Exam 1 Review
Physics 253, Fall 2020

Important Topics

- Magnetic needles and current loops
- Calculating probabilities
- Using Dirac Notation/operators to analyze quantum systems
- Reflection and transmission of light
- Polarization of light
- Stern Gerlach experiment/ analyzer loops
- Bell's Theorem experiment
- Wheeler's delayed choice experiment
- Einstein, Podolsky, Rosen (EPR) Experiment
- Light and slit experiments / quantum mystery
- Mach Zender Interferometer / quantum seeing in the dark experiments
- Hong Ou Mandel Experiment

*This exam will cover Module 1 through to the very beginning of Module 5. In Module 5, content from Lecture 1 will be included but not the disentangling identity.

Module 1 Notes/Math Definitions

- Hermitian matrix $\hat{A}$ satisfies $\hat{A}^{\dagger} = A$ where $\dagger$ denotes the hermitian conjugate (transpose and complex conjugate)
- The hermitian conjugate of a bra is a ket, and vice versa; don't forget to switch the sign of complex numbers too!
- Eigenvectors of hermitian operators form an orthornormal basis
- $|\uparrow\rangle_z = |\downarrow\rangle_{-z}$
- $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$
- A matrix is a representation of an operator in a particular basis

Module 2 Notes/Math Definitions

- Linear operators preserve scalar multiplication and vector addition, that is: $\left(\alpha\hat{A}+\beta\hat{B}\right)|\psi\rangle = \alpha\hat{A}|\psi\rangle+\beta\hat{B}|\psi\rangle$ and $\hat{A}(\alpha|\psi\rangle+\beta|\phi\rangle)=\alpha\hat{A}|\psi\rangle+\beta\hat{A}|\phi\rangle$
- $\hat{S}_z|\uparrow\rangle_z=\frac{\hbar}{2}|\uparrow\rangle_z$ and $\hat{S}_z|\downarrow\rangle_z=-\frac{\hbar}{2}|\downarrow\rangle_z$
- $\hat{S}_+|\downarrow\rangle_z=\hbar|\uparrow\rangle_z$
- $\hat{S}_+|\uparrow\rangle_z=0$
- $\hat{S}_-|\uparrow\rangle_z=\hbar|\downarrow\rangle_z$ and $\hat{S}_-|\downarrow\rangle_z=0$
- $[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$
- $[\hat{A},\hat{B}]=-[\hat{B},\hat{A}]$
- If 2 operators commute, their commutator is equal to 0; if they do NOT commute then their commutator is nonzero. If two operators do not commute then we cannot find a complete set of eigenstates that are simultaneous eigenstates of both operators.
- $[\hat{S}_z,\hat{S}_+]=\hbar\hat{S}_+$
- $[\hat{S}_z,\hat{S}_-]=-\hbar\hat{S}_-$
- $[\hat{S}_+,\hat{S}_-]=2\hbar\hat{S}_z$
- $\hat{S}_x=\frac{1}{2}\left(\hat{S}_++\hat{S}_-\right)$ and $\hat{S}_y=\frac{1}{2i}\left(\hat{S}_+-\hat{S}_-\right)$
- $\hat{S}_+=\hat{S}_x+i\hat{S}_-$ and $\hat{S}_-=\hat{S}_x-i\hat{S}_y$
- $[\hat{S}_i,\hat{S}_j]=i\hbar\sum_k\epsilon_{ijk}\hat{S}_k$
- $\hat{S}^2=\hat{S}_x^2+\hat{S}_y^2+\hat{S}_z^2$
- Leibnitz Identity: $[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$
- $\hat{S}_z=\begin{pmatrix}\frac{\hbar}{2} & 0\\ 0 & -\frac{\hbar}{2}\end{pmatrix}$
- $\hat{S}_z|\uparrow\rangle_z=\frac{\hbar}{2}|\uparrow\rangle_z$ and $\hat{S}_z|\downarrow\rangle_z=-\frac{\hbar}{2}|\downarrow\rangle_z$
- Spin operators in x,y and z are all hermitian, whereas spin raising and lowering operators are NOT hermitian.
- A projection operator $\hat{P}$ satisfies $\hat{P}^2=\hat{P}$, which is called an idempotent operator. If we have a normalized quantum state $|\psi\rangle$, then the operator $|\psi\rangle\langle\psi|$ is a projection operator.
- $|\uparrow\rangle_z=\frac{1}{\sqrt{2}}(|\uparrow\rangle_x+|\downarrow\rangle_x)$
- The $\otimes$ symbol is called the tensor product; we use this to describe quantum states that have different properties, which live in different vector spaces. (For example, position and spin, or spin and an atomic energy state, and so on...)
- $\langle a|\otimes\langle b|\ \cdot\ |c\rangle\otimes|d\rangle=\langle a|c\rangle\times\langle b|d\rangle=\langle a|c\rangle\langle b|d\rangle$
- Superpositioner in Stern Gerlach experiments: transforms $|GS\rangle$ to $\frac{1}{\sqrt{2}}(|GS\rangle+|ES\rangle)$
- De-exciter in Stern Gerlach experiments: behaves as a ground state filter
- Exciter in Stern Gerlach experiments: transforms ground state into the excited state

The general quantum rules

1.) Identify all different **events** for a given experiment.

Events are defined by the initial and final conditions of the apparatus. Things like a photon leaves a source and is detected in one or more specific detectors, and so on.

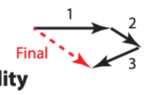
2.) Find all **alternative ways** an event can occur.

This will entail drawing paths for photons that are possible given the results of a particular event.

3.) Assign a **probability arrow** for each alternative way an event can occur by applying all relevant quantum rules.

The quantum rules tell us how to shrink and/or rotate the arrow that represents the particular way the event can occur.

4.) Hook the arrows head to tail to add them together and construct the **final arrow**.

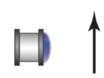


5.) Square the length of the final arrow to find the **probability** that the event will occur.

6.) Repeat steps 2-5 for all other events to find their probabilities. The total probability will be the **sum of the probabilities** for each event.

The quantum rules for light

1.) The amplitude arrow starts at twelve o'clock with unit length when the photon leaves the source.



2.) When moving through air or glass, the arrow rotates according to a clock, whose rate is determined by the color of the light.

Red light rotates at 36,000 revolutions per inch in air.

3.) Reflect off glass coming from air. Shrink by 0.2 and rotate by six hours.

4.) Reflect off glass coming from glass. Shrink by 0.2 only.

5.) Transmitting from glass to air or from air to glass. Shrink by 0.98 only.

Module 4 Notes/Math Definitions

- A 50-50 beam splitter in Mach Zehnder Interferometer apparatus transforms the incident light $|I\rangle$ into a superposition of reflected and transmitted light, denoted by $\frac{1}{\sqrt{2}}(|R\rangle + |T\rangle)$
- When we make measurements on an arbitrary state $|\psi\rangle$ we apply the projection operator (projector) corresponding to our measurement onto the state $|\psi\rangle$. Then, to calculate the probability, we take the modulus squared of our new state, including the projector.
- When light is polarized in the θ direction, we express this as: $|\theta\rangle = \cos\theta|V\rangle + \sin\theta|H\rangle$
- We also define the following state for polarizer problems: $-\theta\rangle = \cos\theta|V\rangle - \sin\theta|H\rangle$
- If we wish to represent the polarization as a spinor (vector) we get: $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = \cos\theta|V\rangle + \sin\theta|H\rangle$
- When a photon goes through a polarizer, it always corresponds to being measured, which we accomplish by applying the appropriate projection operator onto the photon's state.
- When a photon goes through a polarization rotator, on the other hand, it does not correspond to a measurement. It is just a way to manipulate the direction of the polarization of the photon. We write the action of the rotator as an operator that changes the angle of polarization of a polarization state vector as follows $|\theta\rangle \to |\theta + \phi_{\text{rotator}}\rangle$ if the physical rotator rotates the polarization by ϕ_{rotator}.
- The binomial coefficient tells us the general formula for k collections from n objects is: $\frac{n!}{k!(n-k)!} = \begin{pmatrix}n \\ k\end{pmatrix}$
- Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix}n \\ k\end{pmatrix} a^k b^{n-k}$

Exam II Review
Physics 253, Fall 2020

Exam Content

- This exam will cover Modules 5, 6 and 7 (including Problem Set 10).
- This exam will not include questions on the hydrogen atom or central force problems from Week 11 (this week's coursework).
- Focus on: the four operator identities and how to use them; the factorization method; angular momentum; rotations; position eigenstates and spherical harmonics.

Four Fundamental Operator Identities

- Leibnitz Rule:
$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$
- Hadamard:
$$e^{\hat{A}}\hat{B}e^{-\hat{A}} \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\hat{A},...[\hat{A},[\hat{A},\hat{B}]]...]]_n$$
- Weyl Form of BCH identity:
$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$
This holds when $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.
- General Form of BCH identity:
$$e^{\hat{A}}e^{\hat{B}} = e^{\left((\hat{A}+\hat{B})+\frac{1}{2}[\hat{A},\hat{B}]+\frac{1}{12}[\hat{A},[\hat{A},\hat{B}]]+\frac{1}{12}[\hat{B},[\hat{B},\hat{A}]]+...\right)}$$
This formula comes from the lecture video on BCH.
- Exponential Disentangling:

  The form of the identity varies depending on your problem and the section beginning on Module 6, page 6 provides two examples that are worth reviewing. The takeaway is that this procedure lets us rewrite an exponential of a matrix in terms of *different* exponentials of matrices that are often easier to calculate and work with.

  Using this identity usually requires remembering the power series expansion for e^x:
$$e^x = \sum_n 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$
  You can use a power series to expand exponentials of operators (matrices) into a sum with terms that usually either truncate or repeat in a useful pattern.
- Similarity Transformation of Powers (This isn't one of the four identities, but it's worth knowing; you proved this on Module 5, Page 28.)
$$\hat{A}^{-1}\left(\hat{B}\right)^n\hat{A} = (\hat{A}^{-1}\hat{B}\hat{A})^n$$

Tools/formulas from Exam I:

- Hermitian matrix $\hat{A}$ satisfies $\hat{A}^\dagger = A$ where $\dagger$ denotes the hermitian conjugate (transpose and complex conjugate)
- The hermitian conjugate of a bra is a ket, and vice versa; don't forget to switch the sign of complex numbers too!
- $|\uparrow\rangle_z = |\downarrow\rangle_{-z}$
- $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$
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- $\hat{S}_+|\downarrow\rangle_z = \hbar|\uparrow\rangle_z$
- $\hat{S}_+|\uparrow\rangle_z = 0$
- $\hat{S}_-|\uparrow\rangle_z = \hbar|\downarrow\rangle_z$ and $\hat{S}_-|\downarrow\rangle_z = 0$
- $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
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- $\hat{S}_+ = \hat{S}_x + i\hat{S}_-$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$
- $[\hat{S}_i, \hat{S}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{S}_k$
- $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$
- $\hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$
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- A projection operator $\hat{P}$ satisfies $\hat{P}^2 = \hat{P}$, which is called an idempotent operator. If we have a normalized quantum state $|\psi\rangle$, then the operator $|\psi\rangle\langle\psi|$ is a projection operator. The sum (or integral) over all projection operators (also called 'projectors') is the identity matrix. We often use this to 'multiply by 1.'
- $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x + |\downarrow\rangle_x)$
- $\langle a| \otimes \langle b| \ \cdot\ |c\rangle \otimes |d\rangle = \langle a|c\rangle \times \langle b|d\rangle = \langle a|c\rangle\langle b|d\rangle$
- When we make measurements on an arbitrary state $|\psi\rangle$ we apply the projection operator (projector) corresponding to our measurement onto the state $|\psi\rangle$. Then, to calculate the probability, we take the modulus squared of our new state, including the projector.
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Final Exam Review
Physics 253, Fall 2020

Exam Content

- The exam will have 8 problems and 1 short answer question.
- Important concepts: factorization, time dependence, four operator identities, photon detection, angular momentum.
- On the exam, you'll be graded on 8 out of the 9 questions. (I'd suggest trying them all out because partial credit can work in your favor.)
- Exam II covered content up till somewhere in Module 7; so this review sheet will focus on Modules 7, 8, 9, and 10. (It goes without saying that this sheet does not cover everything discussed in class and on homework, but hopefully it helps you to organize the big ideas you learned this semester.)

Module 7

- Angular momentum, $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$
- Any quantity that obeys these cyclical commutator relations is an angular momentum: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, and $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. For example, spin angular momentum also obeys these commutator relations, and therefore has analagous eigenvector-eigenvalue relationships to $\vec{\hat{L}}$.
- Eigenvector-eigenvalue relationships for angular momentum:

$$\hat{\vec{L}}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle \text{ and } \hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$$

- Note that we showed that $\hbar^2[l(l+1) - m^2] \geq 0$, which implies that the maximum value of m is l, and the minimum value of m is $-l$.
- Spin: spin is "spin angular momentum", so we look at the quantum number s associated with $\hat{\vec{S}}^2$ such that the eigenvalue of that operator is $\hbar^2 s(s+1)$.
- Bound states are when a quantum particle is trapped in a potential. The opposite is a continuum state where energy varies continuously and particles can escape to infinity.
- For bound states the following holds:
  - Factorization chains are formed by auxiliary hamiltonians given by $\hat{H}_i + 1 = \hat{A}_i\hat{A}_i{}^\dagger + E_i = \hat{A}_{i+1}^\dagger \hat{A}_{i+1} + E_{i+1}$.
  - Intertwining gives us $\hat{H}_i\hat{A}_i^\dagger = \hat{A}_i^\dagger\hat{H}_{i+1}$.
  - Ground state of the ith auxiliary hamiltonian satisfies $\hat{A}_i|\phi_1\rangle = 0$. This is our subsidiary condition.
  - The ith excited state of $\hat{H}_0$ with every energy E_i is $|\psi_1\rangle = \hat{A}_1^\dagger\hat{A}_2^\cdot..\hat{A}_{i-1}^\dagger|\phi_i\rangle$. Note that this is an unnormalized state. Also note that we *cannot* have $\hat{A}_1 = \hat{A}_0^\dagger$.
  - The superpotential is real and continuous.
  - Factorization is not necessarily unique.
  - Wavefunction nodes occur when $W(k'\hat{x})$ diverges.
  - Node theorem helps us verify that factorization gave us all the solutions.

- We define radial momentum as $\hat{P}_r = \frac{1}{2}\frac{\hat{\vec{r}}}{\hat{r}} \cdot \hat{\vec{P}} + \hat{\vec{P}} \cdot \frac{\vec{r}}{\hat{r}}$. We also have the following commutation relations, which yield $\hat{P}_r = \frac{\hat{\vec{r}}}{\hat{r}} \cdot \hat{\vec{p}} - \frac{i\hbar}{\hat{r}}$.
  - $[\hat{P}_\alpha, \hat{r}] = -i\hbar\frac{\hat{r}_\alpha}{\hat{r}}$
  - $[P_\alpha, \frac{1}{\hat{r}}] = i\hbar\frac{\hat{r}_\alpha}{\hat{r}^3}$
  - $[\hat{r}, \hat{P}_r] = i\hbar$
  - Note that $\hat{P}_r$ is not an observable! It does not have a complete set of eigenstates, and it does not have all real eigenvalues.
- Kinetic Energy separation of variables into radial and angular momentum: $KE = \frac{\hat{P}_r^2}{2m} + \frac{\hat{\vec{L}}^2}{2\hat{r}^2}$
- Isotropic SHO: $\hat{H} = \frac{\hat{P}_r^2}{2m} + \frac{\hat{\vec{L}}^2}{2M\hat{r}^2} + \frac{1}{2}m\omega^2\hat{\vec{r}}^2$.
  - We construct the states $|\psi\rangle = |\psi_r\rangle \otimes |l,m\rangle$ and get $\hat{H} = |\psi\rangle - \hat{H}_l|\psi_r\rangle \otimes |l,m\rangle$ where $\hat{H}_l = \frac{\hat{P}_r^2}{2m} + \frac{\hbar^2 l(l+1)}{2m\hat{r}^2} + \frac{1}{2}m\omega^2\hat{\vec{r}}^2$
  - We have the eigenvector/value relationship: $\hat{H}_l|\psi_{nl} = E_{nl}|\psi_{nl}\rangle$ where n is the principal quantum number.
  - n allows us to distinguish different energy eigenstates with the same l value; the energies do not depend on m.
  - Properties of superpotential: superpotential must be positive as r goes to ∞; as r goes to 0, the superpotential diverges with negative coefficients.
- Hydrogen
  - Hamiltonian for Hydrogen: $\hat{H} = \frac{\hat{\vec{P}}^2}{2\mu} - \frac{e^2}{\hat{r}}$ where $\mu = \frac{m_e m_p}{m_e + m_p}$
  - We can use separation of variables to get: $\hat{H} = \frac{\hat{P}_r^2}{2\mu} + \frac{\hat{\vec{L}}^2}{2\mu\hat{r}^2} - \frac{e^2}{\hat{r}}$
  - In this problem we denote our ladder operators $B_r(l)$ and we say that $\hat{H}_l = \hat{B}_r^\dagger(l)\hat{B}_r(l) + E_l^{aux}$
  - For intertwining we find: $\hat{B}_r(l)\hat{B}_r^\dagger(l) + E_l = \hat{H}_{l+1}$

Module 8

- Virial theorem: $\hat{r}\hat{p}_r$ is what we call the *virial*. This theorem tells us that the expectation value of the potential energy is equal to the expectation value of the kinetic energy for SHO. So we can write, $\langle PE \rangle = \langle KE \rangle = \frac{1}{2}\langle \mathrm{E}_{\text{total}} \rangle$
- First order perterbation theory:
  - The variational method: A method of approximation that we can also use to solve problems exactly. For example, we take the SHO Hamiltonian and perturb it to obtain $H\hat{(}\delta) = \frac{\hat{p}^2}{2m} + \frac{1}{2}k(1+\delta)\hat{x}^2$. We can expand the energy in a Taylor series, and we then say that $E_n(\delta) \approx \langle\psi_n^0|\hat{H}(\delta))|\psi_n^0\rangle$, where the $|\psi_n^0\rangle$ is the unperturbed eigenstate. After solving for $E_n(\delta)$, this gives the correct first order approximation.
  - Feynman - Hellman Theorem: $\frac{dE(\lambda)}{d\lambda} = \langle\psi(\lambda)|\frac{d\hat{H}(\lambda)}{d\lambda}|\psi(\lambda)\rangle$, where λ is some parameter.
- Kramers-Pasternack relation:

$$\frac{2m}{n^2 a_0^2}\langle nl|\hat{r}^{m-1}|nl\rangle = \frac{2(2m-1)}{a_0}\langle nl|\hat{r}^{m-2}|nl\rangle - \frac{1}{2}(m-1)((2l^2+1)^2 - (m-1)^2)\langle nl|\hat{r}^{m-3}|nl\rangle$$

- Hyperfine structure of Hydrogen:

- We calculate the Hamiltonian of the magnetic interactions between the proton and the electron by determining the magnetic field due to the proton at the location of the electron and find it to be $\Delta\hat{H} = \frac{gg_p\mu_B^2 m_e\mu_0}{4\pi m_p\hbar^2}\left(\frac{3\hat{\vec{S}}_e\cdot\hat{\vec{r}}\hat{\vec{S}}_p\cdot\hat{\vec{r}}}{\hat{r}^5} - \frac{\hat{\vec{S}}_e\cdot\hat{\vec{S}}_p}{\hat{r}^3} + \frac{8\pi}{3}\hat{\vec{S}}_e\cdot\hat{\vec{S}}_p\delta(\hat{\vec{r}})\right)$
- We then calculate the expectation value of $\Delta\hat{H}$ with the state $|Psi\rangle = |nl\rangle\otimes|lm\rangle\otimes|s^e\rangle\otimes|s_p\rangle$, which works out to be $\Delta\hat{H} = \frac{2gg_p\mu_B^2 m_e\mu_0}{3\pi m_p\hbar^2 a_0^3}\langle s_e|\otimes\langle s_p|\hat{\vec{S}}_e\cdot\hat{\vec{S}}_p|s_e\rangle\otimes|s_p\rangle$.
- Using the eigenvalues of $\hat{\vec{S}}_e\cdot\hat{\vec{S}}_p$ for the singlet and triplet states, we conclude that $\Delta E_{\text{singlet-triplet}} = \frac{2gg_p\mu_B^2 m_e\mu_0}{3\pi m_p a_0^3}$
- In astronomy, hyperfine hydrogen transitions can tell us how fast objects in space are moving using the red shift from the Doppler effect
- Energy splitting in hydrogen is one of the most accurately measured results with about 13 digits of accuracy. It is measured in a hydrogen maser, an atomic clock similar to a laser.

- Second-order perturbation theory:
  - Now, we write our perturbed Hamiltonian as $H\hat{(}\lambda) = \hat{H}_0 + \lambda V(\hat{x})$, which has energy eigenvalues $E_n(\lambda)$
  - Then we expand the energy as $E_n(\lambda) = E_n(0) + \lambda\left.\frac{dE_n(\lambda)}{d\lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2}\left.\frac{d^2E_n(\lambda)}{d\lambda^2}\right|_{\lambda=0} + ...$
  - We then must calculate that second derivative and use it to find the second-order approximation of the energy, which is done in detail in lecture
- Proton charge radius: In our Hydrogen calculations, we assumed the nucleus was a point charge, but this is not quite accurate. We are able to calculate how the size of the nucleus affects the energy levels of Hydrogen and how they shift, telling us how accurate our measurements must be to determine the size of the nucleus.
- Schrodinger equation: $\frac{-\hbar^2}{2m}\psi''(\hat{x}) + V(\hat{x})\psi(\hat{x}) = E\psi(\hat{x})$
- Note that all one-dimensional potentials that approach 0 for large arguments, are nonpositive, and are piecewise continuous have a bound state

Module 9

- Time dependence of coherent states: We showed that the time evolution of of $|\alpha\rangle$, which we denote as $|\alpha(t)\rangle$, follows by simply time evolving the parameter α via $\alpha \to \alpha e^{-i\omega t}$ plus an additional time-dependent complex phase, but that phase ends up not contributing to expectation values, so it is not very important).
- Maxwell's Equations and Waves
  - Recall that we can write standing waves in the form $Ae^{i\vec{k}x - i\omega t}$
  - We denote the classical electric field as $\vec{E}(\vec{r},t) = \sum_l\left(\vec{\varepsilon}_l E_l(t)e^{i\vec{k}_l\cdot\vec{r}} + c.c.\right)$, where index l denotes the mode for the wave, $\vec{k}_l$ is its wavevector, $\vec{\varepsilon}_l$ is its polarization and $E_l(t)$ is its amplitude (which can be complex).
  - We arrived at the final results for the electric and magnetic fields: $\vec{E}(\vec{r},t) = \sum_l \vec{\varepsilon}_l E_l(0)e^{i\vec{k}_l\cdot\vec{r} - i\omega_l t} + cc$ and $\vec{B}(\vec{r},t) = \sum_l \frac{\vec{k}_l\times\vec{\varepsilon}_l}{\omega_l}E_l(0)e^{i\vec{k}_l\cdot\vec{r} - i\omega_l t} + cc$
  - In class, you used these results to calculate the energy stored by quantized light in a very large box. Refer to Module 9, Page 13.

Module 10

- General principles of the photon:
  - The have definite particle number, but not definite energy/frequency.
  - They have a finite extent in time.
  - They can be detected only once.
  - They are detected as a discrete package, always the same size.
  - They show interference patterns after many are collected.
- Heterodyne detection: used to amplify a weak signal and also to reduce the oscillation rate by using the phenomenon of beats
  - Refer to page Module 10, page 7 for the experimental set up.
  - Initial state: $|\psi_{in}\rangle = |\alpha_1(t)\rangle \otimes |\alpha_2(t)\rangle$
  - Ultimately, we find the following final expression for the electrical current output: $i_4(t) = 2|t|^2 e\eta\Phi_1^{\text{phot}}(1 - \cos((\omega_1 - \omega_2)t - \phi_1 + \phi_2)$
- Homodyne detection: Here we pick $\omega_1 = \omega_2$, $\epsilon_1 = \epsilon_2$, and $r = t = \frac{1}{\sqrt{2}}$.
  - Refer to Module 10, page 8 for the experimental set up.
  - Using the experimental set up shown we can write the raising and lowering operators at each output port: $\hat{a_3} = r\hat{a_1} + t\hat{a_2}$ and $\hat{a_4} = t\hat{a_1} - r\hat{a_2}$, where r and t are the reflection and transmission coefficients, respectively. Note that when light is reflected through glass, we accumulate a 180^o phase difference, hence the negative sign.
  - We then have that $\hat{N_3} = \hat{a_3}^\dagger\hat{a_3}$ and $\hat{N_4} = \hat{a_4}^\dagger\hat{a_4}$, where these are the number operators at the output ports.
- LIGO (laser interferometry gravitational-wave observatory) uses two Michelson-Morely type interferometers. LIGO uses squeezed light to yield high precision measurements: to get high accuracy, we want to time our measurements to precisely the points in time where the uncertainty is minimal.
- LIGO uses the $\hat{P}$ quadrature operator which measures light near its zero (the $\hat{Q}$ quadrature operator measures light near its maximum amplitude).

Tools/Formulas from Exam II: *Four Fundamental Operator Identities*

- Leibnitz Rule:

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

- Hadamard:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\hat{A},...[\hat{A},[\hat{A},\hat{B}]]...]]_n$$

- Weyl Form of BCH identity:

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}+\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

This holds when $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.

- General Form of BCH identity:

$$e^{\hat{A}}e^{\hat{B}} = e^{\left((\hat{A}+\hat{B})+\frac{1}{2}[\hat{A},\hat{B}]+\frac{1}{12}[\hat{A},[\hat{A},\hat{B}]]+\frac{1}{12}[\hat{B},[\hat{B},\hat{A}]]+...\right)}$$

This formula comes from the lecture video on BCH.

- Exponential Disentangling:

  The form of the identity varies depending on your problem and the section beginning on Module 6, page 6 provides two examples that are worth reviewing. The takeaway is that this procedure lets us rewrite an exponential of a matrix in terms of *different* exponentials of matrices that are often easier to calculate and work with.

  Using this identity usually requires remembering the power series expansion for e^x:

$$e^x = \sum_n 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$

  You can use a power series to expand exponentials of operators (matrices) into a sum with terms that usually either truncate or repeat in a useful pattern.

- Similarity Transformation of Powers (This isn't one of the four fundamental operator identities, but it's worth knowing; you proved this on Module 5, Page 28.)

$$\hat{A}^{-1}\left(\hat{B}\right)^n\hat{A} = (\hat{A}^{-1}\hat{B}\hat{A})^n$$

Tools/formulas from Exam I:

- Hermitian matrix $\hat{A}$ satisfies $\hat{A}^\dagger = A$ where $\dagger$ denotes the hermitian conjugate (transpose and complex conjugate)
- Hermitian operators are not necessarily self adjoint.
  - An operator is hermitian if it is equal to its hermitian conjugate, as writen above.
  - A hermitian operator can act on states to the left and to the right.
  - If you are given an operator in matrix form, you can easily tell whether or not it is hermitian: If the matrix has real numbers along the diagonal and zeroes elsewhere, then it corresponds to a Hermitian operator whose eigenvalues are listed along the diagonal.
  - On the other hand, if an operator is self-adjoint, then the operator has a complete set of real eigenvalues.
- The hermitian conjugate of a bra is a ket, and vice versa; don't forget to switch the sign of complex numbers too!
- A bra can be represented by a row vector, whereas a ket can be represented by a column vector. Operators can be represented by matrices.
- $|\uparrow\rangle_z = |\downarrow\rangle_{-z}$
- $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$
- $\hat{S}_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z$ and $\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}|\downarrow\rangle_z$
- $\hat{S}_+|\downarrow\rangle_z = \hbar|\uparrow\rangle_z$
- $\hat{S}_+|\uparrow\rangle_z = 0$
- $\hat{S}_-|\uparrow\rangle_z = \hbar|\downarrow\rangle_z$ and $\hat{S}_-|\downarrow\rangle_z = 0$
- $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
- $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$
- If 2 operators commute, their commutator is equal to 0; if they do NOT commute then their commutator is nonzero. If two operators do not commute then we cannot find a complete set of eigenstates that are simultaneous eigenstates of both operators.
- $[\hat{S}_z,\hat{S}_+] = \hbar\hat{S}_+$
- $[\hat{S}_z,\hat{S}_-] = -\hbar\hat{S}_-$
- $[\hat{S}_+,\hat{S}_-] = 2\hbar\hat{S}_z$
- $\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right)$ and $\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$
- $\hat{S}_+ = \hat{S}_x + i\hat{S}_-$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$
- $[\hat{S}_i,\hat{S}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{S}_k$
- $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$
- $\hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$
- $\hat{S}_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z$ and $\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}|\downarrow\rangle_z$

- A projection operator $\hat{P}$ satisfies $\hat{P}^2 = \hat{P}$, which is called an idempotent operator. If we have a normalized quantum state $|\psi\rangle$, then the operator $|\psi\rangle\langle\psi|$ is a projection operator. The sum (or integral) over all projection operators (also called 'projectors') is the identity matrix. We often use this to 'multiply by 1.'

- $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x + |\downarrow\rangle_x)$

- $\langle a| \otimes \langle b| \ \cdot \ |c\rangle \otimes |d\rangle = \langle a|c\rangle \times \langle b|d\rangle = \langle a|c\rangle\langle b|d\rangle$

- When we make measurements on an arbitrary state $|\psi\rangle$ we apply the projection operator (projector) corresponding to our measurement onto the state $|\psi\rangle$. Then, to calculate the probability, we take the modulus squared of our new state, including the projector.

- When light is polarized in the θ direction, we express this as: $|\theta\rangle = \cos\theta|V\rangle + \sin\theta|H\rangle$. Note that $|H\rangle$ and $|V\rangle$ form an orthonormal basis which means that $\langle H|H\rangle = \langle V|V\rangle = 1$ and $\langle V|H\rangle = \langle H|V\rangle = 0$.

- We also define the following state for polarizer problems: $|-\theta\rangle = \cos\theta|V\rangle - \sin\theta|H\rangle$

- If we wish to represent the polarization as a spinor (vector) we get: $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = \cos\theta|V\rangle + \sin\theta|H\rangle$

- When a photon goes through a polarizer, it always corresponds to being measured, which we accomplish by applying the appropriate projection operator onto the photon's state.

- When a photon goes through a polarization rotator, on the other hand, it does not correspond to a measurement. It is just a way to manipulate the direction of the polarization of the photon. We write the action of the rotator as an operator that changes the angle of polarization of a polarization state vector as follows $|\theta\rangle \rightarrow |\theta + \phi_{\text{rotator}}\rangle$ if the physical rotator rotates the polarization by ϕ_{rotator}.

- The binomial coefficient tells us the general formula for k collections from n objects is: $\frac{n!}{k!(n-k)!} = \begin{pmatrix}n \\ k\end{pmatrix}$

- Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix}n \\ k\end{pmatrix} a^k b^{n-k}$