

Intro to Topology - Overview

1 Key Definitions

- Given maps $f, g : X \times I \rightarrow Y$, a **homotopy** from f to g is a continuous map $F : X \times I \rightarrow Y$ such that $f_0 = f$ and $f_1 = g$. If such a map exists we write $f \simeq g$.
- Given paths $f, g : I \rightarrow X$. We say **f is homotopic to g relative to $\{x, y\}$** and write $f \stackrel{\partial}{\simeq} g$ if
 - (i) $f(0) = g(0) = x$
 - (ii) $f(1) = g(1) = y$
 - (iii) There is a homotopy $F : I \times I \rightarrow X$ such that $f_0 = f$, $f_1 = g$ and for all $t \in I$, $f_t(0) = x$ and $f_t(1) = y$.
- A **pair** of spaces (X, A) is a topological space together with a subspace $A \subseteq X$ using the subspace topology.

Assume we have a pair (X, A) .

- A is a **retract** if there is a continuous map $r : X \rightarrow A$ such that $r|_A = id_A$.
- X **deformation retracts** to A if there exists a homotopy $F : X \times I \rightarrow X$ such that $f_0 = id_X$, $f_1(X) = A$ and $f_t|_A = id_A$ for all $t \in I$.

Assume we have topological space X and Y then,

- X is **homotopy equivalent** to Y there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

$$g \circ f \simeq id_X \quad , \quad f \circ g \simeq id_Y$$

- X is **contractible** if it is homotopy equivalent to $\{pt\}$.

Note:

$$X \text{ deformation retracts to } A \implies X \text{ contractible}$$

$$\nLeftarrow$$

The reverse does not hold because the contraction does not necessarily restrict to id_A .

2 Fundamental Group

Given a pointed space (X, x_0) , a **loop** is a path $f : I \rightarrow X$ such that $f(0) = f(1) = x_0$. We can then define an equivalence class for every loop f :

$$[f] := \left\{ g \mid g(0) = g(1) = x_0, \quad f \stackrel{\partial}{\simeq} g \right\}$$

We then get the **fundamental group** defined to be

$$\pi_1(X, x_0) := \{[f] \mid f \text{ a loop based at } x_0\}$$

This forms a group with the operation $[f] \cdot [g] = [f * g]$ and the identity element being the constant loop.

Theorem 2.1. *If X is path connected and $x_0, x_1 \in X$ then*

$$\pi_1(X, x_0) = \pi_1(X, x_1)$$

Proof. There exists a path $h : I \rightarrow X$ from x_0 to x_1 . Define $\bar{h}(s) := h(1 - s)$. We can then define a base point change homomorphism which we claim is in fact an isomorphism.

$$\beta_h : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1), \quad [f] \mapsto [\bar{h} * f * h]$$

We can see this is in fact an isomorphism because $\beta_{\bar{h}}$ is a left and right inverse. □

A map $p : \tilde{X} \rightarrow X$ is a **covering map** if there is an open cover $\{U_\alpha\}$ of X such that

$$p^{-1}(U_\alpha) = \bigsqcup_{\beta} V_\alpha^\beta$$

with each V_α^β open and such that $p|_{V_\alpha^\beta} : V_\alpha^\beta \rightarrow U_\alpha$ is a homeomorphism.

The covering map is called **n -fold** if each $p^{-1}(x_0)$ has n elements for every x_0 .

Let $p : Y \rightarrow X$ and $q : Z \rightarrow X$ be coverings. These called **isomorphic** if there is a homeomorphism $h : Y \rightarrow Z$ such that

$$q \circ h = p$$

Let $p : \tilde{X} \rightarrow X$ be a cover then a **deck transformation** is an isomorphism $\tau : \tilde{X} \rightarrow \tilde{X}$ such that $p \circ \tau = p$.

$$\text{Deck}(p) := \left\{ \tau : \tilde{X} \rightarrow \tilde{X} \mid \tau \text{ is a deck transformation} \right\}$$

Given a covering $p : \tilde{X} \rightarrow X$ and a map $f : Y \rightarrow X$, a **lift** of f is a map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $f = p \circ \tilde{f}$.

Here are some useful properties of lifts:

- (i) $\tilde{f} : Y \rightarrow \tilde{X}$ then \tilde{f} is a lift of $p \circ \tilde{f}$.
- (ii) $\tilde{f}, \tilde{g} : Y \rightarrow \tilde{X}$ and $f \simeq g \implies p \circ \tilde{f} \simeq p \circ \tilde{g}$ (homotopies descends).
- (iii) $\alpha, \beta : I \rightarrow \tilde{X}$ such that $\alpha(1) = \beta(0)$ then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ (concatenation descends).

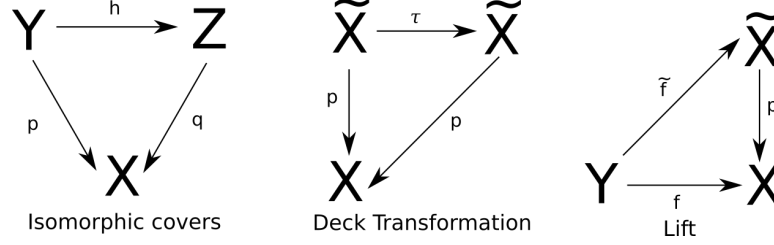


Figure 1: Commutative diagrams defining various concepts

3 Homotopy Lifting Property

A map $p : Z \rightarrow X$ has the **homotopy lifting property** if given any homotopy $F : Y \times I \rightarrow X$ and any lift $g : Y \times \{0\} \rightarrow Z$ there exists a unique homotopy $\tilde{F} : Y \times I \rightarrow Z$ satisfying

- (i) $\tilde{f}_0 = g$
- (ii) $p \circ \tilde{F} = F$

i.e. given a homotopy and a lift of one endpoint, there exists a unique lift of that homotopy. As a special case where $Y = \{pt\}$, a map $p : Z \rightarrow X$ has the **path lifting property** if for any path $f : I \rightarrow X$, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$ there exists a unique path $\tilde{f} : I \rightarrow Z$ such that

- (i) $\tilde{f}(0) = \tilde{x}_0$
- (ii) $p \circ \tilde{f} = f$

Lemma 3.1 (Local Homotopy Lifting Property). *Let $p : \tilde{X} \rightarrow X$ be a covering map and $F : Y \times I \rightarrow X$ a homotopy. Suppose we have $g : Y \times \{0\} \rightarrow \tilde{X}$. Then for every $y \in Y$*

- (a) *There exists an open neighbourhood $N \subseteq Y$ and a unique homotopy $\tilde{F}_N : N \times I \rightarrow \tilde{X}$ such that*

- (i) $(\tilde{f}_N)_0 = g$.
- (ii) $p \circ \tilde{F}_N = F|_{N \times I}$.

- (b) *If $M \subseteq Y$ with $y \in M$ is another open neighbourhood for which (a) holds then (a) also holds for $M \cap N$ and*

$$\tilde{F}_N|_{(M \cap N) \times I} = \tilde{F}_M|_{(M \cap N) \times I} = \tilde{F}_{M \cap N}$$

Proof. This has a very long proof. □

Proposition 3.2. *Covering maps $p : \tilde{X} \rightarrow X$ have the homotopy lifting property.*

Proof. Let $P : \tilde{X} \rightarrow X$ be a covering map. Let $F : Y \times I \rightarrow X$ be a homotopy and choose some arbitrary starting $g : Y \times \{0\} \rightarrow \tilde{X}$.

We can cover $Y = \cup_{\alpha} N_{\alpha}$ such that (a) and (b) hold from the lemma.

We can then define a new homotopy by stitching these together:

$$\tilde{F} : Y \times I \rightarrow \tilde{X}, \quad \tilde{F}(y, t) := \tilde{F}_{N_{\alpha}}(y, t) \text{ if } y \in N_{\alpha}$$

We do not get any ambiguity here thanks to property (b) from the lemma. The continuity of this construction follows from the pasting lemma. □

Theorem 3.3. Let $\omega_N : I \rightarrow S^1$ be defined by $\omega_n(s) = e^{2\pi i ns}$. Then

$$\pi_1(S^1, 1) = \{[\omega_n] \mid n \in \mathbb{Z}\}$$

Proof. Define $\Phi : \mathbb{Z} \rightarrow \pi_1(S^1, 1)$ by $n \mapsto [\omega_n]$. We claim this is an isomorphism. For this define the following useful maps

$$\begin{aligned} p(t) &= e^{2\pi i t} \\ \omega_n(t) &= e^{2\pi i n t} \\ \tilde{\omega}_n(t) &= nt \\ \tau_m(t) &= t + m \end{aligned}$$

- Φ is a group homomorphism.

Then we can see that indeed $\tilde{\omega}_n : \mathbb{R} \rightarrow \mathbb{R}$ is a lift of ω_n . One can also see through the linear homotopy that

$$\tilde{\omega}_{m+n} \stackrel{\partial}{\simeq} \tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n)$$

Now given any $m, n \in \mathbb{Z}$ we have

$$\begin{aligned} \Phi(m+n) &= [\omega_{m+n}] \\ &= [p \circ \tilde{\omega}_{m+n}] && \downarrow \text{lift} \\ &= [p \circ (\tilde{\omega}_m * (\tau_m \circ \tilde{\omega}_n))] && \downarrow \text{homotopies descend} \\ &= [p \circ \tilde{\omega}_m] \cdot [p \circ \tau_m \circ \tilde{\omega}_n] && \downarrow \text{deck transformation} \\ &= [p \circ \tilde{\omega}_m] \cdot [p \circ \tilde{\omega}_n] && \downarrow \text{lift} \\ &= [\omega_m] \cdot [\omega_n] \\ &= \Phi(m) \cdot \Phi(n) \end{aligned}$$

- Φ is surjective.

Choose any $[\alpha] \in \pi_1(S^1, 1)$, we aim to find $n \in \mathbb{N}$ such that $\alpha \stackrel{\partial}{\simeq} \omega_n$. We certainly know that $\alpha(0) = \alpha(1) = 1$ and hence $p^{-1}(1) = \mathbb{Z}$ and in particular $0 \in p^{-1}(1) = p^{-1}(\alpha(0))$.

So by the path lifting property there exists a unique lift $\tilde{\alpha} : I \rightarrow \mathbb{R}$ such that

$$(i) \quad \tilde{\alpha}(0) = 0.$$

$$(ii) \quad p \circ \tilde{\alpha} = \alpha.$$

Now, $\alpha(1) = 1 \implies p(\tilde{\alpha}(1)) = 1 \implies \tilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$. Suppose $\tilde{\alpha}(1) = n \in \mathbb{Z}$. So $\tilde{\alpha}$ is a path from 0 to n in \mathbb{R} . By the linear homotopy we can see that $\tilde{\alpha} \stackrel{\partial}{\simeq} \tilde{\omega}_n$. But homotopies descend and hence

$$\alpha = p \circ \tilde{\alpha} \stackrel{\partial}{\simeq} p \circ \tilde{\omega}_n = \omega_n$$

- Φ is injective.

Assume that $\Phi[\omega_n] = [e]$. We aim to show that in fact $n = 0$.

To start, $\omega_n \stackrel{\partial}{\simeq} e$ and hence we have a homotopy $F : I \times I \rightarrow S^1$, $(s, t) \mapsto F(s, t)$ such that $f_0 = \omega_n$, $f_1 = e$ and $f_t(0) = f_t(1) = 1$.

Now by the HLP we see that there is a unique homotopy $F : I \times I \rightarrow \mathbb{R}$ satisfying

- (i) $\tilde{f}_0 = \tilde{\omega}_n$.
- (ii) $p \circ \tilde{F} = F$.

Now since the left, top and bottom edges were identically 1 in F we must have that the same edges lie in \mathbb{Z} in the lifted homotopy. But consider the bottom edge $\tilde{\omega}_n$. On the left side it is 0 but on the right it is n . By continuity along the left, top and bottom edges of \tilde{F} we must have that $n = 0$. This can be seen in Figure 3.

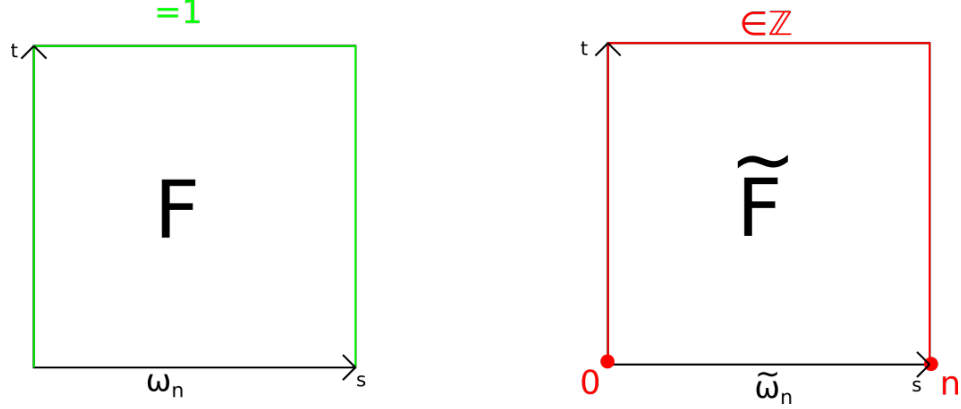


Figure 2: Diagrammatic explanation of continuity argument

□

4 Applications

A **map of pairs** $f : (X, A) \rightarrow (Y, B)$ is a map $f : X \rightarrow Y$ such that $f(A) \subseteq B$.

The **induced homomorphism** of $f : (X, x_0) \rightarrow (Y, y_0)$ is the map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, y_0) \\ [\alpha] &\mapsto [f \circ \alpha] \end{aligned}$$

We need to show that this is well-defined and is in fact a group homomorphism.

Lemma 4.1 (Functoriality). $(g \circ f)_* = g_* \circ f_*$

Corollary 4.2. *If f is a homeomorphism then f_* is a group isomorphism.*

Theorem 4.3. *let $f : X \rightarrow Y$ be a homotopy equivalence and $x_0 \in X$. Then*

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. Let $y_0 := f(x_0)$ and $g : Y \rightarrow X$ be the homotopy inverse. Denote $x_1 := g(y_0)$. We get a homotopy a $K : X \times I \rightarrow X$ such that $k_0 = id_X$ and $k_1 = g \circ f$.

We define path in X by following that path of x_0 under this homotopy, i.e.

$$h : I \rightarrow X \quad t \mapsto K(x_0, t)$$

Claim: $\beta_h[\gamma] = (g \circ f)_*[\gamma]$ for all loops γ based at x_0 .

We have already seen that base point change homomorphisms are in fact isomorphisms and hence we have that $(g \circ f)_* = g_* \circ f_*$ is an isomorphism. This implies that g_* is surjective and f_* is injective. Repeating this argument with the other homotopy then yields the result.

Proof of claim:

Define a new homotopy by

$$H(s, t) := \begin{cases} h(t) & \text{for } s \leq t \\ h(s) & \text{for } s \geq t \end{cases}$$

which has the properties that $h_0 = h$ and $h_1 = h(1) = x_1$.

Also define $K(\gamma(s), t)$ so that $\gamma_0 = \gamma$ and $\gamma_1 = g \circ f \circ \gamma$.

Finally, one can check that $\alpha_t := \bar{h}_t * \gamma_t * h_t$ gives a well-defined path for every t . This gives a homotopy between

$$\begin{aligned} \gamma_0 &= \bar{h} * \gamma * h & \text{and} \\ \gamma_1 &= e_{x_1} * (g \circ f \circ \gamma) * e_{x_1} \end{aligned}$$

□

Proposition 4.4. *Consider the sequence of maps induced by a retract $r : X \rightarrow A$ and the ‘inverse’ inclusion for a point $x_0 \in A$*

$$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0)$$

1. i_* is injective.
2. r_* is surjective.
3. If $r \stackrel{A}{\simeq} id_X$ then the induced maps are in fact isomorphisms.

Proof. (1) and (2) follow immediately from the fact that $r \circ i = id_A$ and functoriality.

For (3) it remains to show that r_* is injective and i_* is surjective. For now we just show that r_* is injective. Suppose $r_*[\gamma] = [e]$ then we wish to show that in fact $[\gamma] = [e]$.

We know $r \circ \gamma \stackrel{\partial}{\simeq} e$ and $r \stackrel{A}{\simeq} id_X$. We have a homotopy $F : X \times I \rightarrow X$ where $f_t|_A = id_A$, $f_0 = r$ and $f_1 = id_X$. We define a new homotopy $G : I \times I \rightarrow X$ by $G(x, t) = F(\gamma(x), t)$ which satisfies.

- (i) $g_t(0) = g_t(1) = f_t(\gamma(0)) = f_t(x_0) = x_0$ since $x_0 \in A$.
- (ii) $g_0(x) = f_0(\gamma(x)) = (r \circ \gamma)(x)$.
- (iii) $g_1(x) = f_1(\gamma(x)) = \gamma(x)$.

Hence we have $r \circ \gamma \stackrel{\partial}{\simeq} \gamma$ and hence $\gamma \stackrel{\partial}{\simeq} e$. The proof that i_* is surjective is very similar. □

Theorem 4.5 (No Retract Theorem). *There is no retract $R : D^2 \rightarrow S^1$.*

Proof. Assume that such a retract exists then there is a surjective homomorphism r_* :

$$0 = \pi_1(D^2, 1) \rightarrow \pi_1(S^1, 1) = \mathbb{Z}$$

which is obviously nonsense. □

Theorem 4.6 (Brouwer Fixed Point Theorem). *Any map $f : D^2 \rightarrow D^2$ has a fixed point.*

Proof. Assume that f has no fixed point, then we construct the following map for every x in S^1 :

$$L_x(t) := tx + (1 - t)f(x) \quad \forall t \geq 0$$

Then we can define a map $\phi : D^2 \rightarrow S^1$ by

$$\phi(x) = L_x(\mathbb{R}_{\geq 0}) \cap S^1$$

which we claim is a retraction. Certainly, $\phi|_{S^1} = id_{S^1}$ but what about continuity?

Well $\phi(x) = L_x(t)$ for the unique t which solves $|L_x(t)|^2 = 1$ and is bigger than 0. The equation for t is quadratic and so t can be shown to be a continuous function of x . Then clearly L_x is continuous so ϕ is continuous.

The no retract theorem yields a contradiction. □

Note: The Brouwer Fixed Point Theorem also holds for sets $S \cong D^2$ and their boundary.

4.1 Involutions and Borsuk-Ulam Theorem

Given a topological space X , an **involution** is a map $h : X \rightarrow X$ such that for every $x \in X$

$$h(h(x)) = x.$$

We often write $h(x) = -x$ for convenience.

Given spaces X and Y each with involutions and a map $f : X \rightarrow Y$, we say f is

$$\begin{aligned} \text{odd} & \quad \text{if} \quad f(-x) = -f(x) \\ \text{even} & \quad \text{if} \quad f(-x) = f(x) \end{aligned}$$

for all x in X .

A map $f : (X, x_0) \rightarrow (Y, y_0)$ is **null-homotopic** if f is homotopic to a constant map.

A map $f : (X, x_0) \rightarrow (Y, y_0)$ is **null-homotopic relative to base points** if there is a homotopy between $F : X \times I \rightarrow Y$ such that $f_0 = f$, $f_1 = e_{y_0}$ and $f_t(x_0) = y_0$ for all t . We then write $f \stackrel{x_0}{\simeq} e_{y_0}$.

Proposition 4.7. *If $f : S^1 \rightarrow S^1$ is odd then f is not null-homotopic.*

Proof. Very long. □

Corollary 4.8. *If $f : S^2 \rightarrow \mathbb{R}^2$ is odd then there is a point $x \in S^2$ such that $f(x) = 0$.*

Proof. Worth doing. □

Theorem 4.9 (Borsuk-Ulam). *Given a map $f : S^2 \rightarrow \mathbb{R}^2$ there is a point $x \in S^2$ such that $f(-x) = f(x)$.*

Proof. The map $g(x) := f(x) - f(-x)$ is by construction odd and hence has a vanishing point. □

5 Product Spaces

Given pointed spaces (X, x_0) and (Y, y_0) then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Also homotopies of paths in $X \times Y$ correspond to pairs of homotopies in X and Y .

5.1 Fundamental group of S^n for $n \geq 2$

Theorem 5.1. *For all $n \geq 2$ the fundamental group of S^n is trivial.*

Idea: Use stereographic projection into \mathbb{R}^n and then use the linear homotopy.

The main issue with this is that the loop may go around or through the north pole, at which point our projection breaks down. However, in these big spheres we should be able to find a point that isn't inside the loop and project from there. Note that in S^1 we cannot do this because every non-trivial curve uses every point.

Proof. Cover the sphere with two open sets such that U_1, U_2 such that $x_0 \in U_1 \cap U_2$ and $U_1 \cap U_2$ is path connected. Now take any loop $\gamma : I \rightarrow S^n$ and subdivide the interval as

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that for all i there is a j such that $\gamma[t_{i-1}, t_i] \subseteq U_j$. So we can now view γ is the concatenation

$$\gamma = \bigstar_{i=1}^m \gamma|_{[t_{i-1}, t_i]}$$

Now since x_0 and $\gamma(t_i)$ are both in $U_1 \cap U_2$, we can let α_i be a path in $U_1 \cap U_2$ from $\gamma(t_i)$ to x_0 . Now we can put these paths in between the subdivisions of γ :

$$\beta := \left[\gamma|_{[t_0, t_1]} * \alpha_1 \right] * \left[\bigstar_{i=2}^{m-1} \bar{\alpha}_{i-1} * \gamma|_{[t_{i-1}, t_i]} * \alpha_i \right] * \left[\bar{\alpha}_{m-1} * \gamma|_{[t_{m-1}, t_m]} \right]$$

Now each $\bar{\alpha}_{i-1} * \gamma|_{[t_{i-1}, t_i]} * \alpha_i$ lies in one U_j and so is homotopic to a constant loop by using the linear homotopy in \mathbb{R}^n . So $\beta \simeq e_{x_0}$ and $\beta \simeq \gamma$ and hence $\gamma \simeq e_{x_0}$. \square

6 Some Algebra

Proposition 6.1. *Let $p : \tilde{X} \rightarrow X$ be a covering, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$.*

1. *The induced map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.*
2. *If $[\alpha] \in \pi_1(X, x_0)$ and $\tilde{\alpha}$ is a lift of α such that $\tilde{\alpha}(0) = \tilde{x}_0$, then*

$$[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \iff \tilde{\alpha} \text{ is a loop}$$

Proof. 1. We wish to show that $p_*([\tilde{\alpha}]) = [p \circ \tilde{\alpha}] = [e_{x_0}] \implies [\tilde{\alpha}] = [e_{\tilde{x}_0}]$.

Given $[\tilde{\alpha}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ such that $[p \circ \tilde{\alpha}] = [e_{x_0}]$, define $\alpha := p \circ \tilde{\alpha}$ and then $\tilde{\alpha}$ is the unique lift of α with the property $\tilde{\alpha}(0) = \tilde{x}_0$. Now, by assumption, $\alpha \simeq e_{x_0}$ and hence there is a homotopy $F : I \times I \rightarrow X$ with $f_0 = \alpha$, $f_1 = e_{x_0}$.

By the HLP, there exists a unique lift $\tilde{F} : I \times I \rightarrow X$ such that $\tilde{f}_0 = \tilde{\alpha}$. Now $F(0, t)$, $F(1, t)$ and $F(s, 1)$ are all constant paths at x_0 and hence lift to constant paths at \tilde{x}_0 . So \tilde{F} tells us that $\tilde{\alpha} \stackrel{\partial}{\simeq} e_{\tilde{x}_0}$.

2. We really only have the forward direction to prove.

Suppose $[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ then $[\alpha] = [p \circ \tilde{\gamma}]$ for some $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Suppose $\tilde{\alpha}$ is a lift of α with $\tilde{\alpha}(0) = \tilde{x}_0$. We have

$$\alpha = p \circ \tilde{\alpha} \stackrel{\partial}{\simeq} p \circ \tilde{\gamma} =: \gamma$$

We know $\tilde{\gamma}$ is a loop at \tilde{x}_0 . So we can lift the homotopy between α and γ to one between $\tilde{\alpha}$ and $\tilde{\gamma}$.

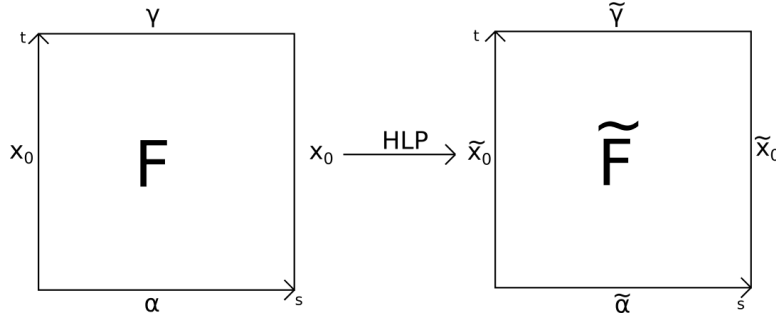


Figure 3: Diagrammatic explanation that $\tilde{\alpha}$ is a loop at \tilde{x}_0 .

□

Given a group G and a subgroup $H \leq G$ we have the **quotient space**

$$\frac{G}{H} := \{Hg \mid g \in G\} = \{\text{right cosets}\}$$

and then the **index** is $[G : H] = |\frac{G}{H}| = \#$ of cosets.

Let $p : \tilde{X} \rightarrow X$ be a cover and assume that both spaces are path connected. Then we define the **degree**

$$\deg(p) = |p^{-1}(x)| \quad \text{for any } x \in X$$

Note: Definition of cover $\implies \deg(p)$ is at least locally constant.

Path connected \implies globally constant.

Proposition 6.2. Let $p : \tilde{X} \rightarrow X$ be a cover with \tilde{X}, X path connected, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Then

$$\deg(p) := \left[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \right].$$

Proof. Let $G := \pi_1(X, x_0)$ and $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Given any $[\alpha] \in G$ we get a unique lift $\tilde{\alpha}$ with $\tilde{\alpha}(0) = \tilde{x}_0$ and then $\tilde{\alpha}(1) = \tilde{x}_1 \in p^{-1}(x_0)$.

Now let $[\gamma] \in H$ then we have

$$\begin{aligned} [\gamma] \in H &\implies [\gamma] \cdot [\alpha] = [\gamma * \alpha] \in H[\alpha] \\ &\implies \tilde{\gamma} \text{ (a lift of } \gamma) \text{ is a loop at } \tilde{x}_0 \\ &\implies (\gamma * \alpha) \text{ lifts to } (\tilde{\gamma} * \tilde{\alpha}) \text{ from } \tilde{x}_0 \text{ to } \tilde{x}_1 \end{aligned} \quad \Bigg\} \text{unique lift}$$

And hence we can construct a map

$$\Phi : \frac{G}{H} \rightarrow p^{-1}(x_0), \quad H[\alpha] \mapsto \tilde{\alpha}(1)$$

which we just showed is independent of the representative of $H[\alpha]$. We wish to show that Φ is bijective.

Injective: Assume that $\Phi(H[\alpha]) = \Phi(H[\beta])$ then $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Write $\gamma := \alpha * \bar{\beta}$ then $\tilde{\gamma} = \tilde{\alpha} * \tilde{\beta}$ is a loop because $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{\beta}(1) = \tilde{x}_0$ and the assumption means that $\tilde{\alpha}$ and $\tilde{\beta}$ join up in the middle.

Hence $[\gamma] = [\alpha] \cdot [\beta]^{-1} \in H$ so $H[\alpha] = H[\beta]$.

Surjective: This follow from path connectivity. □

Let $\{X_\alpha, x_\alpha\}$ be a collection of pointed topological spaces. The **wedge product** is then defined to be

$$\bigvee_{\alpha} X_{\alpha} = \bigsqcup_{\alpha} \frac{X_{\alpha}}{x_{\alpha} \sim x_{\beta}}$$

Then $\{G_\alpha\}_\alpha$ be some groups. Then a **word** on $\{G_\alpha\}_\alpha$ is a finite sequence

$$g = g_1 \dots g_m$$

for some $g_i \in G_{\alpha_i}$. Then m is the **length** of the word g .

We can concatenate words $g = g_1, \dots, g_m$ and $h = h_1, \dots, h_n$ by

$$g * h = g_1 \dots g_m h_1 \dots h_n$$

A word is said to be **reduced** if

- (a) $g_i \neq e_{\alpha_i} \quad \forall i$
- (b) $\alpha_i \neq \alpha_{i+1} \quad \forall i$

We can define the set

$$\bigstar_{\alpha} G_{\alpha} := \{\text{reduced words on } \{G_{\alpha}\}_{\alpha}\}$$

which we can given a product in the following way. Given $g, h \in \bigstar_{\alpha} G_{\alpha}$

1. Take the concatenation $g * h = g_1 \dots g_m h_1 \dots h_n$.
2. If g_m and h_1 lie in different G_{α} then $g \cdot h := g * h$.
3. Else replace $g_m h_1$ by the product.
4. If $g_m h_1 = e$ then remove it and repeat from step 1 with the word $g_1 \dots g_{m-1} h_2 \dots h_n$.

Theorem 6.3. So defined, $(*_\alpha G_\alpha, \cdot)$ forms a group, called the **free product** of $\{G_\alpha\}_\alpha$.

Proof. The only difficult part is associativity, worth reading up on. \square

Lemma 6.4. Let $\{\phi_\alpha : G_\alpha \rightarrow G\}_\alpha$ be a collection of group homomorphisms. Then $\exists!$ group homomorphism $*_\alpha \phi_\alpha : G_\alpha \rightarrow G$ such that $(*_\alpha \phi_\alpha) \circ i_\alpha = \phi_\alpha \quad \forall \alpha$

Proof. Really the only thing to prove is that this definition doesn't depend on how the word is reduced. \square

Theorem 6.5 (Seifert-van Kampen). Let $X = \cup_\alpha A_\alpha$ be an open cover with $x_0 \in \cap_\alpha A_\alpha$, then

(i) If $A_\alpha \cap A_\beta$ is path connected for all α, β then the induced map

$$\Phi := \bigstar_\alpha (i_\alpha)_* : \bigstar_\alpha \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$$

is surjective

(ii) If $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for all α, β, γ then

$$\ker(\Phi) = N = \langle \{ (i_{\alpha\beta})_*(\omega) \cdot (i_{\beta\alpha})_*(\omega)^{-1} \mid \omega \in \pi_1(A_\alpha \cap A_\beta, x_0) \text{ for some } \alpha, \beta \} \rangle$$

and hence

$$\pi_1(X, x_0) \cong \frac{\bigstar_\alpha \pi_1(A_\alpha, x_0)}{N}$$

Lemma 6.6. If $X = \cup_\alpha A_\alpha$ is an open cover with $A_\alpha \cap A_\beta$ path connected for every pair α, β and $x_0 \in \cap_\alpha A_\alpha$, then every loop γ based at x_0 in X factors as

$$[\gamma] = [\gamma_1] \cdot [\gamma_2] \cdots [\gamma_m]$$

with each γ_i being a loop in A_{α_i} .

Proof. Note that we can cover I by the $\gamma^{-1}(A_\alpha)$ and since I is compact we can actually reduce this to a finite sub cover. Then the Lebesgue covering lemma means that we can find

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $\gamma[t_i, t_{i+1}] \subseteq A_{\alpha_{i+1}}$ for each $i \in [0, m)$. Then for each t_i notice that $\gamma(t_i) \in A_{\alpha_i} \cap A_{\alpha_{i+1}}$ which is path connected and hence there is a path g_i from $\gamma(t_i)$ to x_0 in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$.

Now we can define a new path $\hat{\gamma}$ by going back to x_0 along these paths at each t_i and then returning to continue the loop. Since $g * \bar{g}$ is homotopic to $e_{\gamma(t_i)}$ we have that $\hat{\gamma} \stackrel{\partial}{\simeq} \gamma$ and hence we can decompose γ as required. \square

These essentially proves part (i) of the Seifert-van Kampen theorem.

A **factorisation** of $[f] \in \pi_1(X, x_0)$ is a sequence

$$[f_1], \dots, [f_m] \quad \text{with} \quad [f_i] \in \pi_1(A_{\alpha_i}, x_0)$$

such that $f \simeq f_1 * \dots * f_m$ (although this is not necessarily in reduced form).

If $[f_i], [f_{i+1}] \in \pi_1(A_\alpha, x_0)$ then a **reduction** is

$$[f_1] \cdots [f_m] \mapsto [f_1] \cdots [f_i + 1] \cdots [f_i * f_{i+1}] \cdots [f_m]$$

(or an **expansion** if you go in the opposite direction).

If $[f_i] = (i_{\alpha\beta})_*(\omega)$ and $[g_i] = (i_{\beta\alpha})_*(\omega)$ for some $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$ then an **exchange** is

$$[f_1] \dots [f_i] \dots [f_m] \mapsto [f_1] \dots [g_i] \dots [f_m]$$

Two factorisations of f are said to be **equivalent** if one can be transformed to the other by a sequence of the above operations.

Lemma 6.7. *Under the conditions of Seifert-van Kampen (ii) any two factorisations of f are equivalent.*

Proof. In lecture notes. □

This can then be used to prove Seifert-van Kampen (ii).

7 CW Complexes

Given a space X we can build up the **CW complex** recursively as follows:

1. X_0 is a discrete set called the **0-skeleton**.
2. Given X^{n-1} and a collection of **n -cells** $\{D_\alpha^n\}_\alpha$ and attaching maps $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$ we subsequently define X_n by

$$X^n := \frac{[X^{n-1} \sqcup (\bigsqcup_\alpha D_\alpha^n)]}{x \sim \phi_\alpha(x)}$$

3. Then $X = \cup_n X^n$, bestowed with the **weak topology** is a **CW complex**.

Then X is called **finite dimensional** if $X = X^n$ for some $n \in \mathbb{N}$.

Lastly, X is **finite** if $\{D_\alpha^n\}_{\alpha,n}$ is finite (i.e. it is composed of finitely many cells).

Note: $U \subseteq X$ is open in the weak topology $\iff \forall n \quad U \cap X^n$ is open in X^n