedge \cdots \wedge dx_m \right)$$

for some smooth function $\lambda : U \to \mathbb{R}$.

5.5 Pull-backs

Suppose we have a smooth function between manifolds

$$f : M \to N$$

Given a p-form ω on N we can define a pull-back p-form $f^*\omega$ on M as follows. Given $x \in M$ we have the derivative map $d_x f$ and hence a dual map

$$(d_x f)^* : T^*_{f(x)} N \to T^*_x M, \qquad \eta \mapsto \eta \circ d_x f \quad \text{where } \eta : T_{f(x)} N \to \mathbb{R} \text{ is linear}$$

This in turn gives rise to a linear map

$$\Lambda^p (d_x \phi)^* : \Lambda^p T^*_{fx} N \to \Lambda^p T^*_x M$$

Then our pull-back is defined by

$$(f^*\omega)(x) := \left(\Lambda^p (d_x \phi)^* \right) \left[\omega(f(x)) \right]$$

One takes on blind faith that this is smooth and hence a p-form. In particular, we can pull back p-forms to any manifold embedded within a larger manifold (such as $\mathbb{R}^n$). This is a load of gobbledygook so let's go step by step.

1. $x \in M$
2. $f(x) \in N$
3. ω is a p-form on N so we get some linear maps $\eta_i : T_{fx} N \to \mathbb{R}$, then
$$\omega(f(x)) = \eta_1 \wedge \cdots \wedge \eta_p$$
4. Then we take the induced p'th exterior power map which just does $(d_x \phi)^*$ on each of the η_i
5. Hence we can write
$$(f^*\omega)(x) = (\eta_1 \circ d_x f) \wedge \cdots \wedge (\eta_p \circ d_x f)$$

Example:

1. Consider S^1; the unit circle in $\mathbb{R}^2$. Let θ by the angle coordinate so that $x = \cos\theta$ and $y = \sin\theta$. Then the pull back of dx, dy is obtained by differentiating these formulae:
$$-\sin\theta d\theta \quad \text{and} \quad cos\theta d\theta$$
From this we can pull back an arbitrary 1-form by linear extension.
2. Consider S^2; the unit 2-sphere in $\mathbb{R}^3$. Consider spherical polar coordinates θ, ϕ away from

the poles.

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\theta$$

Then the pull backs of dx, dy and dz repressively are

$$\cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi$$
$$\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi$$
$$-\sin\theta d\theta$$

One can see this by writing out the Jacobian and then composing on the left with dx which is $(0, 0, 1)$ and remembering that $d\theta = \binom{1}{0}$ and $d\phi = \binom{0}{1}$.

So then the pull back of $dx \wedge dy = (\cos\theta\sin\theta)(d\theta \wedge d\phi)$. We can see this by writing out the full expression, using multi linearity and alternating-ness of the wedge product and then trigonometric identities.

5.6 Integration of m-forms

Take an atlas $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$. Given $\alpha, \beta \in \mathcal{A}$, then on the overlap $U_\alpha \cap U_\beta$ we get

$$dx_1{}^\alpha \wedge \cdots \wedge dx_m{}^\alpha = \Delta_{\alpha\beta}(x) dx_1{}^\beta \wedge \cdots \wedge dx_m{}^\beta$$

where $\Delta_{\alpha\beta}$ is the determinant of the Jacobian of the transition function $\phi_\alpha \circ \phi_\beta^{-1}$. Note if our atlas is oriented then $\Delta_{\alpha\beta}(x) > 0$. Hence we have the following result.

Theorem 5.3. *An m-manifold is orientable if and only if it admits a nowhere vanishing m-form.*

Proof. Worth looking over. □

Let M be an oriented manifold. Given an m-form ω on M we define

$$\text{supp}(\omega) := \overline{\{x \in M \mid \omega(x) \neq 0\}}$$

We say that a cover $\{U_\alpha\}$ of a Hausdorff space X is locally finite if

$$\forall x \in X \quad \exists O \subseteq X \text{ open, s.t. } x \in O \text{ and } |\{\alpha \mid O \cap U_\alpha \neq \emptyset\}| < \infty$$

that is around every point there is an open set which meets at most finitely many members of the cover.

For now, suppose that $\text{supp}(\omega)$ is compact. Let $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ be a locally finite, oriented atlas. Suppose that η is an n-form such that $\text{supp}(\eta) \subseteq U_\alpha$ for some $\alpha \in \mathcal{A}$. Then write in local coordinates $\eta = \lambda_\alpha(dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha)$ where $\lambda_\alpha : U_\alpha \to \mathbb{R}$ is smooth and compactly supported. Then we set

$$I_\alpha(\eta) := \int_{V_\alpha} \lambda_\alpha \circ \phi_\alpha^{-1}(x)\, dx$$

A partition of unity subordinate to $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}$ such that

1. $\mathrm{supp}(\rho_\alpha) \subseteq U_\alpha$ for all α,
2. $\sum_\alpha \rho_\alpha(x) = 1$ for all $x \in M$.

Theorem 5.4. *Any locally finite open cover of M has a subordinate partition of unity.*

So choose a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and set

$$\int_M \omega := \sum_{\alpha \in \mathcal{A}} I_\alpha(\rho_\alpha \omega)$$

Note: This is a finite sum because only finitely many U_α meet the support of ω.

Lemma 5.5. *This integral is well-defined. That is, its independent of choice of atlas and partition.*

6 Riemannian Manifolds

Using the integral defined in the previous section we can define the volume of a compact, orientable Riemannian manifold. Choose any orientation and let ω be the volume form (that is any m-form, I think). Then the volume is

$$\mathrm{vol}(M) := \int_M \omega$$

In fact, if $f : M \to \mathbb{R}$ is any smooth function then we can integrate f with respect to volume. That is, integrate the m-form $f\omega$. The result $\int_M f\omega$ is often denoted informally as $\int_M f\, dV$. We shouldn't use this notation because exterior derivatives will confuse things.

A Riemannian metric on M is a smooth map $f : TM \oplus TM \to \mathbb{R}$ such that

$$\forall x \in M \quad f\big|_{T_xM \oplus T_xM} \text{ is an inner product on } T_xM$$

Note that f is not a metric in the usual sense. Given $v, w \in T_xM$ we usually denote $f(v,w) =: \langle v, w \rangle$. If $v \in T_xM$ we define $||v|| := \sqrt{\langle v, v \rangle}$. In particular, a Riemannian metric gives us a way to measure norms of tangent vectors in a nice smooth way.

Theorem 6.1. *Every manifold admits a Riemannian metric.*

Proof. Let $\{\phi_\alpha\}$ be an atlas for M which gives rise to a trivialising atlas $\{\psi_\alpha\}$ for TM. Then, by the paracompactness of M, we can assume the cover $\{U_\alpha\}$ is locally finite. Let ρ_α be a partition of unity subordinate to the U_α. Given α, and $v, w \in T_xM$ with $x \in U_\alpha$, we set

$$\langle v, w \rangle_\alpha := (\psi_\alpha v) \cdot (\psi_\alpha w)$$

Now we can set

$$\langle v, w \rangle := \sum_\alpha \rho_\alpha(x) \langle v, w \rangle_\alpha$$

This is smooth and its restriction to each tangent space is an inner product. □

Given $\gamma : [a, b] \to M$ a smooth curve we define its Riemannian length as

$$\int_a^b ||\gamma'(t)||\, dt$$

where $\gamma'(t) \in T_{\gamma(t)}M$ is the tangent as previously defined.