

Complex Analysis Notes

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1 Complex Algebra

- The **principal value of the argument** is the unique $\theta \in (-\pi, \pi]$. This is a continuous function on \mathbb{C} without any half-line (including 0).
- $\xi + i\eta$ is the **logarithm** of $re^{i\theta}$ id

$$\xi = \log(r) \quad \eta = \theta + 2\pi n \quad n \in \mathbb{Z}$$

- The **principal value of the logarithm** corresponds to $n = 0$.
- We say that $\xi + i\eta$ is an element of $z_0^{z_1}$ if

$$\xi + i\eta \in e^{z_1 \log(z_0)}$$

- The **extended complex plane** is $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.
- We can extend inversion to the $\hat{\mathbb{C}}$ by setting

$$\frac{1}{0} := \infty \quad \frac{1}{\infty} := 0$$

1.1 Riemann Sphere

To represent the complex plane, we use stereographic projection of $S^2 \setminus \{\text{north pole}\}$ into \mathbb{C} and then send the north pole to ∞ .

$$\begin{aligned} \pi : S^2 \setminus \{(0, 0, 1)\} &\rightarrow \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 - x_3} + i \frac{x_2}{1 - x_3} \end{aligned}$$

Lemma 1.1. A **circle on S^2** is the intersection of S^2 with some plane. The image of every non-vanishing circle on S^2 , under π is a line or circle in \mathbb{C} .

In this proof we notice that circles through the north pole go to lines and circles not through the north pole go to circles. So we can define $\pi(\text{north pole}) := \infty$ and see that $\pi(S^2) = \hat{\mathbb{C}}$.

We can use this to define a metric on $\hat{\mathbb{C}}$.

$$\forall z, w \in \mathbb{C} \quad d(z, w) := \|\pi^{-1}(z) - \pi^{-1}w\|$$

where $\|\cdot\|$ is the Euclidean norm on S^2 .

Note: We can compute everything in this definition in terms of complex algebra to find

$$d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} + \sqrt{1 + |w|^2}}$$

$$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}$$

When doing complex algebra we stick to the following conventions

- $\infty + z = z + \infty = \infty \quad \forall z \in \mathbb{C}$
- $\infty \cdot z = z \cdot \infty = \infty \quad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$
- $\frac{z}{\infty} = 0 \quad \forall z \in \mathbb{C}$
- $\frac{\infty}{0} = \infty \quad \forall z \in \hat{\mathbb{C}} \setminus \{0\}$

2 Mobius Transformations

Given $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ we can define a **Mobius transformation**

$$f(z) := \frac{az + b}{cz + d} \quad \forall z \in \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\}$$

We can extend this to $\hat{\mathbb{C}}$ by defining $\hat{f}(-\frac{d}{c}) = \infty$ and $\hat{f}(\infty) = \frac{a}{c}$.

Notice we can multiply a, b, c, d by any non-zero complex number and recover the same function.

We say that f is **normalised** if $ad - bc = 1$.

It can be noticed that composing two Mobius transformations yields another Mobius transformation.

We can calculate the coefficients of the transformation by multiplying the corresponding matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lemma 2.1. *Extended Mobius transforms are invertible and their inverse is another Mobius transform.*

2.1 Decomposing Mobius transformations

Let inv be the inversion map $z \mapsto \frac{1}{z}$.

Lemma 2.2. *Let \mathcal{C} be a circle or a line then $\text{inv}(\mathcal{C})$ is a circle or a line.*

Proof. Worth going over. □

The **elementary Mobius transformations** are

(a)	Inversion:	$\text{inv}(z) = \frac{1}{z}$
(b)	Translation:	$z \mapsto z + b$
(c)	Rotation:	$z \mapsto az$ for $a = e^{i\theta}$
(d)	Expansion/Contraction:	$z \mapsto rz$ for $z \in \mathbb{R}, z > 0$

Lemma 2.3. *Every Mobius transformation can be written as a composition of elementary Mobius transformations.*

Proof. **Case 1:** $c \neq 0$

We can write

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}$$

Case 2: $c = 0$

$c = 0$ and $ad - bc \neq 0 \implies d \neq 0$ and hence we can write

$$\frac{az + b}{cz + d} = \frac{a}{d}z + \frac{b}{d}$$

In both cases these transformations can be easily decomposed. □

Theorem 2.4. *The image of a circle or line in $\hat{\mathbb{C}}$ under a Mobius transformation is another circle or line.*

Theorem 2.5. *Given 3 distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and three other distinct points $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ there exists a unique Mobius transform f with $f(z_i) = w_i$ for all i .*

Proof. **Existence:** We define two helper functions, assuming that none of the points are ∞

$$S(z) := \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

and if any z_i is ∞ then we simply remove any term containing that z_i . Notice

$$S(z_1) = 1 \quad S(z_2) = 0 \quad S(z_3) = \infty$$

We define T in the same way but replacing each z_i with w_i . Then we can notice that defining $f := T^{-1}S$ yields a function with the desired properties.

Uniqueness: It suffices to check the cases when $w_1 = 1$, $w_2 = 0$ and $w_3 = \infty$ because we can always compose with T . Then we can just pick two suitable Mobius transformations f_1 and f_2 , then show that $g := f_1 \circ f_2^{-1}$ is the identity Mobius transformation. □

Note: Look up the cross ratio.

- A non-identity Mobius transformation has at most two fixed points because

$$z = \frac{az + b}{cz + d} \iff 0 = cz^2 + (d - a)z - b$$

3 Complex Differentiability

Given $D \subseteq \mathbb{C}$ open, a function $f : D \rightarrow \mathbb{C}$ is **complex differentiable at $z_0 \in \mathbb{C}$** if

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}$$

Note: This definition of f' can be restated as

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |z - z_0| < \delta \implies |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|$$

Prop 3.1. $f : D \rightarrow \mathbb{C}$ complex differentiable at $z_0 \in D$ implies f is continuous at z_0 .

The complex derivative also satisfies all of the usual algebra of derivative functions from real analysis, including the chain rule

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Theorem 3.2 (Cauchy-Riemann Equations). *The following are equivalent, given $f : D \rightarrow \mathbb{C}$ and $z_0 = x_0 + iy_0 \in D$*

- (a) f is \mathbb{C} -differentiable at z_0 .
- (b) f is \mathbb{R} -differentiable at (x_0, y_0) and $df(z_0)$ is complex linear
- (c) f is \mathbb{R} -differentiable at (x_0, y_0) and the CR equations hold:

$$u_x = v_y \quad u_y = -v_x$$

Proof. (i) \iff (ii) is somewhat immediate. Consider the alternative definition given in the notes. We see that being \mathbb{C} -differentiable is equivalent to the existence of a complex number ξ such that

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0) - \xi \cdot h}{h} = 0$$

We can view thus view the derivative as a \mathbb{C} -linear function $h \mapsto \xi \cdot h$. This is equivalent to the definition of \mathbb{R} -differentiability with the additional requirement that the map is \mathbb{C} -linear. In practice this means that the Jacobian matrix is some real number multiplied by a rotation matrix.

This explains (ii) \iff (iii) as well because the Jacobian must be given by

$$r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Alternatively, writing out the Jacobian we see that the derivative as a \mathbb{C} -linear map

$$M(h) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

and then the condition $M(ih) = iM(h) \forall h \in \mathbb{C}$ is equivalent to the Cauchy-Riemann equations. \square

Theorem 3.3 (Power Series Expansion). *Given a sequence $(a_k)_{k \in \mathbb{N}_0}$ with $a_k \in \mathbb{C}$, consider the power series*

$$\sum_{k=0}^{\infty} a_k z^k \tag{1}$$

(a) There exists a **radius of convergence** $r \in [0, \infty]$ such that for all z with $|z| < r$ the series (1) converges, and for all z with $|z| = r' > r$ the series (1) does not converge.

(b) The series

$$\sum_{k=1}^{\infty} k a_k z^{k-1} \quad (2)$$

has the same radius of convergence.

(c) $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $\mathcal{B}_r(0) = \{|z| < r\}$.

Proof. (a) If we have convergence for some z_0 then $(a_k z_0^k) \rightarrow 0$ is bounded by $C > 0$ say and hence for all z with $|z| < |z_0|$ we have

$$\sum_{k=0}^{\infty} |a_k z^k| = \sum_{k=0}^{\infty} |a_k z_0^k| \frac{|z|^k}{|z_0|^k} \leq C \frac{1}{|z_0| - |z|}$$

and hence we get convergence.

The radius of convergence is therefore $\sup \{\eta \geq 0 \mid \exists z \text{ with } |z| = \eta \text{ s.t. (1) converges}\}$

(b) We now consider (2). Suppose $|z| < \hat{r} < r$, then we have

$$\sum_{k=1}^{\infty} |k a_k z^{k-1}| \leq \frac{1}{\hat{r}} \underbrace{\sum_{k=1}^{\infty} k \left(\frac{|z|^{k-1}}{\hat{r}^{k-1}} \right)}_{\rightarrow 0} \underbrace{|a_k \hat{r}^k|}_{\text{convergent}}$$

and hence the sum converges. Likewise the sum diverges wherever the other one does.

(c) Confusing proof. □

4 Cauchy's Collection of Complex Corollaries

4.1 Complex Integration

Let $f : D \rightarrow \mathbb{C}$ be continuous and γ a smooth curve with $\Gamma = \gamma[a, b] \subseteq D$

$$\int_{\Gamma} f(z) dz = \int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \dot{\gamma}(t) dt$$

The length of a curve is defined to be

$$L(\gamma) := \int_a^b |\dot{\gamma}| dt$$

Two curves $\gamma : [a, b] \rightarrow \mathbb{C}$ and $\lambda : [c, d] \rightarrow \mathbb{C}$ are **smoothly equivalent parametrisations** for Γ if there is a smooth function $\rho : [a, b] \rightarrow [c, d]$ such that

(i) $\dot{\rho}(t) \neq 0 \quad \forall t$.

(ii) $\rho^{-1} \in \mathcal{C}^1$ and is never zero.

(iii) $\gamma = \lambda \circ \rho$.

(iv) $\rho(a) = c$ and $\rho(b) = d$.

Lemma 4.1. *The complex line integral is invariant under change of parametrisation.*

Lemma 4.2. *If γ and λ are smoothly equivalent then $L(\gamma) = L(\lambda)$.*

Lemma 4.3. *$f : D \rightarrow \mathbb{C}$ holomorphic and $\gamma \in \mathcal{C}^1([a, b])$ such that $\Gamma \subseteq D$ then*

$$\begin{aligned}\int_{\gamma} f'(z) dz &= \int_a^b f'(\gamma(t)) \dot{\gamma}(t) dt \\ &= \int_a^b \frac{d}{dt} f(\gamma(t)) dt \\ &= f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

4.2 My First Sony Cauchy's Theorem

Theorem 4.4 (Goursat's Theorem). *Take $D \subseteq \mathbb{C}$ open and $f : D \rightarrow \mathbb{C}$ holomorphic. Take a rectangle $Q \subseteq D$ such that $Q \cup \partial Q = \overline{Q} \subseteq D$. Take a \mathcal{C}^1 parametrisation $\gamma : [a, b] \rightarrow \mathbb{C}$ such that $\gamma[a, b] = \partial Q$ and γ circles around Q exactly once in the positive direction. Then*

$$\int_{\gamma} f(z) dz = 0$$

Proof. We split the proof into a number of steps:

1. $f \equiv 1$.

This proof follows easily from the FTC.

2. $f(z) = z$.

This proof also follows easily from the FTC because

$$\int \gamma(t) \dot{\gamma}(t) dt = \frac{1}{2} \int_a^b \frac{d}{dt} (\gamma(t))^2 dt = \frac{1}{2} [\gamma(b)^2 - \gamma(a)^2]$$

3. f holomorphic in D .

Divide into rectangles, this is a very long proof in Lecture 9.

□

Corollary 4.5 (Cauchy's Theorem for images of rectangles). *Given $D \subseteq \mathbb{C}$ open and $f : D \rightarrow \mathbb{C}$ holomorphic such that $\overline{Q} \subseteq D$. Suppose $\phi : \overline{Q} \rightarrow D$ is \mathcal{C}^1 . Let γ be a \mathcal{C}^1 parametrisation of ∂Q then*

$$\int_{\phi \circ \gamma} f(z) dz = 0$$

We say $D \subseteq \mathbb{C}$ is

- a **region** if it is non-empty and connected.
- **polygonally connected** if between every two points are joined by a path consisting of a finite collection of straight lines all contained within D .

A **contour** is a simple closed curve.

Theorem 4.6. Given a non-empty open set $D \subseteq \mathbb{C}$

$$D \text{ is a region} \iff D \text{ is polygonally connected}$$

Theorem 4.7 (Jordan Curve Theorem). Let γ be a contour and $\Gamma = \gamma[a, b]$ then ${}^c\gamma$ consists of

$$I(\gamma) \cup O(\gamma)$$

where $I(\gamma)$ is bounded and $O(\gamma)$ is unbounded and the two regions are disjoint.

Note: Jordan Curve Theorem \implies Cauchy's theorem for contours

4.3 Cauchy's Integral Formula

Theorem 4.8 (Cauchy's Integral Formula). Given $D \subseteq \mathbb{C}$ open and $f : D \rightarrow \mathbb{C}$ holomorphic, suppose that $\overline{\mathcal{B}_r(a)} \subseteq D$ for some $a \in D$ and $r > 0$. Then for all $z_0 \in \mathcal{B}_r(a)$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(a)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Not too difficult, worth going over (Lecture 11) □

4.4 Applications

Theorem 4.9 (Taylor's Theorem). Given $D \subseteq \mathbb{C}$ open and polygonally connected and $f : D \rightarrow \mathbb{C}$ holomorphic. Assume $\exists R > 0$ and $z_0 \in D$ such that $\mathcal{B}_R(z_0) \subseteq D$ then for all $z \in \mathcal{B}_R(z_0)$ we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i} \int_{\partial \mathcal{B}_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Corollary 4.10. Every holomorphic function on D is in fact $C^\infty(D)$.

Corollary 4.11. $D \subseteq \mathbb{C}$ is open and polygonally connected and $f : D \rightarrow \mathbb{C}$ then the following are equivalent:

- (i) f is holomorphic in D .
- (ii) f is real differentiable on D and the CR equations hold.
- (iii) f can be expressed in a power series.

Corollary 4.12. Suppose $f(z) = \sum_{k \in \mathbb{N}} a_k z^k$ is holomorphic on $\mathcal{B}_R(0)$ for some $R > 0$ and suppose f is bounded in that ball, say by M . Then for all $k \in \mathbb{N}$

$$|a_k| \leq \frac{M}{R^k}$$

where R is the radius of convergence.

Theorem 4.13 (Liouville's Theorem). Any entire, bounded function is constant.

Proof. Pick $z_0 \in \mathbb{C}$ and $M > 0$ such that $|f(z)| \leq M \forall z \in \mathbb{C}$. Define

$$m(f, R, z_0) := \max_{z \in \partial \mathcal{B}_R(z_0)} |f(z)|$$

Then by Taylor's theorem we see that

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} m(f, R, z_0) \leq \frac{n!}{R^n} M$$

and in particular $|f'(z_0)| \leq \frac{M}{R} \rightarrow 0$ as $R \rightarrow \infty$. Hence $f'(z_0) = 0$. \square

Corollary 4.14 (Fundamental Theorem of Algebra). *Every non-constant polynomial has at least one zero in \mathbb{C} .*

Proof. Take some polynomial $P(z) := a_n z^n + \dots + a_1 z + a_0$ such that $a_n \neq 0$. Then for any $\epsilon > 0$ there is a radius R such that $\forall |z| > R$ we have

$$(1 - \epsilon) |a_n| |z|^n \leq |P(z)| \leq (1 + \epsilon) |a_n| |z|^n$$

Suppose $P(z)$ has no zeros in \mathbb{C} then $\frac{1}{P(z)}$ is complex differentiable in \mathbb{C} and there is an $R > 0$ such that for all $|z| > R$

$$\frac{1}{2} |a_n| |z| \leq |P(z)| \implies \left| \frac{1}{P(z)} \right| \leq \frac{2}{|a_n| |z|^n} \leq \frac{2}{|a_n| R^n}$$

and hence $\frac{1}{P}$ is bounded on $\{|z| > R\}$ and it is obviously bounded inside by compactness. Hence by Liouville's Theorem P is constant. This is a contradiction. \square

Theorem 4.15 (Morea's Theorem). *Given a region $D \subseteq \mathbb{C}$ and a $f : D \rightarrow \mathbb{C}$ continuous, suppose given any triangle T with $T \cup \partial T \subseteq D$ we have $\int_{\partial T} f(z) dz = 0$. Then f is holomorphic in D .*

Theorem 4.16 (Schwarz Reflection Principle). *Suppose D is open in $\overline{H^+} := \{z \in \mathbb{C} \mid \mathcal{I}(z) \geq 0\}$ and $f : D \rightarrow \mathbb{C}$ is continuous on D and holomorphic on D° . Then*

$$\tilde{f}(z) := \begin{cases} f(z) & z \in D \\ \overline{f(\bar{z})} & z \in \tilde{D} \end{cases}$$

where \tilde{D} is the complex conjugate of D , is well-defined and holomorphic on $D \cup \tilde{D}$.

Proof. By composition of reflections we can easily show that \tilde{f} is holomorphic on \tilde{D}° so that only the lines remain. We show that the integral of f over any triangle with one edge on the line is 0 by a continuity argument, approaching from both sides. \square

5 Zeros of Holomorphic Functions

Given $D \subseteq \mathbb{C}$ open and connected and $f : D \rightarrow \mathbb{C}$ holomorphic, the **order** of any zero $z_0 \in D$ is

$$\text{ord}(f, z_0) := \inf \left\{ k \in \mathbb{N} \mid f^{(k)}(z_0) \neq 0 \right\} \in \mathbb{N} \cup \{\infty\}$$

We say that $f : D \rightarrow \mathbb{C}$ is a **conformal mapping** if f is holomorphic in D and its derivative is non-vanishing on D .

We say that f is **biholomorphic** if f is a conformal mapping such that f^{-1} exists and is also

conformal.

Prop 5.1. *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \rightarrow \mathbb{C}$ holomorphic, suppose we have a zero $z_0 \in D$ of order $k \in \mathbb{N}$. Then there is a neighbourhood U_0 of z_0 and a holomorphic function $h : U_0 \rightarrow V_0$ such that $h(z_0) = 0$, $\text{ord}(f, z_0) = 1$ and*

$$f(z) = (h(z))^k \quad \forall z \in U_0$$

Proof. WLOG we may assume that $z_0 = 0$, then we apply Taylor's theorem to write f as

$$f(z) = \sum_{n=k}^{\infty} c_n z^n$$

because $\text{ord}(f, z_0) = k$ and hence the first k terms vanish. For simplicity we can also assume that $c_k = 1$. Hence we can write

$$f(z) = z^k \left(1 + \underbrace{\sum_{n=k+1}^{\infty} c_n z^{n-k}}_{=:g(z)} \right) = \left(\underbrace{z \sqrt[k]{1+g(z)}}_{=:h(z)} \right)^k$$

Note that g is holomorphic and $g(0) = 0$, and $h(0) = 0$. Moreover, $h'(0) = \sqrt[k]{1+g(0)} + 0 \left(\sqrt[k]{1+g(z)} \right)' = 1 \neq 0$. Hence $\text{ord}(h, 0) = 1$. Read up on making it holomorphic (Lecture 14). \square

Note: This implies that all zeros of finite order are isolated.

Theorem 5.2. *If $\text{ord}(f, z_0) = k \in \mathbb{N}$ for $f : D \rightarrow \mathbb{C}$ holomorphic then $\forall \epsilon > 0$ there exists a $U_\epsilon \subseteq D$ with $z_0 \in U_\epsilon$ such that $f(U_\epsilon) = \mathcal{B}_\epsilon(0)$ and $f|_{U_\epsilon}$ takes every w with $0 < |w| < \epsilon$ exactly k times and 0 for z_0 .*

Proof. Without loss of generality we may assume that $z_0 = 0$.

If $f(z) = z^k$ then any $w = re^{i\theta}$ has exactly k roots.

In the general case we can write $f(z) = (h(z))^k$ for $h : U \rightarrow V$ holomorphic such that $h(0) = 0$ and $h'(0) \neq 0$. Moreover, h is locally biholomorphic around a neighbourhood of 0. Choose $\epsilon > 0$ sufficiently small that

$$A := \{\xi \in \mathbb{C} \mid |\xi| \leq \sqrt[k]{\epsilon}\} \subseteq V$$

Then define $U_\epsilon := h^{-1}(A)$. This set has the desired properties because the original roots of $z \mapsto z^k$ lie in A . \square

Note: Every bijective holomorphic function is biholomorphic.

Theorem 5.3 (Identity Theorem). *Given $D \subseteq \mathbb{C}$ open and connected with $f_1, f_2 : D \rightarrow \mathbb{C}$ holomorphic, assume that $\{f_1 = f_2\}$ has at least one accumulation point in D . Then $f_1 = f_2$ on D .*

Proof. Define $g := f_1 - f_2$ and let z_0 be one of the accumulation points. Then z_0 is a zero of infinite order for g . Apparently this is a proof. \square

Theorem 5.4 (Open Mapping Theorem). *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \rightarrow \mathbb{C}$ holomorphic and non-constant, $f(D)$ is open and connected.*

Theorem 5.5 (Maximum Modulus Principle). *Given $D \subseteq \mathbb{C}$ open and connected and $f : D \rightarrow \mathbb{C}$ holomorphic and non-constant, $|f|$ does not have any maxima.*

Lemma 5.6 (Schwarz Lemma). *Suppose $f : \Delta \rightarrow \Delta$ is holomorphic such that $f(0) = 0$ then*

- (i) $|f(z)| \leq |z|$ for all $z \in \Delta$.
- (ii) $|f'(0)| \leq 1$.
- (iii) *If for some $z \in \Delta \setminus \{0\}$ we have $|f(z)| = |z|$ or $|f'(z)| = 1$ then $\exists \theta \in \mathbb{R}$ such that $f(\tilde{z}) = e^{i\theta}\tilde{z}$ for all $\tilde{z} \in \Delta$.*

6 Singularities

Given $D \subseteq \mathbb{C}$ open and connected and $f \in \mathcal{H}(D)$,

- f has an **isolated singularity** at $z_0 \notin D$ if there is an $\epsilon > 0$ such that f is defined on $\mathcal{B}_\epsilon(z_0) \setminus \{z_0\}$.
- $z_0 \in D$ is a **regular point** if f is complex differentiable at $z_0 \in D$.

Given an isolated singularity z_0 then it has **order**

$$\text{ord}(f, z_0) := -\inf \left\{ n \in \mathbb{Z} \mid \lim_{z \rightarrow z_0} (z - z_0)^n f(z) \text{ exists and } < \infty \right\}$$

then we say z_0 is a

- **removable singularity** if $\text{ord}(f, z_0) \geq 0$.
- **pole of order $n \in \mathbb{N}$** if $\text{ord}(f, z_0) = -n \in (-\infty, -1]$.
- **essential singularity** if $\text{ord}(f, z_0) = -\infty$.

Let $S \subseteq D$ be a discrete set, then a holomorphic function $f : D \setminus S \rightarrow \mathbb{C}$ is called **meromorphic on D** if none of the isolated singularities in S are essential.

Prop 6.1. *Let \mathcal{Z}_f and \mathcal{P}_f be the set of zeros and poles respectively of $f : D \rightarrow \mathbb{C}$ meromorphic, $f \neq 0$. Then neither set has an accumulation point in D .*

Proof. Certainly any pole of f is an isolated singularity and hence cannot be an accumulation point of \mathcal{P}_f . Any other $z \in D$ where f is holomorphic cannot be an accumulation point of poles either. Suppose now that \mathcal{Z}_f has an accumulation point at $z_0 \in D$ then z_0 cannot be a pole otherwise we'd be able to write

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{with } m \in \mathbb{N}, g(z_0) \neq 0$$

and hence for some $\epsilon > 0$ we have that $f(z) \neq 0$ for all $0 \leq |z - z_0| < \epsilon$ which means that z_0 is not an accumulation point.

So any accumulation point must be a complex differentiable point. We are left to show that $D \setminus \mathcal{P}_f$ is open and connected because then the identity theorem tells us that $f \equiv 0$ because $\{f = 0\}$ has an accumulation point in $D \setminus \mathcal{P}_f$. \square

Lemma 6.2. *Suppose $D \subseteq \mathbb{C}$ is open and connected. Suppose $M \subseteq D$ has no accumulation point in D . Then $D \setminus M$ is open and connected.*

Proof. Openness is immediate, for connectedness just draw a picture. \square

6.1 Laurent Series

A **Laurent series** is a series of the form

$$\sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

such that the positive terms converge inside some ball around z_0 and the negative terms converge outside some larger ball around z_0 . Hence the Laurent series converges in an annulus around the point z_0 .

Theorem 6.3 (Cauchy's Theorem for annuli). *Given $0 \leq R_1 < R_2 < \infty$ and $D \subseteq \mathbb{C}$ open and connected such that $\overline{A} := \overline{A(R_1, R_2, z_0)} \subseteq D$, for any $z \in A$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_{R_2}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi - \frac{1}{2\pi i} \int_{\partial B_{R_1}(z_0)} \frac{f(\xi)}{\xi - z_0} d\xi$$

Proof. Do normal Cauchy on a small ball contained in the annulus then do some appropriate contour integration. \square

Theorem 6.4 (Laurent's Theorem). *Given f holomorphic on a neighbourhood of an annulus $A = A(R_1, R_2, z_0)$ and any $z \in A$,*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

where for all $\rho \in [R_1, R_2]$ we can write

$$a_k = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

Corollary 6.5. *Under the same assumption, if f is bounded on $\{|z - z_0| = \rho\}$ for some $\rho \in [R_1, R_2]$ then*

$$|a_k| \leq \frac{M}{\rho^k} \quad \text{for all } k \in \mathbb{Z}$$

6.2 Classification of Singularities

Theorem 6.6 (Riemann's removable singularity theorem). *Gain an isolated singularity z_0 for a function $f \in \mathcal{H}(D \setminus \{z_0\})$, assume that $|f|$ is bounded in a neighbourhood of z_0 . Then there is a holomorphic function $\tilde{f} \in \mathcal{H}(D)$ which extends to f . Moreover, z_0 was a removable singularity.*

Proof. In the neighbourhood we can expand in a Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

and then for all sufficiently small $\rho > 0$ we have $|a_k| \leq \frac{M}{\rho^k}$ for all $k \in \mathbb{Z}$. Taking $\rho \rightarrow 0$ we see that $a_k = 0$ for all $k < 0$. So we can extend f by taking $\tilde{f}(z_0) = a_0$. \square

Corollary 6.7. *Given $f : D \rightarrow \mathbb{C}$ which is holomorphic except for an isolated singularity at z_0 , the following are equivalent:*

- (i) f has a pole at z_0 .
- (ii) At least coefficient of negative order in the Laurent series around z_0 is non-zero, but at most finitely many.
- (iii) $\lim_{z \rightarrow z_0} |f(z)| = +\infty$.

Theorem 6.8 (Casorati-Weierstrass). *Given $f \in \mathcal{H}(D \setminus \{z_0\})$ with an isolated, essential singularity at z_0 , for all $\epsilon > 0$ the set $f(\mathcal{B}_\epsilon(z_0) \setminus \{z_0\})$ is dense in \mathbb{C} .*

Proof. Suppose it's not dense. Then $\exists \delta > 0$ and $w \in \mathbb{C}$ such that $|f(z) - w| > \delta$ for all $z \in \mathcal{B}_\epsilon(z_0) \setminus \{z_0\}$. So we can define a function

$$g(z) := \frac{1}{f(z) - w}$$

and it may be extended to a function which is holomorphic on $\mathcal{B}_\epsilon(z_0)$ because the bottom is always bigger than δ hence g is bounded. So we can write $f(z) = \frac{1}{g(z)} + w$ which clearly doesn't have an essential singularity $\Rightarrow \Leftarrow$. \square

Note: Suppose z_0 is an essential singularity for f , then certainly $\lim_{z \rightarrow z_0} f(z)$ cannot have any finite value $c \in \mathbb{C}$. Suppose $\lim_{z \rightarrow z_0} f(z) = \infty$ then there would be some ϵ ball around z_0 such that $|z - z_0| < \epsilon \implies |f(z)| > 1$. Hence the image of this ball could not be dense. This shows that essential singularities exhibit no limiting behaviour.

7 Residual Theory

7.1 Winding Numbers

Given $z_0, z_1 \in \mathbb{C} \setminus \{0\}$ such that $\frac{z_0}{|z_0|} \neq \frac{-z_1}{|z_1|}$ then there is a unique $\theta \in (-\pi, \pi i)$ such that

$$\frac{z_0}{|z_0|} e^{i\theta} = \frac{z_1}{|z_1|}$$

then we define $\angle(z_0, z_1) := \theta$.

We say $\gamma[t_0, t_1]$ is a **half-plane curve** if the image of γ is contained entirely in one half plane.

Then we define $\angle \gamma := \angle(\gamma(t_0), \gamma(t_1))$.

Given any other piecewise \mathcal{C}^1 curve we can break it up into a sequence of half-plane curves and then sum the angles to define $\angle \gamma$.

Lemma 7.1. Given a closed \mathcal{C}^1 curve $\gamma; [t_0, t_1] \rightarrow \mathbb{C} \setminus \{0\}$, there exists a unique integer $\text{inf}(\gamma, 0)$ called the *index* or *winding number* of γ around 0 such that $\angle \gamma = 2\pi \text{ind}(\gamma, 0)$.

Prop 7.2. Given a closed \mathcal{C}^1 curve γ with $a \notin \gamma[t_0, t_1]$ we have

$$\text{ind}(\gamma, a) = \int_{\gamma} \frac{dz}{z - a}$$

Proof. Assume $a = 0$, $t_0 = 0$ and $t_1 = 1$ then we can split up into the half-plane decomposition:

$$0 = \tau_0 \leq \dots \leq \tau_n = 1$$

Let α_i be the straight line connected $\gamma(\tau_i)$ with $\frac{\gamma(\tau_i)}{|\gamma(\tau_i)|}$ and let β_i be the counter-clockwise path along the unit circle connecting $\frac{\gamma(\tau_{i-1})}{|\gamma(\tau_{i-1})|}$ with $\frac{\gamma(\tau_i)}{|\gamma(\tau_i)|}$. Then we can see

$$\int_{\gamma[\tau_{i-1}, \tau_i]} \frac{dz}{z} = \int_{\alpha_{i-1}} \frac{dz}{z} + \int_{\beta_i} \frac{dz}{z} - \int_{\alpha_i} \frac{dz}{z} \quad \forall i$$

Then summing over i and noting $\alpha_0 = \alpha_n$ obtains the result. \square

A *cycle* is a formal linear combination of closed curves $\gamma = \sum_{i=1}^n \alpha_i \gamma_i$ with $\alpha_i \in \mathbb{Z}$ then we define

$$\text{ind}(\gamma, a) = \sum_{i=1}^n \alpha_i \text{ind}(\gamma_i, a)$$

Then γ is called *homologous to 0 in D* if for every $a \in \mathbb{C} \setminus D$ we have

$$\text{ind}(\gamma, a) = 0$$

Theorem 7.3 (Cauchy's Theorem (homotopy version)). Let $D \subseteq \mathbb{C}$ be open and connected and γ a \mathcal{C}^1 cycle that is homologous to 0 in D . Then $\forall f \in \mathcal{H}(D)$

$$\int_{\gamma} f(z) dz = 0$$

7.2 Residual Theorem

Given f holomorphic with an isolated singularity at z_0 , the *residue of f at z_0* is defined to be

$$\text{res}(f, z_0) := \frac{1}{2\pi i} \int_{\partial B_{\epsilon}(z_0)} f(\xi) d\xi$$

There are a number of convenient ways of calculating the residue.

1. For a Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k$$

we have $\text{res}(f, z_0) = a_{-1}$.

2. Suppose that f has a pole of order n at z_0 . Define $g(z) := (z - z_0)^n f(z)$ then g is holomorphic at z_0 and in fact

$$\text{res}(f, z_0) = a_{-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!} = \lim_{z \rightarrow z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)) \right]$$

3. If $f(z) = \frac{h(z)}{k(z)}$ and $k(z)$ has a simple zero at z_0 then

$$\text{res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{h(z)}{\left(\frac{k(z) - k(z_0)}{z - z_0} \right)} = \frac{h(z_0)}{k'(z_0)}$$

Given $\gamma; [t_0, t_1] \rightarrow \mathbb{C}$ a closed curve with image Γ

$$\text{Int}(\gamma) := \{z \in \mathbb{C} \setminus \Gamma \mid \text{ind}(\gamma, z) \neq 0\}$$

$$\text{Ext}(\gamma) := \{z \in \mathbb{C} \setminus \Gamma \mid \text{ind}(\gamma, z) = 0\}$$

Lemma 7.4. (a) $[a \mapsto \text{ind}(\gamma, a)]$ is locally constant in $\mathbb{C} \setminus \Gamma$.

(b) $\text{Int}(\gamma)$ is bounded.

(c) $\text{Ext}(\gamma)$ is non-empty and unbounded.

Theorem 7.5 (The Residue Theorem). Given $D \subseteq \mathbb{C}$ open and connected and $f \in \mathcal{H}(D \setminus S)$ for some discrete set S of isolated singularities, suppose γ is a closed \mathcal{C}^1 curve homologous to 0 in D such that $\Gamma \cap S = \emptyset$ and $\Gamma \subseteq D$. Then γ winds around at most a finite number of singularities in S and

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{a \in S} \text{ind}(\gamma, a) \text{res}(f, a)$$

Proof. First things first, $A := \{a \in S \mid \text{ind}(\gamma, a) \neq 0\}$ is bounded. Assume that γ winds around infinitely many points in S then A is infinite. Hence there is a sequence of points (a_n) in A such that $a_n \rightarrow a$.

Case 1: $a \in \Gamma$

Case 2: $\text{ind}(\gamma, a) \neq 0$

Both lead to a contradiction apparently.

Now suppose a_1, \dots, a_N are the points around with γ winds and define $\alpha_i := \text{ind}(\gamma, a_i)$.

Choose $\epsilon > 0$ small such that $\overline{\mathcal{B}_{\epsilon}(a_i)} \cap \Gamma = \emptyset$ for all i . Then define $\gamma_i(t) := a_i + \epsilon e^{i2\pi t}$ for $t \in [0, 1]$.

Let β_i be some little paths joining γ to γ_i and then concatenate all the γ_i s and β_i s with γ in some appropriate way to form $\tilde{\gamma}$ Then

$$\int_{\tilde{\gamma}} = 0 \implies \int_{\gamma} f(z) dz = 2\pi i \sum_{a \in S} \text{ind}(\gamma, a) \text{res}(f, a)$$

□

Theorem 7.6 (Argument Principle). Given $D \subseteq \mathbb{C}$ open and connected and f meromorphic on D , let $A \subseteq D$ be open with boundary ∂A being a closed \mathcal{C}^1 curve γ . Assuming $\partial A \subseteq D$ and $\Gamma \cap \mathcal{P}(f) = \Gamma \cap \mathcal{Z}(f) = \emptyset$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \mathcal{Z}_A(f) - \mathcal{P}_A(f)$$

where

$$\mathcal{Z}_A(f) := \sum_{z \in f^{-1}(0) \cap A} \text{ord}(f, z) \quad \text{and} \quad \mathcal{P}_A(f) := \sum_{z \in \mathcal{P}(f) \cap A} |\text{ord}(f, z)|$$

Proof. The proof uses the residual theorem to calculate the integral on the left hand side. The isolated singularities of $\frac{f'}{f}$ are the poles and zeros of f . We need to understand the residue of $\frac{f'}{f}$ (the **logarithmic derivative**) at these points.

To this end, let z_0 be a zero or a pole of f . Then for some $n \in \mathbb{Z}$ and function g which is holomorphic in neighbourhood of z_0 with $g(z_0) \neq 0$ we can write

$$f(z) = (z - z_0)^n g(z)$$

Then we can calculate the logarithmic derivative as follows

$$\frac{f'(z)}{f(z)} = \frac{1}{f(z)} (n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)) = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Since $g(z_0) \neq 0$ we see that $\frac{f'}{f}$ has a simple pole at z_0 and hence $\text{res}\left(\frac{f'}{f}, z_0\right) = n$.

So when we integrate $\frac{f'}{f}$ we sum $\text{ord}(f, z_0)$ for all zeros and poles of f with a factor of $2\pi i$ out front. This counts zeros positively and poles negatively, each with multiplicity respecting the order. \square

8 Rouches Theorem

Theorem 8.1 (Rouches Theorem). *Given $D \subseteq \mathbb{C}$ open and connected and γ a closed \mathcal{C}^1 curve with $\Gamma = \text{im}(\gamma) \subseteq D$. Suppose $f, g \in \mathcal{H}(D)$ satisfy*

$$|f(\xi) - g(\xi)| < |g(\xi)| \quad \forall \xi \in \Gamma \quad (3)$$

Then f and g have the same number of zeros in $\text{Int}(\gamma)$.

Proof. Thanks to the strict inequality in (3), there is a neighbourhood $U \supseteq \Gamma$ such that $h := \frac{f}{g}$ is holomorphic in U . Then again by (3), we see

$$|h(z) - 1| = \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \forall z \in U$$

and hence $h(U) \subseteq \mathcal{B}_1(1) \subseteq \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathcal{R}(z) \leq 0, \mathcal{I}(z) = 0\}$ and so $\log(h)$ is well-defined on U and we can write

$$(\log(h))' = \frac{h'}{h} = \frac{f'}{f} - \frac{g'}{g}$$

then since h is holomorphic on U we can integrate its derivative over γ and get

$$0 = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\xi)}{f(\xi)} d\xi - \frac{1}{2\pi i} \int_{\gamma} \frac{g'(\xi)}{g(\xi)} d\xi$$

\square

Note: This can alternatively be restated as follows:

Given $h, w \in \mathcal{H}(D)$, such that we can write $h = f + g$ and $w = f$, suppose that

$$|g(\xi)| < |f(\xi)| \quad \forall \xi \in \Gamma$$

then $f + g$ and f have the same number of zeros in $\text{Int}(\gamma)$.

Proof. Define the meromorphic function

$$F(z) := \frac{f(z) + g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$$

Note that we want to show that $Z_A(F) = P_A(F)$. By the argument principle we have that $Z_A(F) - P_A(F)$ is the winding number of $F \circ \gamma$ about 0 since

$$\int_{\gamma} \frac{F'(z)}{F(z)} dz = \int_{F \circ \gamma} \frac{1}{z} dz$$

But by assumption we have that for all z in the image of γ $|F(z) - 1| < 1$. So the image of γ under F is contained \square

9 Functional Convergence

Suppose $f_n : D \rightarrow \mathbb{C}$ where $D \subseteq \mathbb{C}$ is open. We say that f_n **converges locally uniformly** to f as $n \rightarrow \infty$ if

$$\forall \text{compact } K \subseteq D, \quad f_n|_K \rightarrow f|_K \text{ uniformly}$$

Theorem 9.1 (Weierstrass Convergence Theorem). *Given $D \subseteq \mathbb{C}$ open and connected and a sequence (f_n) of holomorphic functions on D such that f_n converges locally uniformly to f , then $f \in \mathcal{H}(D)$.*

Proof. Pick $z_0 \in D$ and $\delta > 0$ sufficiently small that $\overline{\mathcal{B}_\delta(z_0)} \subseteq D$. We will prove that f is holomorphic on all of this ball by use of Morea's Theorem. So take γ closed curve in $\mathcal{B}_\delta(z_0)$. We can write

$$\int_{\gamma} f = \int_{\gamma} f_n + \int_{\gamma} (f - f_n)$$

But $f(z) - f_n(z) \rightarrow 0$ uniformly on γ and hence $\int_{\gamma} (f - f_n) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, f_n is holomorphic on D and hence the integral over γ is 0 and thus $\int_{\gamma} f_n = 0$. We may conclude $\int_{\gamma} f = 0$ and so by Morea's theorem f is holomorphic in the ball. \square

Theorem 9.2. *Given $D \subseteq \mathbb{C}$ open and connected and a sequence of functions $f_n \in \mathcal{H}(D)$, if f_n converges locally uniformly to f then the derivatives also converge locally uniformly.*

Theorem 9.3 (Hurwitz Theorem). *$D \subseteq \mathbb{C}$ open and connected, $f_n : D \rightarrow \mathbb{C}$ and $f_n \in \mathcal{H}(D)$. Suppose that f_n converges locally uniformly to f and that none of the f_n have more than some $k \in \mathbb{N}$ zeros. Then either f is constant or has at most k zeros.*

Proof. Suppose that f is not constant. Suppose that f has K zeros of multiplicity m_1, \dots, m_k at distinct $z_1, \dots, z_K \in D$. Fix $\delta > 0$ such that for all no other z_j lies in the δ ball around z_i . Define

$$\epsilon := \inf_{i=1, \dots, k} \inf_{\xi \in \partial \mathcal{B}_\delta(z_i)} |f(\xi)| > 0$$

By compactness we have that $\epsilon > 0$. Then by locally uniform convergence we know there is a n_0 such that

$$\sup_{i=1, \dots, k} \sup_{\xi \in \partial \mathcal{B}_\delta(z_i)} |f_n(\xi) - f(\xi)| < \frac{\epsilon}{2} \quad \forall n > n_0$$

Rouche's theorem tells us that f has exactly m_i zeros in $\mathcal{B}_\delta(x_i)$ since

$$|f_n(\xi) - f(\xi)| < |f(\xi)| \quad \forall \xi \in \mathcal{B}_\delta(z_i)$$

□

In words ϵ was the 'minimum' value of f on the boundary of these balls. Then thanks to locally uniform convergence we were able to ensure that the difference between f_n and f was always less than this minimum value and thus we could use Rouche's theorem.

10 Special Functions

10.1 The Gamma Function

To begin we define the function for $\mathcal{R}(z) > 0$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$$

Theorem 10.1. *The Gamma-integral defines a holomorphic function and the k 'th derivative is*

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1} (\log t)^k e^{-t} dt$$

One can easily show directly from the integral that Γ satisfies the functional relationship

$$\Gamma(z+1) = z\Gamma(z)$$

which we can use to extend Γ to the rest of \mathbb{C} . Given any $z \in \mathbb{C}$ we just choose $n \in \mathbb{N}$ such that $\mathcal{R}(z+n) > 0$ and then define

$$\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1)\dots(z+n)}$$

which is well-defined and does not depend on n . Of course this gives us ∞ at the negative integers and so

Theorem 10.2. *The gamma function extends via. the functional relationship to a meromorphic function with poles at $z = 0, -1, -2, \dots$, each of order 1 and satisfies*

$$\text{res}(\Gamma, -n) = \frac{(-1)^n}{n!}$$

10.2 Infinite Products

We say $\prod_{k=1}^\infty w_k$ **converges** if

- only finitely many $w_n = 0$
- The sequence of partial products converge with non-zero limit

Note that if the first few terms are 0 then we just ignore them.

Then $\prod_{k=1}^{\infty} w_k$ **converges absolutely** if $\exists n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \log w_k$$

converges.

Note:

$$w_n = \frac{\prod_{k=1}^n w_k}{\prod_{k=1}^{n-1} w_k} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

Prop 10.3. Defining $C^- := \mathbb{C} \setminus \{y = 0, x \leq 0\}$

(a) $\prod_{n=1}^{\infty} w_n$ with $w_n \in \mathbb{C}^- \forall n \in \mathbb{N}$ converges $\iff \sum_{n=1}^{\infty} \log w_n$ converges.

(b) $\prod_{n=0}^{\infty} (1 + a_n)$ converges absolutely $\iff \sum_{k=1}^{\infty} |a_k|$ converges

We can use infinite products to further characterise the Gamma function.

Lemma 10.4. The infinite product

$$H(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

converges and absolutely and defines an entire function.

Corollary 10.5.

$$\begin{aligned} G_N(z) &:= z e^{-z \log(N)} \prod_{n=1}^N \left(1 + \frac{z}{n}\right) \\ &= z e^{-z(\log N - \sum_{n=1}^N \frac{1}{n})} \prod_{n=1}^N \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \end{aligned}$$

Theorem 10.6. For any $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ we can write

$$\frac{1}{\Gamma(z)} = G(z) = \lim_{N \rightarrow \infty} G_N(z)$$

10.3 The Zeta Function

Given any $z \in \mathbb{C}$ with $\mathcal{R}(z) > 1$ we can define the **Riemann zeta function** by

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Theorem 10.7. We can express this as an infinite product over the primes $\mathbb{P} = \{2, 3, 5, 7, \dots\}$ by

$$\frac{1}{\zeta(z)} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^z}\right)$$

Note: The ζ -function does not have any zeros in $\{\Re(z) > 1\}$.

Lemma 10.8. We can also relate the ζ -function to the Γ -function by

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-t}}{1 - e^{-t}} dt$$

Note: The ζ -function can also be characterised by the Hankel contour but we really didn't do much on this. It might be worth reading over.

11 Riemann Mapping Theorem

Two open set $U, V \subseteq \mathbb{C}$ are said to be **conformally equivalent** if there exists $\phi : U \rightarrow V$ such that

- ϕ is holomorphic.
- ϕ is a bijection.
- The inverse map ϕ^{-1} is holomorphic.

If ϕ can be given by $z \mapsto az + b$ for some complex a, b then we say U and V are **congruent**. Given $D \subseteq \mathbb{C}$ and loops $\gamma_1, \gamma_2 : [t_0, t_1] \rightarrow D$ then they are **homotopic** if there exists a continuous mapping $h : [0, 1] \times [t_0, t_1] \rightarrow D$ such that for all $t \in [t_0, t_1]$ and $s \in [0, 1]$ we have

- $h(0, t) = \gamma_1(t)$
- $h(1, t) = \gamma_2(t)$
- $h(s, t_0) = \gamma_1(t_0) = \gamma_2(t_0)$
- $h(s, t_1) = \gamma_1(t_1) = \gamma_2(t_1)$

Then we define $h_\tau(t) := (\tau, t)$ so that $h_0 = \gamma_1$ and $h_1 = \gamma_2$.

Finally, a connected set $D \subseteq \mathbb{C}$ is called **simply connected** if any two continuous curves with the same base point are homotopic to one another.

Note: Congruent \implies Conformally equivalent \implies Homeomorphic

Theorem 11.1. Given γ_1, γ_2 homotopic and $f : D \rightarrow \mathbb{C}$ holomorphic

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

Theorem 11.2 (Cauchy's theorem for simply connected set). Given a closed curve γ in D an open and simply connected domain along with any $f \in \mathcal{H}(D)$,

$$\int_{\gamma} f(z) dz = 0$$

Proof. Using the previous theorem, denote by e the constant path at any point along γ then $\int_{\gamma} f(z)dz = \int_e f(z)dz = 0$ \square

Lemma 11.3. *Given $D \subseteq \mathbb{C}$ open and simply connected, $f : D \rightarrow \mathbb{C} \setminus \{0\}$ holomorphic there exists a holomorphic function $g : D \rightarrow \mathbb{C}$ such that*

$$f(z) = e^{g(z)} \quad \forall z \in D$$

moreover, that g is unique up to an additive constant $2\pi n$ for $n \in \mathbb{Z}$.

Proof. Choose $z_0 \in D$ then $f(z_0) \neq 0$ so there exists $w_0 \in \mathbb{C}$ such that $f(z_0) = e^{w_0}$. Now given any other $z \in D$ choose a path γ in D which connects z_0 to z . Now define

$$g(z) := w_0 + \int_{f \circ \gamma} \frac{1}{z} dz$$

This is well-defined because our set is simply connected. Now we can see that in fact

$$g'(z) = w_0 + \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

and so g is holomorphic with derivative $g'(z) \frac{f'(z)}{f(z)}$. Hence we have that

$$\left(f(z)e^{-g(z)} \right)' = f'(z)e^{-g(z)} - f(z)e^{-g(z)} \frac{f'(z)}{f(z)} = 0$$

and so $f(z)e^{-g(z)}$ is a constant say α and moreover

$$\alpha = f(z_0)e^{-g(z_0)} = 1$$

\square

Theorem 11.4 (Riemann Mapping Theorem). *Given $D \subseteq \mathbb{C}$ open and simply connected such that $D \neq \emptyset$ and $D \neq \mathbb{C}$, then D is conformally equivalent to $\Delta = \{|z| < 1\}$.*

Proof. Bit of a mess. \square

THE END