Manifolds Notes

1 Differential Forms

Given vector spaces E, F and $p \in \mathbb{N}$ we denoted

$$A(E^p, F) := p - \text{linear alternating maps } E^p \to F$$

where by alternating we mean that swapping any two coordinates negates the output. Equivalently, if two coordinates are the same then the output is 0.

Lemma 1.1. *Given E and p there is a vector space V together with a surjective map $\mu \in A(E^p, V)$ with the property that if $\theta \in A(E^p, F)$ then there is a linear map $\hat{\theta} : V \to F$ such that $\theta = \hat{\theta} \circ \mu$*

$$\begin{tikzcd}
 & V \arrow[dr, dashed, "\hat{\theta}"] & \\
E^p \arrow[ur, "\mu"] \arrow[rr, "\theta"'] & & F
\end{tikzcd}$$

Note: The $\hat{\theta}$ is unique given θ and V. The V is unique up to isomorphism.

We write $V = \Lambda^p E$ and given $v_1, \ldots, v_p \in E$ we write

$$v_1 \wedge \cdots \wedge v_p := \mu(v_1, \ldots, v_p)$$

We say $\Lambda^p E$ is the p-th exterior power of E.

1.1 Basis for $\Lambda^p E$

Let $e_1, \ldots, e_m$ be a basis for E. Since μ is surjective $\Lambda^p E$ is spanned by

$$\left\{ e_{i_1} \wedge \cdots \wedge e_{i_p} \;\middle|\; i_k \in I(m) \right\}$$

where we can assume that the i_k are distinct else their image would be null. We can also assume that the indices are in order up to sign.

Lemma 1.2. *These elements are linearly independent and hence form a basis.*

Therefore we can say $\dim(\Lambda^p E) = \binom{m}{p}$.

1.2 Wedge Product

Given $p, q \in \mathbb{N}$ with $p, q \geq 1$ we can define the bilinear wedge product

$$\cdot \wedge \cdot : (\Lambda^p E \times \Lambda^q E) \to \Lambda^{p+q} E$$

First we define on it on a basis. So take a basis $e_1, \dots, e_m$ of E and then define

$$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_q}) = e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$$

This can then be extended linearly to arbitrary elements and hence doesn't depend on our initial choice of basis.

1.3 Induced maps

Suppose we have a linear map between finite dimensional vector spaces

$$\phi : E \to F$$

then we get a multi linear map in the natural way

$$\phi^p : E^p \to F^p$$

By composing with the surjective map μ_F we get an alternating map

$$\mu_F \circ \phi^p : E^p \to \Lambda^p F$$

Hence by the defining property of $\Lambda^P E$ we get a linear map

$$\Lambda^p \phi : \Lambda^p E \to \Lambda^p F$$

with the property that the outer diamond in the below diagram commutes.

$$\begin{tikzcd}
& \Lambda^p E \arrow[dr, dashed, "\Lambda^p \phi"] & \\
E^p \arrow[ur, "\mu_E"] \arrow[rr, "\mu_f \circ \phi^p"'] \arrow[dr, "\phi^p"'] & & \Lambda^p F \\
& F^p \arrow[ur, "\mu_F"'] &
\end{tikzcd}$$

Essentially, if $e_1, \dots, e_m$ is a basis for E, then we can describe $\Lambda^p \phi$ by

$$(\Lambda^p \phi)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (\phi e_{i_1}) \wedge \cdots \wedge (\phi e_{i_p}).$$

1.4 ☠ The dreaded p-form ☠

Let M be an m-manifold. Given $x \in M$ we can form the p-th exterior power of the cotangent space

$$\Lambda^p(T_x^* M)$$

We can assemble these together into a vector bundle $\Lambda^p(T^*M)$. Subsequently, a p-form on M is define to be a section of the bundle $\Lambda^p(T^*M)$

What the fuck does this mean???

A more natural way to think about p-forms is to take local coordinates. Let $\phi : U \to \mathbb{R}^m$ be a chart yielding local coordinates $x_1, \ldots, x_m$. We have locally defined 1-forms $dx_1, \ldots, dx_m$ which form a basis for the cotangent space

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}$$

Then given $I \in \mathcal{I}(m,p)$ we write $\mathbf{d}x_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Thus $\{\mathbf{d}x_I \mid I \in \mathcal{I}(m,p)\}$ forms a basis for $\Lambda^P(T^*M)$. It follows that any p-form ω on U can be uniquely written in the form

$$\omega = \sum_{I \in \mathcal{I}(m,p)} \lambda_I \mathbf{d}x_I$$

where each $\lambda_I : U \to \mathbb{R}$ is a locally-defined smooth function.

Note: This is all we really need from the bundle structure of $\Lambda^p(T^*M)$.

In particular, if $p = m$ then an m-form locally looks like

$$\lambda \left(dx_1 \wedge \cdots \wedge dx_m \right)$$

for some smooth function $\lambda : U \to \mathbb{R}$.

1.5 Pull-backs

Suppose we have a smooth function between manifolds

$$f : M \to N$$

Given a p-form ω on N we can define a pull-back p-form $f^*\omega$ on M as follows. Given $x \in M$ we have the derivative map $d_x f$ and hence a dual map

$$(d_x f)^* : T^*_{f(x)} N \to T^*_x M, \qquad \eta \mapsto \eta \circ d_x f \quad \text{where } \eta : T_{f(x)} N \to \mathbb{R} \text{ is linear}$$

This in turn gives rise to a linear map

$$\Lambda^p (d_x \phi)^* : \Lambda^p T^*_{fx} N \to \Lambda^p T^*_x M$$

Then our pull-back is defined by

$$(f^*\omega)(x) := \left(\Lambda^p (d_x \phi)^* \right) \left[\omega(f(x)) \right]$$

One takes on blind faith that this is smooth and hence a p-form. In particular, we can pull back p-forms to any manifold embedded within a larger manifold (such as $\mathbb{R}^n$). This is a load of gobbledygook so let's go step by step.

1. $x \in M$

2. $f(x) \in N$

3. ω is a p-form on N so we get some linear maps $\eta_i : T_{fx}N \to \mathbb{R}$, then

$$\omega(f(x)) = \eta_1 \wedge \cdots \wedge \eta_p$$

4. Then we take the induced p'th exterior power map which just does $(d_x\phi)^*$ on each of the η_i

5. Hence we can write

$$(f^*\omega)(x) = (\eta_1 \circ d_x f) \wedge \cdots \wedge (\eta_p \circ d_x f)$$

Example:

1. Consider S^1; the unit circle in $\mathbb{R}^2$. Let θ by the angle coordinate so that $x = \cos\theta$ and $y = \sin\theta$. Then the pull back of dx, dy is obtained by differentiating these formulae:

$$-\sin\theta d\theta \quad \text{and} \quad cos\theta d\theta$$

From this we can pull back an arbitrary 1-form by linear extension.

2. Consider S^2; the unit 2-sphere in $\mathbb{R}^3$. Consider spherical polar coordinates θ, ϕ away from the poles.

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned}$$

Then the pull backs of dx, dy and dz repressively are

$$\begin{aligned} &\cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi \\ &\cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi \\ &-\sin\theta d\theta \end{aligned}$$

One can see this by writing out the Jacobian and then composing on the left with dx which is $(0, 0, 1)$ and remembering that $d\theta = \binom{1}{0}$ and $d\phi = \binom{0}{1}$.

So then the pull back of $dx \wedge dy = (\cos\theta\sin\theta)(d\theta \wedge d\phi)$. We can see this by writing out the full expression, using multi linearity and alternating-ness of the wedge product and then trigonometric identities.

1.6 Integration of m-forms

Take an atlas $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$. Given $\alpha, \beta \in \mathcal{A}$, then on the overlap $U_\alpha \cap U_\beta$ we get

$$dx_1{}^\alpha \wedge \cdots \wedge dx_m{}^\alpha = \Delta_{\alpha\beta}(x) dx_1{}^\beta \wedge \cdots \wedge dx_m{}^\beta$$

where $\Delta_{\alpha\beta}$ is the determinant of the Jacobian of the transition function $\phi_\alpha \circ \phi_\beta^{-1}$. Note if our atlas is oriented then $\Delta_{\alpha\beta}(x) > 0$. Hence we have the following result.

Theorem 1.3. *An m-manifold is orientable if and only if it admits a nowhere vanishing m-form.*

Proof. Worth looking over. □

Let M be an oriented manifold. Given an m-form ω on M we define

$$\text{supp}(\omega) := \overline{\{x \in M \mid \omega(x) \neq 0\}}$$

We say that a cover $\{U_\alpha\}$ of a Hausdorff space X is locally finite if

$$\forall x \in X \quad \exists O \subseteq X \text{ open, s.t. } x \in O \text{ and } |\{\alpha \mid O \cap U_\alpha \neq \emptyset\}| < \infty$$

that is around every point there is an open set which meets at most finitely many members of the cover.

For now, suppose that $\text{supp}(\omega)$ is compact. Let $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ be a locally finite, oriented atlas. Suppose that η is an n-form such that $\text{supp}(\eta) \subseteq U_\alpha$ for some $\alpha \in \mathcal{A}$. Then write in local coordinates $\eta = \lambda_\alpha(dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha)$ where $\lambda_\alpha : U_\alpha \to \mathbb{R}$ is smooth and compactly supported. Then we set

$$I_\alpha(\eta) := \int_{V_\alpha} \lambda_\alpha \circ \phi_\alpha^{-1}(x)\, dx$$

A partition of unity subordinate to $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0, 1]\}$ such that

1. $\text{supp}(\rho_\alpha) \subseteq U_\alpha$ for all α,
2. $\sum_\alpha \rho_\alpha(x) = 1$ for all $x \in M$.

So choose a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and set

$$\int_M \omega := \sum_{\alpha \in \mathcal{A}} I_\alpha(\rho_\alpha \omega)$$

Note: This is a finite sum because only finitely many U_α meet the support of ω.

Lemma 1.4. *This integral is well-defined. That is, its independent of choice of atlas and partition.*

Using this we can define the volume of a compact, orientable Riemannian manifold. Choose any orientation and let ω be the volume form (that is any m-form, I think). Then the volume is

$$\text{vol}(M) := \int_M \omega$$

In fact, if $f : M \to \mathbb{R}$ is any smooth function then we can integrate f with respect to volume. That is, integrate the m-form $f\omega$. The result $\int_m f\omega$ is often denoted informally as $\int_M f\, dV$. We shouldn't use this notation because exterior derivatives will confuse things.