Solution Strategies for ODEs and DEs - MA133

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Contents

1 Overview

The following scalar equations will be covered:

Type	LCCH	Linear, non-homogeneous	non-linear
1st Order ODEs	✓	✓	(✓)
2nd Order ODEs	✓	✓	(✓)
Higher Order ODEs	(✓)	×	(✓)
1st Order DEs	✓	(✓)	(✓)
2nd Order DEs	✓	×	(✓)
Higher Order DEs	(✓)	×	(✓)

LCCH stands for Linear, Constant Coefficient, Homogeneous.
Non-linear equations will be studied so far as to identify soluble specials cases (i.e. via separation of variables and substitution) and investigating fixed points. We will also consider how to transform these equations into systems of differential equations which we will then attempt to solve.

2 Definitions

Ordinary An equations which involves only one independent variable.

Partial An equation which involves more than one independent variable.

Scalar An equation which involves only one dependent variable.

Vectorvalued An equation which inolves more than one dependent variable.

Order Consider an ODE $O = F\Big(t, x'(t), x''(t), ..., x^{(n)}(t)\Big)$ then O is of order n.

Autonomous The ODE $F(t, x', ..., x^{(n)}) = 0$ is autonomous if F does not depend on t (i.e. no explicit dependence on independent variable).

Linear If an equation can be written in the form

$$a_n(t)\frac{d^n y}{dt^n} + ... + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t)$$

then it is said to be linear. (Linear combination of a function and its derivatives where the coefficients are functions of t).

Homogeneous If an equation can be written in the above, linear form such that f(t)=0, then it is said to be homogeneous.

Equation	Order	Autonomous	Linear	Homogeneous
$u' = e^t u$	1	×	✓	✓
$u' + u = e^t$	1	×	✓	×
$u'' + u' - u^2 = 0$	2	✓	×	Undefined

3 First Order ODEs

3.1 The IVP, and Existence and Uniqueness

Given an open interval I, that contains t_0, a solution of *the* initial value problem (IVP)

$$\frac{dx}{dt}(t) = f(x,t) \quad : \quad x(t_0) = x_0$$

on I is a continuous function $x(t)$ with $x(t_0) = x_0$ and

$$\frac{dx}{dt} = f(x,t) \quad \forall t \in I$$

Theorem 3.1 (Theorem of Existence and Uniqueness)**.** *If $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous for $a < x < b$ and for $c < t < d$, then for any $x_0 \in (a,b)$ and $t_0 \in (c,d)$, the initial value problem above has a unique solution on some open interval I containing t_0.*

3.2 First-Order LCCH ODEs

These are equations of the form

$$\frac{dx}{dt} + px = 0$$

where p is some constant. We can notice by inspection that

$$\boxed{x(t) = Ae^{-pt}}$$

is a solution. We can then set the value of A with initial conditions, and hence arrive at a unique solution.

3.3 First-Order Linear Homogeneous ODEs (with non-constant coefficients)

These are more general equations of the form

$$\frac{dx}{dt} + r(t)x = 0$$

We can use a similar approach to guess the solution

$$x(t) = Ae^{-R(t)} \implies \frac{dR}{dt} = r(t)$$

and so the unique solution is of the form

$$\boxed{x(t) = Ae^{-\int r(t)dt}}$$

3.4 First-Order Linear Inhomogeneous ODEs

These are even more general equations of the form

$$\frac{dx}{dt} + r(t)x = g(t)$$

We want the LHS to be in a form where we can easily find an anti-derivative so we multiply through by $\underline{I(t) = e^{\int r(t)dt}}$, which gives

$$I(t)\frac{dx}{dt} + r(t)I(t)x = \frac{d}{dt}(I(t)x) = I(t)g(t)$$

This leads to the general solution

$$\boxed{x(t) = \underbrace{\frac{1}{I(t)}\int I(t)g(t)dt}_{x_p(t)} + \underbrace{\frac{A}{I(t)}}_{x_h(t)}}$$

We call $I(t)$ the *integrating factor* and so we have a solution which is explicit if we can solve this integration. The general solution consists of:

1. A particular solution $x_p(t)$ which is particular to $g(t)$
2. A complementary solution $x_h(t)$ which is the solution of the homogeneous case

$$\frac{dx}{dt} + r(t)xz = 0$$

3.5 Separable Equations

These are equations of the form

$$\frac{dx}{dt} = f(x)g(t)$$

The informal idea behind solving these equations is we transform the equation to

$$\int \frac{dx}{f(x)} = \int g(t)dt$$

which may have an explicit solution. Formal justification of this approach is available in the notes.

3.6 Substitution Methods

Exact Equations

These are equations of the form

$$M(x,y) + N(x,y)y' = 0 \qquad \text{where} \qquad M_y(x,y) = N_x(x,y)$$

If we let $\Psi(x,y)$ be such that $\Psi_x = M$ and $\Psi_y = N$ then from the chain rule we see that

$$\frac{d}{dx}(\Psi(x,y(x)) = 0$$

and so we get the implicit solution

$$\Psi(x,y) = c$$

Type 1

These are equations of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

We use the substitution $u = \frac{y}{x} \implies \frac{dy}{dx} = \frac{du}{dx}x + u$ and so the equation becomes

$$x\frac{du}{dx} = F(u) - u$$

which is separable.

Type 2

These are equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

which are known as "Bernoulli Equations". We have already solved the cases $n = 0, 1$ and so for $n \geq 2$ we use the substitution $u = y^{1-n}$ which eventually leads to

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

which can be solved with integrating factors.

3.7 Direction Fields

It is worth reading the notes for examples of enlarged phase spaces and integral curves which provide graphical representations of ODEs of the form

$$\frac{dx}{dt} = f(x, t)$$

3.8 First-Order Autonomous ODEs

Sometimes we are unable to find explicit solutions to ODEs but we can still investigate the properties of solutions, such as ODEs of the form

$$\frac{dx}{dt} = f(x)$$

$f(x)$ may be any function, and need not be linear, take for example $f(x) = x^2 - 1$. If we consider $x(t)$ at the position of a particle on the x-axis at time t, then $\frac{dx}{dt}$ corresponds to the velocity of this particle at time t. Given a current x position we can then determine in which direction the particle moves which we represent with a phase line diagram (Figure 1).
This diagram highlights two important points $x = \pm 1$ where $\frac{dx}{dt} = 0$, which are called *fixed points* since a particle at this position would remain here indefinitely. In general, we can categorise fixed points into three categories:

Stable FPs Points x_* where $f'(x_*) < 0$ so a particle starting close to x_* get pulled closer towards it (e.g. $x = -1$).

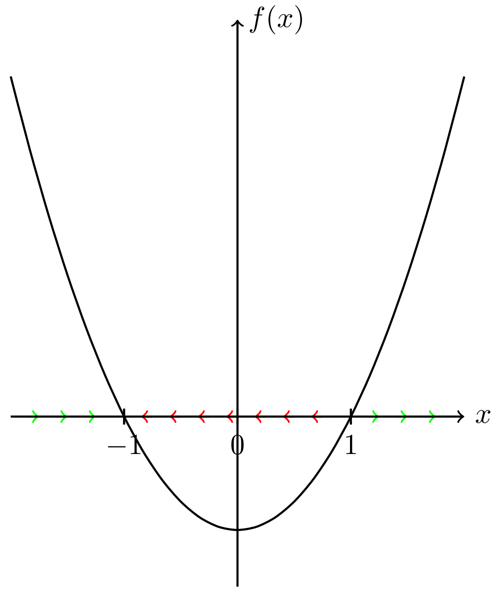

Figure 1: Phase line diagram for $f(x) = x^2 - 1$

Unstable FPs Points x_* where $f'(x_*) > 0$ so a particle starting close to x_* get pushed away from it (e.g. $x = 1$).

Structurally Unstable FPs Points x_* where $f'(x_*) = 0$ so a small change to the equation can make the fixed point stable or unstable.

4 Second Order Linear ODEs

4.1 Linearity and the IVP

The initial value problem in the second-order case is largely the same however we require two initial values to identify a particular solution. Moreover, the underlying assumption of linearity means that

$$x_1(t), x_2(t) \text{ are solutions } \implies \alpha x_1(t) + \beta x_2(t) \text{ is a solution}$$

In order to form a general solution, we need precisely two linearly independent solutions to our ODE (in this case, it is sufficient that they are not scalar multiples) which we combine in a linear combination as above. The proof of this is available in the notes.

4.2 Second-Order LCCH ODEs

These are equations of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

where a, b, c are some constants. Similar to first-order ODEs, we can guess a solution is of the form $x(t) = e^{kt}$ which leads to the *auxiliary equation*:

$$ak^2 + bk + c = 0$$

This leads to three cases:

1. Two distinct real roots k_1, k_2
2. One repeated root k
3. A complex conjugate pair of roots $p \pm iq$

Case 1 - Two distinct real roots k_1, k_2

This automatically gives us our two linearly independent solutions, so our general solution is

$$\boxed{x(t) = Ae^{k_1 t} + Be^{k_2 t}}$$

where A, B are constants which will be fixed by initial conditions.

Case 2 - One repeated root k

In this case we only have one solution $x(t) = Ae^{kt}$, where $k = -\frac{b}{2a}$. To find the second solution, we attempt a trial solution of the form $x(t) = u(t)e^{kt}$, where $u(t)$ is an unknown function. Differentiating and substituting into our original ODE reveals that $u(t) = A + Bt$ and hence we have our general solution

$$\boxed{x(t) = (A + Bt)e^{kt}}$$

Here our two LI solutions are $x_1(t) = e^{kt}$ and $x_2(t) = te^{kt}$

Case 3 - A complex conjugate pair of roots $p \pm iq$

As in **Case 1** our general solution is

$$x(t) = Ae^{(p+iq)t} + Be^{(p-iq)t}$$

However this gives complex solutions, which we often wish to avoid so we can re-arrange as followed:

$$\begin{aligned} x(t) &= Ae^{(p+iq)t} + Be^{(p-iq)t} \\ &= e^{pt}(Ae^{iqt} + Be^{-iqt}) \\ &= e^{pt}[(A + B)\cos qt + (A - B)i \sin qt] \end{aligned}$$

using Euler's formula and letting $C = A + B, D = A - B$

$$\boxed{x(t) = e^{pt}(C \cos qt + D \sin qt)}$$

Note - Using standard trigonometric identities, we can rearrange this solution to

$$x(t) = Ee^{pt} \cos(qt - \phi)$$

where $\tan \phi = \frac{D}{C}$ and $E = \sqrt{C^2 + D^2}$, which is sometimes more informative.

4.3 Mass/Spring Systems

For a mass-spring system where we assume Hooke's Law and the friction to be proportional to velocity, we can arrive at the equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This leads to the auxiliary equation

$$my^2 + cy + k = 0 \quad \text{with discriminant} \quad c^2 - 4mk$$

This gives rise to four possibilities based on the value of this discriminant:

- undamped, $c = 0$
- underdamped, $c^2 - 4mk < 0$
- critically damped, $c^2 - 4mk = 0$
- overdamped, $c^2 - 4mk > 0$

For more details and diagrams, see the lecture notes.

4.4 Second-Order Linear Inhomogeneous ODEs

These are equations of the form

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = f(t)$$

Similar to the first-order case, observe that if $x_p(t)$ is a solution to the differential equation (*the particular integral*) and $Ax_1(t) + Bx_2(t)$ is a solution to the homogeneous case (*the complementary function*)

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = 0$$

then $Ax_1(t) + Bx_2(t) + x_p(t)$ is a solution to the equation due to linearity. So to obtain a solution to this equation we follow the process:

1. Solve the homogeneous case
2. Find a particular integral
3. Add the two parts together

Finding a particular integral is a difficult problem and is often done by the method of "inspired guesswork". Observe the following inspiring table:

f(t)	Trial solution $x_p(t) =$
ae^{kt} (k not a root of the aux. eq.)	Ae^{kt}
ae^{kt} (k is a root of the aux. eq.)	Ate^{kt} or At^2e^{kt}
$a\sin(\omega t)$ or $a\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
at^n where $n \in \mathbb{N}$	$P(t)$
at^ne^{kt} where $n \in \mathbb{N}$	$P(t)e^{kt}$
$t^n\Big(a\sin(\omega t) + b\cos(\omega t)\Big)$	$P_1(t)\sin(\omega t) + P_2(t)\cos(\omega t)$
$e^{kt}\Big(a\sin(\omega t) + b\cos(\omega t)\Big)$	$e^{kt}\Big(\sin(\omega t) + \cos(\omega t)\Big)$

where $P(t), P_1(t), P_2(t)$ are general polynomials of degree n.

5 Introduction to Difference Equations

5.1 First-Order Homogeneous Linear DEs

These are equations of the form

$$x_{n+1} = ax_n$$

where a is a constant. We must also specify an initial value x_0. We see inductively that the solution is

$$x_n = a^n x_0$$

We can make some observations:

- If $x_0 = 0$ then it is a fixed point since this given $x_n = x_0 = 0 \forall n \in \mathbb{N}$.
- If $|a| < 1$, then for any x_0 we see $\lim_{n\to\infty} x_n = 0$ so $x_0 = 0$ is a *stable fixed point.*
- If $|a| > 1$, then $x_0 = 0$ is an *unstable fixed point.*
- If $|a| = 1$, then $x_0 = 0$ demonstrates *structural instability.*

There are obvious parallels here with First-Order Autonomous ODEs.

5.2 Second-Order LCCH DEs

These are equations of the form

$$x_{n+2} + ax_{n+1} + bx_n = 0$$

Drawing similarities with the First-Order case and ODEs, we can attempt the solution $x_n = Ak^n$, which leads to the *auxiliary equation*

$$k^2 + ak + b = 0$$

As before, this leads to three cases:

1. Two distinct real roots k_1, k_2
2. One repeated root k
3. A complex conjugate pair of roots $p \pm iq$

Case 1 - Two distinct real roots k_1, k_2

As with ODEs, this automatically gives us our two linearly independent solutions, so our general solution is

$$\boxed{x_n = Ak_1^n + bk_2^n}$$

Case 2 - One repeated root k

Again appealing to our previous solutions to the ODEs, we attempt a solution of the form

$$\boxed{x_n = (A + Bn)k^n}$$

Checking with the original DE, we see that Ak^n and Bnk^n both yield a solution so by linearity, this solution works.

Case 3 - A complex conjugate pair of roots $p \pm iq$

Writing the solutions in the form $k = re^{\pm i\theta}$ where $r^2 = p^2 + q^2$ and $\tan\theta = \frac{p}{q}$, we get the general solution

$$\boxed{x_n = r^n(A\cos n\theta + B\sin n\theta)}$$

which is easily checked in the original DE (see the lecture notes for more hints).

5.3 Second-Order Linear Constant Coefficient Inhomogeneous DEs

These are equations of the form

$$x_{n+2} + ax_{n+1} + bx_N = f_n$$

We follow the same process as with Linear Inhomogeneous ODEs:

1. Solve the homogeneous case
2. Find a particular solution
3. Add the two parts together

Particular solutions are found in a similar way to the ODEs; by trailing solutions of similar forms to f_n. For more detail and specific examples, see the lecture notes.

5.4 First-Order Autonomous Nonlinear DEs

In general, these are equations of the form

$$x_{n+1} = f(x_n)$$

which is autonomous since f does not depend on n. We can see inductively that the solution is

$$x_n = f^n(x_0)$$

Note if $f(x_n)$ is nonlinear, then you are often unable to write an explicit solution. However, we are able to study fixed points of these equations:

Fixed Point A fixed point of the difference equation

$$x_{n+1} = f(x_n)$$

is a point x_* such that $f(x_*) = x_*$ hence $x_n = x_*$ $\forall n$.

Stable FP If $|f'(x_*)| < 1$, then x_* is a stable fixed point (i.e. $f^n(x_* + h) \to x_*$).

Unstable FP If $|f'(x_*)| > 1$, then x_* is an unstable fixed point

5.5 Cobweb Diagrams

In ODEs, between two unstable fixed points there must be another fixed point, however, since difference equations need not be monotonic, this is not the case. This can give way to more complicated behaviour around fixed points which we can study using cobweb diagrams.

In Figure 2 we plot the graphs $x_{n+1} = x_n$ and $x_{n+1} = f(x_n)$ and complete the following procedure:

1. Start at x_0 on the x_n axis.
2. Go up to to the graph of f and across to the x_{n+1} axis to get x_1.
3. Draw a line from x_1 to the graph $x_{n+1} = x_n$ then down to the x_n axis.
4. Repeat from step 2.

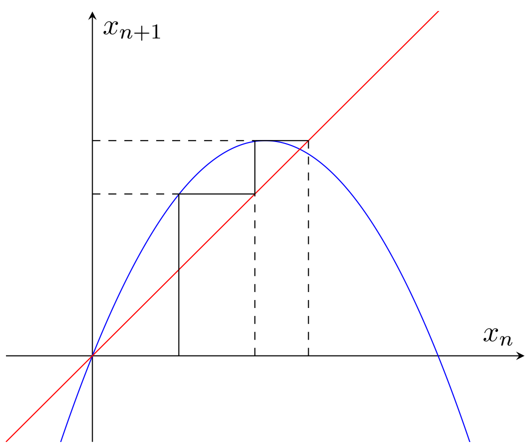

Figure 2: Cobweb diagram for $f(x_n) = \frac{5}{2}x_n(1 - x_n)$

Period k orbit A point in a period k orbit is a fixed point of $f^k(x)$, i.e. a point x_* such that $f^k(x_*) = x_*$.

For more examples of cobweb diagrams and orbits (including plots of orbits in the Verhulst equation) see the lecture notes.

6 Systems of Linear First-Order ODEs

An $n \times n$ system of first-order ODEs is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \qquad : \qquad \mathbf{x}(t) \in \mathbb{R}^n, \quad \mathbf{f} : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}, t) = \mathbf{f}(x_1, x_2, ..., x_n, t) = \begin{pmatrix} f_1(x_1, x_2, ..., x_n, t) \\ f_2(x_1, x_2, ..., x_n, t) \\ \vdots \\ f_n(x_1, x_2, ..., x_n, t) \end{pmatrix}$$

A solution of the IVP

$$\frac{d}{dt}(\mathbf{x}(t)) = \mathbf{f}(\mathbf{x}, t) \qquad : \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}_0 \in \mathbb{R}^n$$

on an open interval I that contains t_0 is a continuous function $\mathbf{x} : I \to \mathbb{R}^n$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) \forall t \in I$. The Jacobian matrix of a function function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Theorem 6.1 (Existence and Uniqueness)**.** *If $\boldsymbol{f}(\boldsymbol{x}, t)$ and $D\boldsymbol{f}(\boldsymbol{x}, t)$ are continuous functions for $\boldsymbol{x} \in$ some set $U \subseteq \mathbb{R}^n, a < t < b$ then for any $\boldsymbol{x}_0 \in U$ and $t_0 \in (a, b)$, there exists a unique solution to the IVP on some open interval containing t_0.*

6.1 Homogeneous Linear 2×2 Systems with Constant Coefficients

These are systems of the form

$$\begin{aligned} \frac{dx}{dt} &= px + qy \\ \frac{dy}{dt} &= rx + sy \end{aligned}$$

for some constants p, q, r, s, which can be written as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \qquad \textit{where} \quad A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

We try solutions of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{v} = e^{\lambda t}\begin{pmatrix} a \\ b \end{pmatrix}$$

Which leads to the equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

so finding solutions $\mathbf{x}$ is equivalent to finding eigenvalues λ and eigenvectors $\mathbf{v}$ of A. Again this gives rise to three cases:

1. **Distinct real eigenvalues -** Due to the assumption of linearity, if A has two distinct real eigenvalues λ_1, λ_2 with eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ respectively then the solution is

$$\boxed{\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}}$$

2. **Complex conjugate eigenvalue pair -** It can be shown that if λ and $\mathbf{v}$ are an eigenvalue and eigenvector of A then so too are $\bar{\lambda}, \bar{\mathbf{v}}$ so a solution is

$$\mathbf{x}(t) = c\mathbf{v}e^{\lambda t} + \bar{c}\bar{\mathbf{v}}e^{\bar{\lambda} t} = 2\Re[c\mathbf{v}e^{\lambda t}]$$

choosing c such that $\mathbf{x}(t)$ is real. Now we let $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2, \lambda = p + iq, c = \alpha + i\beta$ gives

$$\mathbf{x}(t) = 2e^{pt}[(\alpha \cos qt - \beta \sin qt)\mathbf{v}_1 - (\beta \cos qt + \alpha \sin qt)\mathbf{v}_2]$$

which leads to the solution

$$\boxed{\mathbf{x}(t) = e^{pt}[(a \cos qt + b \sin qt)\mathbf{v}_1 + (b \cos qt - a \sin qt)\mathbf{v}_2]}$$

where a, b are constants determined by the initial conditions.

3. **Repeated real root -** The solution is

$$\boxed{\mathbf{x}(t) + Be^{\lambda t}\mathbf{v} + Ce^{\lambda t}(\mathbf{a} + t\mathbf{v})}$$

where B, C are arbitrary constants and $\underline{(A - \lambda I)\mathbf{a} = \mathbf{v}}$. For proof, see the lecture notes.

6.2 Phase Diagrams

These are covered in the lecture notes in good detail.