Chapter 1

Mathematical Modelling of Fluid Flow

In this chapter, we will derive the equations of fluid dynamics that describe the motion of liquids and gases. The rest of the course will involve applying these equations to a remarkably wide range of phenomena.

⚠ Overview.ppt + website

1.1 What is Fluid Dynamics?

- Fluids, unlike solids, can 'flow' by changing their shape. They include not only liquids (water, oil, etc) but also gaseous media (air, plasma, etc.).
- Fluid flow occurs across huge variations in scale: from the very small (e.g. microdrops from a 3D printer) to the huge (tornados) and from the incredibly slow (motion of glaciers) to the very fast (aeronautics).

Figure 1.1: Top: Small (inkjet microdrops) → Large (torndao). Bottom: Slow (glacial flow) → Fast (aeronautics)

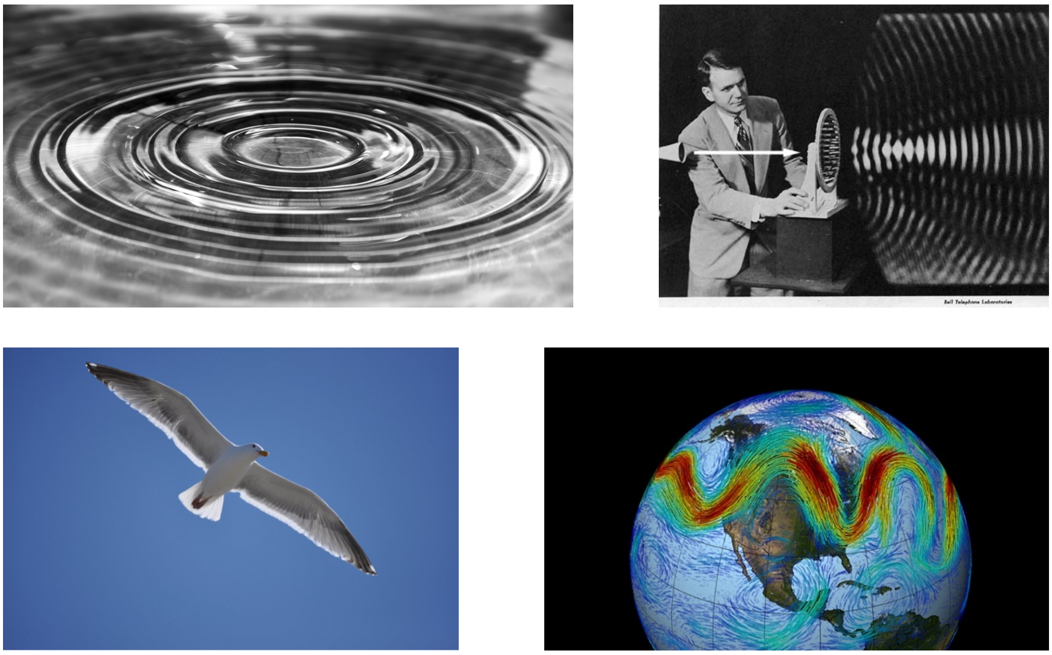


Figure 1.2: Everyday fluid phenomena - water waves, sound, flight and weather

- Such phenomena can be seen (and heard!) all around us, see figure 1.2.
- Motion at extreme scales often appear counter-intuitive.
- Remarkably, the same set of equations, the Navier-Stokes equations, can be used to describe the flow of both gases and liquids in all of the aforementioned situations.
- The study of fluids using these equations is often referred to as 'fluid mechanics', 'fluid dynamics' or 'hydrodynamics' and it assumes that the fluid can be idealised as a continuous medium, i.e. it uses continuum mechanics.
- The continuum approach is accurate when the length scale ℓ characterising molecular motion is much smaller than the dimensions of the flow L, so that $\ell/L \ll 1$ (the continuum limit takes $\ell/L \to 0$). Note that ℓ/L is a 'dimensionless number/parameter', which compares the size of two dimensional quantities - we will encounter more of these later [1].
- When $\ell/L \approx 1$ or $\ell \gg L$, microscopic methods (molecular dynamics for liquids and kinetic theory for gases) are required, see figure 1.3.

1.2 Fluids in Continuum Mechanics

Our focus will be on using continuum mechanics to describe the state of a fluid flow at coordinate $\mathbf{x} \in \mathbb{R}^d$ of d-dimensional space, $d = 1, 2$ or 3 at time t. In general we will have $d = 3$, but we will encounter $d = 2$, e.g. in atmospheric dynamics where vertical motion is suppressed, and $d = 1$, e.g. in flow through a pipe.

[1]For gases, if ℓ is taken to be the mean free path (i.e. the average distance between molecular collisions) then ℓ/L is called the Knudsen number.

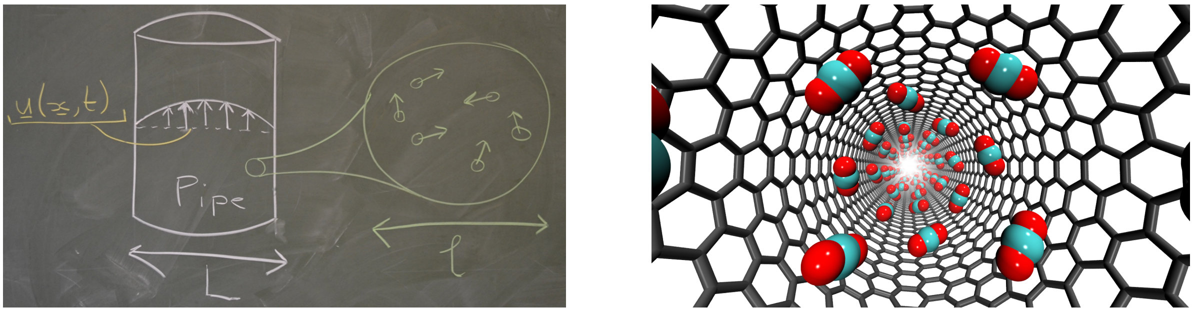


Figure 1.3: Left: sketch of flow through a pipe (or tube) of diameter L which will be accurately described by continuum mechanics if $\ell \ll L$, where ℓ is a characteristic length scale of the molecular flow. Right: an example where continuum mechanics is unlikely to work - a molecular dynamics simulation of a desalination membrane, where water molecules flow through a carbon nanotube (see http://www.micronanoflows.ac.uk).

- The state of the fluid flow is characterised by its velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, pressure $p = p(\mathbf{x}, t)$ and density $\rho = \rho(\mathbf{x}, t)$. In more complex cases, other relevant fields enter the description (e.g. temperature, electromagnetism, chemical composition, etc).
- Our main task is to find these fields (especially $\mathbf{u}$ [2]) and thus be able to predict the flow behaviour in space and time.
- Much of our focus will be on 'incompressible' fluids and we will assume ρ is constant, i.e. a material parameter.
- The incompressible Navier-Stokes equations we will derive are based on conservation of mass and momentum.
- The derivation is complicated because a fluid flows, so that conservation laws must be applied to a material that moves and deforms in time.
- The complexity of the Navier-Stokes equations makes solving them an extremely difficult task. One can either (a) use mathematical modelling to simplify the problem (our focus) and/or (b) use computational approaches (the field of Computational Fluid Dynamics - CFD).

1.3 Notation

Sometimes it will be convenient to use index notation (described/recapped below) rather than vector notation [3]. When using index notation, the summation convention and some additional shorthand provide a succinct way to write PDEs.

[2] As it tells us where 'stuff' is going.

[3] We reserve the right to use either, whenever it suits.

1.3.1 Index Notation

- Vectors and tensors will be written by their components *relative to a set of basis vectors* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. For example, the position vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = (x_1, x_2, x_3)$ will be written as $\mathbf{x} = x_i$, where 'i' is the *free index* which will take values $i = 1, 2, 3$ and we repeat that the x_i are components relative to a set of specified basis vectors. We will only use subscripts for indices.
- Usually, we will be considering Cartesian coordinates, so that $\mathbf{x} = x_i = (x, y, z)$, and it will be made clear when this is not the case. The specification $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ would be written as $u_i = u_i(x_j, t)$
- For matrices and tensors the same procedure will be used. For example, the components of the stress tensor are T_{ij} where i and j are free indices, meaning that in general this tensor has $3 \times 3 = 9$ components.
- The Kronecker delta is $\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$. One use it will have is to confine the gravitational acceleration $g\,\delta_{i3}$ to a single direction (usually aligned with the z-axis).
- Index notation can also be used to represent different operators, e.g. the gradient applied to the density ρ gives
$$\nabla\rho = \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right) = \frac{\partial\rho}{\partial x_i}$$
where we recall that $i = 1, 2, 3$ is implied in this notation.

1.3.2 Summation

- Where *repeated indices* are used, automatic summation over these is implied. For example, the dot product
$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \xcancel{\sum_{j=1}^{3}} a_jb_j = a_jb_j$$
is written in index notation succinctly as a_jb_j, with summation over j implied. The choice of repeated index is arbitrary, as this is a dummy variable which does not appear in the final result (e.g. a_kb_k would be equally valid for the dot product).
- Repeated indices can also be used for differential operators. Examples in Cartesian coordinates are:
  - The divergence $\nabla\cdot$ applied to a vector $\mathbf{u}$ is
$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \xcancel{\sum_{j=1}^{3}} \frac{\partial u_j}{\partial x_j} = \frac{\partial u_j}{\partial x_j}$$

- The Laplacian ∇^2 applied to a function f can be written as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \xcancel{\sum_{j=1}^{3}} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j} = \frac{\partial^2 f}{\partial x_j^2}$$

  where the repeated index is in the denominator ∂x_j^2.

- The curl of a vector $\mathbf{u}$ is

$$\omega_i = \nabla \times \mathbf{u} = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}, \qquad \epsilon_{ijk} = \begin{cases} 0 & \text{if } \ i = j, j = k \ \text{ or } \ i = k \\ 1 & \text{for } \ (i,j,k) = (1,2,3), (2,3,1) \ \text{ or } \ (3,1,2) \\ -1 & \text{for } \ (i,j,k) = (1,3,2), (2,1,3) \ \text{ or } \ (3,2,1) \end{cases}$$

  where we will later see that $\boldsymbol{\omega} = \omega_i$ is the *vorticity* and ϵ_{ijk} is the Levi-Civita symbol.

- When applied to the term $\mathbf{d} = \nabla^2 \mathbf{u}$ (the label $\mathbf{d}$ is arbitrary) in the Navier-Stokes equations we have

$$\mathbf{d} = d_i = \frac{\partial^2 u_i}{\partial x_j^2}$$

  with one free index i.

> **Turning Point Example**: How would you write the y-component of $(\mathbf{u} \cdot \nabla)\,\mathbf{u}$?
>
> The same rules apply for matrix-vector products such as $\mathbf{c} = (\mathbf{u} \cdot \nabla)\,\mathbf{u}$ (the label $\mathbf{c}$ is arbitrary) appearing in the Navier-Stokes equations which in index notation is
>
> $$\mathbf{c} = c_i = \xcancel{\sum_{j=1}^{3}} u_j \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j}.$$
>
> Note that the j's are repeated indices and the i is the sole free index. Therefore, in Cartesian coordinates, the $i = 2$ (y-) component would be
>
> $$c_2 = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}.$$
>
> Observe also that the brackets in $(\mathbf{u} \cdot \nabla)$ indicate that the dot product is between the velocity $\mathbf{u}$ and the gradient operator ∇.

- Always check that each term in an equation formed using index notation has to contain the same free indices (e.g. $a_{ij} = b_{ijk}e_k + f_{ij}$ is fine, but $a_{ij} = b_{ijk}e_j + f_{ik}$ would not be!).

1.3.3 Shorthand

- A shorthand we will sometimes use (often in the text) is to shorten partial differentials (e.g. $\frac{\partial}{\partial t}$) by replacing the denominator by a subscript which indicates what the differential is with respect to (e.g. $\partial_t = \frac{\partial}{\partial t}$).

- This shorthand can be used in conjunction with indices. For example,

$$\frac{\partial \rho}{\partial x_i} = \partial_{x_i}\rho \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x_j^2} = \partial_{x_j x_j} f.$$

1.4 Lagrangian and Eulerian Descriptions

There is a video in the Complementary Material on 'Lagrangian and Eulerian descriptions of fluid flow'.

The equations of fluid dynamics will be formulated in Eulerian coordinates; but, it will also be important to appreciate Lagrangian coordinates which 'follow the fluid'.

- In classical mechanics, we know the motion of a material point if we can specify its Eulerian coordinates x_i as a function f_i of time

$$x_i = f_i(t).$$

  To extend this to a fluid modelled as a continuous medium we need to identify and follow the (infinitely many) material points which form the continuum.

- Material points in a fluid will be referred to as *fluid elements* or *fluid particles*, i.e. infinitesimal volumes of fluid which are advected with the fluid's velocity $\mathbf{u}$ and hence are always composed of the same material.

- One way to uniquely identify each fluid element x_i is by its Eulerian coordinates $x_j(t = t_0) = X_j = (X_1, X_2, X_3)$ at a reference (or initial) time t_0 and then allow these coordinates to move and deform with the fluid throughout its motion. The X_j are then Lagrangian coordinates (see figure 1.4) and the position of a fluid element $x_i(t)$ at a time t is given by

$$x_i = f_i(X_j, t)$$

- Whilst in the Eulerian description one sits at a fixed position and watches the flow pass (like standing at the bank of a river), in the Lagrangian description we 'follow the fluid' (as in a boat flowing with the river or on a weather balloon).

- It is possible to formulate the equations of fluid mechanics in Lagrangian coordinates (so that $u_i = u_i(X_j, t)$) but this is quite un-natural for fluid flow as it is impossible to identify a sensible reference configuration (not such an issue in solid mechanics).

- The notion of 'following the fluid' will be an important one, as the laws of nature (e.g. Newton II) are formulated for material substances (i.e. fluid elements), rather than to arbitrary points in space (which are composed of different fluid elements at every instant). This consideration will lead us to the notion of the material derivative, which allows us to express the rate of change inside a material element (i.e. a Lagrangian derivative) in terms of Eulerian coordinates.

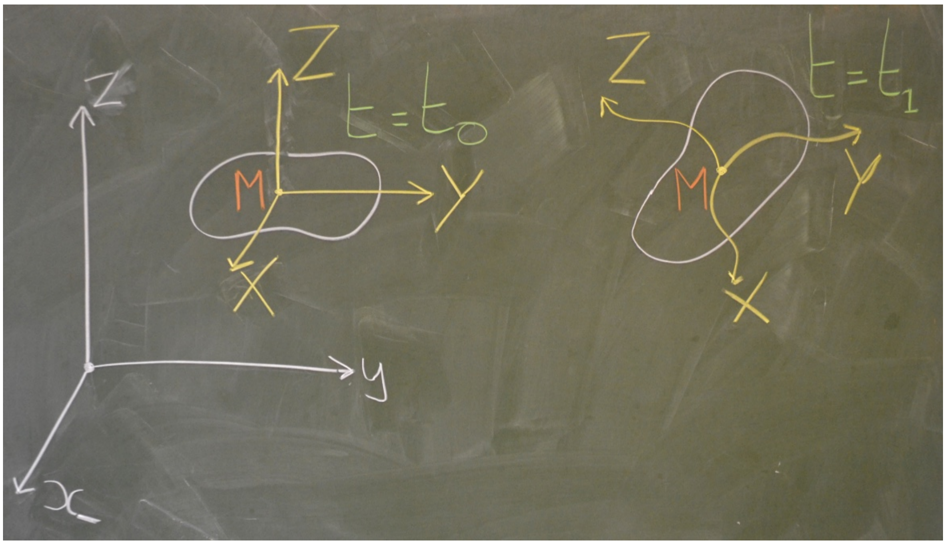


Figure 1.4: The motion of a material fluid element M whose properties (e.g. velocity) can be described by Eulerian coordinates (x, y, z) or Lagrangian ones (X, Y, Z). The Lagrangian coordinates remain frozen inside M as it evolves and deforms in time (from t_0 to t_1), i.e. they follow material points as advected by the fluid velocity.

1.5 Flow Visualisation

There is a video in the Complementary Material on 'Flow Visualisation'.

The most common way to visualise a flow, theoretically (once we have $\mathbf{u}$) or experimentally, is to consider the paths which particles take inside it. Alternatively, we may be interested in the velocity field at an instant in time (the streamlines). Only for a steady flow ($\partial_t \mathbf{u} = \mathbf{0}$) do these curves coincide.

1.5.1 Particle Paths

The paths of fluid elements ('*particle paths*') $x_i^p(t)$ are defined by

$$\left.\frac{\partial x_i^p(t)}{\partial t}\right|_{\text{fixed } X_k} = u_i \tag{1.1}$$

with *initial positions fixed* $x_i^p(t = 0) = X_i$. Upon integration, eliminating t gives the particle paths.

This also provides a way to switch from the Eulerian description to the Lagrangian one, with the three constants of integration (X, Y, Z) specifying the Lagrangian coordinates of a particular trajectory (i.e. a specific fluid element).

1.5.2 Streamlines

In general, the particle paths differ from the *streamlines* defined by curves which are tangential to the velocity at every point at a *fixed time*. Mathematically, if we parameterise these curves by an independent parameter s then our streamlines $(x, y, z) = (x(s), y(s), z(s))$ are obtained by solving

$$\left.\frac{\partial x_i(s)}{\partial s}\right|_{\text{fixed } t} = u_i. \tag{1.2}$$

Turning Point Example (from Acheson Ex 1.8): Consider the two-dimensional flow given by

$$\mathbf{u} = (u_0, kt, 0), \qquad \text{with} \qquad u_0, k > 0.$$

What is the shape of the particle paths?

The particle paths are obtained from solving

$$\frac{\partial x^p}{\partial t} = u_0, \qquad \frac{\partial y^p}{\partial t} = kt \qquad \frac{\partial z^p}{\partial t} = 0$$

for fixed initial position $(x^p, y^p, z^p) = (X, Y, Z)$ at $t = 0$. Therefore,

$$(x^p, y^p, z^p) = (u_0 t + X, \frac{1}{2}kt^2 + Y, Z)$$

which, notably, could be used to transform from Eulerian to Lagrangian coordinates (by expressing (x, y, z) in terms of (X, Y, Z)). We can eliminate t to find the particle paths

$$y^p = \frac{k}{2}\left(\frac{x^p - X}{u_0}\right)^2 + Y$$

which are parabolas (figure 1.5).
The streamlines come from

$$\frac{\partial x}{\partial s} = u_0, \qquad \frac{\partial y}{\partial s} = kt, \qquad \frac{\partial z}{\partial s} = 0$$

for a *fixed* t. Therefore

$$(x, y, z) = (u_0 s + a, kts + b, c)$$

for a, b, c constant. Eliminating s we find that

$$y = \frac{kt}{u_0}x + (b - kta/u_0)$$

showing that the streamlines are straight lines whose gradient will increase in time. The result is figure 1.5.

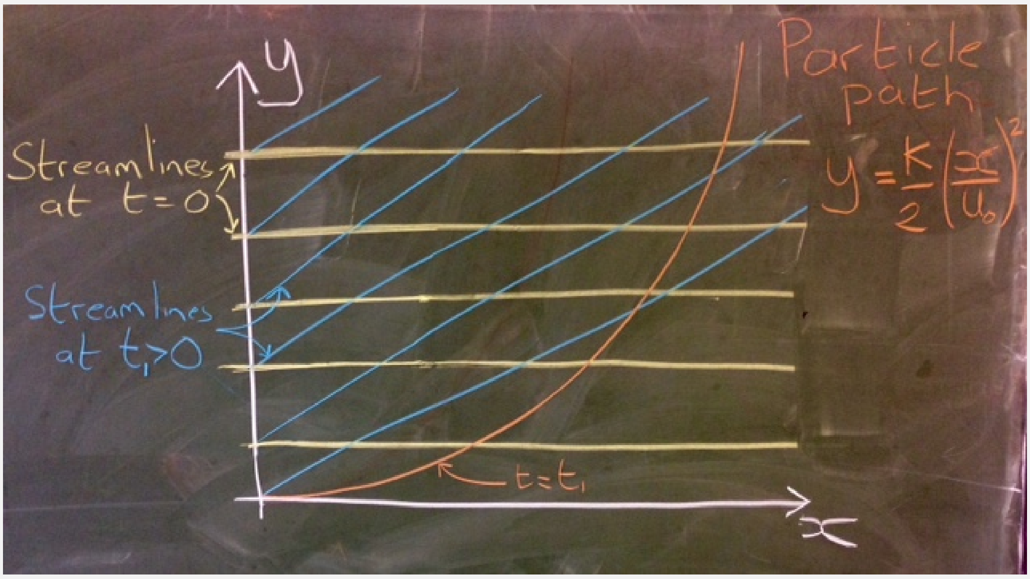


Figure 1.5: Sketch showing how streamlines change in time, differing at $t = 0$ and $t = t_1$, and the particle path $y = \frac{k}{2}(x/u_0)^2$ of a fluid element starting from $X = Y = 0$ at $t = 0$.

1.5.3 Experimental Visualisation

To visualise a flow field one usually places illuminated passive tracers (e.g. dye, bubbles or beads that don't alter the flow) into the flow (figure 1.6). Then, to obtain particle paths one simply tracks the position of these tracers as a function of time, whilst to obtain the streamlines one could take a short exposure photo so that the illuminated tracers show short streak lines (whose length will be proportional to the fluid speed). In a steady flow, this streamline pattern will be fixed for all times and a fluid particle starting on a streamline will remain on it, whereas for unsteady flow the particle paths and streamlines will not coincide.

Notably, even in a steady flow, as fluid particles travel along the streamlines their velocity will change, i.e. they will accelerate and decelerate. To formalise this rate of change caused by the evolution of fluid particles, we have to introduce the material derivative.

1.6 The Material Derivative

There is a video in the Complementary Material on 'Material Derivative applied to the concentration of a pollutant in a river'.

The *material derivative* $D_t = \frac{D}{Dt}$ gives the rate of change of a quantity which 'follows the fluid' (i.e. moves under the fluid's velocity field) as opposed to ∂_t which gives the rate of change of a quantity at a point fixed with respect to Eulerian coordinates. In other words, it gives the time derivative moving along a fluid particle trajectory described by (1.1).

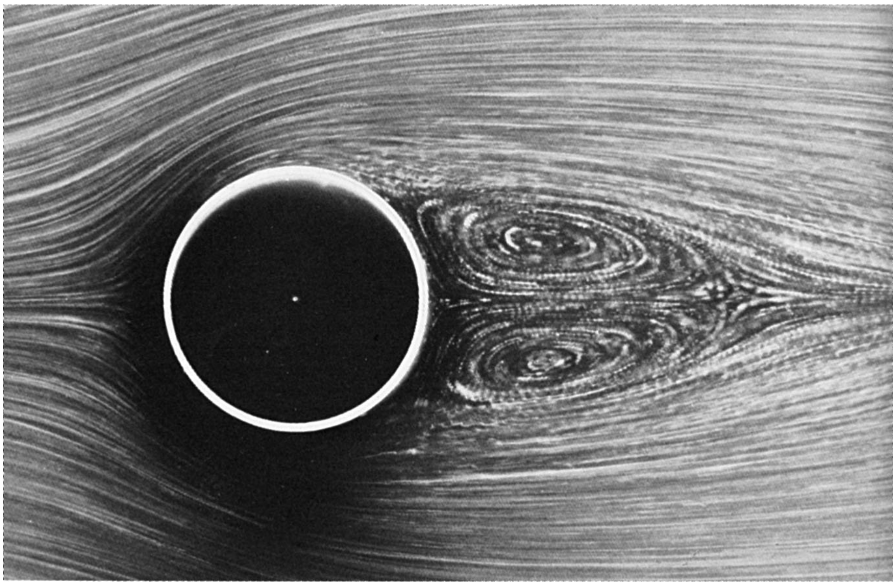


Figure 1.6: Steady laminar flow (from left to right) around a solid cylinder with streamlines/particle paths showing the trajectories of fluid elements

When applied to a quantity (scalar or vector) $f(x_i^p(t), t)$, written in Eulerian coordinates, a simple application of the chain rule gives:

$$\frac{Df}{Dt} = \frac{d}{dt} f(x_i^p(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{\partial x_i^p(t)}{\partial t} \overset{(1.1)}{=} \left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \right) f$$

so that symbolically the material derivative operator is

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \tag{1.3}$$

- The first term is the local rate of change at a fixed Eulerian position $\mathbf{x}$ due to temporal changes.
- The second is the 'convective rate of change' caused by the velocity $\mathbf{u}$ driving fluid elements through spatial gradients in the quantity of interest.

Turning Point Example: If there is a concentration of pollutant $c = c(x)$ in a river that flows steadily with constant speed $\mathbf{u} = (u_0, 0, 0)$, how does the concentration change in a fluid element that 'follows the fluid'?

Note that (a) the concentration at a fixed point does not change with time $\partial_t c = 0$ but (b) the concentration inside a fluid element that 'follows the fluid' will change in time, at a rate given by the material derivative

$$\frac{Dc}{Dt} = \cancelto{0}{\frac{\partial c}{\partial t}} + u \frac{\partial c}{\partial x} + \cancelto{0}{v \frac{\partial c}{\partial y}} + \cancelto{0}{w \frac{\partial c}{\partial z}} = u_0 \frac{\partial c}{\partial x}.$$

which is non-zero if and only if there is flow $u_0 \neq 0$ and spatial gradients in $\partial_x c \neq 0$.

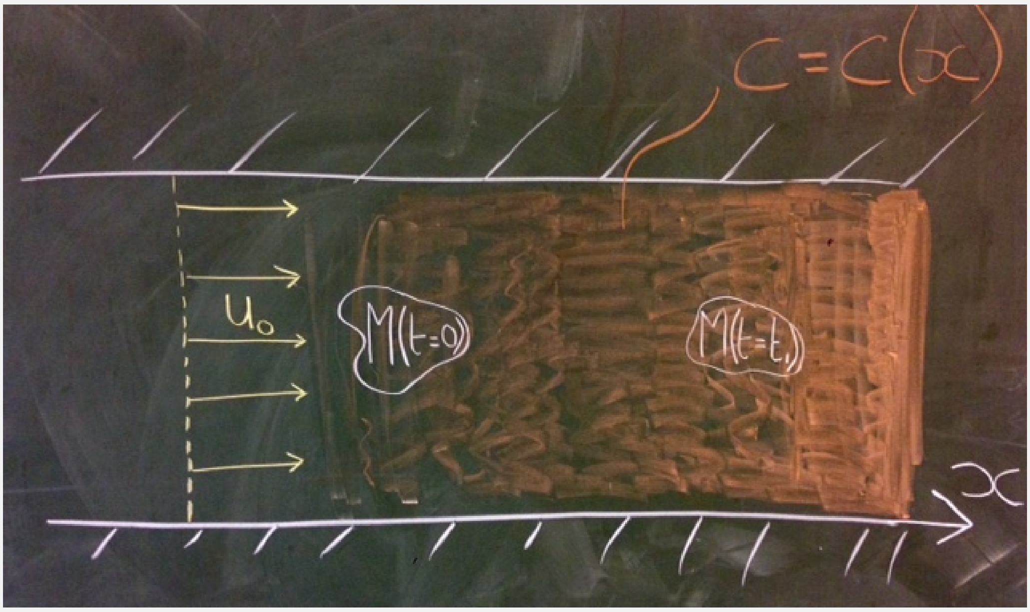


Figure 1.7: Evolution of a fluid element through a spatial gradient in concentration $c = c(x)$ showing that the concentration inside the fluid element will change from $t = 0$ to $t = t_1$ even though the flow is steady $\partial_t \mathbf{u} = \mathbf{0}$.

- The material derivative will be important when it comes to a consideration of the acceleration a_i of a fluid element.
- If $D_t f = 0$, then clearly the quantity f is a constant *in a particular fluid element*. However, this constant can take different values in different fluid elements - all we know is that f retains the value it started with.
- If $\mathbf{e}_s$ is a unit vector parallel to a streamline and in the direction of flow, then

$$\mathbf{u} \cdot \nabla f = |\mathbf{u}| \mathbf{e}_s \cdot \nabla f = |\mathbf{u}| \frac{\partial f}{\partial s}$$

  with s the distance along a streamline. This shows that the convective term is the rate of change of f along a streamline $\partial_s f$ multiplied by the speed $|\mathbf{u}|$ at which it travels along that streamline, i.e. it is the rate of change of f with time.
- It follows that if $(\mathbf{u} \cdot \nabla) f = 0$, then f is constant *along a streamline*. However, it may take different values on each streamline.

1.6.1 Acceleration

It is the material derivative which gives us the rate of change of velocity, i.e. the acceleration $\mathbf{a} = a_i$, of the fluid element

$$a_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \qquad \text{or} \qquad \mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{1.4}$$

Notably, even when the flow is *steady*, so that $\partial_t \mathbf{u} = 0$, the acceleration still has a component $(\mathbf{u} \cdot \nabla)\mathbf{u}$ due to the spatial variation of the velocity field—this is the convective acceleration.

Turning Point Example: If we take a one-dimensional steady flow $\mathbf{u} = (u(x), 0, 0)$, then what is the acceleration of fluid particles?

$$a_1 = \cancelto{0}{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\cancelto{0}{\frac{\partial u}{\partial y}} + w\cancelto{0}{\frac{\partial u}{\partial z}}\,, \qquad a_2 = a_3 = 0,$$

so that where there is a spatial gradient in u there is an acceleration.

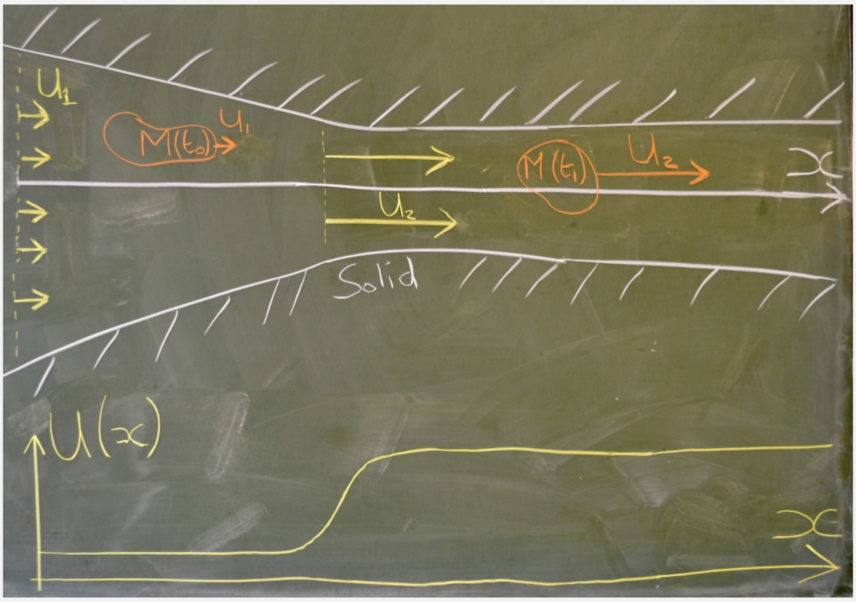


Figure 1.8: The acceleration of a fluid element M as it travels from an area of low speed (left) at time t_0 to a region of high speed (right) at $t = t_1$. Therefore, despite the flow being steady, so that at a fixed point $\partial_t u = 0$, the fluid inside M feels an acceleration due to its motion.

In the Examples Class you will consider a *steady* flow in uniform rotation with angular velocity Ω with velocity given by $(u, v, w) = (-\Omega y, \Omega x, 0)$. As we know, fluid particles must be accelerating inwardly (centrifugal force) and you will see it is the convective acceleration that accounts for this term.

1.7 Vorticity and the Rate of Strain Tensor

The study of the deformation of continuous media (including solid and fluid dynamics) is a course in its own right (used to be a 4th year course at Warwick), so we will only consider the concepts we will require to formulate the equations of fluid mechanics. In particular, we will attempt to understand how flow deforms fluid elements; as it is this (rate of) deformation that will generate the internal stress (and hence forces) required by our momentum balance equations (Newton II).

The velocity can be decomposed into fundamental components which specify the kinematics of the flow (the geometry of the motion), i.e. how small fluid elements are deformed by the flow. To do so, consider the flow at an infinitesimal distance δx_j from a reference point x_j, see figure 1.9.

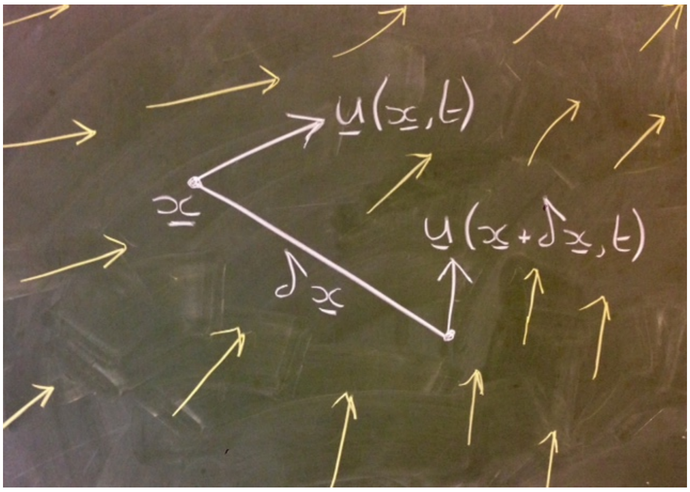


Figure 1.9: Flow kinematics - how the flow velocity varies in the vicinity of (a distance $\delta \mathbf{x}$ from) a point $\mathbf{x}$.

The Taylor expansion of $u_i(x_j + \delta x_j)$, keeping only the terms up to the first power in δx_j inclusive, gives

$$u_i(x_j+\delta x_j, t) = u_i(x_j, t) + \frac{\partial u_i}{\partial x_j}\,\delta x_j = \underbrace{\underbrace{u_i(x_j)}_{\text{Translation}} + \underbrace{r_{ij}\,\delta x_j}_{\text{Rotation}}}_{\text{Rigid Body Motions}} + \underbrace{e_{ij}\,\delta x_j}_{\text{Shearing and Extension}} \tag{1.5}$$

where we have decomposed $\partial_{x_j} u_i$ into an anti-symmetric ($r_{ij} = -r_{ji}$) *rate of rotation tensor*

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

and the symmetric ($e_{ij} = e_{ji}$) *rate of strain tensor*

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Each of the terms contributing to the velocity

$$u_i(x_j+\delta x_j, t) = u_i^T + u_i^R + u_i^S, \qquad \text{where} \qquad u_i^T = u_i(x_j), \quad u_i^R = r_{ij}\,\delta x_j, \quad u_i^S = e_{ij}\,\delta x_j \tag{1.6}$$

can now be analysed

- The first term in (1.5) is due to rigid body translation $u_i^T = u_i(x_j, t)$. If $r_{ij} = e_{ij} = 0$, then all fluid elements have the same velocity.

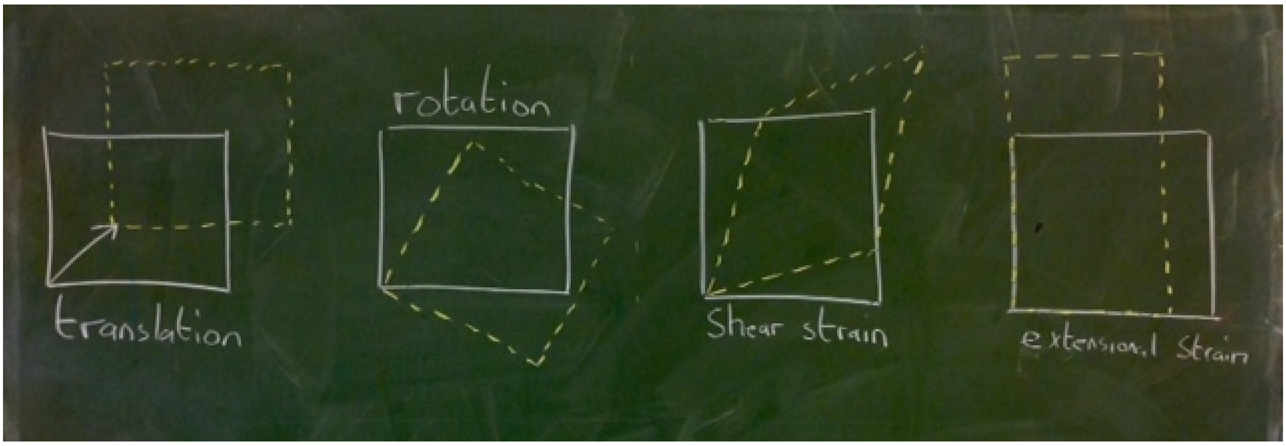


Figure 1.10: Different modes of deformation to a fluid element (in 2D).

- Due to the antisymmetry of r_{ij}, there are only 3 independent components (all diagonal elements zero) and we can rewrite the term $u_i^R = r_{ij}\,\delta x_j$ as a cross product $u_i^R = \epsilon_{ijk}\Omega_j\delta x_k$ or $\mathbf{u}^R = \mathbf{\Omega} \times \delta\mathbf{x}$ (see figure 1.11), where $\Omega_j = (r_{32}, r_{13}, r_{21})$ is the rate of rotation *vector* (you can check this). Thus we see that this term is associated with rigid body rotation about the reference point with rotational rate $\mathbf{\Omega}$.

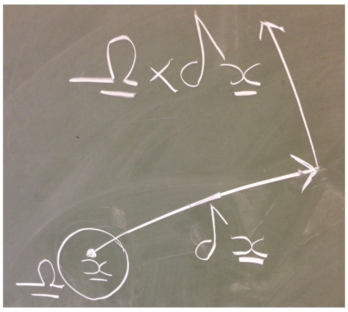


Figure 1.11: Sketch illustrating the rigid body rotation $\mathbf{u}^R = \mathbf{\Omega} \times \delta\mathbf{x}$ induced a distance $\delta\mathbf{x}$ away from a point $\mathbf{x}$ by the rotation rate $\mathbf{\Omega}$. Here, $\mathbf{\Omega}$ points out of the paper.

Notably, the rate of rotation vector $\mathbf{\Omega}$ is equal to half the *vorticity* vector

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = 2\mathbf{\Omega}, \qquad \text{so that} \qquad \mathbf{u}^R = \frac{1}{2}\boldsymbol{\omega} \times \delta\mathbf{x} \tag{1.7}$$

which is an important fluid mechanical quantity we will repeatedly encounter - now we can recognise it as a measure of the local rate of rotation in the fluid. In Cartesian coordinates it is given by

$$\boldsymbol{\omega} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

> **Turning Point Example**: How many components of vorticity does a 2D flow have?
>
> For a two-dimensional flow $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$, the vorticity has only one component $(0, 0, \omega)$ where $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, with a rotation axis in the z-direction.

- As the first two terms in (1.5) are associated with rigid body motion, it is the third term $u_i^S = e_{ij}\delta x_j$ which represents the straining motions which distinguish the flow through the relative motion of adjacent fluid elements. The rate of strain tensor e_{ij} can be further decomposed into (a) diagonal elements which are associated with extensional motion and (b) off-diagonal elements give the shear rate of strain.
- Consider how the components of e_{ij} deform an infinitesimal rectangular fluid element of size $\delta x \times \delta y$ (figure 1.12) in a two-dimensional flow $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$. For simplicity, we will remove translation of the fluid element by considering flow relative to the point A, so that a Taylor expansion gives the flow components at adjacent points to A shown in figure 1.12.

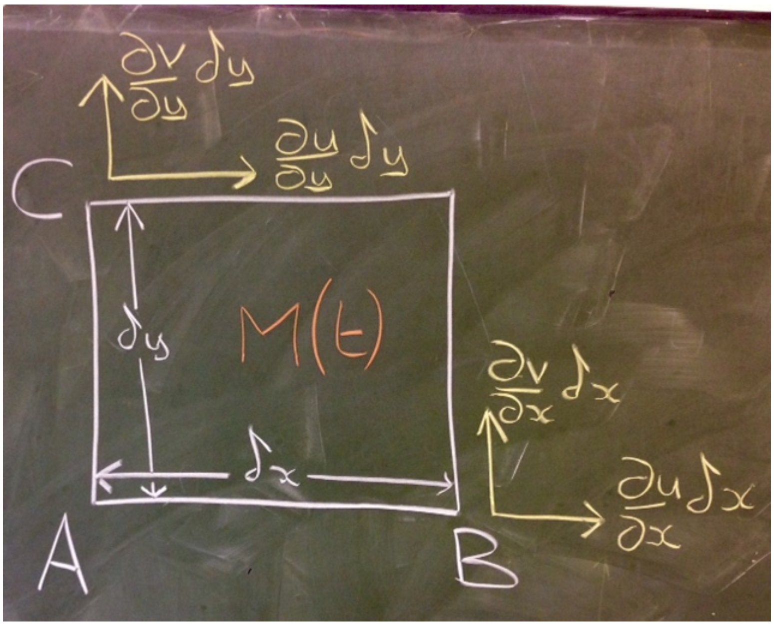


Figure 1.12: The velocity components relative to a point A in a material element M at time t.

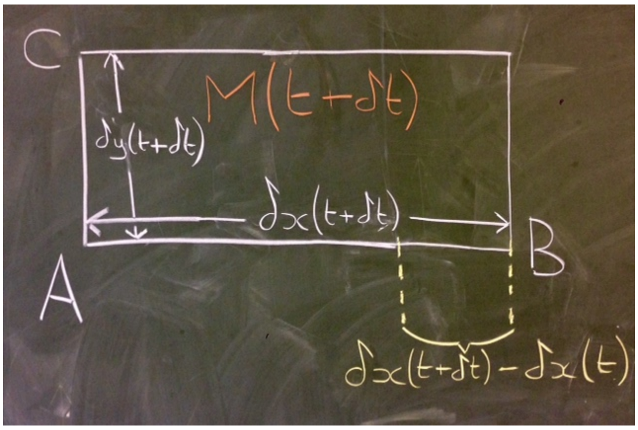
  - From figure (1.13) we can see that *extensional rate of strain* acts to increase the lengths of the sides. For example, taking the line AB, in a time δt the point B moves so that the new length of AB at time $t + \delta t$ is

$$\delta x(t + \delta t) = \delta x(t) + \underbrace{\frac{\partial u}{\partial x}}_{\text{speed}} \delta x \underbrace{\delta t}_{\text{time}}. \tag{1.8}$$

Therefore, the (infinitesimal) strain (extension/original length)

$$\delta S_x = (\delta x(t+\delta t) - \delta x(t))/\delta x(t) \overset{(1.8)}{=} \frac{\partial u}{\partial x}\delta t \quad \text{so the rate of strain is} \quad \frac{\partial S_x}{\partial t} = \frac{\partial u}{\partial x} = e_{11}$$

as hoped/expected.



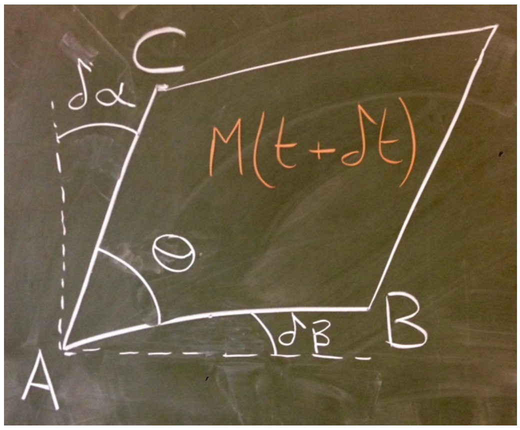


Figure 1.13: Left: deformation due to extensional strain on the material element at a time $t + \delta t$. Right: influence of shear strain on the material element at a time $t + \delta t$.

– It can be shown that the extensional components change the volume V of the fluid element as

$$\frac{1}{V}\frac{DV}{Dt} = e_{kk} \qquad \text{symbolically} \qquad \frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \mathbf{u} \tag{1.9}$$

so that for an incompressible fluid, where the volume of the fluid element does not change in time ($D_t V = 0$), we have $e_{kk} = e_{11} + e_{22} + e_{33} = 0$. In other words, extension of a fluid element in one direction must lead to contraction in one of the other directions.

– The *shear* rate of strain acts to drive the element from its rectangular shape. This can be quantified by measuring the angle between the sides. It can be shown that (see Examples Sheet 2) the angle θ at CAB in figure 1.12 evolves according to

$$-\frac{\partial \theta}{\partial t} = 2e_{12}.$$

So we have seen that it is the vorticity and rate of strain that give us all of the information we require to understand how fluid elements are locally deformed, with the former telling us about the rotation of the flow and the latter about the relative motion of fluid elements (which generates viscous stress).

An important class of flows we will encounter are those without vorticity $\boldsymbol{\omega} = \mathbf{0}$, and they are said to be *irrotational*. In such flows, each fluid element has no angular velocity. For example, we will see that fluid elements in the viscous boundary layer near a solid are rotational, so that these elements rotate end over end as they move parallel to the boundary, whereas elements outside this later are irrotational (figure 1.14).

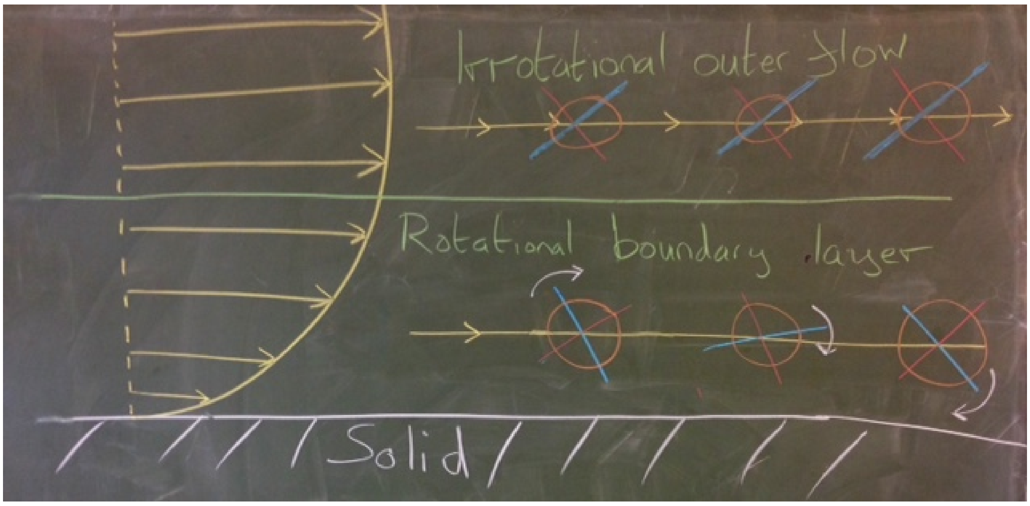


Figure 1.14: The difference between rotational (in the boundary layer) and irrotational flow (outside the layer). The crosses represent markers placed into the flow, which would rotate inside the boundary layer and not outisde.

Turning Point Example (Adapted from Acheson §1.4): Consider the 'shear flow' $\mathbf{u} = (\beta y, 0, 0)$. How does this simple flow deform fluid elements? E.g., what is the contribution from deformation to (1.6)?

Let's consider all the different contributions to (1.6).

- The translational term is just $u_i^T = (\beta y, 0, 0)$.
- The vorticity will only have a component in the z-direction $\boldsymbol{\omega} = (0, 0, \omega)$ which is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\beta.$$

  So there is rotation in the flow, despite the streamlines and particle paths being straight. This is because the velocity at the point C is greater than that at A so the line CA rotates clockwise (figure 1.15). The rate of rotation will be $\omega/2 = -\beta/2$.

  Therefore, the contribution to (1.6) is

$$u_i^R = \frac{1}{2}\epsilon_{ijk}\omega_j \delta x_k = \frac{1}{2}(\beta dy, -\beta dx, 0)$$

- The only non-zero entries in the rate of strain tensor will be due to $\partial_y u = \beta$. Therefore, extensional strains are all zero (ensuring incompressibility) and the only non-zero entries are $e_{12} = e_{21} = \beta/2$. Consequently, the contribution to (1.6) is

$$u_i^S = \frac{1}{2}(\beta dy, \beta dx, 0).$$

- Therefore, even in this simple flow there are rigid body motions of translation and rotation as well as deformation.

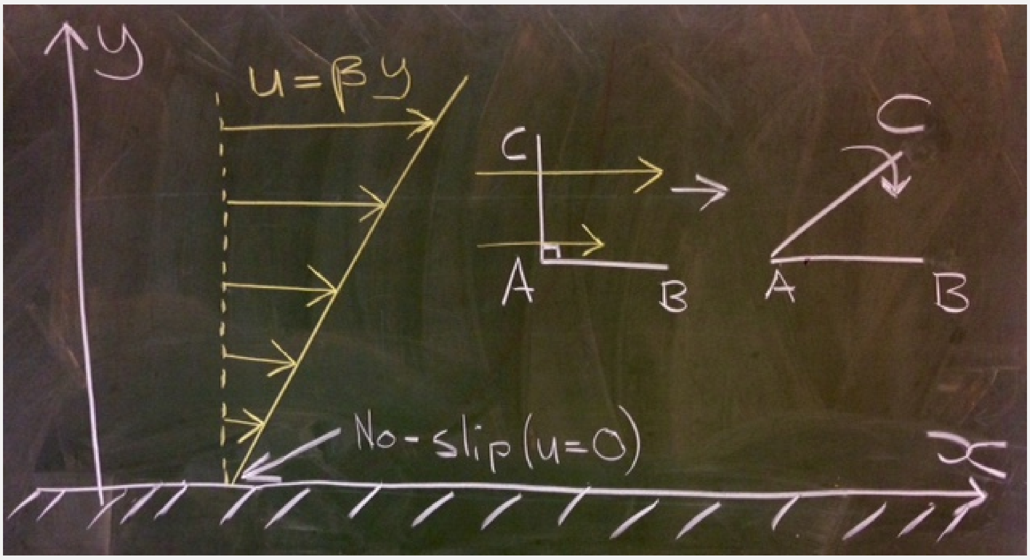


Figure 1.15: The deformation of two initially perpendicular fluid line element (i.e. they 'follow the fluid') in a shear flow showing how AC rotates - hence the flow contains vorticity.

1.8 Conservation of Mass

There are a variety of ways to derive the equations of fluid mechanics, some of which are based on integral representations (see Batchelor & Examples Sheet) and some which directly form the required PDEs, as considered here. Our approach is chosen to be fit for purpose and will be based on considering the conservation of mass and momentum over an infinitesimal cube in the fluid.

- Consider an infinitesimal volume fixed in space (i.e. it does not follow the fluid), which for simplicity is chosen to be a cube, with sides of length d aligned with a Cartesian axes (see figure 1.16), so that its bottom corner is at $\mathbf{x_0} = (x_0, y_0, z_0)$.

  The law of conservation of mass tells us that the rate of increase of fluid mass (volume d^3 times rate of change of density $\partial_t \rho$) inside this cube is equal to the total mass flux in through its six sides:

$$d^3 \frac{\partial \rho}{\partial t} = d^2 [\underbrace{-\rho(x_0 + d, y, z)u(x_0 + d, y, z)}_{\text{Term in Figure 1.16}} + \rho(x_0, y, z)u(x_0, y, z)$$

$$-\rho(x, y_0 + d, z)v(x, y_0 + d, z) + \rho(x, y_0, z)v(x, y_0, z)$$

$$-\rho(x, y, z_0 + d)w(x, y, z_0 + d) + \rho(x, y, z_0)w(x, y, z_0)] =$$

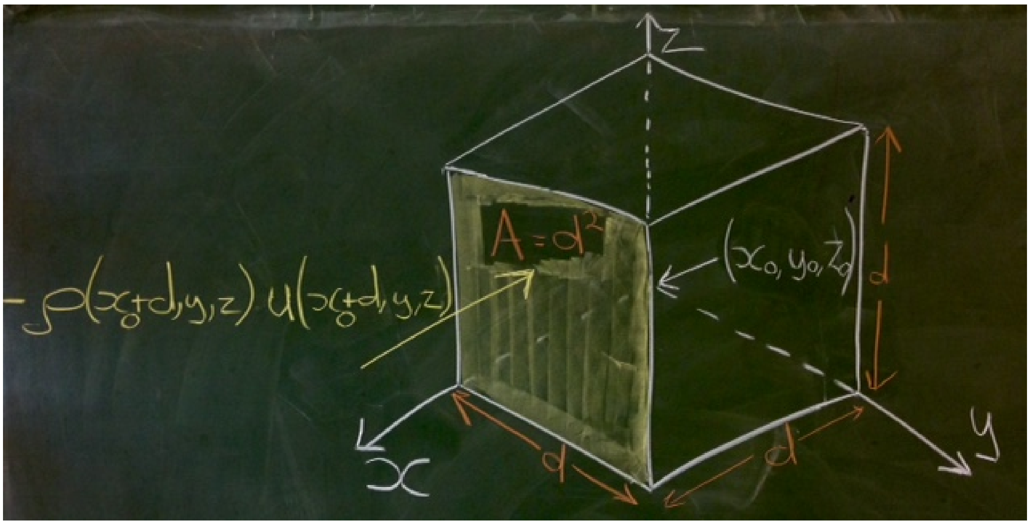


Figure 1.16: An infinitesimal fluid volume aligned with Cartesian axes. The mass flux *into* the face with outward normal $\mathbf{n}_x = (1,0,0)$ is given by -$\rho(x_0+d,y,z)u(x_0+d,y,z)$ multiplied by the area $A = d^2$, where the minus sign accounts for the fact that the flow into this face is $\mathbf{u}\cdot(-\mathbf{n})$.

> **Turning Point Example**: How can we rewrite the right hand side of this expression?
>
> $$-d^3\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+O(d)\right] = -d^3\,\nabla\cdot(\rho\mathbf{u})+O(d^4).$$
>
> where $\mathbf{u} = (u,v,w)$ is the velocity field.

Dividing both sides by d^3, in the limit of small d we get the desired equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u}) = 0. \tag{1.10}$$

Now that the equation is in vector form, we can apply it to any coordinate system (despite being derived in Cartesian).

- Alternatively, we could have considered the conservation of mass applied to an infinitesimal fluid element of volume V which 'follows the flow'. In this case, the mass inside the element must remain unchanged (as there is no flux in or out of it) as ρV so that

$$\frac{D(\rho V)}{Dt} = V\frac{D\rho}{Dt}+\rho\frac{DV}{Dt} = 0.$$

But we know from (1.9) that $\frac{DV}{Dt} = V\nabla\cdot\mathbf{u}$ so that (noting V is arbitrary) we have

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{u} = 0, \tag{1.11}$$

which is just an alternative way of writing (1.10). This form shows that variations in the density of a fluid particle ($\frac{D\rho}{Dt}\neq 0$) are caused by the velocity field having a non-zero divergence ($\nabla\cdot\mathbf{u}\neq 0$).

- In general, for compressible flows we will also require an energy balance relation and an equation of state specifying the properties of the medium. However, there is a simpler class of flows where we can assume $p = p(\rho)$ to close the system, known as barotropic flows, where for the simplest case of *isentropic gas* we have

$$p \propto \rho^{\gamma} \tag{1.12}$$

where γ is a constant whose value depends on the properties of the gas.

- Compressible flows will only be considered in our treatment of sound waves. For many cases, especially for liquids, we will see that the velocity is approximately divergence-free, so that conservation of mass is simply

$$\nabla \cdot \mathbf{u} = 0. \tag{1.13}$$

In §1.11.2, we will determine when this approximation of incompressibility is valid.

1.9 Conservation of Momentum

Here, we will formulate conservation of momentum for a fluid element (which follows the flow), which involves applying Newton's second law to these material elements. As we already know how to calculate the acceleration using the material derivative, all we require is the forces acting on fluid element. These will split into two contributions - body forces (we will only consider gravitational forces in this course) and internal forces (pressure and stress).

1.9.1 Stress

Definition. Stress is force per unit area (e.g. pressure).

Definition. Stress tensor T_{ij} is the ith component of the stress on the surface element which has outward normal $\mathbf{n}_j$ pointing in the j-th direction (figure 1.17). For example, in Cartesian coordinates $T_{32} = T_{zy}$ would be the z-component of stress acting on a face with normal $(0, 1, 0)$ (i.e. it would be a 'shear' or 'tangential' stress component). We will consider cases in which $T_{ij} = T_{ji}$ is symmetric.

Definition. The stress at a point on a surface with outwards normal n_j is $T_{ij}n_j$ and this generates a force $\delta F_i = T_{ij}n_j\delta A$ over an area δA, with $t_i = T_{ij}n_j$ known as the stress vector.

Let us consider forces acting on a infinitesimal cubic fluid element (that follows the fluid) with side d and bottom corner at $\mathbf{x} = (x_0, y_0, z_0)$; see figure 1.17. We will start with the x-component of this force which will involve us calculating $T_{1j}n_j\delta A$ (with $\delta A = d^2$) over all six sides of the cube.

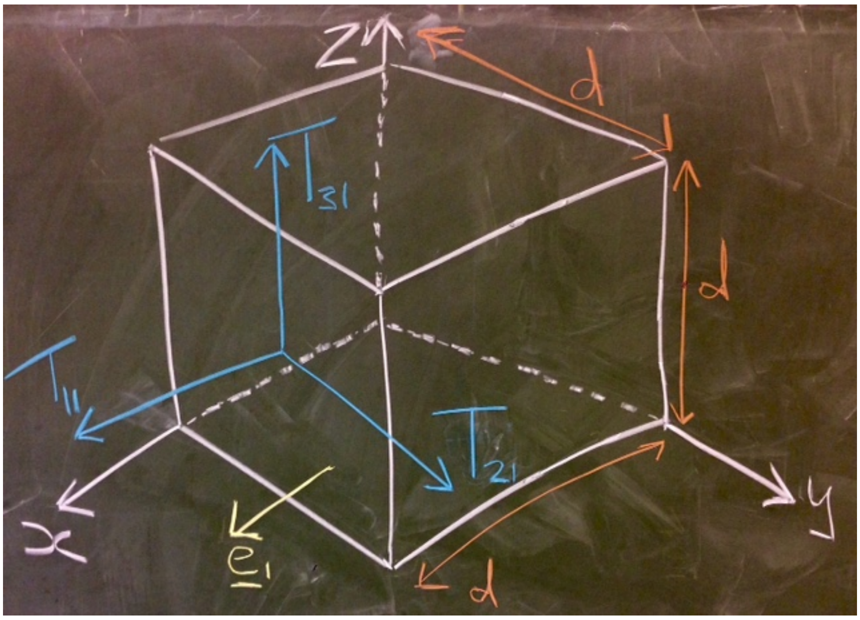


Figure 1.17: An infinitesimal cubic fluid element. Shown are the components of the stress tensor acting on one face whose normal aligns with the x-axis.

$$\begin{gathered}\delta F_1 = d^2[T_{11}(x_0+d,y,z) - T_{11}(x_0,y,z)+ \\ T_{12}(x,y_0+d,z) - T_{12}(x,y_0,z)+ \\ T_{13}(x,y,z_0+d) - T_{13}(x,y,z_0)].\end{gathered}$$

Taylor expanding this expression in small d and keeping the leading order only, we have:

$$\delta F_1 = d^3 \left[\frac{\partial T_{11}(x,y,z)}{\partial x} + \frac{\partial T_{12}(x,y,z)}{\partial y} + \frac{\partial T_{13}(x,y,z)}{\partial z}\right] = d^3\ \frac{\partial T_{1j}(x,y,z)}{\partial x_j}$$

where we have used the summation convention (with j the repeated index).

Generalising to the other force components, we have:

$$\delta F_i = d^3\ \frac{\partial T_{ij}(x,y,z)}{\partial x_j}.$$

symbolically this is the divergence of the stress tensor $\delta \mathbf{F} = d^3\ \nabla \cdot \mathbf{T}$.

1.9.2 Cauchy's Momentum Equation

Let us now write Newton's second law: 'net force on material element = mass x acceleration', where the net force will be composed of internal forces (δF_i) and a body force from gravity (g will be the acceleration due to gravity), which will align with the z-axis (our choice). Then

$$\underbrace{(\rho d^3)}_{\text{Mass}}\ \underbrace{\frac{Du_i}{Dt}}_{\text{Acceleration}} = \underbrace{d^3\,\frac{\partial T_{ij}}{\partial x_j}}_{\text{Force from Stresses}} - \underbrace{(\rho d^3) g\,\delta_{i3}}_{\text{Gravitational Body Force}},$$

where $\delta_{i3} = 1$ if $i = 3$ and 0 otherwise. Dividing this equation by mass ρd^3, we finally have:

$$\frac{Du_i}{Dt} = \frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j} - g\,\delta_{i3} \qquad \text{or symbolically} \qquad \frac{D\mathbf{u}}{Dt} = \frac{1}{\rho}\nabla \cdot \mathbf{T} - g\mathbf{e}_z, \tag{1.14}$$

which is called Cauchy's momentum equation.

Turning Point Example: How many unknowns and how many equations do we have when we combine Cauchy's momentum equation with conservation of mass?

Even for an incompressible fluid we currently have 4 equations (mass and momentum) for 9 unknowns (velocity and stress) and thus our system is not closed.

Cauchy's equation is valid for a broad range of continuous media (not only fluids and gases) – e.g. crystal or amorphous solid, rubber, a pile of sand (in the continuum limit) – but to make it specifically a fluid equation, one has to specify a particular relevant form of the stress tensor.

We will start with the simplest case of an inviscid fluid and after consider the more complex case of a viscous fluid.

1.9.3 Stress Tensor for an Inviscid fluid

In an inviscid fluid there is no friction between fluid elements, so that the stress (a pressure force) is normal to the surface and inward (hence the minus sign). Therefore, our particular form of the stress tensor (known as a constitutive relation) is

$$T_{ij} = -p\delta_{ij}.$$

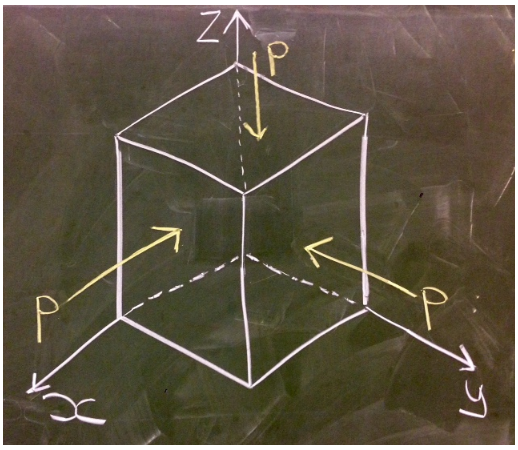


Figure 1.18: In an inviscid fluid the stress is generated entirely by the pressure, which acts with equal magnitude on all faces of our infinitesimal cubic fluid element. Clearly then, flow must be driven by this pressure changing from point to point, i.e. by spatial gradients in pressure.

Substituting into Cauchy's equation (1.14) we arrive at the Euler equation which describes inviscid fluid flow

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\,\delta_{i3} \qquad \text{or symbolically} \qquad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z. \tag{1.15}$$

Notably, in the absence of gravitational forces we can see clearly that gradients in pressure (∇p) accelerate fluid elements from regions of high to low pressure. This is because at any point the pressure is the same in all directions, so there is only a net force on a fluid element if the pressure varies spatially.

In the absence of fluid motion $D_t\mathbf{u} = 0$ we recover a hydrostatic balance $p = p_0 - \rho g z$.

1.9.4 Stress Tensor for a Viscous Fluid

To describe a viscous fluid, we separate the pressure contribution to the stress tensor from the remaining terms — the latter will be associated with an internal friction (viscosity):

$$T_{ij} = -p\delta_{ij} + \sigma_{ij}, \tag{1.16}$$

where σ_{ij} is called the *viscous stress tensor* [4].

In §1.7, we saw that it is the rate of strain tensor e_{ij} which describes the rate of deformation of the fluid, and it is this which generates the viscous stress σ_{ij}. A *constitutive relation* provides the missing link between σ_{ij} and e_{ij}. When the relationship between these is linear, the fluid is called Newtonian.

For an incompressible flow ($e_{kk} = 0$) [5], this simplifies considerably to

$$\sigma_{ij} = 2\,\mu\,e_{ij} \qquad \text{so that} \qquad T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.18}$$

We see that for an incompressible fluid ($e_{kk} = 0$) the pressure $p = -\frac{1}{3}(T_{11} + T_{22} + T_{33})$ is the mean of the three normal stresses at a point[6].

The dynamic viscosity μ (units kg m^{-1} s^{-1}) is the coefficient of proportionality between the rate of strain tensor and the stress tensor which describes how easily a fluid moves under a

[4] Note that different texts may use alternative symbols for the stress tensor, viscous stress tensor and rate of strain tensors.

[5] For a *compressible flow,* where σ_{ij} can depend both on e_{ij} and the identity tensor, the most general form of such a tensor is

$$\sigma_{ij} = \underbrace{2\,\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}\,e_{kk}\right)}_{\text{Strain without volume change}} + \underbrace{\xi\,e_{kk}\delta_{ij}}_{\text{Strain due to volume change}}, \tag{1.17}$$

where μ is the viscosity ('shear viscosity' or 'first viscosity coefficient'), ξ is the second viscosity coefficient, and $e_{kk} = e_{11} + e_{22} + e_{33} = \nabla\cdot\mathbf{u}$ is the trace of tensor e_{ij}.

[6] For compressible flow, things are more complicated as the thermodynamic pressure and mean pressure are not necessarily the same; however, such issues are beyond the scope of this course

shear force. Fluids of high viscosity, like honey, flow less easily than those of low viscosity, like water (the density of each of these is similar).

The flow of incompressible Newtonian fluids, i.e. those satisfying (1.18), will be the focus of this course and will allow us to consider a huge range of phenomena. However, there are many cases in which non-Newtonian behaviour is encountered so that more complex constitutive relations than (1.18) are required (e.g. the flow of toothpaste or mayonaise), but these are beyond the scope of this course.

Substituting (1.18) into Cauchy's equation (1.14), we have the *incompressible Navier-Stokes equations*:

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2} - g\delta_{i3} \qquad \text{or symbolically} \qquad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{e}_z, \quad (1.19)$$

where constant $\nu = \mu/\rho$ is the kinematic viscosity coefficient.

Turning Point Example: What are the SI units of the kinematic viscosity coefficient?

There are numerous ways we can figure this out from terms in the equations whose units we are familiar with. For example, we know that the material derivative is an acceleration and thus has units of m s^{-2}. Therefore $\nu\nabla^2\mathbf{u}$ must have units of acceleration too. Given that $|\nabla^2\mathbf{u}| \sim U/L^2 \sim \text{m}^{-1}\ \text{s}^{-1}$ we must have m $\text{s}^{-2} \sim \nu\text{m}^{-1}\ \text{s}^{-1}$.
Therefore, ν has units of $\text{m}^2\ \text{s}^{-1}$ and can be seen as a diffusion coefficient for momentum transfer.

1.10 Summary - The Governing Equations

Our main focus will be on incompressible flow in which case the incompressible Navier-Stokes equations are

$$\nabla\cdot\mathbf{u} = 0, \qquad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{e}_z \qquad (1.20)$$

In this case, (1.20) forms a closed system, independent of the thermodynamic properties of the fluid.

Note that fluids can be incompressible and yet the density of the fluid particles may vary in physical space. In this case we need an extra equation describing the conservation of ρ along the fluid particle trajectories, which from (1.10) we can see will be:

$$\frac{D\rho}{Dt} = 0. \qquad (1.21)$$

Then, in the general case of $d = 3$, we have 5 equations for 5 unknowns $(\rho, p, \mathbf{u})$. However, unless stated otherwise, ρ will be taken as a material constant, giving us 4 equations for 4 unknowns

If the fluid is incompressible and inviscid, then the viscous term ($\nu\nabla^2\mathbf{u}$) in (1.20) is taken to be zero and we have the Euler equations

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z \tag{1.22}$$

Now we must consider (a) when it is valid to use the Euler equations rather than the Navier-Stokes (to make our life easier) and (b) when our assumption of incompressible flow is a valid one.

1.11 Controlling Flow Parameters

In many cases, certain terms in the Navier-Stokes equations (associated with particular physical effects) are negligible, so that our system of equations can be (often vastly) simplified. It is only with these simplifications that we are able to make progress using mathematical analysis, giving great insight into numerous flow configurations, without turning to brute-force computational approaches.

Below we will compare the magnitude of different terms in the Navier-Stokes equations to find two controlling parameters—which will indicate when the flow can be regarded as viscous/inviscid and compressible/incompressible. The key point here is that these parameters are dimensionless, so that it makes sense to talk about what is 'large' [7].

1.11.1 Reynolds number

Consider a flow whose velocity field varies over a characteristic length L with a typical velocity magnitude U. For example, for the air flow around an aeroplane, a typical scale for L could be the wing span and a typical velocity magnitude would be the speed at which the plane flies U (often it's not so easy to establish these values). For what values of L, U and kinematic viscosity coefficient ν can one neglect viscous forces? For what values of these parameters can one ignore the inertial forces associated with fluid particle acceleration?

The typical value of the fluid particle acceleration is

$$|(\mathbf{u} \cdot \nabla)\mathbf{u}| \sim U^2/L,$$

whereas the typical value of the viscous term is

$$|\nu\nabla^2\mathbf{u}| \sim \nu U/L^2.$$

[7] When a quantity is dimensional, say a length, it makes no sense to say whether it is large or small, as it depends on what we are comparing it to (e.g. a pen is large when compared to the atomic scale, but small compared to that of the universe). In other words, it is only when we make this length dimensionless by a characteristic length of the system, that we are able to establish whether that quantity is large or small. The same holds for the terms in our equations, it only makes sense to say whether a term is large relative to another. Here, we have only touched on the extremely useful area of dimensional analysis.

The ratio of these two typical values makes the Reynolds number

$$Re = \frac{UL}{\nu}. \tag{1.23}$$

Therefore, it appears one can neglect the influence of viscosity if $Re \gg 1$ (inviscid flow), whereas the fluid particle acceleration term can be ignored if $Re \ll 1$ (Stokes flow or 'creeping flow'). The latter is associated with laminar flow, and is critical for small-scale flows (small L) such as those occurring in microfluidics (e.g. look up 'lab-on-a-chip' devices), whilst the former can create laminar or turbulent flows and are commonly observed all around us (e.g. weather).

However, we will see that things are more complex: even if $Re \gg 1$ when estimated based on the size L of the flow, the local length-scale in some parts of the flow may get much smaller and, therefore, the local Re may get strongly reduced, so that the flow is no longer inviscid. This situation is typical for thin layers close to the solid boundaries—the so-called *boundary layers* which will be treated later in this course. This situation is also frequent in turbulent flows, e.g. in wakes behind moving bodies. Both cases are illustrated in figure 1.19.

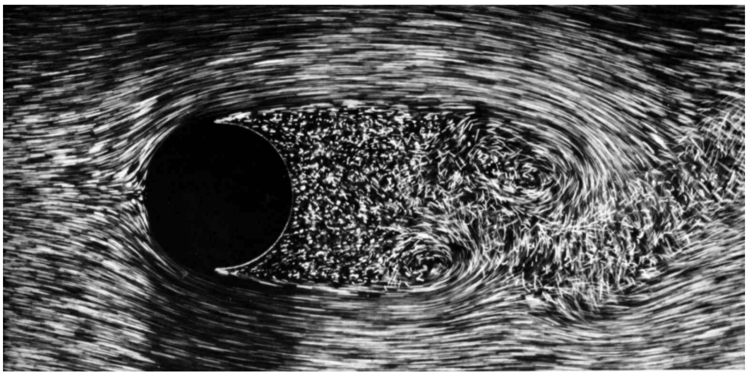


Figure 1.19: Flow past cylinder at Re= 3900 $\gg$ 1. Upstream of the cylinder the flow is laminar, large-scale and almost inviscid (except for a thin boundary layer close to the cylinder). Downstream, the flow contains small scales and its viscosity is important.

This number is named after Osborne Reynolds (1842-1912) who realised that this combination of dimensional parameters controls the transition from a laminar to turbulent flow in a pipe. The apparatus used to conduct these experiments (in Manchester) shown in figure 1.20.

Stokes Flow

Another limit to consider is when the Reynolds number is small $Re \to 0$ so that the viscous forces dominate inertial forces. Then the equations of motion are

$$0 = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{e}_z, \qquad \nabla\cdot\mathbf{u} = 0 \tag{1.24}$$

This is commonly referred to as 'Stokes flow', 'Creeping Flow' or 'Viscous-Dominated flow' and is often encountered in small-scale flows (where L is small, e.g. of the order of micrometres).

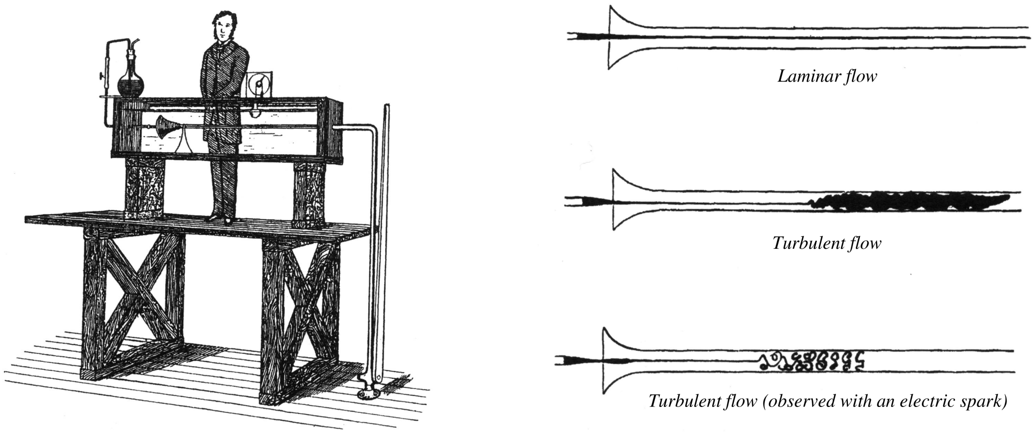


Figure 1.20: Reynolds' experiments on flow through a pipe showing that as you increase Re (e.g. by increasing the flow rate and hence U) you transition from a laminar state to a turbulent one in which dye injected into the fluid at the entrance is mixed with the surrounding fluid.

The study, and mathematical analysis, of such flows is a topic in its own right which we will have little opportunity to look into. A quick read of E.M.Purcell's 'Life at Low Reynolds Number', concerned with how small organisms are able to swim, will give an insight into this counter-intuitive world.

1.11.2 Mach number

Consider a compressible isentropic flow with a typical velocity magnitude U. For what values of U can the compressibility of the fluid be neglected?

To work this out, we must establish an estimate for relative changes in density $\delta\rho/\rho$ from our equations. Assuming that the main balance is between the inertia term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and the pressure force term in the Euler (or Navier-Stokes) equation (1.22), we obtain an estimate for :

$$(\mathbf{u}\cdot\nabla)\mathbf{u} \sim \frac{\nabla p}{\rho} \sim \frac{\nabla \rho}{\rho}\,\frac{\partial p}{\partial \rho} \quad \rightarrow \quad U^2/L \sim \frac{\delta\rho}{\rho L}\,\frac{\partial p}{\partial \rho}.$$

Recognising (as we'll see later in the course) that the speed of sound is $c_s = \sqrt{\frac{\partial p}{\partial \rho}}$ we have

$$\frac{\delta\rho}{\rho} \sim \frac{U^2}{c_s^2} = Ma^2 \tag{1.25}$$

where we have defined the Mach number as :

$$Ma = \frac{U}{c_s}, \tag{1.26}$$

Therefore, the fluid compressibility is negligible ($\delta\rho \ll \rho$) when the Mach number is small, $Ma \ll 1$.

Moving cars or small aircraft (like Cessna) are strongly subsonic and air flow past them is nearly incompressible; see figure 1.21. On the other hand, Boeing passenger jets of 7xx series have a high-subsonic cruise speed of Mach 0.74–0.85 (780–920 km/h) so the flow compressibility is essential; see figure 1.21. Obviously it is even more essential for supersonic and hypersonic fighter jets and rockets.

Figure 1.21: Cessna (left) with $Ma \ll 1$ and Boeing 777 (right) with $Ma \sim 0.8$

Notably, whilst the *fluid* may be demonstrably compressible (e.g. air), the Mach number can still indicate that the *flow* is approximately incompressible.

1.11.3 Similarity

Knowing the parameters that govern a flow allows us to unlock powerful methods of similarity. For example, if we consider incompressible flow and assume that gravitational effects are negligible, then the sole dimensionless parameter formed from the Navier-Stokes equations is the Reynolds number[8]. If we then consider two different flows which are geometrically similar and have the same boundary conditions, then at the same Reynolds numbers the fluid flows will be identical.

It is this phenomenon which allows small-scale models (e.g. of ships) to be built and tested before constructing full scale models - if the flows are dynamically and geometrically similar, then the flows will be too. For example, if the model is reduced in size by a factor of 10 ($L \to 0.1L$), then one could increase the characteristic speed by a factor of 10 ($U \to 10U$) to ensure Re remains unchanged.

1.12 Boundary and initial conditions

As usual for PDEs, in order to find a unique solution one has to specify appropriate boundary conditions and initial conditions.

[8]Such information can be established rigorously using the Buckingham Pi Theorem (not considered here).

1.12.1 Initial conditions

The set of relevant fields must be specified at $t = 0$ at each point $\mathbf{x}$ in the domain occupied by the fluid, $\mathbf{x} \in V \subset \mathbb{R}^d$. For incompressible fluids one only has to specify the initial velocity field $\mathbf{u}_0$ (as there are no time derivatives on ρ in this case)[9]

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0.$$

Obviously, such an initial velocity field must satisfy relevant boundary conditions (see later) and be divergence-free.[10]

1.12.2 Boundary conditions

The number of boundary conditions required depends on the bulk equations (Euler or Navier Stokes in our case) and their form is determined by the properties of the boundaries (i.e. the physics at the boundary).

Viscous Fluids

At solid impenetrable boundaries, it has been found empirically that the fluid velocity at the boundary ∂V of the retaining volume V adjusts itself to the boundary's velocity $\mathbf{U}_b$: this is the famous *no-slip boundary condition*:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}_b(\mathbf{x}, t), \quad \mathbf{x} \in \partial V. \tag{1.29}$$

Inviscid Fluids

For inviscid fluids, where one only has first derivatives of fluid velocity (no $\nabla^2 \mathbf{u}$ term), one can only enforce inpenetrability of the boundary, i.e. that the normal to the boundary velocity component has to match to the one of the boundary, whereas the parallel velocity component

[9]In compressible fluids, one has to additionally specify the initial density and entropy, or chose another couple of thermodynamic fields which would be most relevant to the specific problem (temperature, pressure, etc.).

[10]Note that in this case the pressure field is not independent and can be expressed in terms of the velocity field by taking $\nabla \cdot$ if the Navier-Stokes (1.20) (kills $\partial_t \mathbf{u}$ and $\nabla^2 \mathbf{u}$) to give:

$$\frac{1}{\rho} \nabla^2 p = \nabla \cdot \left[-(\mathbf{u} \cdot \nabla) \mathbf{u} - g \mathbf{e}_z \right], \tag{1.27}$$

or

$$p = \rho \nabla^{-2} \nabla \cdot \left[-(\mathbf{u} \cdot \nabla) \mathbf{u} - g \mathbf{e}_z \right], \tag{1.28}$$

where ∇^{-2} is the inverse Laplacian (operator representing solution of the Laplace equation). Thus, the pressure field is uniquely determined by the velocity field and the gravity term at any time including the initial moment.

remains arbitrary, since the fluid can slip freely along the boundary in the absence of internal friction. This is the so-called *free-slip boundary conditions* [11]:

$$\mathbf{u}_{\perp}(\mathbf{x}, t) = \mathbf{U}_{b\perp}(\mathbf{x}, t), \quad \mathbf{x} \in \partial V. \tag{1.30}$$

At free surfaces

Where a liquid meets a passive gas (e.g. the surface of a pond) there is a 'free surface' or 'free boundary' whose position must often be found as part of the solution (as considered in Chapter 8). For now, all we need to know is that in addition to (1.30), we may assume that the pressure of the liquid at such a boundary is equal to the (usually constant) atmospheric value p_a in the gas

$$p(\mathbf{x}, t) = p_a \quad \mathbf{x} \in \partial V. \tag{1.31}$$

[11] These conditions often do not ensure that the flow solution is unique: as we will see later, there are typically infinitely many solutions to the Euler equations satisfying the free-slip boundary conditions, and one has to seek an additional prescription on how to select the physically relevant solution out of the infinite set. This uncertainty and non-uniqueness is a "tragic" consequence of the drastic idealisation of assuming the fluid to be inviscid. In reality, no matter how high the Reynolds number is, there is always a thin layer of fluid close to the boundary where viscosity is important and must be taken into account: this is the so-call *boundary layer*, which will be considered in detail later in Chapter 7.