Dynamical Notes - Proofs to Remember

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1 Sharkovskii's Theorem

Theorem 1.1 (Sharkovskii's Theorem)**.** *If $f : I \to I$ is continuous and there is a point of prime period 3. Then for each $n \in \mathbb{N}$ there is a periodic point of prime period n.*

The proof proceeds by a number of lemmata.

Lemma 1.2. *Given $I \subseteq [0, 1]$ a closed interval ,if $f(I) \supseteq I$ or $f(I) \subseteq I$ then I contains a fixed point for f.*

Proof. Use the ITV on $g(x) = f(x) - x$ and consider the endpoints. □

Lemma 1.3 (Whittling down intervals)**.** *If $I, I' \subseteq [0, 1]$ are closed intervals and $f(I) = I'$, then $\exists$ a closed interval $I_0 \subseteq I$ such that $f(I_0) = I'$.*

Proof. Suppose $I' = [a, b]$ then let

$$A := f^{-1}(a) \cap I$$
$$B := f^{-1}(b) \cap I$$

then take $x_0 = \sup(A)$ and $y_0 = \inf(B)$. Then $I_0 := [x_0, y_0]$ will do the job. □

Lemma 1.4. *Assume that we have closed intervals $I_1, \dots, I_n \subseteq [0, 1]$ such that*

- $f(I_n) \supseteq I_1$,
- $f(I_j) \supseteq I_{j+1}$ *for all appropriate* j,

then there is a fixed point x for f^n such that

$$x \in I_1, f(x) \in I_2, \dots, f^{n-1} \in I_n$$

Proof. We can just apply the whittling lemma to the intervals in reverse order so

$$\begin{array}{lll} \exists I'_n \subseteq I_n & s.t. & f(I'_n) = I_1 \\ \exists I'_{n-1} \subseteq I_{n-1} & s.t. & f(I'_{n-1}) = I'_n \\ & \vdots & \\ \exists I'_1 \subseteq I_1 & s.t. & f(I_1)' = I'_2 \end{array}$$

In particular we have that $f^n(I'_1) = I_1 \supseteq I'_1$ and hence the first lemma gives us the desired fixed point. □

Proof. of Theorem 1.1.
Let $f^3(x) = x$ be our point of prime period 3. For now we will assume that

$$\{x, f(x), f^2(x)\} = \{x_1, x_2, x_3\}$$

where $0 \leq x_1 < x_2 < x_3 \leq 1$. We also assume $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$. Other cases are similar. Let $I_0 := [x_1, x_2]$ and $I_1 := [x_2, x_3]$.
Observe that

(a) $f(I_0) \supseteq I_1$, and

(b) $f(I_1) \supseteq I_0 \cup I_1$.

We now split the proof into a number of cases:
Case 1: ($n = 3$) This follows from the assumption.
Case 2: ($n = 1$) This follows from the first lemma thanks to *(b)*.
Case 3: ($n = 2$ or $n \geq 4$)
Note that we can make a chain of intervals as in the last Lemma.

$$\begin{tikzcd} I_0 \arrow[r, rightsquigarrow, "f"] \arrow[rrrrr, dashed, bend right, "n-1 \text{ times}"'] & I_1 \arrow[r, rightsquigarrow, "f"] & I_1 \arrow[r, rightsquigarrow, "f"] & \dots \arrow[r, rightsquigarrow, "f"] & I_1 \arrow[r, rightsquigarrow, "f"] & I_0 \end{tikzcd}$$

where $A \rightsquigarrow B$ means $f(A) \supseteq B$. Hence there is a fixed point for f^n which starts in I_0 spends $n-1$ in I_1 and then returns to I_0. Because the earliest return is at time n we can be sure that this is our prime period. $\square$

2 Independence of Lifts

3 Dense Irrational Orbits

Theorem 3.1. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then for any $z \in \mathcal{K}$ we have*

$$\{R^n_\alpha(x) \mid n \in \mathbb{N}\}$$

is a dense set in the circle $\mathcal{K}$.

Proof. Get yourself some pigeons. If you put a pigeon on the circle for every point in the orbit up to time $\frac{1}{\epsilon} + 1$ then two pigeons, Kenny k and Lenny l, must be ϵ close.

$$d(R^l_\alpha(p), R^k_\alpha(p)) < \epsilon$$

Without loss of generality, assume that Kenny is further along the orbit then Lenny so that

$$m := k - l > 0.$$

Then for any $x \in \mathcal{K}$ we have $d(R^m_\alpha(x), x < \epsilon)$. Hence the orbit $\{x, R^m_\alpha(x), R^{2m}_\alpha(x), R^{3m}_\alpha(x), \dots\}$ is ϵ dense in the circle. $\square$

4 Rational Points and Periodic Points

Theorem 4.1. *If $f : \mathcal{K} \to \mathcal{K}$ has a periodic point x_0 of period m then $\alpha(f) \in \mathbb{Q}$.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be some lift then we must have

$$F^m(x) - x = k \in \mathbb{Z}$$

where $\rho(x) = x_0$. Then we can write any integer as $n = pm + r$ where $p \geq 0$ and $r \in [0, m)$. Hence

$$F^n(x) = F^{pm+r}(x) = F^r(x) + pk$$

Then we can conclude

$$\lim_{n\to\infty} \frac{1}{n} F^n(x) = \lim_{p\to\infty} \frac{1}{pm+r} \left(F^r(x) + pk\right) = \frac{k}{m} \in \mathbb{Q}$$

□

Theorem 4.2. *If $f : \mathcal{K} \to \mathcal{K}$ has 0 rotation number then f has a fixed point.*

Proof.
- Take a lift $\widetilde{F}$ that gives $\lim_{n\to\infty} \frac{\widetilde{F}^n(x)}{n} = m$.
- Create a nicer lift $F := \widetilde{F} - m$ so that the limit becomes 0.
- Assume d has no fixed point then by the IVT we can assume WLOG $F(y) > y$ for all $y \in \mathbb{R}$.
- Hence $(F^n(0))_{n\in\mathbb{N}}$ is increasing so we just need to show boundedness.
- Suppose unbounded then $|F^{n_0}(0)| > 1$ and hence for all m we have $|F^{mn_0}(0)| > m$.

  $$\frac{|F^{mn_0}(0)|}{mn_0} > \frac{m}{mn_0} = \frac{1}{n_0}$$

  which cause a contradiction with the earlier 0 limit.
- It can be seen that the limit of this sequence is a fixed point.

□

Note: As a corollary if the rotation number is $\frac{a}{b} \in \mathbb{Q}$ then f^b has 0 rotation number and hence fixed point. Therefore, f has a periodic point.

5 Pointeré's Theorem and Minimality

A homeomorphism is called minimal if every orbit is dense.

Example: Any irrational rotation R_α is minimal.

Theorem 5.1 (Poincaré's Theorem)**.** *Any minimal circle homeomorphism is topologically conjugate to an irrational rotation.*

Given a circle homeomorphism $f : \mathcal{K} \to \mathcal{K}$ and some lift F we define the following countable sets

$$\Lambda_{x_0} := \{F^n(x_0) + m \mid m, n \in \mathbb{Z}\}$$
$$\Omega := \{n\rho + m \mid m, n \in \mathbb{Z}\}$$

for some fixed $x_0 \in \mathbb{R}$ and where $\rho = \rho(f)$ is the rotation number. Note that $\Lambda_{x_0} = \pi^{-1}\{f^n(\pi x_0)\}$ and $\Omega = \pi^{-1}\left\{R_\rho^n(0)\right\}$ where π is the usual projection.

Lemma 5.2. *Let f be a circle homeomorphism and $x_0 \in \mathcal{K}$. If the rotation number ρ is irrational then the map $T : \Lambda_{x_0} \to \Omega$ given be*

$$T(F^n(x_0) + m) = n\rho + m$$

is a bijection. Moreover,

1. *T is strictly increasing*
2. *$T(x+1) = T(x) + 1$*
3. *$T(F(x)) = T(x) + \rho$ for all $x \in \Lambda_{x_0}$.*

Proof. This is omitted but might be worth glancing over. $\square$

Proof. of Poincaré's Theorem Since f is minimal, it has no periodic points because their orbits would be finite and hence not dense. So the rotation number ρ is irrational.
Take a lift F of f and $x_0 \in \mathbb{R}$ and write $\Lambda = \Lambda_{x_0}$. The sets Ω and Λ are dense in $\mathbb{R}$ due to the minimality of R_ρ and f respectively.
Thus $\pi(\Omega)$ and $\pi(\Lambda)$ must be dense in $\mathcal{K}$. Moreover, the Lemma tells us that $T : \Lambda \to \Omega$ is strictly increasing. Consequently, we can extend to a unique continuous function $H : \mathbb{R} \to \mathbb{R}$ (which restricts to T on Λ). Moreover, H is strictly increasing, H is continuous and so is its inverse.

Note: This is non-trivial. It is an exercise to show that given dense sets $X, Y \subseteq \mathbb{R}$ and $f : X \to Y$ a bijection, there exists a unique homeomorphism extension to $\mathbb{R}$.

By continuity H inherits the properties *(2)* and *(3)* in the previous Lemma. The first says that H is a lift of circle homeomorphism h. The second say that $h \circ f = R_\rho \circ h$. $\square$

So we now know that if f is a circle homeomorphism then there is a unique homeomorphism h satisfying

$$h(f(x)) = h(x) + \rho \quad (mod 1) \quad \forall x \in \mathcal{K}$$

Note that this is a linear equation on h. We can conclude that a solution to this equation is unique up to adding a constant corresponding to choosing with point in $\mathcal{K}$ is sent to zero. For a hand-wavey explanation of this, see the lecture notes.

6 Expanding Maps and Shift Spaces

Theorem 6.1. *If $f : \mathcal{K} \to \mathcal{K}$ is an expanding map, preserves orientation and has degree 2 then there is a semi-conjugacy $h : \Sigma \to \mathcal{K}$ to the full shift on two symbols.*

$$\begin{tikzcd}
\Sigma \arrow[r, "\sigma"] \arrow[d, "h"'] & \Sigma \arrow[d, "h"] \\
\mathcal{K} \arrow[r, "f"] & \mathcal{K}
\end{tikzcd}$$

Proof. Take any $n \in \mathbb{N}$. Then $\deg f^n = (\deg f)^n$ so there are w^n pre-images of p under f^n. These are numbered p_j starting with $p_0 = p$ and number consecutively anticlockwise. These points define intervals which we denote $A_{\omega_0 \dots \omega_{n-1}}$ where the sequence of ω_i is just the binary representation of the position in the circle.
Let K denote the uniform bound away from 1 of the derivative. We have a number of results:

1. $f^n(A^{\circ}_{\omega_0 \dots \omega_{n-1}}) = \mathcal{K} \setminus \{p\}$
2. $A_{\omega_0 \dots \omega_{n-1}}$ is a closed interval of length $< K^{-n}$.
3. $A_{\omega_0 \dots \omega_{n-1}\omega_n} \subseteq A_{\omega_0 \dots \omega_{n-1}}$.
4. $f^n(A_{\omega_0 \dots \omega_n}) = A_{\omega_n}$.
5. $f(A_{\omega_0 \dots \omega_n}) = A_{\omega_1 \dots \omega_n}$.

Now we can define our conjugacy $h : \Sigma \to \mathcal{K}$. Given $\omega = (\omega_k)_{k=0}^{\infty} \in \Sigma$ let $B_n(\omega) = A_{\omega_0 \dots \omega_{n-1}}$. These are the points in the circle that start in the ω_0 interval then go to ω_1, then to ω_2 and after f^{n-1} are in the ω_{n-1} interval. The properties implies that $B_{n+1}(\omega) \subseteq B_n(\omega)$. The sets are also closed and their diameters go to 0. Hence their infinite intersection is a single points which we define to be $h(\omega)$. The proof of their desired properties is discussed below in vague detail but is written in the lecture notes with more rigour. □

7 Finding semi-conjugacies/conjugacies

If you can partition your space X into n subsets $I_1, \dots, I_n$ where one could conceivably go from any partition element I_a to any other I_b, then you might be able to find a semi-conjugacy to the full shift on n symbols.
The trick is to define a map $\pi : \Sigma \to X$ by

$$\pi(\mathbf{x}) = \bigcap_{n=1}^{\infty} T^{-n} I_{x_n}$$

If the sets $I(x_0, \dots, x_n) := \cap_{k=0}^{n} T^{-k} I_{x_k}$ are closed and nested and their diameter tends to zero as $n \to \infty$ then this map is well-defined because the infinite intersection contains one point. Moreover, it is continuous because if $\mathbf{x}$ and $\mathbf{y}$ agree up to N places then they both lie in $I(x_0, \dots x_{N-1})$ whose diameters goes to 0 as $N \to \infty$.
The commutative relationship $T \circ \pi = \pi \circ \sigma$ then follows rather quickly. To get surjectivity, it suffices to show that the image of Σ is dense. This usually involves taking in point $x \in X$ such that no $T^n x$ lies on the boundary between any I_j for some $n \geq 0$ and then this points orbit will describe its pre-image in Σ.

Note: Shift spaces are totally disconnected, i.e. the connected components are one-point sets. In particular, they are disconnected and so this can often be used to rule out the existence of conjugacies to more familiar sets.

8 Transitivity and Mixing

Note: A compact metric space has a countable dense set of points!

Theorem 8.1 (Baire's Theorem)**.** *Given a compact metric space X, the intersection of countably many open, dense subsets of X is itself dense in X.*

Theorem 8.2. *If a map $T : X \to X$ on a compact metric space X is topologically transitive then there exists a dense orbit.*

Proof. There is a countable dense set of points $\{x_k\}$ so if we can find an orbit the gets ϵ close to every x_k for arbitrary ϵ then we are done. So we want x such that for every x_k and $m \geq 1$ there is an $n \in \mathbb{Z}$ such that

$$x \in T^{-n}\mathbb{B}\left(x_k, \frac{1}{m}\right)$$

or equivalently we want to find

$$x \in \bigcap_{k,m} \bigcup_{n \in \mathbb{Z}} T^{-n}\mathbb{B}\left(x_k, \frac{1}{m}\right)$$

which is a countable intersection (over m) of open dense sets. By Baire's Theorem our desired point exists. $\square$

9 Arithmetic Progressions

We say a subset $C \subseteq \mathbb{Z}$ contains arithmetic progressions of arbitrary length if

$$\forall k \geq 1 \quad \exists c \in \mathbb{Z} \text{ and } d \in \mathbb{N} \quad \text{such that}$$

$$c, c+d, c+2d, \ldots, c+(k-1)d \in C$$

Similarly we say a map $T : X \to X$ is multiple mixing if for any non-empty open set $U \subseteq X$ and $k \geq 1$ there exists $d \geq 1$ such that

$$U \cap T^{-d}U \cap T^{-2d}U \cap \cdots \cap T^{-(k-1)d}U \neq \emptyset$$

Theorem 9.1 (van der Waerden's Theorem)**.** *Given any finite integer partition $\mathbb{Z} = \cup_{i=1}^{M} C_i$ there is an i such that C_i contains arithmetic progressions of arbitrary length.*

To prove this via a dynamical approach we must create a dynamical formulation. To a partition of $\mathbb{Z}$ we associate a single infinite sequence $\mathbf{x} = (x_n) \in \{1, \ldots, M\}^{\mathbb{Z}}$ defined by

$$x_n = i \quad \text{if} \quad n \in C_i$$

Let $X = \overline{\cup_{n \in \mathbb{Z}} \sigma^n \mathbf{x}}$ be the closure of the orbit of $\mathbf{x}$ where σ is the shift map.

Lemma 9.2 (Dynamic Formulation)**.** *Assume that for some $[i]$ (cylinder set) we have that*

$$X \cap [i] \cap \sigma^{-d}[i] \cap \sigma^{-2d}[i] \cap \ldots \sigma^{-(k-1)d}[i] \neq \emptyset$$

for some $k, d \geq 1$ then C_i contains an arithmetic progression of length k.

Proof. The space is the closure of the orbit of $\mathbf{x}$ and this set is non-empty and open. The orbit itself is dense in X and hence intersects our open set. So there is $n \in \mathbb{Z}$ such that $\sigma^n x$ is in our set. This means that $x_{n+jd} = i$ for $j = 0, \ldots, k-1$ and hence $n + jd \in C_i$ for these j. $\square$

Proposition 9.3 (Multiple Recurrence)**.** *The shift map is multiple mixing when restricted to a minimal subset $Y \subseteq X$.*

Proof. of van der Waerden's Theorem Take a minimal subset $Y \subseteq X$. Taking $U = [i]$ where i is chosen such that $[i] \cap Y \neq \emptyset$, we see the set from the dynamical formulation is open and hence non-empty by multiple recurrence so we have arbitrary arithmetic progressions. □

10 Hyperbolic Toral Automorphisms

Fixed points and mixing

11 Entropy

of shift maps and toral automorphisms.