Chapter 13 – Commutators

Definition 13.1.1 If g, h are two elements of a group G, then we call the element

$$[g,h] = ghg^{-1}h^{-1}$$

the **commutator** of g and h.

Note: If g and h commute then

$[g,h] =$

$[g,h]$ provides a measure for how much g and h fail to commute.

If α, β are permutations, and α, β fail to commute by "just a little bit" then $[\alpha,\beta]$ will be "close" to ε i.e. it will only permute a few numbers.

Example: In S_3 let $\alpha = (1\ 3)$, $\beta = (1\ 2\ 3)$

Creating Puzzle Moves with commutators:

We will concentrate on permutations in S_n.

Definitions: For $\alpha \in S_n$, define the fixed set of α by

$$\mathrm{fix}(\alpha) = \{ m \in [n] \mid \alpha(m) = m \}$$

(This is just the set of numbers that would appear as 1-cycles in the disjoint cycle form of α.)

The moved set of α is the complement of $\mathrm{fix}(\alpha)$:

$$\mathrm{mov}(\alpha) = \overline{\mathrm{fix}(\alpha)} = \{ m \in [n] \mid \alpha(m) \neq m \}$$

(This is the set of all numbers which appear in cycles of length ≥ 2 in the disjoint cycle form of α.)

For $A \subset [n]$, and $\alpha \in S_n$ we define

$$\alpha A = \{ \alpha(a) \mid a \in A \}$$

called the <u>image of A under α</u>.

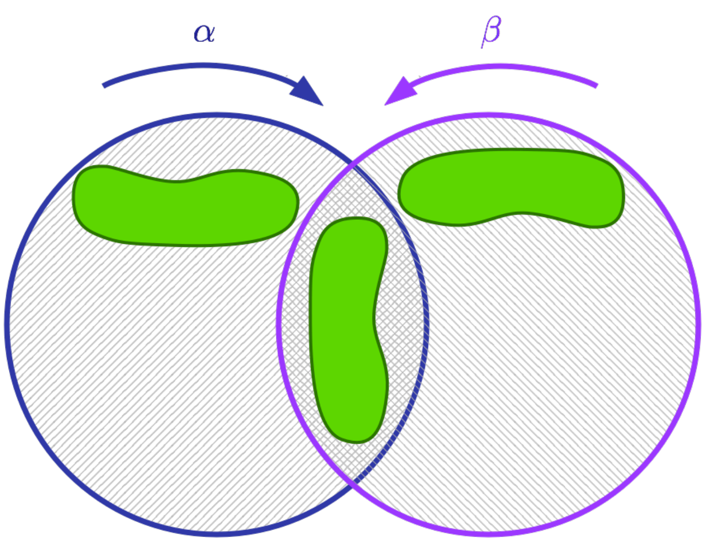
Note: $|\alpha A| = |A|$ since α is injective.

Example: Let $\alpha = (1\ 5\ 3\ 9)(4\ 11) \in S_{13}$

<u>When is $[\alpha, \beta]$ close to the identity:</u>

We'll look at conditions for which $\mathrm{mov}([\alpha,\beta])$ is small.

First, consider the diagram:



$$\mathrm{mov}([\alpha,\beta]) \subset \mathrm{mov}(\alpha,\beta) \cup \alpha^{-1}\mathrm{mov}(\alpha,\beta) \cup \beta^{-1}\mathrm{mov}(\alpha,\beta)$$

If $m \in [n]$ is moved by $[\alpha,\beta]$, i.e. $m \in mov([\alpha,\beta])$ then both:

(a) $m \in mov(\alpha)$ or $\beta(m) \in mov(\alpha)$; and
(b) $m \in mov(\beta)$ or $\alpha(m) \in mov(\beta)$.

In other words,

$$(*) \qquad mov([\alpha,\beta]) = \left(mov(\beta) \cup \alpha^{-1} mov(\beta)\right) \cap \left(mov(\alpha) \cup \beta^{-1} mov(\alpha)\right)$$

Proof of (a), (b) :

(a)

(b) Proof similar to part (a) . $\square$

Another way to write $(*)$ is

$$\boxed{mov([\alpha,\beta]) \subset \underbrace{mov(\alpha,\beta)}_{\substack{\text{pieces moved by both } \alpha \text{ and } \beta \\ \text{(common intersection)}}} \cup \alpha^{-1}\underbrace{mov(\alpha,\beta)}_{\substack{\text{pieces moved by } \alpha \text{ to common} \\ \text{intersection}}} \cup \beta^{-1}\underbrace{mov(\alpha,\beta)}_{\substack{\text{pieces moved by } \beta \text{ to} \\ \text{common intersection.}}}.} \qquad (13.2)$$

This says :

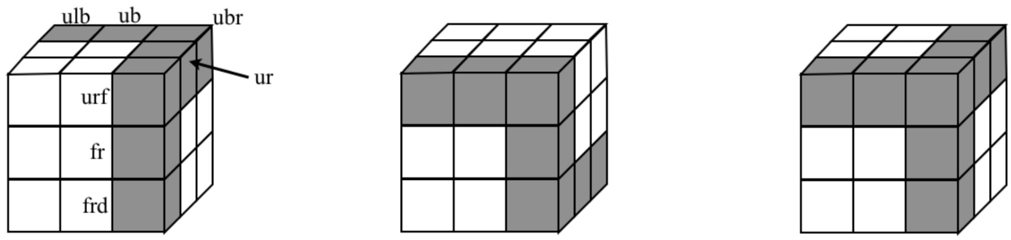
If α, β are puzzle moves then $[\alpha,\beta]$ only affects pieces that are in, or moved to, locations that are moved by both α and β.

To create a move which only affects a few pieces choose α and β to have very little overlap.

Creating Moves on Rubik's Cube :

Puzzle moves $[x,y]$:

$x = U$, $y = R$ | $x = F$, $y = R$ | $x = F$, $y = R^{-1}$



(a) Possible cubies moved by $URU^{-1}R^{-1}$.

(b) **Z-commutator**: Shading indicates locations changed by $FRF^{-1}R^{-1}$

(c) **Y-commutator**: Shading indicates locations changed by $FR^{-1}F^{-1}R$

Figure 13.1: Y- and Z- commutators

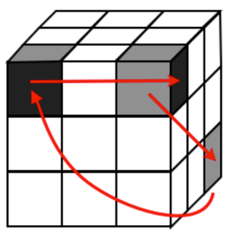


Figure 13.2: cubies moved by $[LD^2L^{-1}, U]$.

Consider

$x = LD^2L^{-1}$ (swaps rdb and lfu)

$y = U$

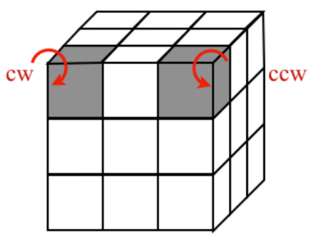
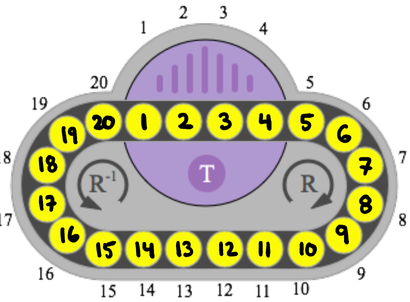


Figure 13.3: cubies moved by $[LD^2L^{-1}F^{-1}D^2F, U]$.

$x = LD^2L^{-1}F^{-1}D^2F$ (twists ufl)

$y = U$

Oval Track Puzzle :

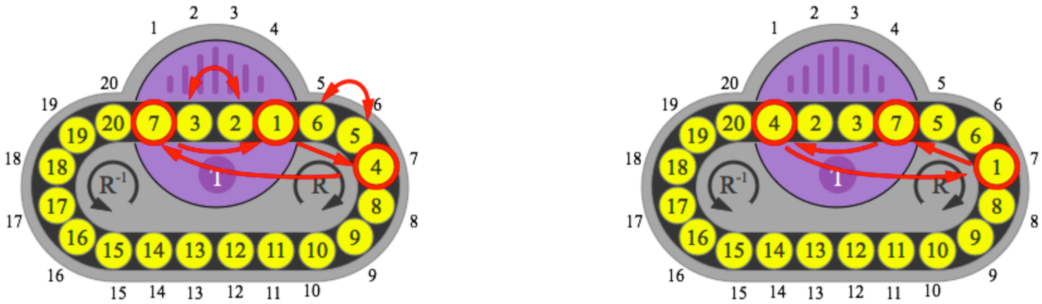


$mov(T) = \{1, 2, 3, 4\}$

$mov(R) = \{1, 2, \ldots, 20\}$

$[R^{-3}, T] = (1\,4\,7)(2\,3)(5\,6)$

$[R^{-3}, T]^2 = (1\,7\,4)$



(a) $R^{-3}TR^3T^{-1} = (1\,4\,7)(2\,3)(5\,6)$

(b) $(R^{-3}TR^3T^{-1})^2 = (1\,7\,4)$

Figure 13.8: Basic commutators on the Oval Track puzzle