Dynamical Notes - Proofs to Remember

Thomas Chaplin

1 Sharkovskii's Theorem

Theorem 1.1 (Sharkovskii's Theorem)**.** *If $f : I \to I$ is continuous and there is a point of prime period 3. Then for each $n \in \mathbb{N}$ there is a periodic point of prime period n.*

The proof proceeds by a number of lemmata.

Lemma 1.2. *Given $I \subseteq [0,1]$ a closed interval ,if $f(I) \supseteq I$ or $f(I) \subseteq I$ then I contains a fixed point for f.*

Proof. Use the ITV on $g(x) = f(x) - x$ and consider the endpoints. □

Lemma 1.3 (Whittling down intervals)**.** *If $I, I' \subseteq [0,1]$ are closed intervals and $f(I) = I'$, then $\exists$ a closed interval $I_0 \subseteq I$ such that $f(I_0) = I'$.*

Proof. Suppose $I' = [a,b]$ then let

$$\begin{aligned} A &:= f^{-1}(a) \cap I \\ B &:= f^{-1}(b) \cap I \end{aligned}$$

then take $x_0 = \sup(A)$ and $y_0 = \inf(B)$. Then $I_0 := [x_0, y_0]$ will do the job. □

Lemma 1.4. *Assume that we have closed intervals $I_1, \ldots, I_n \subseteq [0,1]$ such that*

- $f(I_n) \supseteq I_1$,
- $f(I_j) \supseteq I_{j+1}$ *for all appropriate* j,

then there is a fixed point x for f^n such that

$$x \in I_1, f(x) \in I_2, \ldots, f^{n-1} \in I_n$$

Proof. We can just apply the whittling lemma to the intervals in reverse order so

$$\begin{array}{lll} \exists I'_n \subseteq I_n & s.t. & f(I'_n) = I_1 \\ \exists I'_{n-1} \subseteq I_{n-1} & s.t. & f(I'_{n-1}) = I'_n \\ & \vdots & \\ \exists I'_1 \subseteq I_1 & s.t. & f(I_1)' = I'_2 \end{array}$$

In particular we have that $f^n(I'_1) = I_1 \supseteq I'_1$ and hence the first lemma gives us the desired fixed point. □

Proof. of Theorem 1.1.
Let $f^3(x) = x$ be our point of prime period 3. For now we will assume that

$$\{x, f(x), f^2(x)\} = \{x_1, x_2, x_3\}$$

where $0 \leq x_1 < x_2 < x_3 \leq 1$. We also assume $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$. Other cases are similar. Let $I_0 := [x_1, x_2]$ and $I_1 := [x_2, x_3]$.
Observe that

(a) $f(I_0) \supseteq I_1$, and

(b) $f(I_1) \supseteq I_0 \cup I_1$.

We now split the proof into a number of cases:
Case 1: $(n = 3)$ This follows from the assumption.
Case 2: $(n = 1)$ This follows from the first lemma thanks to *(b)*.
Case 3: $(n = 2$ or $n \geq 4)$
Note that we can make a chain of intervals as in the last Lemma.

$$\underbrace{I_0 \overset{f}{\rightsquigarrow} I_1 \overset{f}{\rightsquigarrow} I_1 \overset{f}{\rightsquigarrow} \dots \overset{f}{\rightsquigarrow} I_1 \overset{f}{\rightsquigarrow} I_0}_{n-1 \text{ times}}$$

where $A \rightsquigarrow B$ means $f(A) \supseteq B$. Hence there is a fixed point for f^n which starts in I_0 spends $n-1$ in I_1 and then returns to I_0. Because the earliest return is at time n we can be sure that this is our prime period. ◻

2 Dense Irrational Orbits

Theorem 2.1. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then for any $z \in \mathcal{K}$ we have*

$$\{R_\alpha^n(x) \mid n \in \mathbb{N}\}$$

is a dense set in the circle $\mathcal{K}$.

Proof. Get yourself some pigeons. If you put a pigeon on the circle for every point in the orbit up to time $\frac{1}{\epsilon} + 1$ then two pigeons, Kenny k and Lenny l, must be ϵ close.

$$d(R_\alpha^l(p), R_\alpha^k(p)) < \epsilon$$

Without loss of generality, assume that Kenny is further along the orbit then Lenny so that

$$m := k - l > 0.$$

Then for any $x \in \mathcal{K}$ we have $d(R_\alpha^m(x), x < \epsilon)$. Hence the orbit $\{x, R_\alpha^m(x), R_\alpha^{2m}(x), R_\alpha^{3m}(x), \dots\}$ is ϵ dense in the circle. ◻

3 Rational Points and Periodic Points

Theorem 3.1. *If $f : \mathcal{K} \to \mathcal{K}$ has a periodic point x_0 of period m then $\alpha(f) \in \mathbb{Q}$.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be some lift then we must have

$$F^m(x) - x = k \in \mathbb{Z}$$

where $\rho(x) = x_0$. Then we can write any integer as $n = pm + r$ where $p \geq 0$ and $r \in [0, m)$. Hence

$$F^n(x) = F^{pm+r}(x) = F^r(x) + pk$$

Then we can conclude

$$\lim_{n\to\infty} \frac{1}{n} F^n(x) = \lim_{p\to\infty} \frac{1}{pm+r} \left(F^r(x) + pk\right) = \frac{k}{m} \in \mathbb{Q}$$

□

Theorem 3.2. *If $f : \mathcal{K} \to \mathcal{K}$ has 0 rotation number then f has a fixed point.*

Proof.

- Take a lift $\widetilde{F}$ that gives $\lim_{n\to\infty} \frac{\widetilde{F}^n(x)}{n} = m$.
- Create a nicer lift $F := \widetilde{F} - m$ so that the limit becomes 0.
- Assume d has no fixed point then by the IVT we can assume WLOG $F(y) > y$ for all $y \in \mathbb{R}$.
- Hence $(F^n(0))_{n\in\mathbb{N}}$ is increasing so we just need to show boundedness.
- Suppose unbounded then $|F^{n_0}(0)| > 1$ and hence for all m we have $|F^{mn_0}(0)| > m$.

  $$\frac{|F^{mn_0}(0)|}{mn_0} > \frac{m}{mn_0} = \frac{1}{n_0}$$

  which cause a contradiction with the earlier 0 limit.
- It can be seen that the limit of this sequence is a fixed point.

□

Note: As a corollary if the rotation number is $\frac{a}{b} \in \mathbb{Q}$ then f^b has 0 rotation number and hence fixed point. Therefore, f has a periodic point.