## Manifolds Notes

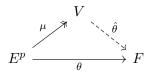
### 1 Differential Forms

Given vector spaces E, F and  $p \in \mathbb{N}$  we denoted

$$A(E^p, F) := p - \text{linear alternating maps } E^p \to F$$

where by alternating we mean that swapping any two coordinates negates the output. Equivalently, if two coordinates are the same then the output is 0.

**Lemma 1.1.** Given E and p there is a vector space V together with a surjective map  $\mu \in A(E^p, V)$  with the property that if  $\theta \in A(E^p, F)$  then there is a linear map  $\hat{\theta} : V \to F$  such that  $\theta = \hat{\theta} \circ \mu$ 



**Note:** The  $\hat{\theta}$  is unique given  $\theta$  and V. The V is unique up to isomorphism.

We write  $V = \Lambda^p E$  and given  $v_1, \dots, v_p \in E$  we write

$$v_1 \wedge \cdots \wedge v_p := \mu(v_1, \dots, v_p)$$

We say  $\Lambda^p E$  is the p-th exterior power of E.

#### 1.1 Basis for $\Lambda^p E$

Let  $e_1, \ldots, e_m$  be a basis for E. Since  $\mu$  is surjective  $\Lambda^p E$  is spanned by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid i_k \in I(m)\}$$

where we can assume that the  $i_k$  are distinct else their image would be null. We can also assume that the indices are in order up to sign.

**Lemma 1.2.** These elements are linearly independent and hence form a basis.

Therefore we can say  $\dim(\Lambda^p E) = \binom{m}{p}$ .

## 1.2 Wedge Product

Given  $p, q \in \mathbb{N}$  with  $p, q \ge 1$  we can define the bilinear wedge product

$$\cdot \wedge \cdot : (\Lambda^p E \times \Lambda^q E) \to \Lambda^{p+q} E$$

First we define on it on a basis. So take a basis  $e_1, \ldots, e_m$  of E and then define

$$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_q}) = e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}$$

This can then be extended linearly to arbitrary elements and hence doesn't depend on our initial choice of basis.

## 1.3 Induced maps

Suppose we have a linear map between finite dimensional vector spaces

$$\phi: E \to F$$

then we get a multi linear map in the natural way

$$\phi^p: E^p \to F^p$$

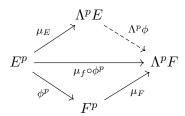
By composing with the surjective map  $\mu_F$  we get an alternating map

$$\mu_F \circ \phi^p : E^p \to \Lambda^p F$$

Hence by the defining property of  $\Lambda^P E$  we get a linear map

$$\Lambda^p \phi : \Lambda^p E \to \Lambda^p F$$

with the property that the outer diamond in the below diagram commutes.



Essentially, if  $e_1, \ldots, e_m$  is a basis for E, then we can describe  $\Lambda^p \phi$  by

$$(\Lambda^p \phi)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (\phi e_{i_1}) \wedge \cdots \wedge (\phi e_{i_p}).$$

# 1.4 🙎 The dreaded p-form 🙎

Let M be an m-manifold. Given  $x \in M$  we can form the p-th exterior power of the cotangent space

$$\Lambda^p(T_r^*M)$$

We can assemble these together into a vector bundle  $\Lambda^p(T^*M)$ . Subsequently, a *p*-form on M is define to be a section of the bundle  $\Lambda^p(T^*M)$ 

#### What the fuck does this mean????

A more natural way to think about p-forms is to take local coordinates. Let  $\phi: U \to \mathbb{R}^m$  be a chart yielding local coordinates  $x_1, \ldots, x_m$ . We have locally defined 1-forms  $dx_1, \ldots, dx_m$  which form a basis for the cotangent space

$$dx_i \left( \frac{\partial}{\partial x_i} \right) = \delta_{ij}$$

Then given  $I \in \mathcal{I}(m,p)$  we write  $\mathbf{d}x_I := dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ . Thus  $\{\mathbf{d}x_I \mid I \in \mathcal{I}(m,p)\}$  forms a basis for  $\Lambda^P(T^*M)$ . It follows that any *p*-form  $\omega$  on U can be uniquely written in the form

$$\omega = \sum_{I \in \mathcal{I}(m,p)} \lambda_I \mathbf{d} x_I$$

where each  $\lambda_I: U \to \mathbb{R}$  is a locally-defined smooth function.

**Note:** This is all we really need from the bundle structure of  $\Lambda^p(T^*M)$ .

In particular, if p = m then an m-form locally looks like

$$\lambda (dx_1 \wedge \cdots \wedge dx_m)$$

for some smooth function  $\lambda: U \to \mathbb{R}$ .

#### 1.5 Pull-backs

Suppose we have a smooth function between manifolds

$$f:M\to N$$

Given a p-form  $\omega$  on N we can define a pull-back p-form  $f^*\omega$  on M as follows. Given  $x \in M$  we have the derivative map  $d_x f$  and hence a dual map

$$(d_x f)^*: T_{f(x)}^* N \to T_x^* M, \qquad \eta \mapsto \eta \circ d_x f \quad \text{where } \eta: T_{f(x)} N \to \mathbb{R} \text{ is linear}$$

This in turn gives rise to a linear map

$$\Lambda^p (d_x \phi)^* : \Lambda^p T_{fx}^* N \to \Lambda^p T_x^* M$$

Then our pull-back is defined by

$$(f^*\omega)(x) := (\Lambda^p (d_x \phi)^*) [\omega(f(x))]$$

One takes on blind faith that this is smooth and hence a p-form. In particular, we can pull back p-forms to any manifold embedded within a larger manifold (such as  $\mathbb{R}^n$ ).