MAST30021 Complex Analysis

- Complex analysis is a core subject in pure and applied mathematics, as well as the physical and engineering sciences. While it is true that physical phenomena are given in terms of real numbers and real variables, it is often too difficult and sometimes not possible, to solve the algebraic and differential equations used to model these phenomena without introducing complex numbers and complex variables and applying the powerful techniques of complex analysis.

- Topics include: the topology of the complex plane; convergence of complex sequences and series; analytic functions, the Cauchy-Riemann equations, harmonic functions and applications; contour integrals and the Cauchy Integral Theorem; singularities, Laurent series, the Residue Theorem, evaluation of integrals using contour integration, conformal mapping; and aspects of the gamma function.

- At the completion of this subject, students should understand the concepts of analytic function and contour integral and should be able to:

• apply the Cauchy-Riemann equations

• use the complex exponential and logarithm

• apply Cauchy's theorems concerning contour integrals

• apply the residue theorem in a variety of contexts

• understand theoretical implications of Cauchy's theorems such as the maximum modulus principle, Liouvilles Theorem and the fundamental theorem of algebra.

Week 1. Complex Numbers and Complex Plane

1. Complex numbers, polar form, principal argument
2. Complex plane, topology of planar sets, including open and closed sets
3. Functions of a complex variable, limits, point at infinity

Week 2. Complex Derivatives and Analytic Functions

4. Complex derivative, Cauchy-Riemann equations
5. Analytic functions, entire functions
6. Harmonic functions, singularities

Week 3. Complex Transcendental Functions

7. Complex exponential, complex logarithm
8. Branches, complex powers
9. Trigonometric/hyperbolic functions, inverse trigonometric functions

Week 4. Complex Sequences and Series

10. Complex sequences, Cauchy convergence
11. Power series, radius of convergence and its calculation
12. Statement of Taylor's theorem, term-by-term integration and differentiation

Week 5. Line and Contour Integrals

13. Line and contour integrals, paths and curves, path dependence
14. Cauchy-Goursat theorem and applications
15. Fundamental theorem of calculus, path independence

Week 6. Cauchy's Integral Formula

16. Deformation of contours about simple poles
17. General Cauchy integral formula
18. Trigonometric integrals

Week 7. Singularities and Laurent Series

19. Isolated zeros and poles, removable and essential singularities
20. Laurent series, definition of residues
21. Analytic continuation

Week 8. Meromorphic Functions and Residues

22. Meromorphic functions, residue theorem
23. Calculation of residues
24. Evaluation of integrals involving rational functions

Week 9. Residue Calculus

25. Evaluation of integrals involving trigonometric functions
26. Evaluation of integrals using indented contours
27. Summation of series using the residue calculus

Week 10. Applications of the Cauchy Integral theorems

28. Mean value theorem, maximum modulus principle, applications to harmonic functions
29. Liouville's theorem, the fundamental theorem of algebra
30. The identity theorem and analytic continuation

Week 11. Conformal Transformations

31. Analytic functions as conformal mappings
32. Möbius transformations and basic properties
33. Conformal transformations from Möbius transformations

Week 12 Gamma and Zeta Functions

34. The Gamma function
35. General discussion of the Zeta function
36. Revision

Week 1: Complex Numbers

- Some simple equations do not admit solutions in the field of real numbers:

$$z^2 = -1, \qquad z \in \mathbb{R}$$

If $z \in \mathbb{R}$, then $z^2 \geq 0$ and hence $z^2 \neq -1$.

Definition: The *imaginary unit* i is a number such that

$$\boxed{i = +\sqrt{-1}, \qquad i^2 = -1, \qquad i \notin \mathbb{R}}$$

A *complex number* z is a number of the form

$$\boxed{z = x + iy, \qquad x, y \in \mathbb{R}}$$

The *real* and *imaginary* parts of z are

$$\boxed{\text{Re } z = x \in \mathbb{R}, \qquad \text{Im } z = y \in \mathbb{R}}$$

- Two complex numbers are equal if and only if they have the same real and imaginary parts:

$$z_1 = z_2 \quad \Leftrightarrow \quad \text{Re } z_1 = \text{Re } z_2 \quad \text{and} \quad \text{Im } z_1 = \text{Im } z_2$$

$$x_1 + iy_1 = x_2 + iy_2 \quad \Leftrightarrow \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

Field of Complex Numbers

Definition: The set of all complex numbers is denoted

$$\boxed{\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}}$$

- This is the set of numbers obtained by appending i to the real numbers. A number of the form iy is called *pure imaginary*.

- The field of complex numbers is the set $\mathbb{C}$ equipped with the arithmetic operations of addition, subtraction, multiplication and division defined by

$$\begin{aligned}
(x_1 + iy_1) + (x_2 + iy_2) &:= (x_1 + x_2) + i(y_1 + y_2), \qquad x_1, x_2, y_1, y_2 \in \mathbb{R} \\
(x_1 + iy_1) - (x_2 + iy_2) &:= (x_1 - x_2) + i(y_1 - y_2), \qquad x_1, x_2, y_1, y_2 \in \mathbb{R} \\
(x_1 + iy_1)(x_2 + iy_2) &:= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \qquad x_1, x_2, y_1, y_2 \in \mathbb{R}
\end{aligned}$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} := \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\,\frac{(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}, \qquad x_2^2 + y_2^2 \neq 0$$

Definition: The *conjugate* $\overline{z}$ of $z = x + iy$ is

$$\boxed{\overline{z} := x - iy}$$

Laws of Complex Arithmetic

Laws of Complex Algebra: $z_1, z_2, z_3 \in \mathbb{C}$

0. Closure:

$$z_1 + z_2 \in \mathbb{C}, \qquad z_1\, z_2 \in \mathbb{C}$$

1. Additive and multiplicative identity:

$$z + 0 = z, \qquad 1\, z = z, \qquad \text{for all } z \in \mathbb{C}$$

2. Commutative laws:

$$z_1 + z_2 = z_2 + z_1, \qquad z_1\, z_2 = z_2\, z_1$$

3. Associative laws:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) = z_1 + z_2 + z_3$$
$$(z_1\, z_2) z_3 = z_1 (z_2\, z_3) = z_1\, z_2\, z_3$$

4. Distributive laws:

$$z_1(z_2 + z_3) = z_1\, z_2 + z_1\, z_3$$

5. Inverses:

$$z_1 + z_2 = 0 \quad \Rightarrow \quad z_2 = -z_1, \qquad z_1\, z_2 = 1 \quad \Rightarrow \quad z_2 = z_1^{-1} = \frac{1}{z_1}, \;\; z_1 \neq 0$$

6. Zero factors:

$$z_1\, z_2 = 0 \quad \Rightarrow \quad z_1 = 0 \;\; \text{or} \;\; z_2 = 0$$

Algebraic Construction of Complex Numbers

- The complex numbers $\mathbb{C}$ can be constructed formally from the set of real numbers $\mathbb{R}$:

Definition: A *complex number* is defined algebraically as the ordered pair $[x, y]$ of real numbers $x, y \in \mathbb{R}$. The complex operations on $\mathbb{C} := \{[x, y] : x, y \in \mathbb{R}\}$ are defined by

1 Equality. $[x, y] = [x', y'] \quad \Leftrightarrow \quad x = x'$ and $y = y'$

2 Addition. $[x, y] + [x', y'] := [x + x', y + y']$

3 Multiplication. $[x, y][x', y'] := [xx' - yy', xy' + x'y]$

Exercise: Show that the operations of addition and multiplication are commutative, associative and distributive.

- Numbers of the form $[x, 0]$ behave like real numbers so we identify $x \equiv [x, 0]$:

1 Addition. $[x, 0] + [x', 0] := [x + x', 0]$

2 Multiplication. $[x, 0][x', 0] := [xx', 0]$

- We prove that $\boxed{[x, y]^2 = [-1, 0] \equiv -1}$

has a solution in $\mathbb{C}$. Identifying $i \equiv [0, 1]$, we verify

$$i^2 = (i)(i) \equiv [0, 1][0, 1] := [(0)(0) - (1)(1), (0)(1) + (1)(0)] = [-1, 0] \equiv -1$$

- It follows that every complex number can be written in the form $[x, y] \equiv x + iy$ since

$$[x, y] = [x, 0] + [0, y] = [x, 0] + [0, 1][y, 0] \equiv x + iy$$

Solving Equations

Example: Solve $z^2 = -1$

$$z^2 = -1 \quad \Rightarrow \quad z = \pm i \quad \text{since} \quad (\pm i)^2 = i^2 = -1 \qquad \blacksquare$$

Example: Solve $z^2 + 2z + 2 = 0$

$$z^2 + 2z + 2 = 0 \quad \Rightarrow \quad z = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

$$\begin{aligned} \text{Check:} \quad \text{LHS} &= z^2 + 2z + 2 \\ &= (-1 \pm i)(-1 \pm i) + 2(-1 \pm i) + 2 \\ &= (1 \mp 2i - 1) + (-2 \pm 2i) + 2 = 0 \\ &= \text{RHS} \qquad \blacksquare \end{aligned}$$

Example: Solve $z^2 = \overline{z}$

$$\begin{aligned} z^2 = \overline{z} \quad &\Rightarrow \quad (x + iy)^2 = x - iy \\ &\Rightarrow \quad x^2 - y^2 + 2ixy = x - iy \end{aligned}$$

Equating real and imaginary parts

$$\begin{aligned} &\Rightarrow \quad x^2 - y^2 = x \quad \text{and} \quad 2xy = -y \\ &\Rightarrow \quad x = -\tfrac{1}{2} \quad \text{and} \quad y^2 = \tfrac{3}{4} \\ &\text{or} \quad y = 0 \quad \text{and} \quad x = 0, 1 \end{aligned}$$

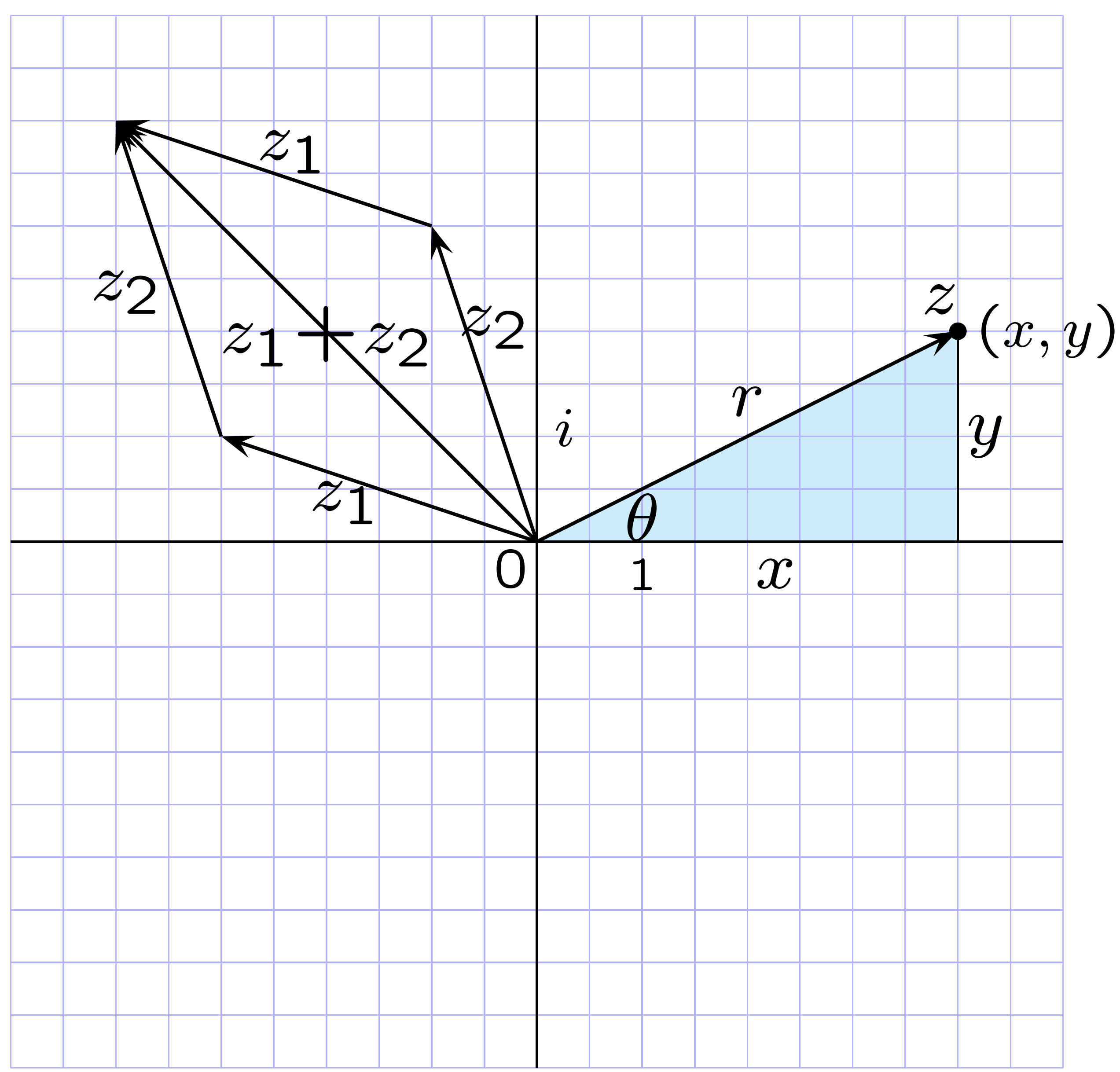
Hence there are four solutions:

$$z = 0, 1, \; -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2} \qquad \blacksquare$$

Complex Plane

- A complex number $z = x + iy \in \mathbb{C}$ can be represented as a point (x, y) in the plane $\mathbb{R}^2$. Such diagrams using cartesian or polar coordinates are called *Argand diagrams*:

- The complex number z can be viewed as a vector in $\mathbb{R}^2$. Addition of complex numbers satisfies the parallellogram rule.



Polar Form

- The use of polar coordinates (r, θ) in $\mathbb{R}^2$: $\qquad (x, y) = (r\cos\theta, r\sin\theta)$

gives the *polar form*

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta)}$$

Definition: The *absolute value* or *modulus* of $z = x + iy$ is

$$\boxed{|z| := \sqrt{x^2 + y^2} = r}$$

- By Pythagoras, this is the distance of the point z or (x, y) from the origin 0.

Definition: The *argument* of $z \neq 0$ is

$$\boxed{\arg z = \theta, \quad \text{where} \quad \cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}}$$

The argument θ is multi-valued

$$\arg z = \theta = \theta_0 + 2k\pi, \quad k \in \mathbb{Z}, \quad -\pi < \theta_0 \leq \pi$$

The *principal argument*

$$\boxed{\operatorname{Arg} z = \theta_0 = \{\text{principal value}\}, \quad -\pi < \theta_0 \leq \pi}$$

is single-valued but discontinuous across the *branch cut* along $(-\infty, 0]$.

- Sometimes it is convenient to choose another principal branch such as $0 \leq \theta_0 < 2\pi$.

Conjugate, Modulus and Argument

The conjugate, modulus and argument satisfy the following properties:

1. $\overline{z} = x - iy$; $\quad \overline{\overline{z}} = z$; $\quad z + \overline{z} = 2 \text{ Re } z$; $\quad z - \overline{z} = 2i \text{ Im } z$

2. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$; $\quad \overline{z_1\, z_2} = \overline{z}_1\, \overline{z}_2$; $\quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}, \quad z_2 \neq 0$

3. $\text{Re } z \leq |\text{Re } z| \leq |z|$; $\quad \text{Im } z \leq |\text{Im } z| \leq |z|$

$|\overline{z}| = |-z| = |z|$; $\quad |z|^2 = z\,\overline{z}$; $\quad |z_1 z_2| = |z_1||z_2|$; $\quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

4. $z_1 z_2 = |z_1||z_2|\Big[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\Big]$

5. If $z \neq 0, z_1 \neq 0, z_2 \neq 0$ then $\arg z_1 z_2 = \arg z_1 + \arg z_2$, $\quad \arg \overline{z} = -\arg z$

$$\text{Arg}\, z_1 z_2 = \text{Arg}\, z_1 + \text{Arg}\, z_2 + 2k\pi, \quad k = 0, \pm 1$$

6. $z_1 = z_2 \quad \Leftrightarrow \quad |z_1| = |z_2|$ and $\text{Arg}\, z_1 = \text{Arg}\, z_2$, $\quad z_1 \neq 0, z_2 \neq 0$

7. Unlike real numbers, the complex numbers are not ordered. So inequalities, such as $z_1 \geq z_2$ or $z_1 > z_2$, only make sense if $z_1, z_2 \in \mathbb{R}$.

Triangle Inequality

- Triangle Inequality: $\boxed{|z_1 + z_2| \le |z_1| + |z_2|}$

Proof:

$$
\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\
&= z_1\overline{z}_1 + (z_1\overline{z}_2 + z_2\overline{z}_1) + z_2\overline{z}_2 \\
&= |z_1|^2 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + |z_2|^2 \\
&= |z_1|^2 + 2\ \mathrm{Re}(z_1\overline{z}_2) + |z_2|^2 \\
&\le |z_1|^2 + 2\,|z_1\overline{z}_2| + |z_2|^2 \\
&= |z_1|^2 + 2\,|z_1||\overline{z}_2| + |z_2|^2 \\
&= |z_1|^2 + 2\,|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}
$$

Now take positive square root. ■

- Triangle Inequality Variant: $\boxed{\big||z_1| - |z_2|\big| \le |z_1 - z_2|}$

Complex Number: Examples

Example: Simplify $(2+i)\overline{(1+3i)}-2+4i$

$$
\begin{aligned}
&(2+i)\overline{(1+3i)}-2+4i=(2+i)(1-3i)-2+4i\\
=\ &(2-3i^2+i-6i)-2+4i=(2+3-5i)-2+4i\\
=\ &5-5i-2+4i=3-i \qquad \blacksquare
\end{aligned}
$$

Example: Simplify $\dfrac{i(3-2i)}{1+i}$

$$\frac{i(3-2i)}{1+i}=\frac{2+3i}{1+i}\,\frac{1-i}{1-i}=\frac{2+3+3i-2i}{1+1+i-i}=\frac{5+i}{2} \qquad \blacksquare$$

Example: Simplify $\left|\dfrac{(3-4i)\overline{(2-i)}}{1+3i}\right|$. Since $|\overline{z}|=|z|$,

$$\left|\frac{(3-4i)\overline{(2-i)}}{1+3i}\right|=\frac{|3-4i||2-i|}{|1+3i|}=\frac{\sqrt{9+16}\sqrt{4+1}}{\sqrt{1+9}}=\frac{\sqrt{25}\sqrt{5}}{\sqrt{10}}=\sqrt{\frac{25}{2}} \qquad \blacksquare$$

Example: Put $z=2-2i$ in polar form and find $\operatorname{Arg} z$

$$z=|z|(\cos\theta+i\sin\theta), \qquad |z|=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\cos\theta=\frac{x}{|z|}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}, \qquad \sin\theta=\frac{y}{|z|}=\frac{-2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}$$

Hence $\theta=-\pi/4+2k\pi,\quad k\in\mathbb{Z}$ (fourth quadrant) and $\operatorname{Arg} z=-\frac{\pi}{4}\in(-\pi,\pi]$ so

$$z=2\sqrt{2}\,e^{-\pi i/4} \qquad \blacksquare$$

Complex Exponential

Definition: The complex exponential is defined as

$$\boxed{e^z := e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)}$$

In particular, this yields Euler's equation

$$\boxed{e^{iy} = \cos y + i \sin y, \qquad y \in \mathbb{R}}$$

The exponential polar form is thus

$$\boxed{z = |z| e^{i\theta}, \qquad \theta = \arg z}$$

- **Special Unimodular Values:** $|e^{i\theta}| = 1, \quad \theta \in \mathbb{R}$

$$e^{\pi i/2} = i, \quad e^{-\pi i/2} = -i, \quad e^{k\pi i} = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$

- **Exponent Laws:**

$$e^{z_1+z_2} = e^{z_1} e^{z_2}, \qquad (e^z)^n = e^{nz}, \quad n \in \mathbb{Z}$$

De Moivre and Complex Trigonometry

- De Moivre's Formula:

$$\boxed{(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta, \qquad \theta \in \mathbb{R},\ n \in \mathbb{Z}}$$

- Trigonometric Functions: $\theta \in \mathbb{R}$

$$\boxed{\cos\theta = \text{Re } e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \text{Im } e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

- Complex Hyperbolic/Trigonometric Functions: $z \in \mathbb{C}$

$$\boxed{\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \cosh z := \frac{e^{z} + e^{-z}}{2}, \qquad \cos z = \cosh iz}$$

$$\boxed{\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \sinh z := \frac{e^{z} - e^{-z}}{2}, \qquad \sin z = -i\sinh iz}$$

Exercise: Prove the fundamental trigonometric identity

$$\boxed{\cos^2 z + \sin^2 z = 1, \qquad z \in \mathbb{C}}$$

Binomial Theorem

Theorem 1 (Binomial Theorem) *For* $z_1, z_2 \in \mathbb{C}$ *and* $n \in \mathbb{N}$

$$\boxed{\begin{aligned}(z_1 + z_2)^n &= z_1^n + \binom{n}{1} z_1^{n-1} z_2 + \cdots + \binom{n}{k} z_1^{n-k} z_2^k + \cdots + z_2^n \\ &= \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k\end{aligned}}$$

Factorials and binomial coefficients are

$$n! := n(n-1)\ldots 1, \quad 0! := 1; \quad \binom{n}{k} := \begin{cases} \frac{n!}{(n-k)!\,k!}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$$

Example: Find $(1+i)^6$, (a) by using the binomial theorem and (b) by using exponential polar form:

$$\begin{aligned}(1+i)^6 &= 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6 \\ &= 1 + 6i - 15 - 20i + 15 + 6i - 1 = -8i \\ (\sqrt{2}e^{\pi i/4})^6 &= 8e^{3\pi i/2} = -8i \qquad \blacksquare\end{aligned}$$

Proof of Binomial Theorem

Proof: By induction. True for $n = 0, 1$ so assume true for n and show true for $n + 1$:

$$
\begin{aligned}
(z_1 + z_2)^{n+1} &= (z_1 + z_2)(z_1 + z_2)^n \\
&= (z_1 + z_2) \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \\
&= \sum_{k=0}^{n} \binom{n}{k} z_1^{n+1-k} z_2^k + \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^{k+1} \\
&= \sum_{k=0}^{n+1} \binom{n}{k} z_1^{n+1-k} z_2^k + \sum_{k=1}^{n+1} \binom{n}{k-1} z_1^{n+1-k} z_2^k \\
&= \sum_{k=0}^{n+1} \left[\binom{n}{k} + \binom{n}{k-1} \right] z_1^{n+1-k} z_2^k \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} z_1^{n+1-k} z_2^k \qquad \blacksquare
\end{aligned}
$$

- In the last step we have used the binomial identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Trigonometric Identity I

Exercise: Sum the geometric series

$$\sum_{k=0}^{n} z^k = 1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad n \in \mathbb{N}, \ z \in \mathbb{C}, \ z \neq 1$$

Example: Use the geometric series to sum the trigonometric series

$$\sum_{k=0}^{n} \cos k\theta = \frac{\cos \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}, \qquad 0 < \theta < 2\pi$$

Use geometric series with $z = e^{i\theta}$ and $0 < \theta < 2\pi$ so $z \neq 1$

$$\sum_{k=0}^{n} \cos k\theta = \mathrm{Re}\left(\sum_{k=0}^{n} e^{ki\theta}\right) = \mathrm{Re}\left(\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}\right)$$

$$= \mathrm{Re}\left(\frac{e^{(n+\frac{1}{2})i\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right) = \frac{\mathrm{Re}\left[\frac{1}{2i}(e^{(n+\frac{1}{2})i\theta} - e^{-i\theta/2})\right]}{\sin \frac{\theta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin(n+\frac{1}{2})\theta + \sin \frac{\theta}{2}\right]}{\sin \frac{\theta}{2}} = \frac{\cos \frac{n\,\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}} \qquad \blacksquare$$

Trigonometric Identity II

Example: Prove the trigonometric identity

$$\sin z_1 + \sin z_2 = 2\cos\tfrac{1}{2}(z_1 - z_2)\,\sin\tfrac{1}{2}(z_1 + z_2), \qquad z_1, z_2 \in \mathbb{C}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{2i}\left(e^{i(z_1-z_2)/2} + e^{-i(z_1-z_2)/2}\right)\left(e^{i(z_1+z_2)/2} - e^{-i(z_1+z_2)/2}\right) \\ &= \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1} + e^{iz_2} - e^{-iz_2}\right) = \text{LHS}\end{aligned}$$

In this way *all* trigonometric identities in z are reduced to algebraic identities in $e^{\pm iz}$. ■

- The previous identity follows by choosing

$$z_1 = (n + \tfrac{1}{2})\theta, \qquad z_2 = \frac{\theta}{2}$$

Roots of Unity

Roots of Unity: Solve $z^n = 1$: Write z^n and 1 in exponential polar form and equate modulus and argument

$$
\begin{aligned}
& z^n = (|z|e^{i\theta})^n = |z|^n e^{ni\theta}, \qquad 1 = e^{2k\pi i}, \qquad k \in \mathbb{Z} \\
\Rightarrow \quad & |z|^n = 1 \quad \text{and} \quad \arg(z^n) = n \arg z = n\theta = 2k\pi \\
\Rightarrow \quad & |z| = 1 \quad \text{and} \quad \operatorname{Arg} z = \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1 \\
\Rightarrow \quad & z = e^{2k\pi i/n} = \omega^k, \quad k = 0, 1, \ldots, n-1
\end{aligned}
$$

where we have chosen the branch $0 \le \operatorname{Arg} z < 2\pi$ and

$$\boxed{\omega = 1^{1/n} = e^{2\pi i/n} = \{\text{primitive } n\text{th root of unity}\}}$$

Example: Solve $w^n = z$, that is, find $w = z^{1/n}$: Write w, z in exponential form and equate modulus and argument

$$
\begin{aligned}
& w^n = |w|^n e^{ni\phi}, \; z = |z|e^{i\theta + 2k\pi i}, \; k \in \mathbb{Z}; \quad \arg w = \phi, \; \arg z = \theta \\
\Rightarrow \quad & |w|^n = |z| \quad \text{and} \quad n\phi = \theta + 2k\pi \\
\Rightarrow \quad & |w| = |z|^{1/n} \quad \text{and} \quad \phi = (\theta + 2k\pi)/n \\
\Rightarrow \quad & w = z^{1/n} = |z|^{1/n} e^{i(\theta + 2k\pi)/n} = \omega^k |z|^{1/n} e^{i\theta/n}, \\
& \qquad k = 0, 1, \ldots, n-1 \qquad \blacksquare
\end{aligned}
$$

Example: Roots of Unity

Example: Find the cube roots of $\sqrt{2}+i\sqrt{2}$ in the cartesian form $x+iy$:

$$
\begin{aligned}
(\sqrt{2}+i\sqrt{2})^{1/3} &= (2e^{\pi i/4+2k\pi i})^{1/3} = 2^{1/3}e^{\pi i/12+2k\pi i/3},\ k=0,1,2 \\
&= 2^{1/3}e^{\pi i/12},\ 2^{1/3}e^{9\pi i/12},\ 2^{1/3}e^{17\pi i/12} \\
&= 2^{1/3}(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}),\ 2^{1/3}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}), \\
&\qquad 2^{1/3}(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}) \qquad \blacksquare
\end{aligned}
$$

Fractional Powers

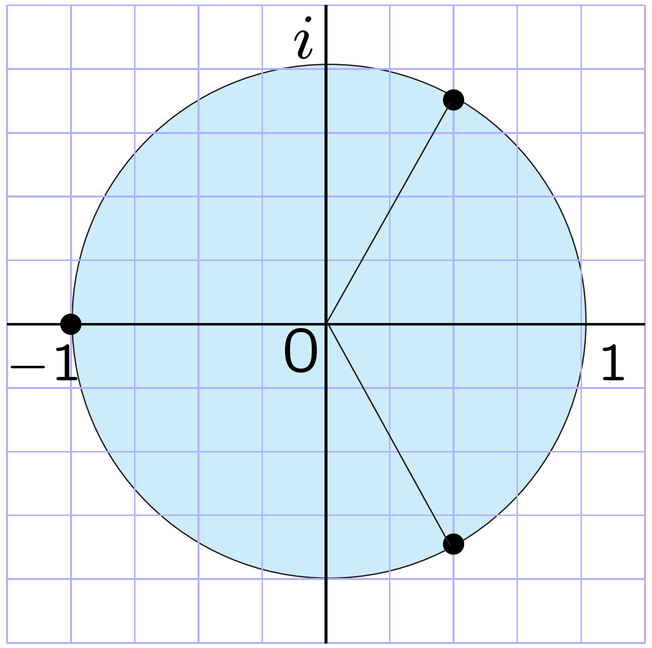
Exercise: Show that the sets of numbers $(z^{1/n})^m$ and $(z^m)^{1/n}$ are the same. We denote this common set by $z^{m/n}$.

Fractional Powers: The fractional power $z^{m/n}$ of the complex number $z = |z|e^{i\theta}$ is given by

$$\boxed{z^{m/n} = |z|^{m/n} e^{mi(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1}$$

Example: Find $i^{2/3}$ in the cartesian form $x + iy$:

$$\begin{aligned} i^{2/3} &= (e^{\pi i/2+2k\pi i})^{2/3} = e^{\pi i/3+4k\pi i/3}, \quad k = 0, 1, 2 \\ &= e^{\pi i/3}, e^{5\pi i/3}, e^{3\pi i} = e^{\pi i/3}, e^{-\pi i/3}, -1 \\ &= -1,\ \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = (i^2)^{1/3} = (-1)^{1/3} \qquad \blacksquare \end{aligned}$$



Argand Diagrams

Example: Indicate graphically, on a single Argand diagram, the sets of values of z determined by the following relations:

(a) Point $z = 1 - 2i$

(b) Line $|z+1+i| = |z-1-i|$

(c) Circle $|z - 1 - i| = 1$

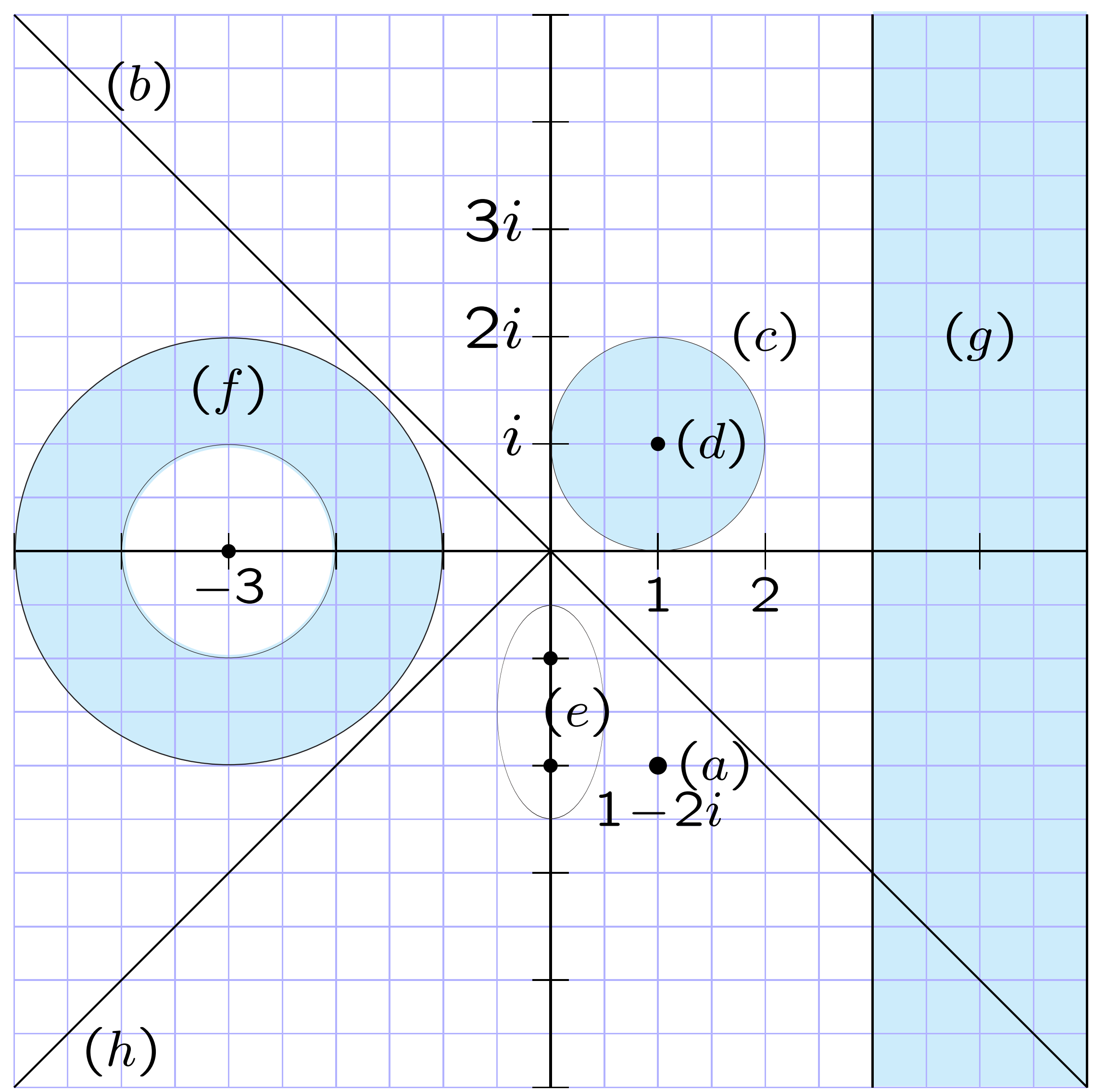
(d) Disk $|z - 1 - i| < 1$

(e) Ellipse $|z + i| + |z + 2i| = 2$

(f) Annulus $1 \leq |z + 3| \leq 2$

(g) Strip $3 \leq \text{Re } z \leq 5$

(h) Ray $\text{Arg } z = -3\pi/4$



Open and Closed Planar Sets

- Open Disks: The set of points

$$\boxed{\text{Open Disk:} \qquad |z - z_0| < r}$$

inside the circle of radius r about $z = z_0$ is called an *open disk* or *neighbourhood* of z_0. The set $|z| < 1$ is the *open unit disk*.

- Open Sets: A point z_0 in a set $S \subset \mathbb{C}$ is an *interior point* of S if there is some open disk about z_0 which is completely contained in S. If every point of S is an interior point of S we say S is *open*. The empty set $\emptyset$ and $\mathbb{C}$ are open sets.

- Closed Disks: The set of points

$$\boxed{\text{Closed Disk:} \qquad |z - z_0| \le r}$$

is the *closed disk* of radius r about $z = z_0$.

- Closed Sets: A point z_0 is said to be a *boundary point* of $S \subset \mathbb{C}$ if every open disk about z_0 contains at least one point in S and at least one point not in S. Note that a boundary point z_0 may or may not be in S. The set ∂S of all boundary points of S is called the *boundary of* S. A set which contains all of its boundary points is called *closed*. A set S is closed if and only if its complement $\mathbb{C}\backslash S$ is open. A point which is not an interior point or boundary point of S is an *exterior point*. The empty set $\emptyset$ and $\mathbb{C}$ are both open and closed sets.

Bounded and Connected Planar Sets

- Bounded Sets: A set $S \subset \mathbb{C}$ is called *bounded* if there exists a real number R such that $|z| < R$ for every $z \in S$. A set $S \subset \mathbb{C}$ which is both closed and bounded is called *compact*.

- Connectedness: An open set $S \subset \mathbb{C}$ is said to be *connected* if every pair of points in S can be joined by a path (of finite or infinite length) that lies entirely in S. An open set $S \subset \mathbb{C}$ is said to be *polygonally-connected* if every pair of points in S can be joined by a polygonal path (finite number of straight line segments) that lies entirely in S. A *region* is an open polygonally-connected set S together with all, some or none of its boundary points. We assume polygonal-connectedness to avoid infinite length paths and fractal-like open sets.

Examples of Planar Sets

- **Open and Closed Disks:**

Disk (d) $|z-1-i|<1$ is an open disk. The Disk (i) $|z-1-i|\le 1$ is a closed disk. Disk (d) is the interior of Disk (i). The exterior of Disk (d) $|z-1-i|\ge 1$ is closed.

- **Regions:**

The Disk (d), Annulus (f) and Strip (g) are regions. So is the open Elliptical Disk (j) $|z-i|+|z-2i|<2$.

- **Boundaries:**

The boundary of Disk (d) is the Circle (c). The boundary of Annulus (f) is the union of the circles $|z+3|=1$ and $|z+3|=2$. The boundary of the Strip (g) is the union of the lines Re $z=3$ and Re $z=5$.

- **Open and Closed Sets:**

The planar sets (d) and the Elliptical Disk (j) $|z-i|+|z-2i|<2$ are open. The sets (a), (b), (c), (e), (f), (g) are closed. The Ray (h) and the strip $3<$ Re $z\le 5$ are neither open nor closed. Note the Ray (h) does not contain the boundary point at the origin since Arg z is not defined there.

- **Bounded and Compact Sets:**

The sets (a), (c), (d), (e), (f) are bounded. The sets (b), (g), (h) are unbounded. The sets (a), (c), (e), (f) are compact.

- **Connected Open Sets**

The Disk (d), the open Elliptical Disk (j) $|z-i|+|z-2i|<2$ and the interiors of the Annulus (f) and Strip (g) are connected. The disjoint union of the open sets (d) and (j) is not connected. Likewise the set $\mathbb{C}\backslash\{|z|=1\}$ is not connected. The Annulus (f) is connected but not *simply connected* because loops around the hole cannot be continuously shrunk to zero.

Week 2: Derivatives and Analytic Functions

4. Complex derivative, Cauchy-Riemann equations
5. Analytic functions, entire functions
6. Harmonic functions, singularities

Example Functions of a Complex Variable

Example: Functions of the complex variable z:

1. $f(z) = z^2 + 1 = (x + iy)^2 + 1 = (x^2 - y^2 + 1) + i(2xy)$

2. $f(z) = \cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$

3. $f(z) = z\,\overline{z} = (x + iy)(x - iy) = x^2 + y^2$

4. $f(z) = (\text{Re } z)^2 + i = x^2 + i$

5. $f(z) = z^{1/2} = \pm|z|^{1/2}e^{i\theta_0/2}$ with $\theta_0 = \text{Arg}\, z$ is multi-valued unless it is restricted to the branch $f_+(z) = +|z|^{1/2}e^{i\theta_0/2}$. It has a *branch cut* along $(-\infty, 0]$ and a *branch point* at $z = 0$.
■

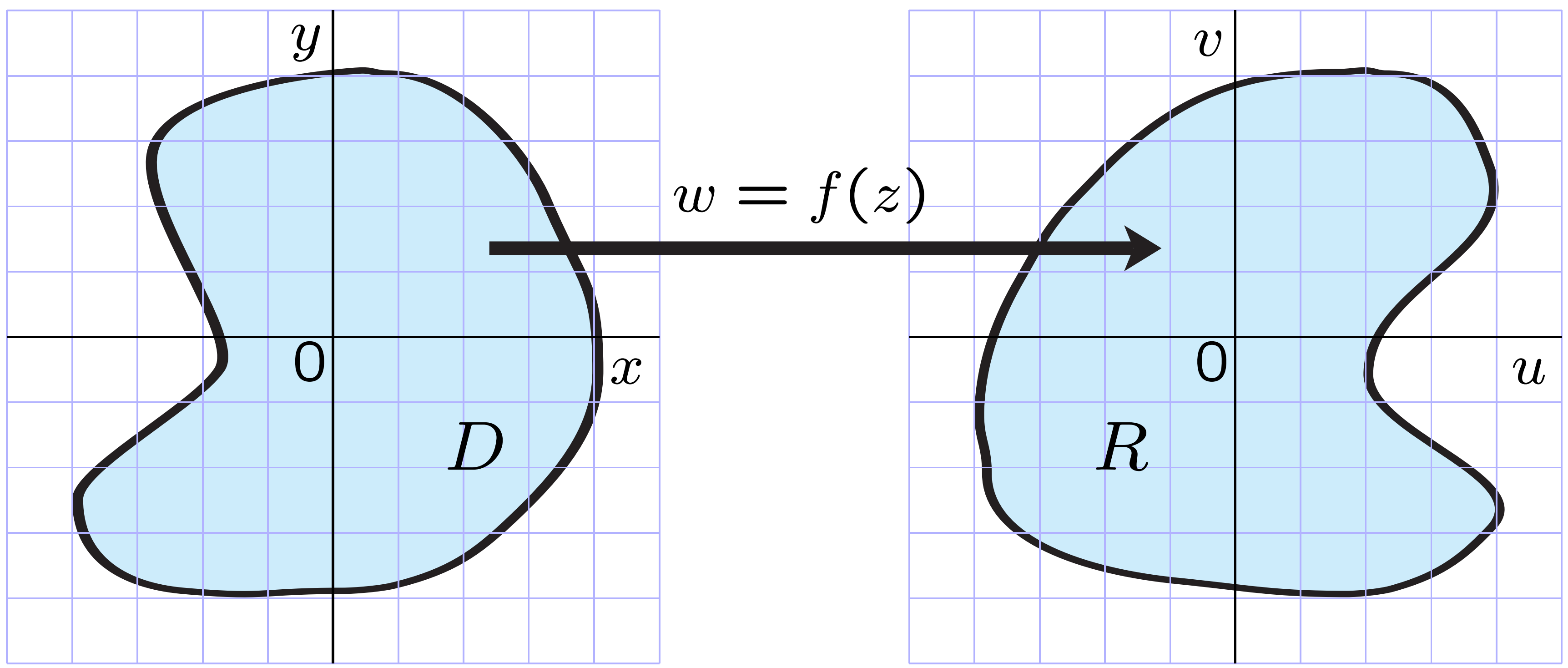
Functions of a Complex Variable

Definition: A *function* of a *complex variable* z is an assignment or rule

$$f : D \to R, \qquad f : z = x + iy \quad \mapsto \quad w = f(z) = u(x,y) + iv(x,y)$$

which assigns to each z in the *domain* $D \subset \mathbb{C}$ a unique image $w = f(z)$ in the *range* $R \subset \mathbb{C}$ so that $f(z)$ is single-valued.

- It is not possible to represent a complex function $f(z)$ by a graph. The complex function $f(z)$ is however determined by the pair of real functions $u(x,y)$, $v(x,y)$ of two real variables x and y. D and R are usually regions in $\mathbb{C}$:



Limits of Complex Functions

Definition: Suppose $f(z)$ is defined in an open disk about $z = z_0$ with the possible exception of the point z_0 itself. We say that the *limit of* $f(z)$ *as* z *approaches* z_0 *is* w_0 and write

$$\boxed{\lim_{z \to z_0} f(z) = w_0}$$

if for any $\epsilon > 0$ there exists a positive number $\delta(\epsilon)$ such that

$$\boxed{|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta(\epsilon)}$$

- Unlike a function of a real variable, z can approach z_0 along many different paths in the complex plane. If the limit exists, it is independent of the way in which z approaches z_0.

Example: Show from the limit definition that $\lim_{z \to i} z^2 = -1$:

$$\begin{aligned} |z^2 - (-1)| &= |z^2 + 1| = |(z-i)(z+i)| = |z-i||(z-i) + 2i| \\ &\leq |z-i|(|z-i| + 2) < \frac{\epsilon}{3}(1+2) = \epsilon \end{aligned}$$

whenever $|z - i| < \delta(\epsilon) = \text{Min}(1, \frac{\epsilon}{3})$ ∎

Example: Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist:

For the limit to exist, it must be independent of the path along which $z = x + iy$ approaches $z_0 = 0$. We show that the limits $z \to 0$ along the x- and y-axes are different

$$\lim_{\substack{z \to 0 \\ y=0}} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x}{x} = 1 \neq \lim_{\substack{z \to 0 \\ x=0}} \frac{z}{\bar{z}} = \lim_{y \to 0} \frac{iy}{(-iy)} = -1, \qquad \text{so the limit does not exist} \quad ∎$$

Limit Theorems

Theorem 2 (Limit Theorems) *If $a, b \in \mathbb{C}$ are constants (independent of z) and $\lim_{z \to z_0} f(z)$, $\lim_{z \to z_0} g(z)$ exist then*

1. **Linear:** $$\lim_{z \to z_0} \Big(a f(z) + b\, g(z) \Big) = a \lim_{z \to z_0} f(z) + b \lim_{z \to z_0} g(z)$$

2. **Product:** $$\lim_{z \to z_0} \Big(f(z)\, g(z) \Big) = \Big(\lim_{z \to z_0} f(z) \Big) \Big(\lim_{z \to z_0} g(z) \Big)$$

3. **Quotient:** $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} \quad \text{if} \quad \lim_{z \to z_0} g(z) \neq 0$$

Theorem 3 (Limits Using Real Variables)
Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Show that

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \quad \textit{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0$$

Proof: Exercise using $|\text{Re } w| \leq |w|$, $|\text{Im } w| \leq |w|$ and the triangle inequality. Note, for these two variable limits to exist they must be independent of the path along which $(x, y) \to (x_0, y_0)$ in $\mathbb{R}^2$. ■

Limit Theorems: Examples

Example: Find the limit $\lim_{z \to 2} \dfrac{z^2+3}{iz}$;

$$\lim_{z \to 2} \frac{z^2+3}{iz} = \frac{\lim_{z \to 2} (z^2+3)}{\lim_{z \to 2} iz} = \frac{7}{2i} = -\frac{7}{2} i \qquad \blacksquare$$

Example: Find the limit $\lim_{z \to i} \dfrac{z^2+1}{z^4-1}$;

$$\lim_{z \to i} \frac{z^2+1}{z^4-1} = \lim_{z \to i} \frac{(z^2+1)}{(z^2-1)(z^2+1)} = \lim_{z \to i} \frac{1}{(z^2-1)} = -\tfrac{1}{2} \qquad \blacksquare$$

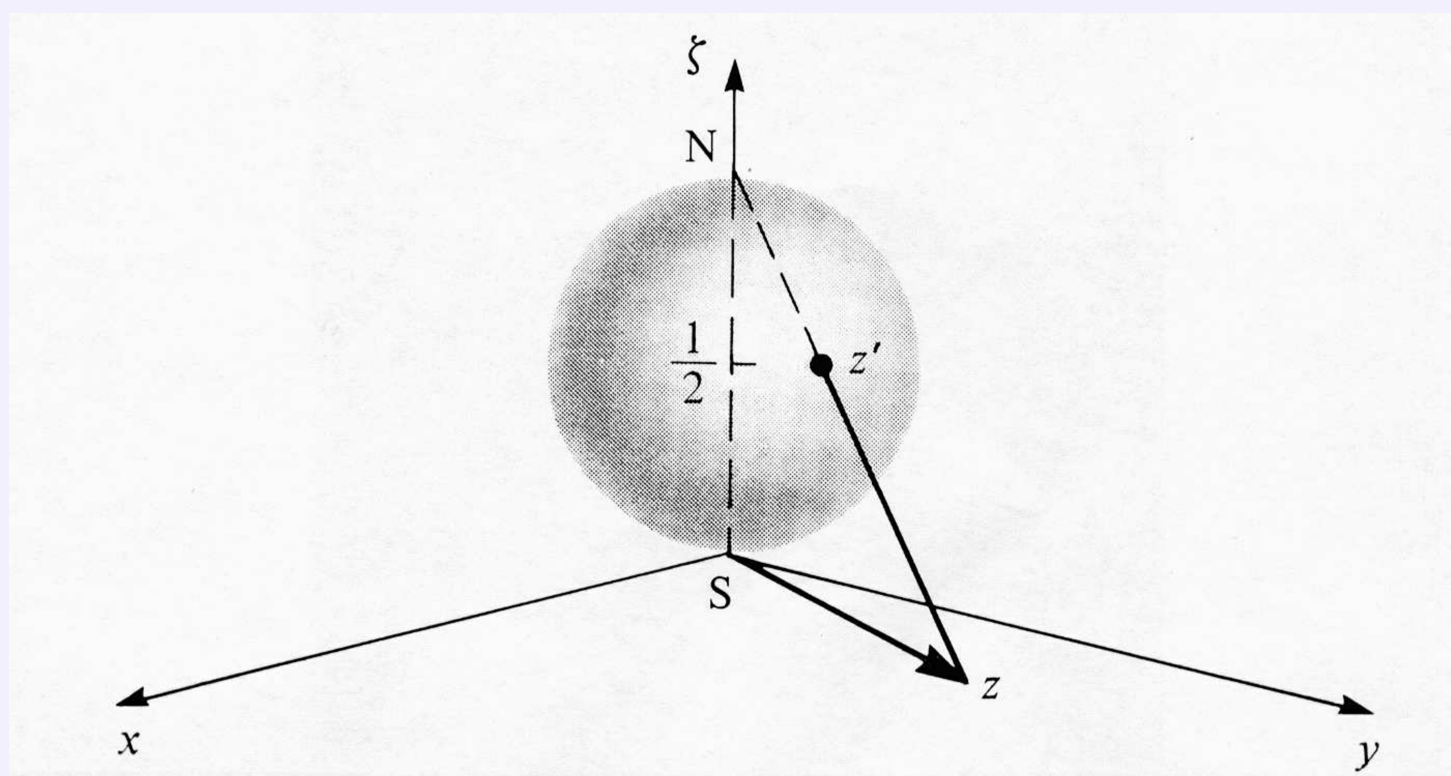
Extended Complex Plane

- There are many directions or paths along which $\frac{1}{z}$ can approach infinity as $z \to 0$. We identify these "limits" with a single number and extend the complex plane by adding a *point at infinity* denoted by the symbol ∞. This forms the *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

- The extended complex plane is closed and identified with the *stereographic projection* of a sphere (*Riemann sphere*) of radius $r = \frac{1}{2}$ onto the horizontal plane $\mathbb{C}$ passing through the south pole ($z = 0$). A line drawn from the north pole of the sphere through the point z' on the sphere maps to the point $z \in \mathbb{C}$. The equator maps onto the unit circle $|z| = 1$, the southern hemisphere maps onto $|z| < 1$ and the northern hemisphere maps onto $|z| > 1$. The south pole corresponds to the origin $z = 0$ and the north pole to the point $z = \infty$ at infinity.

Stereographic Projection:

$$\boxed{\lim_{z\to 0} \frac{1}{z} = \infty}$$



Limit $z \to \infty$

Definition: We define

$$\boxed{\lim_{z\to\infty} f(z) := \lim_{z\to 0} f\left(\frac{1}{z}\right)}$$

An open disk about the point at infinity is $\quad |z| > M, \quad$ for $M > 0$.

Example:

$$\lim_{z\to\infty} \frac{iz^2 - z}{z^2 - 1} = \lim_{z\to 0} \frac{\frac{i}{z^2} - \frac{1}{z}}{\frac{1}{z^2} - 1} = \lim_{z\to 0} \frac{i - z}{1 - z^2} = \frac{\lim\limits_{z\to 0}(i - z)}{\lim\limits_{z\to 0}(1 - z^2)} = i \qquad \blacksquare$$

Example:

$$\lim_{z\to\infty} \frac{z^2 + iz}{z^3 + 1} = \lim_{z\to 0} \frac{\frac{1}{z^2} + \frac{i}{z}}{\frac{1}{z^3} + 1} = \lim_{z\to 0} \frac{z + iz^2}{1 + z^3}$$

$$= \frac{\lim\limits_{z\to 0}(z + iz^2)}{\lim\limits_{z\to 0}(1 + z^3)} = \frac{0}{1} = 0 \qquad \blacksquare$$

Example:

$$\lim_{z\to\infty} \frac{z^2 + 1}{z} = \lim_{z\to 0} \frac{\frac{1}{z^2} + 1}{\frac{1}{z}} = \lim_{z\to 0} (\frac{1}{z} + z) = \infty$$

since

$$\lim_{z\to 0} \frac{1}{\frac{1}{z} + z} = \lim_{z\to 0} \frac{z}{1 + z^2} = \frac{0}{1 + 0} = 0 \qquad \blacksquare$$

Continuity

Definition: Suppose the function $f(z)$ is defined in an open disk about $z = z_0$ and $f(z_0)$ is defined at $z = z_0$. Then $f(z)$ is *continuous at* z_0 if

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0) = f\left(\lim_{z \to z_0} z\right)}$$

- To be continuous at z_0, (i) the function $f(z)$ must be defined at $z = z_0$, (ii) the limit $\lim_{z \to z_0} f(z)$ must exist and (iii) the limit must equal the function value $f(z_0)$.
- A function $f(z)$ is *continuous in a region* if it is continuous at all points in the region.

Theorem 4 (Continuity Theorems)
If $a, b \in \mathbb{C}$ *are constants and* $f(z)$ *and* $g(z)$ *are continuous at* $z = z_0$ *so also are the functions*

1. **Linear:** $a\, f(z) + bg(z)$
2. **Product:** $f(z)\, g(z)$
3. **Quotient:** $\dfrac{f(z)}{g(z)}$ provided $g(z_0) \neq 0$
4. **Composite:** $f(g(z))$ if $f(w)$ is continuous at $w_0 = g(z_0)$

Proof: Follows from the Limit Theorems. ■

Continuity Examples

Example: Show that $f(z) = e^z$ is continuous in $\mathbb{C}$:

$$f(z) = e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

is continuous in $\mathbb{C}$ since

$$u(x,y) = e^x \cos y, \qquad v(x,y) = e^x \sin y$$

are continuous functions of two real variables in $\mathbb{R}^2$ being products of the continuous functions e^x, $\cos y$ and $\sin y$. ■

Example: Show that $f(z) = |z|$ is continuous in $\mathbb{C}$:

$$\begin{aligned}\lim_{z \to z_0} f(z) &= \lim_{z \to z_0} |z| = \lim_{\substack{x \to x_0 \\ y \to y_0}} \sqrt{x^2 + y^2} = \left(\lim_{\substack{x \to x_0 \\ y \to y_0}} (x^2 + y^2) \right)^{1/2} \\ &= \sqrt{x_0^2 + y_0^2} = |z_0| = f(z_0)\end{aligned}$$

for any $z_0 \in \mathbb{C}$ since the function of one real variable $g(r) = \sqrt{r}$ is continuous for $r \geq 0$ where here $r = x^2 + y^2$. ■

Complex Derivative and Analyticity

Definition: Suppose the function $f(z)$ is defined in an open disk about $z = z_0$ and $f(z_0)$ is defined at $z = z_0$. Then $f(z)$ is *differentiable at* z_0 if the limit

$$\boxed{\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}$$

defining the derivative exists.

- The function $f(z)$ is *analytic* (regular or holomorphic) in an open *simply-connected* region R if it has a derivative at every point of R. More precisely, the function is analytic in any open region R that does not encircle a branch point. The function $f(z)$ is *entire* if it is analytic everywhere in $\mathbb{C}$.

- The concept of an analytic function on an open region is very powerful, much more powerful than the concept of an analytic real function on an open interval. For example, we will show later, that if a function $f(z)$ is analytic then its derivatives $f^{(n)}(z)$ of all orders exist and are also analytic. This is not the case for real functions:

$$f(x) = \text{sgn}(x)\, x^2, \quad f'(x) = 2|x|, \quad x \in (-\epsilon, \epsilon) \quad \text{but} \quad f''(x) \text{ does not exist at } x = 0$$

- Note that the real function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

is C^∞ (has derivatives of all orders) but is not analytic since $f^{(n)}(0) = 0$ for all n and the Taylor expansion about $x = 0$ converges to zero and not to $f(x)$.

Complex Derivatives from First Principles

Example: Show by first principles that $\dfrac{d}{dz} z^n = nz^{n-1}$.

Solution: Using the binomial theorem

$$\begin{aligned}\frac{(z+\Delta z)^n - z^n}{\Delta z} &= \frac{nz^{n-1}\Delta z + \binom{n}{2} z^{n-2}(\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z} \\ &= nz^{n-1} + \binom{n}{2} z^{n-2}(\Delta z) + \cdots + (\Delta z)^{n-1}\end{aligned}$$

So

$$\frac{d}{dz} z^n = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} = nz^{n-1}$$

The function $f(z) = z^n$ is thus analytic in $\mathbb{C}$. ■

Example: The function $f(z) = \overline{z}$ is nowhere differentiable:

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z_0+\Delta z)} - \overline{z}_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

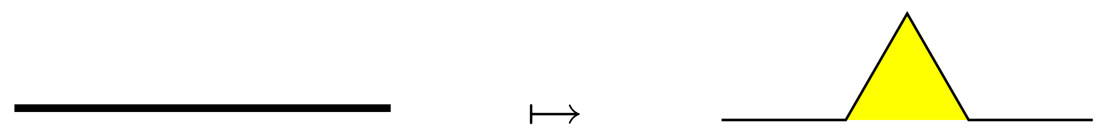
but we saw this limit is path dependent and so does not exist. At first this result is surprising but it says that $\overline{z} = x - iy$ cannot be expressed as an *analytic* function of the indivisible unit $z = x + iy$. ■

Exercise: Show that the functions $f(z) = \text{Re } z$, $f(z) = \text{Im } z$ and $f(z) = |z|$ are continuous but nowhere differentiable.

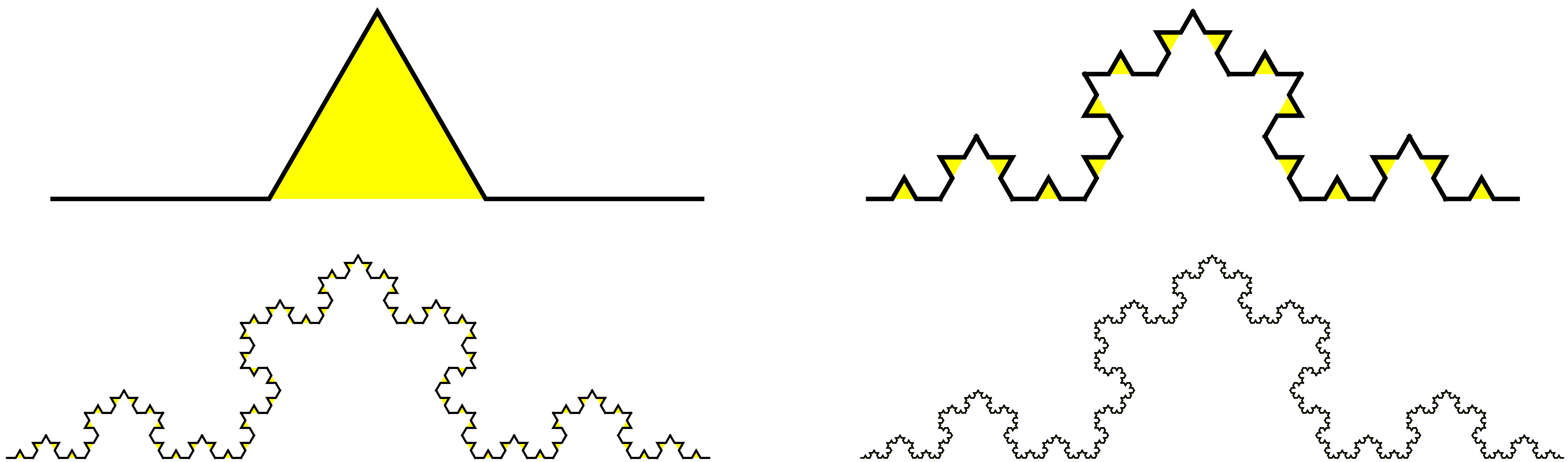
- Examples of continuous but nowhere differentiable functions of a single real variable abound. These include fractals such as the Koch curve and the Katsuura function.

Koch Curve

- Consider the curve defined iteratively starting with the real line segment $[0, 1]$ according to the replacement:



- Iterating gives:



- The limiting curve is a fractal called the Koch (snowflake) curve. This curve is continuous but nowhere differentiable (there are no tangents to the curve). It has an infinite length but the area under the curve is finite. Its fractal dimension is

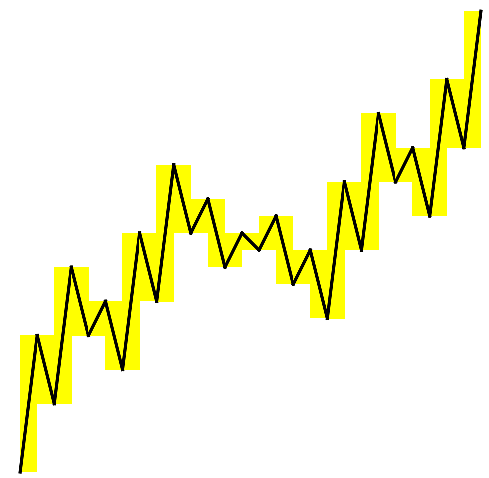
$$d_{\text{fractal}} = \frac{\log 4}{\log 3} \approx 1.26$$

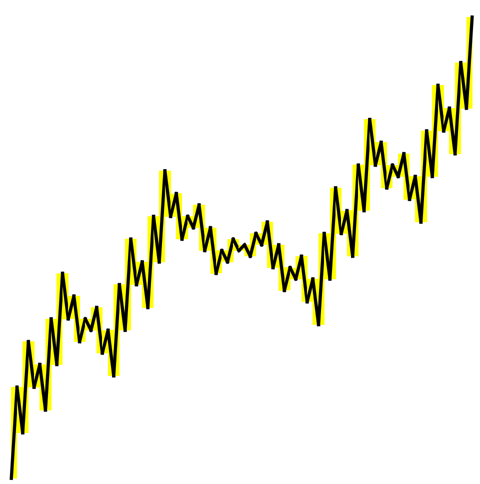
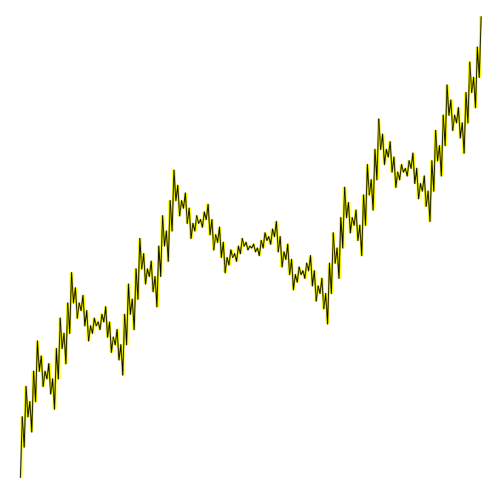
Johan Thim, *Continuous Nowhere Differentiable Functions*, Master's Thesis, Lulea (2003).

Katsuura Function

- Consider the function defined iteratively starting with the function $K_0(x) = x$ on $[0, 1]$ according to the replacements:

- Iterating gives

$K_2(x) \quad =$  $K_3(x) \quad =$ 

$K_4(x) \quad =$  $K_5(x) \quad =$ 

- The Katsuura function $K(x) = \lim_{n\to\infty} K_n(x)$ is continuous but nowhere differentiable on $[0, 1]$.

H. Katsuura, American Mathematical Monthly **98** (1991) 411–416.
Johan Thim, *Continuous Nowhere Differentiable Functions*, Master's Thesis, Lulea (2003).

Differentiable Implies Continuous

Theorem 5 (Differentiable Implies Continuous)
If $f(z)$ is differentiable at $z = z_0$ it must be continuous at z_0.

Proof: Let $h = \Delta z \neq 0$. Then

$$f(z_0 + h) - f(z_0) = \frac{f(z_0 + h) - f(z_0)}{h} h$$

So using the product limit theorem and the fact that the derivative $f'(z_0)$ exists

$$\begin{aligned} \lim_{h \to 0} [f(z_0 + h) - f(z_0)] &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \lim_{h \to 0} h \\ &= f'(z_0) \cdot 0 = 0 \end{aligned}$$

This shows that

$$\lim_{h \to 0} f(z_0 + h) = f(z_0) \quad \text{or} \quad \lim_{z \to z_0} f(z) = f(z_0)$$

that is, $f(z)$ is continuous at $z = z_0$. ■

- The converse is not true. The function $f(z) = \overline{z} = x - iy$ is continuous everywhere because the functions $u(x, y) = x$ and $v(x, y) = -y$ are continuous. But we saw that $f(z) = \overline{z}$ is nowhere differentiable.

Differentiation Rules

Theorem 6 (Differentiation Rules) *If a, b are constants and $f(z)$ and $g(z)$ are differentiable in an open region R, then in this region*

1. **Linear:** $\frac{d}{dz}\Big(af(z) + bg(z)\Big) = a\,f'(z) + bg'(z)$

2. **Product:** $\frac{d}{dz}\Big(f(z)\,g(z)\Big) = f'(z)\,g(z) + f(z)\,g'(z)$

3. **Quotient:** $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{g(z)^2}$ if $g(z) \neq 0$

4. **Chain:** $\frac{d}{dz}\Big(f(g(z))\Big) = \frac{df}{dg}\frac{dg}{dz} = f'(g(z_0))g'(z_0)$ if $f(w)$ is differentiable in an open region about $w = g(z_0)$

5. **Inverse:** $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)}$ with $w = f(z)$, $z = f^{-1}(w)$

Proof of Differentiation Rules

Proof: We prove 2 and 5. Others are exercise (or see text).

- 2. Derivative of a product:

$$\begin{aligned}
\frac{d}{dz}\Big(f(z)\,g(z)\Big) &= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)[g(z+\Delta z) - g(z)] + [f(z+\Delta z) - f(z)]g(z)}{\Delta z} \\
&= \lim_{\Delta z\to 0} f(z+\Delta z)\,\frac{g(z+\Delta z) - g(z)}{\Delta z} + \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}\,g(z) \\
&= f(z)\,g'(z) + f'(z)\,g(z) \qquad \text{(by limit theorems)}
\end{aligned}$$

because $f(z)$ is differentiable and therefore continuous

$$\lim_{\Delta z\to 0} f(z+\Delta z) = f(z) \qquad \blacksquare$$

- 5. If $w = f(z)$ is a bijection, then $f^{-1}(w)$ exists, $f^{-1}(f(z)) = z$, $f(f^{-1}(w)) = w$ and $f^{-1}(w)$ is analytic near w_0 with derivative

$$\frac{df^{-1}}{dw}(w_0) = \lim_{w\to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z\to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)} \qquad \blacksquare$$

Continuity of Rational Functions

Example: Show that *rational functions* of the form

$$R(z) = \frac{P_n(z)}{Q_m(z)} = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}, \quad a_j, b_j \in \mathbb{C}$$

are continuous in $\mathbb{C}$ except at the zeros of the denominator:

Solution: This rational function is not defined at the zeros of $Q_m(z)$. Away from these points the polynomials are differentiable, e.g.

$$\frac{d}{dz}(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

The quotient is therefore differentiable for $g(z) = Q_m(z) \neq 0$ by the quotient rule:

$$\frac{d}{dz}\Big(\frac{f(z)}{g(z)}\Big) = \frac{g(z)\, f'(z) - f(z)\, g'(z)}{g(z)^2} \quad \text{if } g(z) \neq 0 \qquad \blacksquare$$

Since differentiable implies continuous, it follows that $R(z)$ is continuous at any point z where the denominator does not vanish.

L'Hôpital's Rule

Theorem 7 (L'Hôpital's Rule) *If $f(z)$ and $g(z)$ are differentiable at $z = z_0$ and $f(z_0) = 0$, $g(z_0) = 0$ but $g'(z_0) \neq 0$ then*

$$\boxed{\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}}$$

Proof: Using the quotient limit theorem

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0}} = \frac{f'(z_0)}{g'(z_0)} \qquad \blacksquare$$

Theorem 8 (General l'Hôpital's Rule)
If $f(z)$ and $g(z)$ are analytic at $z = z_0$ and $f(z)$, $g(z)$ and their first $n-1$ derivatives all vanish at $z = z_0$ but the nth derivative $g^{(n)}(z_0) \neq 0$ then

$$\boxed{\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)} = \lim_{z \to z_0} \frac{f^{(n)}(z)}{g^{(n)}(z)}}$$

Proof: Consequence of Taylor's theorem proved later. $\blacksquare$

Examples of L'Hôpital's Rule

Example: Evaluate the limit $\displaystyle\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$

Solution: Since $f(z_0)=g(z_0)=0$ the limit is indeterminate so we use l'Hôpital's rule with $f'(z)=10z^9$ and $g'(z)=6z^5$

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1}=\lim_{z\to i}\frac{10z^9}{6z^5}=\frac{10i^9}{6i^5}=\frac{10i}{6i}=\frac{5}{3} \qquad \blacksquare$$

Example: Evaluate the limit $\displaystyle\lim_{z\to 0}\frac{e^z-1-z}{z^2}$

Solution: Since $f(0)=g(0)=f'(0)=g'(0)=0$ and $f''(z)=e^z$, $g''(z)=2\neq 0$

$$\lim_{z\to 0}\frac{e^z-1-z}{z^2}=\lim_{z\to 0}\frac{e^z-1}{2z}=\lim_{z\to 0}\frac{e^z}{2}=\frac{e^0}{2}=\tfrac{1}{2} \qquad \blacksquare$$

- Here we have assumed that

$$\frac{d}{dz}e^z=e^z$$

To prove this we need the Cauchy-Riemann theorem.

Cauchy-Riemann Equations

- Differentiability of $f(z) = u(x,y) + iv(x,y)$ implies strong constraints on $u(x,y)$, $v(x,y)$.

If $f(z)$ is differentiable the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

must exist independent of the path. We conclude that the limit along the x- and y-axes must agree

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$= \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

- Hence *necessary* conditions for $f(z)$ to be differentiable are given by the *Cauchy-Riemann equations*

$$\boxed{f'(z) \;=\; \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \;=\; -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy-Riemann Theorem

Theorem 9 (Cauchy-Riemann Theorem)
Suppose $f(z) = u(x,y) + iv(x,y)$ is defined in an open region R containing z_0. If $u(x,y)$ and $v(x,y)$ and their first partial derivatives exist and are continuous at z_0 (that is $u(x,y)$ and $v(x,y)$ are C^1 at (x_0, y_0)) and satisfy the Cauchy-Riemann equations at z_0, then $f(z)$ is differentiable at z_0. Consequently, if $u(x,y)$ and $v(x,y)$ are C^1 and satisfy the Cauchy-Riemann equations at all points of R then $f(z)$ is analytic in R.

Proof: See a textbook such as Saff and Snider. The proof uses the mean-value theorem. ■

- The Cauchy-Riemann equations can be understood informally using functions of two variables and the chain rule which is valid for C^1 functions:

$$z(x,y) = x + iy, \quad f(z) = f(z(x,y)), \quad f(x,y) = u(x,y) + iv(x,y)$$

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}, \qquad \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}$$

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \qquad \frac{df}{dz} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Cauchy-Riemann and Derivatives

Example: Show that the function $f(z) = e^z = e^x \cos y + ie^x \sin y$ is entire with derivative

$$\boxed{\frac{d}{dz} e^z = e^z}$$

Solution: The first partial derivatives are continuous and satisfy the Cauchy-Riemann equations everywhere in $\mathbb{C}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$$

Hence by the Cauchy-Riemann theorem $f(z) = e^z$ is entire and

$$f'(z) \;=\; \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \;=\; e^x(\cos y + i \sin y) \;=\; e^z \qquad \blacksquare$$

Example: Discuss where the function $f(z) = (x^2 + y) + i(y^2 - x)$ is (a) differentiable and (b) analytic.

Solution: Since $u(x, y) = x^2 + y$ and $v(x, y) = y^2 - x$, we have

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1$$

These partial derivatives are continuous everywhere in $\mathbb{C}$. They satisfy the Cauchy-Riemann equations on the line $y = x$ but not in any open region. It follows by the Cauchy-Riemann theorem that $f(z)$ is differentiable at each point on the line $y = x$ but nowhere analytic. $\blacksquare$

Harmonic Functions

Definition: A real function of two variables $\phi(x, y)$ is *harmonic* in an open connected domain D if it is C^2 (that is continuous with continuous first and second partial derivatives) in D and satisfies the Laplace equation

$$\boxed{\nabla^2\phi \;:=\; \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \;=\; 0}$$

- The Laplace equation occurs in many areas of two-dimensional physics including continuum and fluid mechanics, aerodynamics and the heat equation. We see that the solutions to these equations (harmonic functions) are naturally associated with *analytic functions*.

Theorem 10 (Harmonic Functions) *If $f(z) = u(x, y) + iv(x, y)$ is analytic in an open connected domain D, then $u(x, y)$ and $v(x, y)$ are harmonic in D.*

Proof: Since $f(z)$ is analytic, $u(x, y)$ and $v(x, y)$ are C^∞ (possess continuous partial derivatives of all orders). We will prove this later. In particular, since they are C^2, the mixed second derivatives are equal

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y}, \qquad \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}$$

Substituting for the first partial derivatives from the Cauchy-Riemann equations give

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}, \qquad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \qquad \blacksquare$$

Harmonic Conjugates

Theorem 11 (Harmonic Conjugates)
If $u(x, y)$ is harmonic in a simply-connected open domain D then there exists another harmonic function $v(x, y)$ on D (called the harmonic conjugate of u) such that $f(z) = u(x, y) + iv(x, y)$ is analytic in D. The conjugate is obtained by solving the Cauchy-Riemann equations.

Example: Find an analytic function $f(z)$ by finding the conjugate of the harmonic function $u(x, y) = x^3 - 3xy^2 + y$.

Solution: First check that $u(x, y)$ is harmonic in $\mathbb{C}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

From the Cauchy-Riemann equations, the conjugate $v(x, y)$ must satisfy

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 1$$

Integrating gives

$$v(x, y) = 3x^2 y - y^3 + g(x) = 3x^2 y - x + h(y)$$

where the real functions $g(x)$ and $h(y)$ are arbitrary. It follows that $g(x) = -x + \text{const}$, $h(y) = -y^3 + \text{const}$, and

$$v(x, y) = 3x^2 y - y^3 - x + c, \qquad c = \text{const}$$

Hence the analytic function is

$$f(z) = (x^3 - 3xy^2 + y) + i(3x^2 y - y^3 - x + c) = z^3 - iz + ic \qquad \blacksquare$$

Singular Points

Definition: A point at which $f(z)$ fails to be analytic is called a singular point of $f(z)$.

Types of singularities:

1. Isolated Singularities: The point $z = z_0$ is an isolated singularity of $f(z)$ if there is an open neighbourhood that encloses no other singular point.

2. Poles: If there is a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = \rho \neq 0$$

exists then $z = z_0$ is a pole of order n. If $n = 1$, z_0 is a *simple pole*

3. Branch Points: Multi-valued functions like $f(z) = \log(z - z_0)$ and $f(z) = (z - z_0)^{1/n}$ have a *branch point* at $z = z_0$.

4. Removable Singularities: An apparent singular point z_0 of $f(z)$ is *removable* if $\lim_{z \to z_0} f(z)$ exists.

5. Essential Singularities: A singularity which is not a pole, branch point or removable singularity is an *essential singularity*.

6. Singularities at Infinity: The type of *singularity at infinity* of $f(z)$ is the same as that of $f(\frac{1}{z})$ at $z = 0$.

Example Singular Points

Examples:

1. The singularities at $z = \pm i$ of $\dfrac{z^4 - 1}{z^2 + 1}$ are removable.

2. The function $e^{1/(z-2)}$ has an essential singularity at $z = 2$.

3. The function $\dfrac{z}{(z^2 + 1)(z + i)}$ has a simple pole at $z = i$ and

a pole of order 2 at $z = -i$.

4. The function $f(z) = z^3$ has a pole of order 3 at $z = \infty$. ■

Week 3: Complex Transcendental Functions

7. Complex exponential, complex logarithm
8. Branches, complex powers
9. Trigonometric/hyperbolic functions, inverse trigonometric functions

Exponential Function

Definition: The complex exponential is defined by

$$\boxed{\exp: \ \mathbb{C} \to \mathbb{C}, \qquad \exp(z) \ \equiv \ e^z \ := \ e^x(\cos y + i \sin y)}$$

We have seen that this function is entire with derivative

$$\boxed{\frac{d}{dz} e^z = e^z}$$

- Unlike the real exponential e^x, the complex exponential function e^z is not one-to-one

$$\boxed{e^z = 1 \ \Leftrightarrow \ z = 2k\pi i, \qquad e^{z_1} = e^{z_2} \ \Leftrightarrow \ z_1 = z_2 + 2k\pi i, \qquad k \in \mathbb{Z}}$$

Consequently, the inverse function $\log z$ is multi-valued.

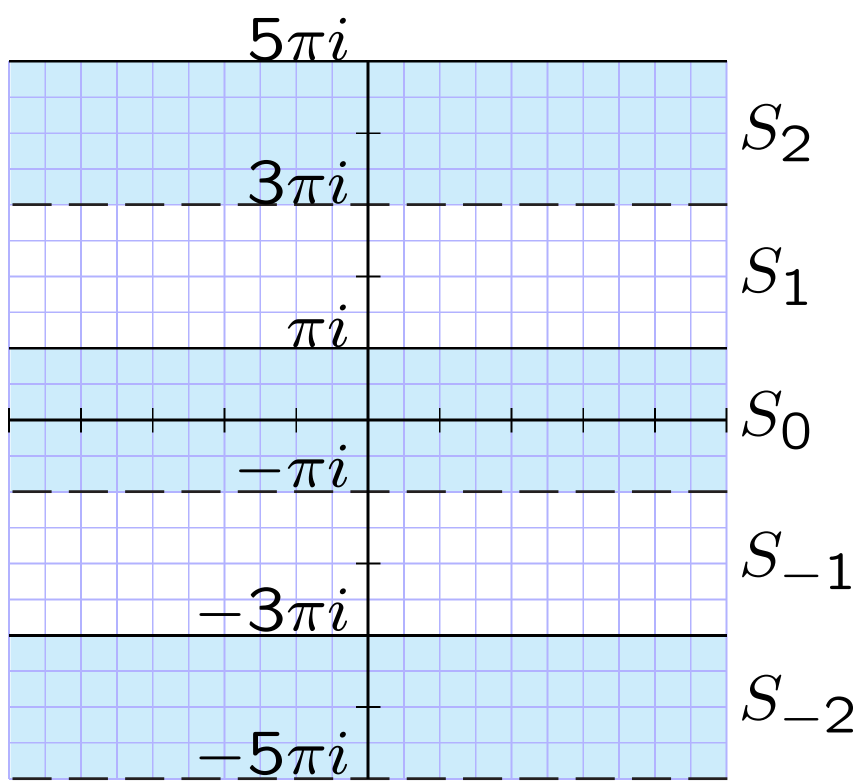
- On the complex domain the exponential function is periodic

$$e^{z+2\pi i} = e^z, \qquad \text{complex period} = 2\pi i$$

We can restrict the domain to one of the fundamental strips

$$S_k: \qquad (2k-1)\pi < \operatorname{Im} z \leq (2k+1)\pi$$

Then $\exp\colon S_0 \to \mathbb{C}\backslash\{0\}$, is *one-to-one* on the *principal domain* S_0 and admits an inverse.



Logarithm Function

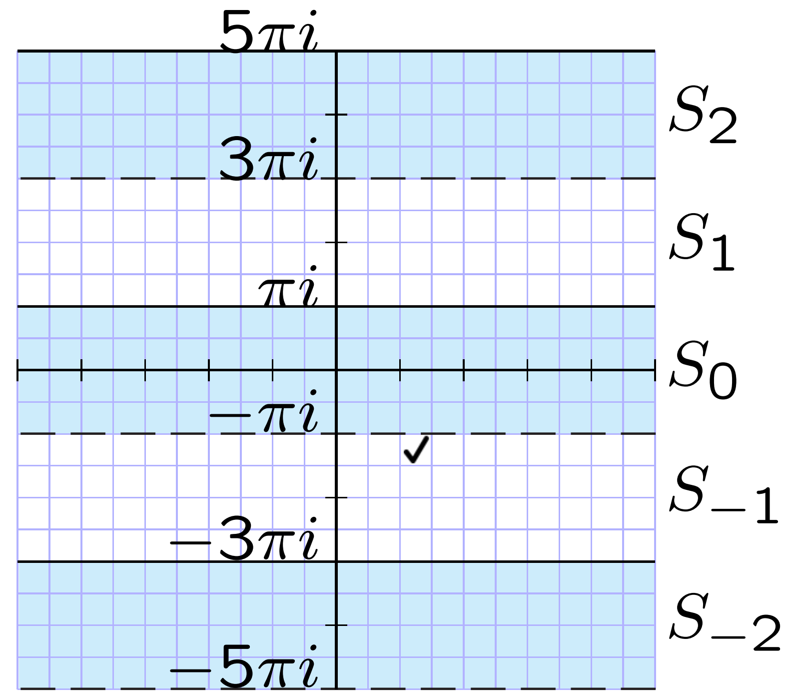
Definition: For $z \neq 0$ we define the logarithm as the inverse of the exponential function

$$\boxed{z = e^w \quad \Leftrightarrow \quad w = \log z}$$

Since e^w is not one-to-one, $\log z$ is multi-valued taking infinitely many values

$$\boxed{\log z := \operatorname{Log}|z| + i \arg z = \operatorname{Log}|z| + i \operatorname{Arg} z + 2k\pi i, \qquad k \in \mathbb{Z}}$$

To obtain a single-valued function (bijection), we define the *principal value of the logarithm*



$$\boxed{\operatorname{Log}: \ \mathbb{C}\backslash\{0\} \to S_0 \qquad \operatorname{Log} z := \operatorname{Log}|z| + i \operatorname{Arg} z}$$

$$\boxed{z = e^{\operatorname{Log} z}, \ z \in \mathbb{C}\backslash\{0\} \qquad \operatorname{Log}(e^w) = w, \ w \in S_0}$$

Exercise: Show that

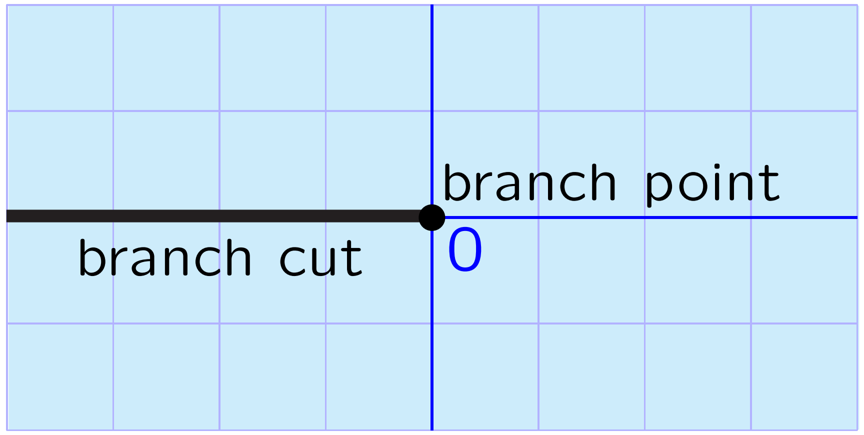
$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2k\pi i, \qquad k = 0, \pm 1$$

Derivative of Logarithm

Theorem 12 (Derivative of Log)
The function $\operatorname{Log} z$ *is analytic in the cut plane* $\mathbb{C}\backslash(-\infty, 0]$ *with derivative*

$$\boxed{\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}, \qquad z \in \mathbb{C}\backslash(-\infty, 0]}$$

Proof: Since $\operatorname{Log} z$ is the inverse of e^w which is analytic in S_0, $\operatorname{Log} z$ is analytic in the cut plane $\mathbb{C}\backslash(-\infty, 0]$. There is a jump discontinuity across the branch cut along the negative real axis:



Applying the chain rule to $\operatorname{Log}(e^w) = w$ gives the derivative

$$\frac{d}{dw}\operatorname{Log}(e^w) = \frac{d}{dz}\operatorname{Log} z \frac{d}{dw} e^w = z \frac{d}{dz}\operatorname{Log} z = 1$$

Hence

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}, \qquad z \neq 0 \qquad \blacksquare$$

Branches and Branch Cuts

Definition: $F(z)$ is a *branch* of a multi-valued function $f(z)$ in an open domain D if $F(z)$ is single-valued and analytic in D and is such that, for each $z \in D$, the value $F(z)$ is one of the values of $f(z)$. A line used to create a domain of analyticity D is called a *branch cut*. The end points of branch cuts are called *branch points*.

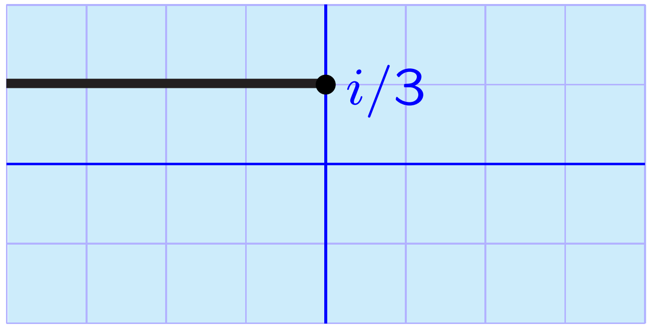
Example: Determine the domain of analyticity of $\text{Log}(3z - i)$:

Solution: This function is analytic by the chain rule except where

$$\text{Re}(3z - i) = 3\ \text{Re}\ z \le 0 \quad \text{and} \quad \text{Im}(3z - i) = 3\ \text{Im}\ z - 1 = 0$$

corresponding to a branch cut with branch point at $z = i/3$

$$z = x + iy, \qquad x \le 0,\ y = \frac{1}{3}$$



The branch cut can be rotated about the branch point. ■

Complex Powers

Definition: If $a, z \in \mathbb{C}$ and $z \neq 0$, we define the complex power

$$\boxed{z^a := e^{a \log z}}$$

so that each value of $\log z$ gives a value of z^a. The *principal branch* of z^a is

$$\boxed{z^a := e^{a \operatorname{Log} z}, \qquad z \neq 0}$$

- Notice that, if a is independent of z,

$$\frac{d}{dz} z^a = \frac{d}{dz} e^{a \operatorname{Log} z} = \frac{a}{z} e^{a \operatorname{Log} z} = a\, e^{(a-1) \operatorname{Log} z} = \frac{a}{z} z^a = a z^{a-1}$$

provided the same branch is used for the logarithm defining z^a and z^{a-1}.

Example: Find the principal value of $z = i^{2i}$:

By definition with $z = i$ and $a = 2i$, the principal value is

$$i^{2i} = e^{2i \operatorname{Log} i} = e^{2i(\operatorname{Log}|i| + i \operatorname{Arg} i)} = e^{2i(\operatorname{Log} 1 + \pi i/2)} = e^{-\pi} \qquad \blacksquare$$

Example: If $z_1 = -1 + i$, $z_2 = i$ and $a = \frac{1}{2}$, show that for principal values $(z_1 z_2)^a \neq z_1^a z_2^a$:

We have $z_1 z_2 = -1 - i = \sqrt{2} e^{-3\pi i/4}$ and

$$\begin{aligned}
(z_1 z_2)^{1/2} &= (-1-i)^{1/2} = e^{\frac{1}{2} \operatorname{Log}(-1-i)} = e^{\frac{1}{2} \operatorname{Log} \sqrt{2} + \frac{1}{2} i \operatorname{Arg}(-1-i)} \\
&= e^{\frac{1}{4} \operatorname{Log} 2 - 3\pi i/8} = 2^{1/4} e^{-3\pi i/8} \\
z_1^{1/2} z_2^{1/2} &= (-1+i)^{1/2} i^{1/2} = (\sqrt{2} e^{3\pi i/4})^{1/2} (e^{\pi i/2})^{1/2} \\
&= 2^{1/4} e^{3\pi i/8} e^{\pi i/4} = 2^{1/4} e^{5\pi i/8} = -2^{1/4} e^{-3\pi i/8} \qquad \blacksquare
\end{aligned}$$

Square Root Function

- The square root function is two-valued

$$w = \sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\mathrm{Log}\,|z| + \frac{1}{2} i \arg z} = |z|^{1/2}\, e^{\frac{1}{2} i \arg z}$$

In particular, $\arg w = \frac{1}{2}\arg z$ and

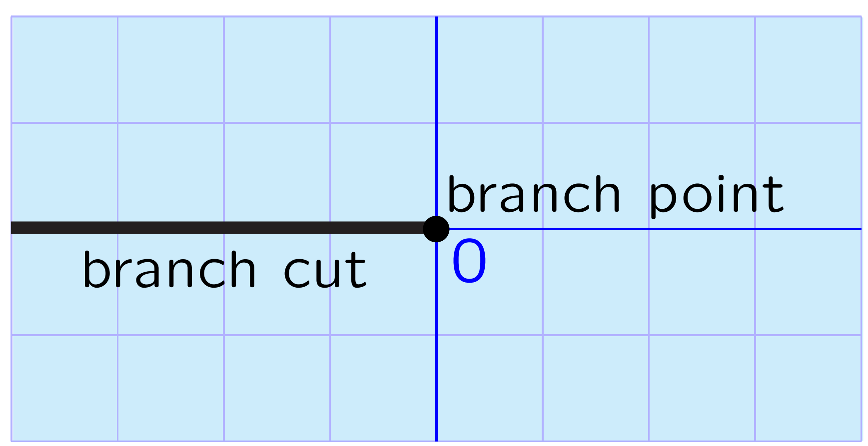
$$-\frac{\pi}{2} < \arg w \le \frac{3\pi}{2} \quad \text{for} \quad -\pi < \arg z \le 3\pi$$

Consequently, increasing $\arg z$ by 2π (which brings you back to the same point in the complex z-plane) changes the sign of w

$$w = z^{1/2} = \begin{cases} +|z|^{1/2}\, e^{\frac{1}{2} i\,\mathrm{Arg}\, z}, & -\pi < \arg\, z \le \pi \quad \text{(branch 1)} \\ -|z|^{1/2}\, e^{\frac{1}{2} i\,\mathrm{Arg}\, z}, & \pi < \arg\, z \le 3\pi \text{ (branch 2)} \end{cases}$$

These branches cover the complex plane $\mathbb{C}$ twice except at the branch point.

- The principal value is the branch 1 value which has the same branch cut as the logarithm on $\mathbb{C}\backslash(-\infty, 0]$

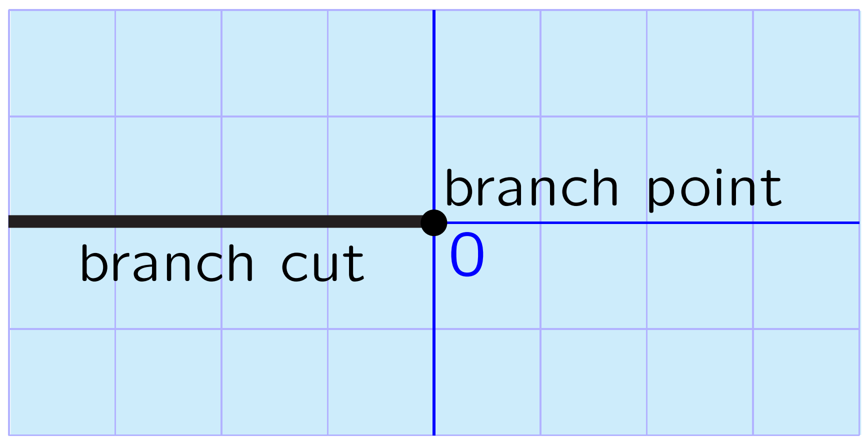


Riemann Surfaces

- The square root function is in fact single-valued and analytic on a *Riemann surface of two sheets*

$$\boxed{\text{Riemann surface} = \mathbb{C}\backslash\{0\} \cup \mathbb{C}\backslash\{0\}}$$

where it is understood that each time we cross the cut we move continuously from one to the other branch (sheet).

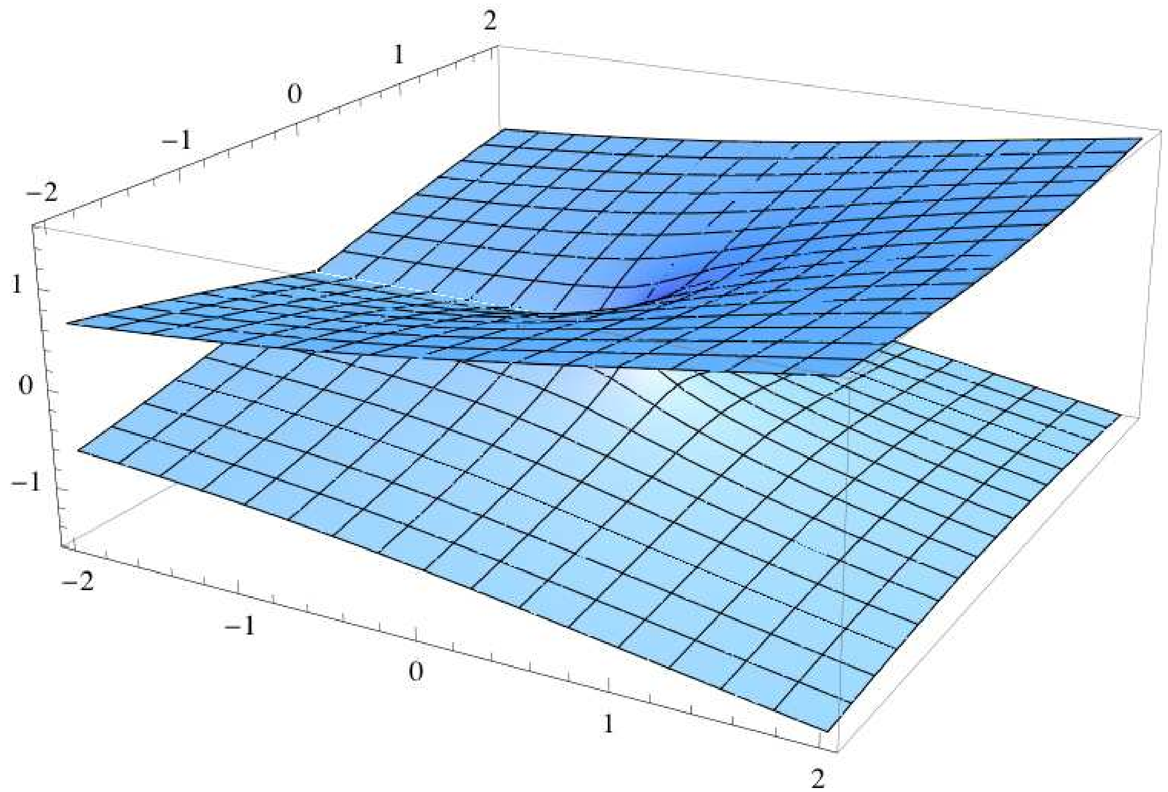


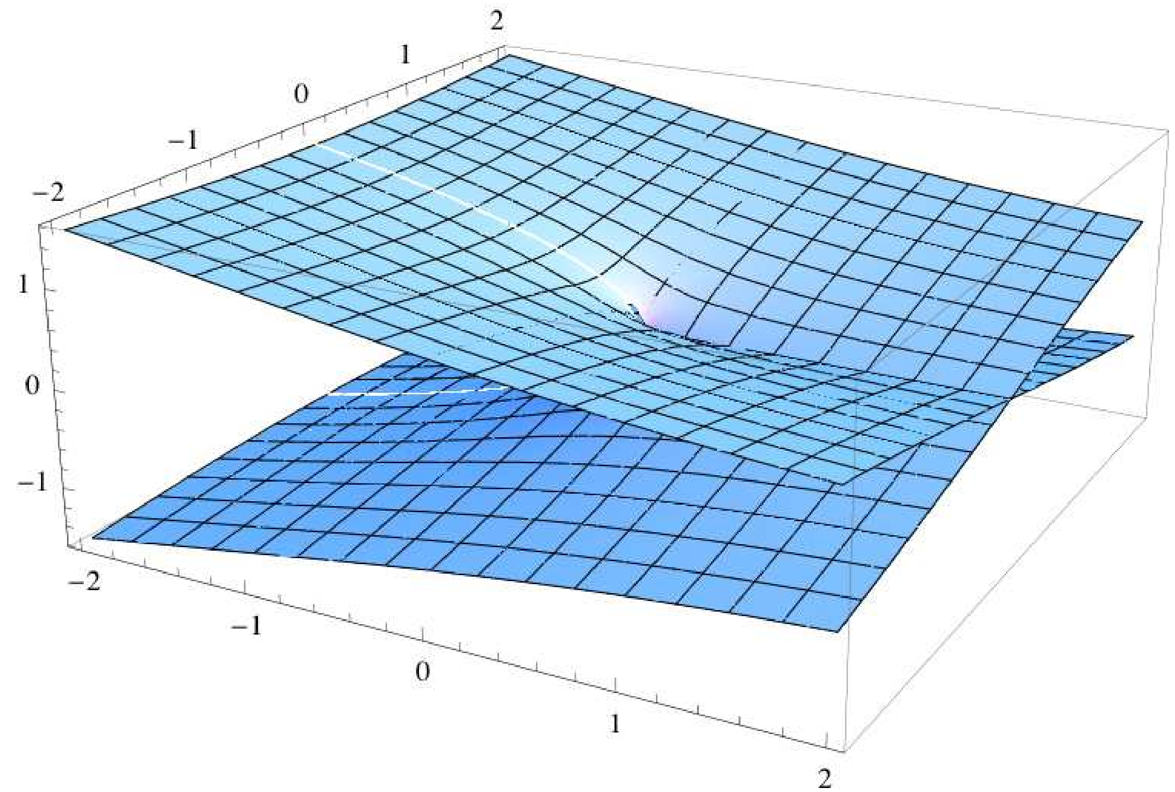
- More generally, $z^{m/n}$ is analytic on a Riemann surface with n sheets and $\log z$ is analytic on a Riemann surface with an infinite number of sheets (winding once anti-clockwise around the branch point $z = 0$ increases the imaginary part by 2π).

Square Root Riemann Surface

- To visualize the square root Riemann surface, let $z = x + iy$ and plot

$$u(x,y) = \pm\,\mathrm{Re}\,\sqrt{z}, \qquad\qquad v(x,y) = \pm\,\mathrm{Im}\,\sqrt{z}$$





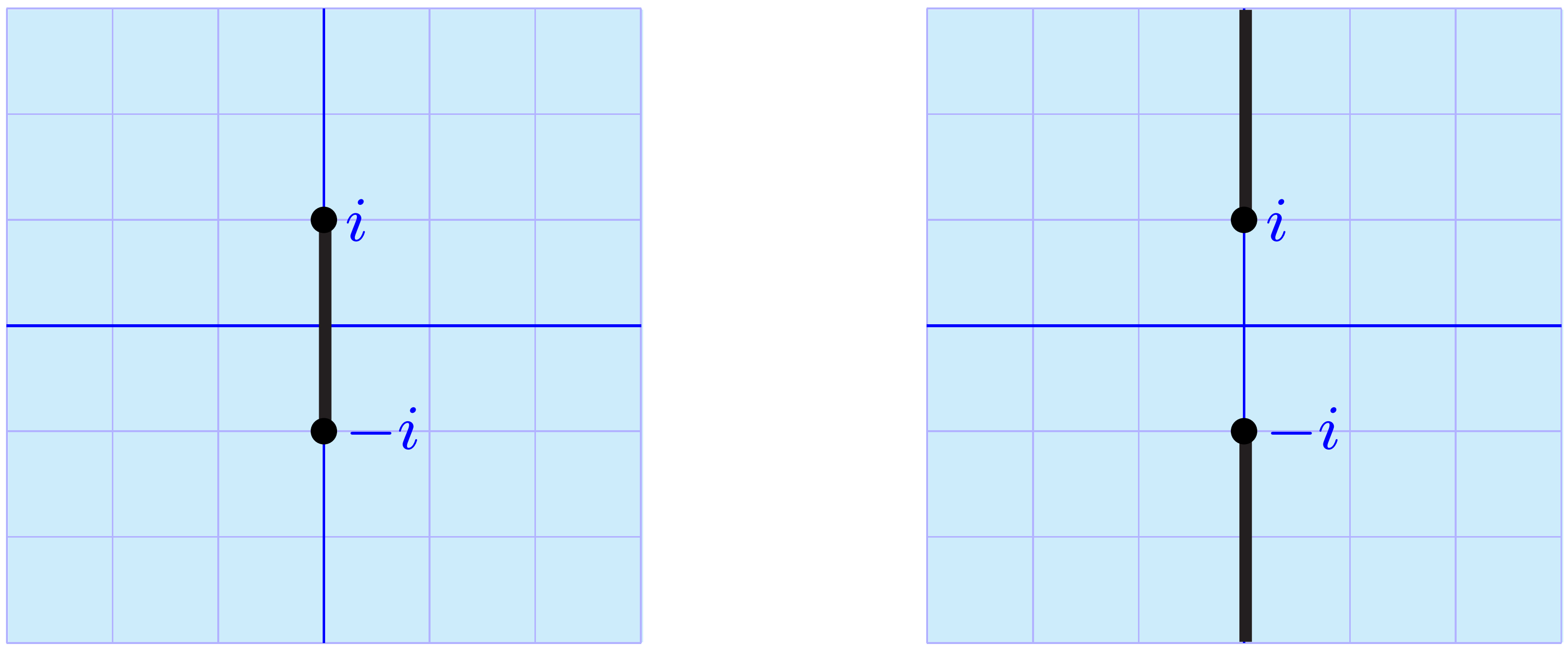
Example Branch Cuts

Example: Determine a branch of $\sqrt{z^2+1}$:

Solution: Using the definition of the square root

$$\sqrt{z^2+1} = (z-i)^{1/2}(z+i)^{1/2} = \exp[\tfrac{1}{2}\log(z-i) + \tfrac{1}{2}\log(z+i)]$$

Clearly, there are branch points at $z = \pm i$. Two possible choices of branch cuts are:



Trigonometric/Hyperbolic Derivatives

- The trigonometric and hyperbolic functions are defined in terms of the complex exponential

$$\boxed{\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \cosh z := \frac{e^{z} + e^{-z}}{2}, \qquad \cos z = \cosh iz}$$

$$\boxed{\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \sinh z := \frac{e^{z} - e^{-z}}{2}, \qquad \sin z = -i \sinh iz}$$

- These functions are all entire. Their derivatives are easily obtained using the derivative of the exponential and the rules for differentiation. For example,

$$\frac{d}{dz} \sin z \; = \; \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\tfrac{1}{2} i (i e^{iz} + i e^{-iz}) = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z$$

- The other standard trigonometric and hyperbolic functions are defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

$$\sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}$$

These functions are not entire — they are meromorphic (exhibit poles).

- The familiar formulas for derivatives of these functions carry over to the complex functions. For example, away from poles

$$\boxed{\frac{d}{dz} \tan z = \sec^2 z, \qquad \frac{d}{dz} \sec z = \sec z \tan z}$$

Inverse Sine

Exercise: Show $\sin : -\frac{\pi}{2} < \text{Re } z < \frac{\pi}{2} \to \mathbb{C}\backslash\{(-\infty, -1] \cup [1, \infty)\}$

Example: Show that the inverse sine function is the multi-valued function

$$\boxed{\arcsin z = -i \log[iz + (1 - z^2)^{1/2}]}$$

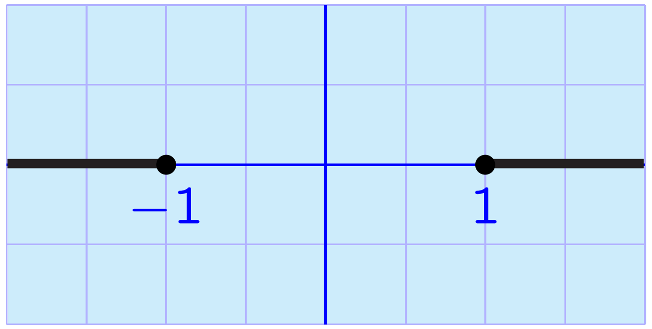
$$\begin{aligned}
& z = \sin w = \frac{1}{2i}(e^{iw} - e^{-iw}) \\
\Rightarrow \quad & e^{2iw} - 2iz\, e^{iw} - 1 = 0 \\
\Rightarrow \quad & e^{iw} = iz + (1 - z^2)^{1/2} \\
\Rightarrow \quad & w = \arcsin z = -i \log[iz + (1 - z^2)^{1/2}]
\end{aligned}$$

where the square root is multivalued. ■

Definition: The principal value of the inverse sine function is

$$\boxed{\text{Arcsin } z = -i \text{ Log}[iz + e^{\frac{1}{2}\text{Log}(1 - z^2)}]}$$

where the branch cuts for $e^{\frac{1}{2}\text{Log}(1 - z^2)}$ are



Inverse Trigonometrics/Hyperbolics

- The principal branch inverse trigonometric and hyperbolic functions are defined by

$$\mathrm{Arcsin}\, z = -i\,\mathrm{Log}(iz + \sqrt{1-z^2}), \qquad \mathrm{Arcsinh}\, z = \mathrm{Log}(z + \sqrt{1+z^2})$$

$$\mathrm{Arccos}\, z = -i\,\mathrm{Log}(z + i\sqrt{1-z^2}), \qquad \mathrm{Arccosh}\, z = \mathrm{Log}(z + \sqrt{z-1}\sqrt{z+1})$$

$$\mathrm{Arctan}\, z = \frac{1}{2}i\,[\mathrm{Log}(1-iz) - \mathrm{Log}(1+iz)], \qquad \mathrm{Arctanh}\, z = \frac{1}{2}[\mathrm{Log}(1+z) - \mathrm{Log}(1-z)]$$

- The multi-valued inverse functions are obtained by replacing the Log with log and the principal branch square root functions with with the multi-valued square root.

- The derivatives of the inverse multi-valued trigonometric functions are

$$\begin{aligned}
\frac{d}{dz}\,\mathrm{arcsin}\, z &= (1-z^2)^{-1/2}, & z &\neq \pm 1 \\
\frac{d}{dz}\,\mathrm{arccos}\, z &= -(1-z^2)^{-1/2}, & z &\neq \pm 1 \\
\frac{d}{dz}\,\mathrm{arctan}\, z &= (1+z^2)^{-1}, & z &\neq \pm i
\end{aligned}$$

Exercise: Find the derivatives of the inverse multi-valued hyperbolic functions

(a) arcsinh z (b) arccosh z (c) arctanh z

Exercises: Complex Powers and Inverses

Exercise: Show that the following identities hold when each complex power is given by its principal value

(a) $z^{-a} = \dfrac{1}{z^a}$ (b) $z^a\, z^b = z^{a+b}$ (c) $\dfrac{z^a}{z^b} = z^{a-b}$

Exercise: Show that in the case z is real

$$-\frac{\pi}{2} < \operatorname{Arcsin} x < \frac{\pi}{2}, \qquad x \in (-1, 1)$$

Exercise: Show using the chain rule that

$$\frac{d}{dz}\operatorname{Arcsin} z = (1 - z^2)^{-1/2}, \qquad z \neq \pm 1$$

where the same branches are used on either side.

Week 4: Complex Sequences and Series

10. Complex sequences, Cauchy convergence
11. Power series, radius of convergence and its calculation
12. Statement of Taylor's theorem, term-by-term integration and differentiation

Complex Sequences

Definition: We say that a *complex sequence* $\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots\}$ *converges* to a limit L in $\mathbb{C}$ and write

$$\boxed{\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty}$$

if for any $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\boxed{|a_n - L| < \epsilon \quad \text{whenever} \quad n > N(\epsilon)}$$

If the sequence $\{a_n\}$ does not converge we say it *diverges*.

Theorem 13 (Limit Theorems for Sequences)
If a_n and b_n are convergent complex sequences, $A, B \in \mathbb{C}$ and $f(z)$ is continuous then

1. **Linear:** $$\lim_{n\to\infty}(Aa_n + Bb_n) = A\lim_{n\to\infty} a_n + B\lim_{n\to\infty} b_n$$

2. **Product:** $$\lim_{n\to\infty}(a_n\, b_n) = \Big(\lim_{n\to\infty} a_n\Big)\Big(\lim_{n\to\infty} b_n\Big)$$

3. **Quotient:** $$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \textit{if} \quad \lim_{n\to\infty} b_n \neq 0$$

4. **Continuity:** $$\lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big)$$

Complex Sequences Exercises

Example: (Geometric Sequence) Use the ϵ-N definition to show that, for $|z| < 1$, the sequence $a_n = z^n$ converges to the limit $L = 0$:

Solution:

$$|a_n - L| = |z|^n < \epsilon \quad \text{whenever} \quad n > N(\epsilon) = \frac{\operatorname{Log} \epsilon}{\operatorname{Log} |z|}, \;\; z \neq 0 \qquad \blacksquare$$

Exercise: Show that a complex sequence converges if and only if the real and imaginary parts converge, that is, if $a_n = x_n + iy_n$

$$\lim_{n\to\infty} a_n = L = A + iB \quad \Leftrightarrow \quad \lim_{n\to\infty} x_n = A \quad \text{and} \quad \lim_{n\to\infty} y_n = B$$

Also show that $a_n \to L$ as $n \to \infty$ implies

(i) $\lim_{n\to\infty} \overline{a_n} = \overline{L}$ (ii) $\lim_{n\to\infty} |a_n| = |L|$

Cauchy Convergence

Definition: A complex sequence a_n is a *Cauchy sequence* if for any $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$|a_n - a_m| < \epsilon \quad \text{for all } m, n > N(\epsilon)$$

- Terms of a Cauchy sequence are arbitrarily close together for n and m sufficiently large.

Theorem 14 (Completeness of $\mathbb{R}$)
(i) A Cauchy sequence of real numbers converges to a limit in $\mathbb{R}$. (ii) A bounded ($|a_n| \le M$) monotonic sequence ($a_{n+1} \ge a_n$ or $a_{n+1} \le a_n$) of real numbers converges to a limit in $\mathbb{R}$.

- See text — deep properties of $\mathbb{R}$ related to closure.

Theorem 15 (Completeness of $\mathbb{C}$)
A complex sequence a_n converges to a limit in $\mathbb{C}$ if and only if it is Cauchy.

Proof: (i) Suppose $a_n \to L$ as $n \to \infty$ then given $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{whenever} \quad n > N(\epsilon)$$

Hence a_n is Cauchy

$$|a_n - a_m| = |(a_n - L) - (a_m - L)| \le |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever} \quad m, n > N(\epsilon)$$

(ii) Conversely, suppose a_n is Cauchy. Then using the inequality $|\operatorname{Re} z| \le |z|$:

$$|\operatorname{Re} a_n - \operatorname{Re} a_m| \;=\; |\operatorname{Re}(a_n - a_m)| \;\le\; |a_n - a_m| < \epsilon \quad \text{for all } m, n > N(\epsilon)$$

which implies $x_n = \operatorname{Re} a_n$ is a real Cauchy sequence. Similarly, $y_n = \operatorname{Im} a_n$ is a real Cauchy sequence. So $x_n \to x$ and $y_n \to y$ are convergent sequences by the completeness of $\mathbb{R}$ and so $a_n = x_n + iy_n \to x + iy$ is a convergent sequence. ■

Euler's Number

Theorem 16 (Euler's Number e)

$$\boxed{e := \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty}\frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots := \eta}$$

Proof: (i) By the binomial theorem and geometric series

$$\begin{aligned}
a_n &= (1+\frac{1}{n})^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\ldots 1}{n!}\frac{1}{n^n} \\
&= 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n}) \\
&\le 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \;=\; b_n \;\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 1+\frac{1}{1-\frac{1}{2}} = 3
\end{aligned}$$

Now a_n, b_n are bounded increasing sequences, so they converge to limits e, η with $e \le \eta$.

(ii) Now suppose $m < n$ and keep the first m terms of a_n

$$a_n > 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \cdots + \frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{m-1}{n})$$

Holding m fixed and letting $n \to \infty$ gives

$$e \ge 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!} = b_m$$

Taking $m \to \infty$ we have both $e \ge \eta$ and $e \le \eta$ so $e = \eta$. ■

- Like π, Euler's number e is a transcendental number. A decimal approximation to e is

$$\boxed{e = 2.71828182845904\ldots}$$

Complex Exponential

Theorem 17 (Complex Exponential)

$$\boxed{e^w = \lim_{n\to\infty}\left(1+\frac{w}{n}\right)^n, \qquad w \in \mathbb{C}}$$

Proof: Let $w = 1/z$ and $h = 1/n$ then

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z} = \lim_{h\to 0+}\frac{\operatorname{Log}(z+h)-\operatorname{Log} z}{h} = \lim_{h\to 0+}\frac{\operatorname{Log}(1+\frac{h}{z})}{h} = \lim_{n\to\infty}\operatorname{Log}(1+\frac{w}{n})^n = w$$

By continuity of e^w, it follows that

$$e^w = e^{\lim_{n\to\infty}\operatorname{Log}(1+\frac{w}{n})^n} = \lim_{n\to\infty} e^{\operatorname{Log}(1+\frac{w}{n})^n} = \lim_{n\to\infty}(1+\frac{w}{n})^n \qquad \blacksquare$$

- Using the binomial expansion and expanding suggests

$$\begin{aligned} e^w &= \lim_{n\to\infty}(1+\frac{w}{n})^n = \lim_{n\to\infty}\left[1+n\frac{w}{n}+\frac{n(n-1)}{2!}\frac{w^2}{n^2}+\cdots+\frac{n(n-1)\ldots 1}{n!}\frac{w^n}{n^n}\right] \\ &= 1+w+\lim_{n\to\infty}\frac{n(n-1)}{n^2}\frac{w^2}{2!}+\lim_{n\to\infty}\frac{n(n-1)(n-2)}{n^3}\frac{w^3}{3!}+\cdots \\ &= 1+w+\frac{w^2}{2!}+\frac{w^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{w^n}{n!} \end{aligned}$$

The result is true. But this derivation cannot be justified — it is not a proof!

- We will prove this exponential series later when we consider complex Taylor series. But first we need to consider the convergence of complex series.

Complex Series

Definition: A *complex series* is an infinite sum of the form $\sum_{n=1}^{\infty} a_n$ where $\{a_n\}$ is a complex sequence. Each series is associated with a *sequence of partial sums* $\{S_n\}$

$$\boxed{S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n}$$

We say that a complex series *converges* to a sum S and write

$$\boxed{S = \sum_{n=1}^{\infty} a_n}$$

if the sequence of partial sums converges to S. Otherwise, the series is said to *diverge*.

- The index for a series need not start at $n = 1$. For example, the *geometric series* is $\sum_{n=0}^{\infty} z^n$.

Example: (Geometric Series) Show, for $|z| < 1$, the geometric series converges with the sum

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad |z| < 1}$$

Solution: If $|z| < 1$ then since $z^{n+1} \to 0$ as $n \to \infty$

$$S_n = 1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1-z} \to \frac{1}{1-z} \qquad \text{as} \quad n \to \infty \qquad \blacksquare$$

Limit Theorems and Divergence Test

Theorem 18 (Limit Theorems for Series)

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent complex series and* $A, B \in \mathbb{C}$ *then*

1. **Linear:** $$\sum_{n=1}^{\infty} (Aa_n + Bb_n) = A \sum_{n=1}^{\infty} a_n + B \sum_{n=1}^{\infty} b_n$$

2. **Parts:** $$\sum_{n=1}^{\infty} a_n = S \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \mathit{Re}\, a_n = \mathit{Re}\, S \text{ and } \sum_{n=1}^{\infty} \mathit{Im}\, a_n = \mathit{Im}\, S$$

Theorem 19 (Divergence Test) *If* $\sum_{n=1}^{\infty} a_n$ *is convergent then* $\lim_{n\to\infty} a_n = 0$. *Equivalently, if* $\lim_{n\to\infty} a_n \neq 0$ *or if* $\lim_{n\to\infty} a_n$ *does not exist then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof: If $\sum_{n=1}^{\infty} a_n$ is convergent then $S_n = \sum_{k=1}^{n} a_k \to S$ as $n \to \infty$. So

$$a_n = S_n - S_{n-1} \to S - S = 0 \quad \text{as} \quad n \to \infty \qquad \blacksquare$$

Example: The geometric series $\sum_{n=0}^{\infty} z^n$ with $a_n = z^n$ converges for $|z| < 1$. For $|z| > 1$ it is clear that $|a_n| = |z|^n$ diverges. Similarly, for $|z| = 1$, $|a_n| = |z|^n = 1 \not\to 0$. Since $\lim_{n\to\infty} a_n \neq 0$ or does not exist in these cases we conclude from the divergence test that the series diverges. In summary, the geometric series converges for $|z| < 1$ and diverges for $|z| \geq 1$. $\blacksquare$

Example Complex Series

- The comparison, ratio and root tests (familiar from real analysis) can be used to establish convergence of complex series. See the supplementay material at the end of the slides for this week.

Example: Compare use of the ratio and root tests to show the exponential series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$:

Solution: Ratio test is easy

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|}{n+1} \to 0 \quad \text{as} \quad n \to \infty$$

But the root test requires Stirling's approximation

$$\boxed{n! \sim \sqrt{2\pi n}\,(n/e)^n}$$

$$|a_n|^{1/n} = \frac{|z|}{(n!)^{1/n}} \sim \frac{e|z|}{n} \to 0 \quad \text{as} \quad n \to \infty \qquad \blacksquare$$

Example: $\sum_{n=1}^{\infty} \frac{(-2)^{3n+1} z^n}{n^n}$ converges absolutely for all $z \in \mathbb{C}$ since:

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2^{3+1/n}|z|}{n} = 0 < 1 \qquad \blacksquare$$

- It is usually easier to use the ratio test (rather than the root test) when it applies.

More Example Complex Series

Example: Determine the largest region in which the following series is convergent and find its sum:

$$\sum_{n=0}^{\infty} \frac{1}{(4+2z)^n}$$

This is a geometric series $\sum_{n=0}^{\infty} w^n$ with $w = (4+2z)^{-1}$

$$\sum_{n=0}^{\infty} \frac{1}{(4+2z)^n} = \frac{1}{1-\frac{1}{4+2z}} = \frac{4+2z}{3+2z}, \qquad |z+2| > \tfrac{1}{2}$$

since $|w| < 1 \quad \Leftrightarrow \quad |z+2| > \frac{1}{2}$. ■

Example: Show that the series $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$ is convergent for all $\theta \in \mathbb{R}$ and find its sum:

Since $|\frac{e^{i\theta}}{2}| = \frac{1}{2} < 1$ we have a convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2}\right)^n = \frac{1}{1-\frac{e^{i\theta}}{2}} = \frac{2}{2-e^{i\theta}} \frac{2-e^{-i\theta}}{2-e^{-i\theta}} = \frac{4-2\cos\theta+2i\sin\theta}{5-4\cos\theta}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \text{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2}\right)^n = \frac{4-2\cos\theta}{5-4\cos\theta} \qquad ■$$

Harmonic Series

Theorem 20 (Harmonic Series) *The harmonic series, given by the Riemann zeta function $\zeta(p)$ with $p \in \mathbb{R}$, converges if $p > 1$ and diverges if $p \leq 1$*

$$\boxed{\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 1}$$

Proof: (i) For $p = 1$ the series diverges since

$$\begin{aligned} S_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots + \left(\frac{1}{2}\right) = 1 + \frac{k}{2} \to \infty \end{aligned}$$

(ii) For $p < 1$ the series diverges by comparison to $p = 1$

$$S_n = \sum_{k=1}^{n} \frac{1}{k^p} > \sum_{k=1}^{n} \frac{1}{k} \to \infty \quad \text{as} \quad n \to \infty$$

(iii) For $p > 1$ we have by the geometric series since $\left|\frac{1}{2^{p-1}}\right| < 1$

$$\begin{aligned} S_{2^k-1} &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots + \left(\frac{1}{(2^{k-1})^p} + \cdots + \frac{1}{(2^k-1)^p}\right) \\ &\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{k-1}}{(2^{k-1})^p} = 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{k-1})^{p-1}} < \frac{1}{1 - \frac{1}{2^{p-1}}} \end{aligned}$$

So the increasing sequence of partial sums S_n converges by the completeness of $\mathbb{R}$. ∎

- We will show later that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, etc.

Power Series

Definition: A series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n(z) = \sum_{n=0}^{\infty} a_n\,(z-a)^n}$$

with coefficients $a_n \in \mathbb{C}$ is called a *power series* around $z = a$.

Theorem 21 (Radius of Convergence)

A power series has a radius of convergence R such that either:
(i) The series converges only at a and $R = 0$.
(ii) The series converges absolutely on $|z-a| < R$ and diverges if $|z-a| > R > 0$.
(iii) The series converges for all z and $R = \infty$.

Proof: See text. ■

Theorem 22 (Analytic Power Series)

A power series converges to an analytic function inside its circle of convergence $|z-a| = R$.

Proof: See text. ■

- A power series may converge at some, all or no points on the circle of convergence $|z-a| = R$. The largest disk on which a power series converges is called its *disk of convergence*. This is either an open or closed disk of the form $|z-a| < R$ or $|z-a| \le R$.

Radius of Convergence

- The radius of convergence R is actually defined by the Cauchy-Hadamard formula

$$\boxed{\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}}$$

In practice, however, the radius of convergence is usually determined by applying the ratio or root test:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}(z)}{a_n(z)}\right| = |z-a| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \;\Rightarrow\; \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$\rho = \lim_{n\to\infty} |a_n(z)|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} < 1 \;\Rightarrow\; \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

Example: Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)^2} = 1 + \frac{z}{4} + \frac{z^2}{9} + \frac{z^3}{16} + \cdots$$

Solution: The convergence is determined by the ratio test with

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = 1$$

So $R = 1$. In fact the series converges absolutely for $|z| < 1$. ■

Taylor and Maclaurin Series

Definition: If $f(z)$ is analytic at $z = a$ then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

is called the *Taylor series of* $f(z)$ *around* a. If $a = 0$, it is called the *Maclaurin series for* $f(z)$.

Example: Find Maclaurin series for e^z, e^{iz}, $\cosh z$, $\sinh z$, $\cos z$, $\sin z$ and $\mathrm{Log}(1+z)$ giving the disk of convergence:

Solution: These can be obtained using the formulas

$$\frac{d^n}{dz^n} e^z \Big|_{z=0} = 1, \quad \frac{d^n}{dz^n} e^{iz} \Big|_{z=0} = i^n$$

$$\frac{d^n}{dz^n} \cosh z \Big|_{z=0} = \frac{1}{2}\frac{d^n}{dz^n}(e^z + e^{-z})\Big|_{z=0} = \frac{1}{2}[1 + (-1)^n], \quad \text{etc}$$

$$\frac{d^n}{dz^n} \mathrm{Log}(1+z)\Big|_{z=0} = (-1)^{n-1}(n-1)!$$

and the ratio test for convergence. ■

Standard Maclaurin Series

$$
\begin{aligned}
e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots && |z| < \infty \\
e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz - \frac{z^2}{2!} - i\,\frac{z^3}{3!} + \cdots + \frac{(iz)^n}{n!} + \cdots && |z| < \infty \\
\cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots && |z| < \infty \\
\sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots && |z| < \infty \\
\cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots && |z| < \infty \\
\sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots && |z| < \infty \\
\mathrm{Log}(1+z) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n+1} \frac{z^n}{n} + \cdots && |z| < 1
\end{aligned}
$$

- Note that $\mathrm{Log}(1+z)$ has a branch point at $z = -1$.

Taylor's Theorem

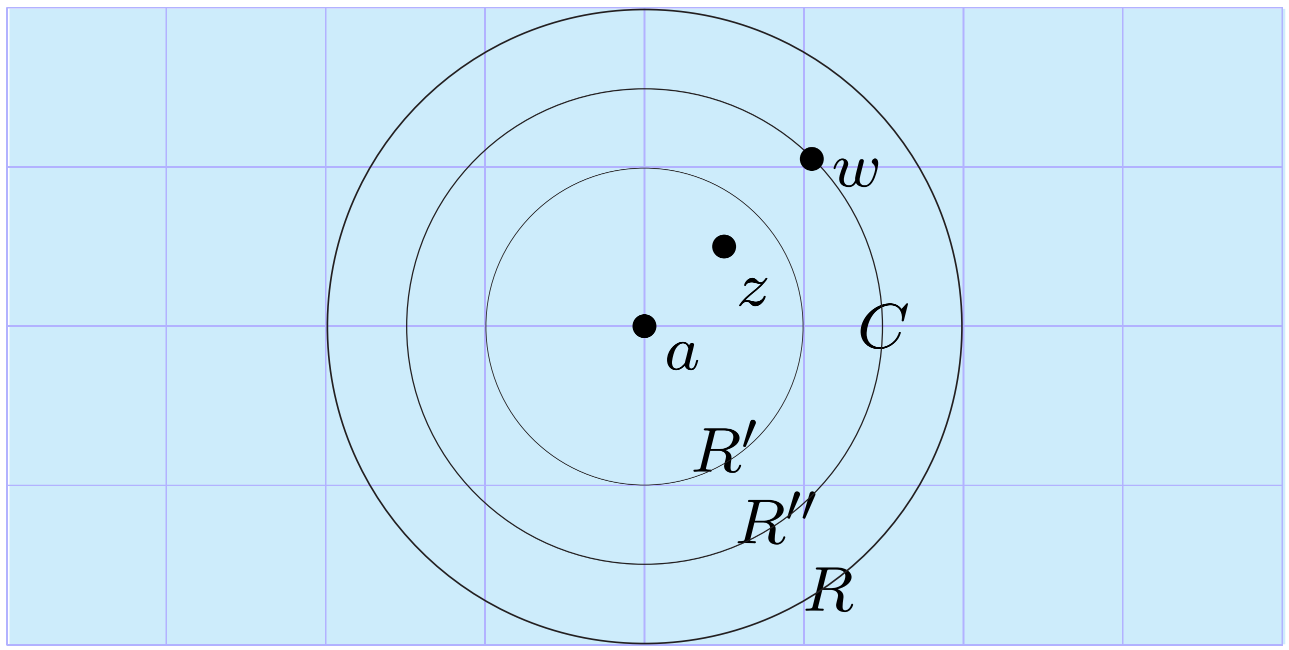
Theorem 23 (Taylor's Theorem)
If $f(z)$ is analytic in the disk $|z-a| < R$ then the Taylor series converges to $f(z)$ for all z in this disk. The convergence is uniform in any closed subdisk $|z-a| \leq R' < R$. Specifically,

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(z-a)^k + R_n(z)$$

where the remainder satisfies

$$\sup_{|z-a|\leq R'} |R_n(z)| \to 0 \quad \text{as } n \to \infty$$

Proof: Given later after Cauchy's integral theorems. ∎



General l'Hôpital's Rule

Theorem 24 (General l'Hôpital's Rule) *If $f(z)$ and $g(z)$ are analytic at $z = a$ and $f(z)$, $g(z)$ and their first $n-1$ derivatives all vanish at $z = a$ but the nth derivative $g^{(n)}(a) \neq 0$ then*

$$\boxed{\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(n)}(a)}{g^{(n)}(a)} = \lim_{z\to a} \frac{f^{(n)}(z)}{g^{(n)}(z)}}$$

Proof: Since $f(z)$ and $g(z)$ are both analytic in an open disk around $z = a$ we can represent them by their Taylor series

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \lim_{z\to a} \frac{\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(z-a)^k} = \lim_{z\to a} \frac{\displaystyle\sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^{k-n}}{\displaystyle\sum_{k=n}^{\infty} \frac{g^{(k)}(a)}{k!}(z-a)^{k-n}}$$

$$= \frac{\displaystyle\lim_{z\to a}\sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^{k-n}}{\displaystyle\lim_{z\to a}\sum_{k=n}^{\infty} \frac{g^{(k)}(a)}{k!}(z-a)^{k-n}} = \frac{f^{(n)}(a)}{g^{(n)}(a)} = \lim_{z\to a} \frac{f^{(n)}(z)}{g^{(n)}(z)}$$

and use the quotient rule, uniform convergence and continuity of the series and continuity of the nth derivatives. ■

Example: General l'Hôpital's Rule

Example: Evaluate (i) $\lim_{z\to 0} \dfrac{2\cosh z - 2 - z^2}{z^4}$, (ii) $\lim_{z\to 0}(\dfrac{1}{z} - \cot z)$:

(i) The first three derivatives vanish at $z = 0$ so by l'Hôpital

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f^{(4)}(a)}{g^{(4)}(a)} = \frac{2\cosh 0}{4!} = \frac{1}{12}$$

(ii) Alternatively you can use Taylor series

$$\lim_{z\to 0}(\frac{1}{z} - \cot z) = \lim_{z\to 0} \frac{\sin z - z\cos z}{z\sin z}$$

$$= \lim_{z\to 0} \frac{(z - z^3/6 + \cdots) - z(1 - z^2/2 + \cdots)}{z(z - z^3/6 + \cdots)} = \lim_{z\to 0} \frac{z^3/3 + \cdots}{z^2 + \cdots} = 0 \qquad \blacksquare$$

Taylor with Removable Singularity

Example: Find the Taylor expansion about $z = 0$ of

$$f(z) = \frac{\sin z}{z}$$

Solution: Note that $f(z)$ has a removable singularity at $z = 0$ and that $f(z)$ is in fact entire. Indeed,

$$f(z) \;=\; \frac{\sin z}{z} \;=\; \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} \;=\; 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

The derivatives $f^{(n)}(z)$ are undefined at $z = 0$, so the coefficients in the Taylor series must be interpreted as limits

$$\boxed{\frac{f^{(n)}(0)}{n!} \mapsto \lim_{z \to 0} \frac{f^{(n)}(z)}{n!}}$$

Hence the Taylor coefficients are

$$\lim_{z \to 0} f(z) \;=\; \lim_{z \to 0} \frac{\sin z}{z} = 1$$

$$\lim_{z \to 0} f'(z) \;=\; \lim_{z \to 0} \frac{z \cos z - \sin z}{z^2} \;=\; \lim_{z \to 0} \frac{z(1 - \frac{z^2}{2} + \cdots) - (z - \frac{z^3}{6} + \cdots)}{z^2} \;=\; \lim_{z \to 0} \frac{-\frac{z^3}{3} + \cdots}{z^2} = 0$$

$$\lim_{z \to 0} \frac{f''(z)}{2!} \;=\; \lim_{z \to 0} \frac{(2 - z^2) \sin z - 2z \cos z}{2z^3}$$

$$= \lim_{z \to 0} \frac{(2 - z^2)(z - \frac{z^3}{6} + \cdots) - 2z(1 - \frac{z^2}{2} + \cdots)}{2z^3} \;=\; \lim_{z \to 0} \frac{-\frac{z^3}{3} + \cdots}{2z^3} = -\frac{1}{6} = -\frac{1}{3!}$$

and so on. ■

Real Taylor with Essential Singularity

Example: Find the Taylor series expansion about $x = 0$ of the function of the real variable x

$$\boxed{f(x) = e^{-1/x^2}}$$

Solution: Note that $f(x)$ is C^∞ (it has continuous partial derivatives of all orders) but it is *not* analytic — it has an essential singularity at $x = 0$. Again since the derivatives $f^{(n)}(x)$ are undefined at $x = 0$ we must interpret the coefficients in the Taylor series as limits

$$\boxed{\frac{f^{(n)}(0)}{n!} \mapsto \lim_{x \to 0} \frac{f^{(n)}(x)}{n!}}$$

Hence the Taylor coefficients are

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-1/x^2} = 0, \qquad \lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{2e^{-1/x^2}}{x^3} = 0$$

$$\lim_{x \to 0} \frac{f''(x)}{2!} = \lim_{x \to 0} \frac{(4 - 6x^2)e^{-1/x^2}}{2x^6} = 0, \qquad \lim_{x \to 0} \frac{f'''(x)}{3!} = \lim_{x \to 0} \frac{4(2 - 9x^2 + 6x^4)e^{-1/x^2}}{6x^9} = 0$$

and so on since for $n \geq 0$

$$\left| \frac{e^{-1/x^2}}{x^n} \right| = \frac{|x|^{-n}}{e^{1/x^2}} \leq \frac{|x|^{-n}}{\frac{(1/x^2)^{n+1}}{(n+1)!}} = \frac{(n+1)!\, x^{2n+2}}{|x|^n} \to 0 \quad \text{as} \quad x \to 0$$

We conclude that the Taylor series of $f(x)$ about $x = 0$ vanishes identically and so it does not converge to $f(x)$ in any open neighbourhood of $x = 0$. Existence of the Taylor series is not sufficient to guarantee convergence to the function — we need *analyticity*! ■

Manipulation of Series

Theorem 25 (Adding Series)
Let A and B be constants and let $f(z)$, $g(z)$ be analytic with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

Then the Taylor series of $Af(z) + Bg(z)$ is

$$Af(z) + Bg(z) = \sum_{n=0}^{\infty} (Aa_n + Bb_n)(z-a)^n$$

Proof: Use $\dfrac{d^n}{dz^n}[Af(z) + Bg(z)] = Af^{(n)}(z) + Bg^{(n)}(z)$. ■

Theorem 26 (Cauchy Product)
Let $f(z)$, $g(z)$ be analytic with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

then the Taylor series of $f(z)g(z)$ around a is given by the Cauchy product

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

Proof: The coefficients c_n are given by Leibnitz formula evaluated at $z = a$

$$\frac{1}{n!}\frac{d^n}{dz^n}\Big[f(z)g(z)\Big] = \sum_{k=0}^{n} \frac{f^{(n-k)}(z)}{(n-k)!}\frac{g^{(k)}(z)}{k!} \qquad ■$$

- When series are added or multiplied, the resultant series converges in the smaller of the two disks of convergence.

Example of Manipulating Series

Example: Use the Cauchy product of series to show that

$$\boxed{e^{z_1}e^{z_2} = e^{z_1+z_2}}$$

Solution: For $|z| < \infty$, use the exponential series around $z = 0$

$$f(z) = e^{zz_1} = \sum_{n=0}^{\infty} \frac{z_1^n}{n!}\, z^n, \qquad a_n = \frac{f^{(n)}(z)}{n!}\bigg|_{z=0} = \frac{z_1^n}{n!}$$

$$g(z) = e^{zz_2} = \sum_{n=0}^{\infty} \frac{z_2^n}{n!}\, z^n, \qquad b_n = \frac{g^{(n)}(z)}{n!}\bigg|_{z=0} = \frac{z_2^n}{n!}$$

$$\begin{aligned} f(z)g(z) &= e^{zz_1}e^{zz_2} = \Big(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\, z^n\Big)\Big(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\, z^n\Big) \\ &= \sum_{n=0}^{\infty} c_n z^n = e^{z(z_1+z_2)} \end{aligned}$$

since by the binomial expansion

$$c_n = \sum_{k=0}^{n} a_{n-k}b_k = \sum_{k=0}^{n} \frac{z_1^{n-k} z_2^k}{(n-k)!k!} = \frac{(z_1+z_2)^n}{n!}$$

The result follows by taking $z \to 1$ in $e^{zz_1}e^{zz_2} = e^{z(z_1+z_2)}$ using continuity which follows from uniform convergence of the series on $|z| \le r$ for any $r > 0$. ■

Another Example of Manipulating Series

Example: Use the Cauchy product to show

$$\boxed{\tanh z = \sum_{n=0}^{\infty} a_n z^n = z - \frac{z^3}{3} + \frac{2z^5}{15} - \cdots}$$

Solution:

$$\begin{aligned}
&\cosh z \tanh z = (1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots)(a_0+a_1 z+a_2 z^2+a_3 z^3+\cdots) \\
&= a_0 + a_1 z + (a_2+\frac{a_0}{2})z^2 + (a_3+\frac{a_1}{2})z^3 + (a_4+\frac{a_2}{2}+\frac{a_0}{4!})z^4 + (a_5+\frac{a_3}{2}+\frac{a_1}{4!})z^5 + \cdots \\
&= \sinh z \;=\; z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots
\end{aligned}$$

$$\Rightarrow \; a_0 = 0,\; a_1 = 1,\; a_2 = 0,\; a_3 = -\frac{1}{3},\; a_4 = 0,\; a_5 = \frac{2}{15},\; \ldots \qquad \blacksquare$$

Limits of Integrals

Lemma 27 (Limits of Integrals)

Let $\{a_n(z)\}$ be a sequence of continuous functions on a region S and let γ be a curve inside S. If $a_n(z)$ converges uniformly to $a(z)$ on γ then

$$\boxed{\lim_{n\to\infty}\int_\gamma a_n(z)dz = \int_\gamma \lim_{n\to\infty} a_n(z)dz = \int_\gamma a(z)dz}$$

Proof: Using integral bounds we find

$$\begin{aligned}
& \left|\int_\gamma a_n(z)dz - \int_\gamma a(z)dz\right| = \left|\int_\gamma (a_n(z) - a(z))dz\right| \\
\le\ & \int_a^b |a_n(z(t)) - a(z(t))||z'(t)|dt \\
\le\ & \sup_{z\in\gamma} |a_n(z) - a(z)| \int_a^b |z'(t)|dt \\
=\ & \sup_{z\in\gamma} |a_n(z) - a(z)| \text{ Length}(\gamma) \to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

since $a_n(z)$ converges uniformly to $a(z)$ on γ. ■

Term-By-Term Integration

Theorem 28 (Term-By-Term Integration)

Let $\{a_n(z)\}$ be a sequence of continuous functions on a region S and let γ be a curve inside S. If $a(z) = \sum_{n=1}^{\infty} a_n(z)$ converges uniformly on γ then

$$\boxed{\int_\gamma a(z)dz = \int_\gamma \sum_{n=1}^{\infty} a_n(z)dz = \sum_{n=1}^{\infty} \int_\gamma a_n(z)dz}$$

Proof: Apply the previous lemma to the sequence of partial sums. ■

Example: Use term-by-term integration to find the Maclaurin series of Log$(1+z)$: On $|z| \le r < 1$ we have

$$\text{Log}(1+z) = \int_0^z \frac{dw}{1+w} = \int_0^z \sum_{n=0}^{\infty} (-1)^n w^n dw = \sum_{n=0}^{\infty} (-1)^n \int_0^z w^n dw = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} \qquad ■$$

Uniform and absolute convergence follows by Weierstrass test.

Term-by-Term Differentiation

Lemma 29 (Term-by-Term Differentiation)

If $a'_n(z)$ exists in $S \subset \mathbb{C}$, $\sum_{n=1}^{\infty} a_n(z)$ converges in S and $\sum_{n=1}^{\infty} a'_n(z)$ converges uniformly in S, then the series can be differentiated term-by-term

$$\frac{d}{dz} \sum_{n=1}^{\infty} a_n(z) = \sum_{n=1}^{\infty} a'_n(z) \quad \textit{for all } z \in S$$

Proof: See text — similar to term-by-term integration. ■

Theorem 30 (Differentiation of Taylor Series)

If $f(z)$ is analytic at $z = a$ so that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

the Taylor series for $f'(z)$ is given by term-by-term differentiation of the Taylor series for $f(z)$.

Proof: The derived series obtained by term-by-term differentiation is

$$f'(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(z-a)^{n-1}$$

Since this is the Taylor series of the *analytic* function $f'(z)$ it converges uniformly in any closed subdisk of the disk of convergence. So the result follows from the lemma. ■

- The derived series for $f'(z)$ converges with the same disk of convergence as for $f(z)$.

Example: Use term-by-term differentiation to find the Maclaurin series of $(1-z)^{-2}$:

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{d}{dz}(z^n) = \sum_{n=1}^{\infty} n\, z^{n-1}, \quad |z| \le r < 1 \qquad ■$$

Suplementary Material on Complex Series

- The remaining slides of Week 4 are supplementary material on complex series. For the most part, this material involves results and concepts obtained by relatively straightforward extensions of the corresponding results and concepts for real series previously encountered in real analysis.

- Knowledge of most of this material will be assumed for completing problems on the problem sheets and assignments as well as in the exams. In particular, you should be familiar with the following topics:
  - conditional and absolute convergence
  - comparison, ratio and root test
  - uniform convergence
  - Weierstrass M-test

- This material we not be included in lectures. Please see the recommended textbooks for more details on this material.

Absolute and Conditional Convergence

Definition: A complex series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent. A complex series that is convergent but not absolutely convergent is called *conditionally convergent*.

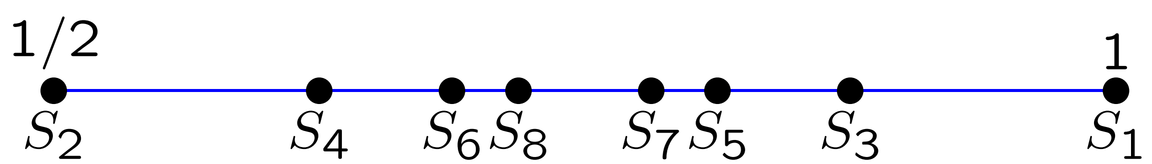
Theorem 31 (Absolute Convergence Implies Convergence)
If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

Proof: Let $S_n = \sum_{k=1}^{n} a_k$, $T_n = \sum_{k=1}^{n} |a_k|$. Since $\{T_n\}$ converges it is a Cauchy sequence. The sequence of partial sums S_n is also Cauchy and therefore converges since for $n > m > N(\epsilon)$

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| = |T_n - T_m| < \epsilon \qquad \blacksquare$$

Example: Conditional Convergence

Example: The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally:



(i) No absolute convergence since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(ii) The even and odd partial sums are monotone and bounded

$$\begin{aligned}
S_{2n} &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right) \\
&= 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\cdots-\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)-\frac{1}{2n} < 1-\frac{1}{2n} \le 1 \\
S_{2n-1} &= 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\cdots-\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right) \\
&= \frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-3}-\frac{1}{2n-2}\right)+\frac{1}{2n-1} \ge \frac{1}{2}
\end{aligned}$$

By completeness, they converge, and to the same limit since

$$\lim_{n\to\infty}(S_{2n-1}-S_{2n}) = \lim_{n\to\infty}\frac{1}{2n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} \qquad \blacksquare$$

Rearranging Series

Theorem 32 (Rearranging Absolutely Convergent Series)
The terms of an absolutely convergent series can be arbitrarily rearranged.

Proof: See text. ■

- Since the previous series is absolutely convergent for $|z| < 1$, we can rearrange its terms

$$\text{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = z + \tfrac{z^3}{3} - \tfrac{z^2}{2} + \tfrac{z^5}{5} + \tfrac{z^7}{7} - \tfrac{z^4}{4} + \cdots \quad |z| < 1$$

$$= \sum_{n=1}^{\infty} \left(\frac{z^{4n-3}}{4n-3} + \frac{z^{4n-1}}{4n-1} - \frac{z^{2n}}{2n} \right), \quad |z| < 1$$

But this is not a power series, Abel's theorem does not apply and the result does not hold at $z = 1$. In fact for $n \geq 1$ we have the identity

$$\tfrac{1}{4n-3} + \tfrac{1}{4n-1} - \tfrac{1}{2n} = \left(\tfrac{1}{4n-3} - \tfrac{1}{4n-2} + \tfrac{1}{4n-1} - \tfrac{1}{4n} \right) + \tfrac{1}{2} \left(\tfrac{1}{2n-1} - \tfrac{1}{2n} \right)$$

This implies that

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right)$$

$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \text{Log}\,2 + \tfrac{1}{2}\,\text{Log}\,2 = \tfrac{3}{2}\,\text{Log}\,2 \qquad ■$$

- Rearranging the terms of a conditionally convergent series is thus invalid.

Comparison Test

Theorem 33 (Comparison Test)

(i) If $|a_n| \leq b_n$ for $n \geq n_0$ and $\sum_{n=n_0}^{\infty} b_n = T$ is convergent, then $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent.

(ii) If $0 \leq c_k \leq d_k$ for $k \geq n_0$ and $\sum_{n=n_0}^{\infty} c_n$ diverges then $\sum_{n=n_0}^{\infty} d_n$ diverges.

Proof: (i) Let $S_n = \sum_{k=n_0}^{n} |a_k|$, $T_n = \sum_{k=n_0}^{n} b_k$. Then

$$S_n \leq T_n < T$$

and the increasing sequence of partial sums S_n converges by the completeness of $\mathbb{R}$. So $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent and so is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n_0-1} a_n + \sum_{n=n_0}^{\infty} a_n$$

(ii) The partial sums are bounded from below

$$U_n = \sum_{k=n_0}^{n} d_k \geq \sum_{k=n_0}^{n} c_k$$

So $U_n \to \infty$ as $n \to \infty$ since $\sum_{k=n_0}^{n} c_k \to \infty$. ■

Ratio Test

Theorem 34 (Ratio Test) *Suppose that*

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then the complex series $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $\rho < 1$ *and diverges if* $\rho > 1$*. The test is inconclusive if* $\rho = 1$*.*

Proof: (i) If $\rho < 1$ and r is such that $0 \leq \rho < r < 1$ then

$$|a_{n+1}| < |a_n| r, \qquad n \geq N$$

$$\Rightarrow \quad |a_{N+1}| < |a_N| r, \qquad |a_{N+m}| < |a_N| r^m, \quad m > 0$$

$$\Rightarrow \quad |a_{N+1}| + |a_{N+2}| + \cdots < |a_N|(r + r^2 + r^3 + \cdots)$$

and the series is absolutely convergent by comparison with the geometric series.

(ii) If $\rho > 1$ and r is such that $1 < r < \rho$ then

$$|a_{n+1}| > |a_n| r > |a_n|, \qquad n \geq N$$

So $\lim_{n\to\infty} |a_n| \neq 0$ and the series diverges.

(iii) $\rho = 1$ for the absolutely convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, the conditionally convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. ■

Examples: Comparison and Ratio Test

Example: Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ is absolutely convergent for $|z| \leq 1$:

$$|a_n| = \left|\frac{z^n}{n(n+1)}\right| = \frac{|z|^n}{n(n+1)} \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2} = b_n$$

So the series converges absolutely for $|z| \leq 1$ by comparison to the convergent harmonic series with $p = 2 > 1$. ■

Example: (i) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ converges absolutely (ii) $\sum_{n=1}^{\infty} \frac{(3)^n}{n^2}$ diverges since:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \lim_{n\to\infty} \dfrac{3}{n+1} = 0 < 1 & \text{(i)} \\ \lim_{n\to\infty} \dfrac{3n^2}{(n+1)^2} = 3 > 1 & \text{(ii)} \end{cases} \quad ■$$

Example: Show that the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{4^n(n+1)^3}$ converges absolutely for all $|z+2| \leq 4$:

Using the ratio test for $|z+2| < 4$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|z+2|}{4}\frac{(n+1)^3}{(n+2)^3} = \frac{|z+2|}{4} < 1$$

so the series converges absolutely for $|z+2| < 4$. If $|z+2| = 4$, the series converges absolutely by comparison to the harmonic series with $p = 3 > 1$:

$$|a_n| = \frac{|z+2|^{n-1}}{4^n(n+1)^3} = \frac{1}{4(n+1)^3} \leq \frac{1}{n^3} \quad ■$$

Root Test

Theorem 35 (Root Test) *Suppose that*

$$\rho = \lim_{n\to\infty} |a_n|^{1/n}$$

Then the complex series $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $\rho < 1$ *and diverges if* $\rho > 1$. *The test is inconclusive if* $\rho = 1$.

Proof: (i) if $\rho < 1$ and r is such that $0 \leq \rho < r < 1$ then

$$|a_n|^{1/n} < r \quad \text{or} \quad |a_n| < r^n, \qquad n \geq N$$

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} r^n$$

So the series is absolutely convergent by comparison with the geometric series. (ii) and (iii) are similar to the ratio test. ■

Uniform Convergence

Definition: The sequence $\{a_n(z)\}$ converges *uniformly* to $a(z)$ on a subset $S \subset \mathbb{C}$ if for any $\epsilon > 0$ there is an $N(\epsilon)$ (independent of $z \in S$) such that

$$\boxed{\sup_{z\in S} |a_n(z) - a(z)| < \epsilon \quad \text{whenever} \quad n > N(\epsilon)}$$

The series $\sum_{n=1}^{\infty} a_n(z)$ converges *uniformly* to $a(z)$ on S if the sequence of partial sums converges uniformly to $a(z)$ on S.

- For *pointwise* convergence $N = N(\epsilon, z)$ can depend on $z \in \mathbb{C}$.

Theorem 36 (Uniform Convergence Implies Convergence)
(i) If the sequence $\{a_n(z)\}$ converges uniformly to $a(z)$ on a subset $S \subset \mathbb{C}$ then $\{a_n(z)\}$ converges pointwise on S.

(ii) If the series $\sum_{n=1}^{\infty} a_n(z)$ converges uniformly to $a(z)$ on a subset $S \subset \mathbb{C}$ then the series converges pointwise on S.

Proof: (i) We need to show that for each $z \in S$

$$|a_n(z) - a(z)| < \epsilon \quad \text{whenever} \quad n > N(\epsilon, z)$$

But now for any $z \in S$, by uniform convergence

$$|a_n(z) - a(z)| \le \sup_{z\in S} |a_n(z) - a(z)| < \epsilon \quad \text{whenever} \quad n > N(\epsilon)$$

So pointwise convergence holds on S with $N(\epsilon, z) = N(\epsilon)$ independent of $z \in S$.

(ii) Apply (i) to the sequence of partial sums. ■

Uniform Convergence and Continuity

Theorem 37 (Uniform Convergence and Continuity)
(i) If $a_n(z)$ converges uniformly to $a(z)$ on $S \subset \mathbb{C}$ and $a_n(z)$ is continuous on S for each n then $a(z)$ is continuous.

(ii) If $S(z) = \sum_{n=1}^{\infty} a_n(z)$ converges uniformly on S and $a_n(z)$ is continuous for each n then $S(z)$ is continuous on S, that is,

$$\boxed{\lim_{z\to z_0} \sum_{n=1}^{\infty} a_n(z) = \sum_{n=1}^{\infty} \lim_{z\to z_0} a_n(z) = \sum_{n=1}^{\infty} a_n(z_0), \qquad z_0 \in S}$$

Proof: (i) We are given that (a) $\{a_n(z)\}$ converges *uniformly* to $a(z)$ on S, (b) $a_n(z)$ is continuous at z_0, that is,

$$\begin{array}{lll} \text{(a)} & \sup_{z\in S} |a_n(z)-a(z)| < \dfrac{\epsilon}{3} & \text{whenever} \quad n > N(\epsilon) \\ \text{(b)} & |a_n(z)-a_n(z_0)| < \dfrac{\epsilon}{3} & \text{whenever} \quad |z-z_0| < \delta(\epsilon, z_0) \end{array}$$

We show $a(z)$ is continuous at $z = z_0$. Suppose $z \in S$, $n > N(\epsilon)$ and use the triangle inequality

$$\begin{aligned} |a(z) - a(z_0)| &\le |a(z)-a_n(z) + a_n(z)-a_n(z_0) + a_n(z_0)-a(z_0)| \\ &\le |a(z)-a_n(z)| + |a_n(z)-a_n(z_0)| + |a_n(z_0)-a(z_0)| \\ &\le \sup_{z\in S} |a(z)-a_n(z)| + |a_n(z)-a_n(z_0)| + \sup_{z\in S} |a_n(z)-a(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{whenever} \quad |z-z_0| < \delta(\epsilon, z_0) \quad \text{by (a) and (b)} \end{aligned}$$

(ii) Follows by applying (i) to the sequence of continuous partial sums. ■

Example of Non-Uniform Convergence

Example: The sequence of real functions $a_n(x) = x^n$ converges pointwise on the interval $x \in [0, 1]$ to $a(x)$ given by

$$a(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$$

The sequence converges uniformly and absolutely on any closed interval $[0, r]$ with $r < 1$ but does not converge uniformly on the closed interval $[0, 1]$:

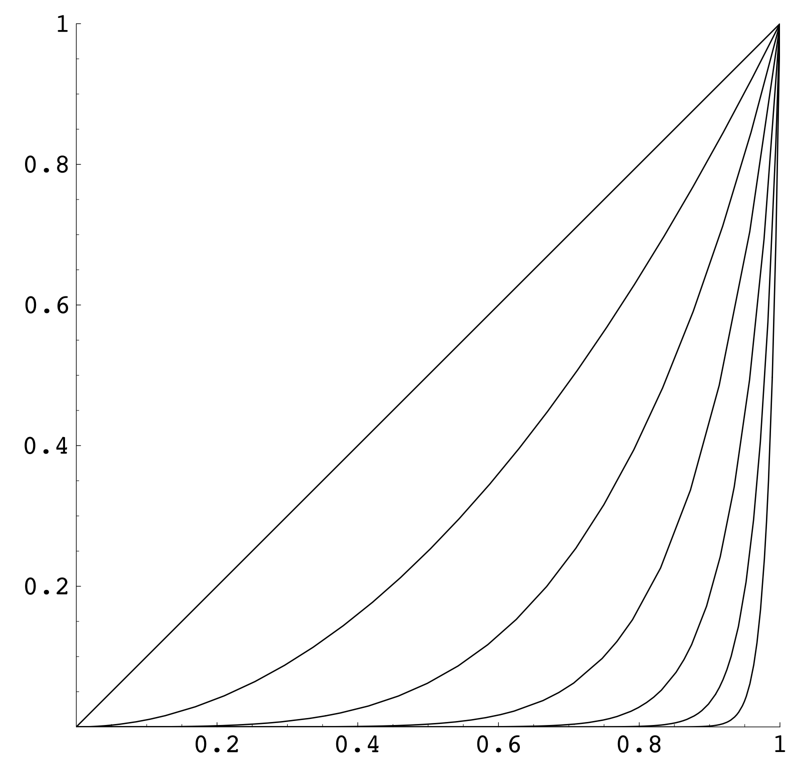
Solution: On the interval $[0, r]$ the convergence is uniform and absolute

$$\sup_{x\in[0,r]} |x^n - a(x)| = \sup_{x\in[0,r]} |x^n| = r^n \to 0 \quad \text{as} \quad n \to \infty$$

But on the interval $[0, 1]$ the convergence is not uniform

$$\sup_{x\in[0,1]} |x^n - a(x)| = \sup_{x\in[0,1]} |x^n| = 1 \not\to 0 \quad \text{as} \quad n \to \infty \qquad \blacksquare$$

- Notice that the limit function $a(x)$ is continuous on any interval $[0, r]$ but not continuous on $[0, 1]$ where uniform convergence fails.



Exercise: Discuss the convergence, uniform or otherwise, of $a_n(z) = |z|^n$ on the square $0 \le \text{Re}\ z \le 1$, $0 \le \text{Im}\ z \le 1$. $\blacksquare$

Weierstrass M-Test

Theorem 38 (Weierstrass M-Test)

Suppose (i) $|a_n(z)| \leq M_n$ for all $z \in S$ and (ii) $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} a_n(z)$ converges uniformly and absolutely on S.

Proof: The absolute (pointwise) convergence of $S(z) = \sum_{n=1}^{\infty} a_n(z)$ follows by comparison with the series $T = \sum_{n=1}^{\infty} M_n$. For $m > n \geq 1$, consider the partial sums

$$S_n(z) = \sum_{k=1}^{n} a_k(z), \qquad T_n = \sum_{k=1}^{n} M_k$$

$$\begin{aligned} |S_m(z) - S_n(z)| &= \left| \sum_{k=n+1}^{m} a_k(z) \right| \leq \sum_{k=n+1}^{m} |a_k(z)| \\ &\leq \sum_{k=n+1}^{m} M_k = |T_m - T_n| \end{aligned}$$

Letting $m \to \infty$ and taking the supremum (maximum), we find

$$\sup_{z \in S} |S(z) - S_n(z)| \leq |T - T_n| < \epsilon \quad \text{for} \quad n > N(\epsilon)$$

where $N(\epsilon)$ is independent of $z \in S$ and uniform convergence follows. ■

Examples of Uniform Convergence

Example: The Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly and absolutely for Re $z \geq p > 1$:

$$
\begin{aligned}
|a_n(z)| &= |n^{-z}| = \left| e^{-(x+iy)\,\mathrm{Log}\, n} \right| = e^{-x\,\mathrm{Log}\, n} \\
&\leq e^{-p\,\mathrm{Log}\, n} = n^{-p} = M_n
\end{aligned}
$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, the result follows from the Weierstrass M-test. ■

Example: Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly and absolutely on $|z| \leq 1$.

Solution: For $|z| \leq 1$, we apply the Weierstrass M-test with

$$
|a_n(z)| = \frac{|z|^n}{n^2} \leq \frac{1}{n^2} = M_n
$$

Since the harmonic series $\sum_{n=1}^{\infty} n^{-p}$ with $p = 2 > 1$ converges, the series converges uniformly and absolutely on $|z| \leq 1$. ■

More Examples of Uniform Convergence

Example: Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges uniformly and absolutely on $|z| \le r$ for any $r < 1$.

Solution: For $|z| \le r < 1$, we apply the Weierstrass M-test with

$$|a_n(z)| = \frac{|z|^n}{n} \le \frac{r^n}{n} \le r^n = M_n$$

Since the geometric series $\sum_{n=1}^{\infty} r^n$ with $r < 1$ converges, the series converges uniformly and absolutely on $|z| \le r < 1$. ■

- In fact, using Taylor series, we will see later that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\operatorname{Log}(1-z), \qquad |z| < 1$$

Notice that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ diverges at $z = 1$ but converges on $|z| < 1$. However, this convergence is not uniform on $|z| < 1$. The convergence is only uniform on the closed sub-disks $|z| \le r < 1$. ■

Abel's Continuity Theorem

Theorem 39 (Abel's Continuity Theorem)
Suppose the power (Taylor) series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

has the disk of convergence $|z-a| < R$ *and converges at* $z = w$ *on the circle of convergence* $|z-a| = R$. *Then*

$$\boxed{f(w) = \lim_{z \to w} f(z) = \sum_{n=0}^{\infty} a_n (w-a)^n}$$

where the limit is taken from inside the circle of convergence.

Proof: See text. ■

Example: From the Taylor series

$$\mathrm{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}, \qquad |z| < R = 1$$

where the power series is conditionally convergent at $z = 1$. Hence, by Abel's theorem,

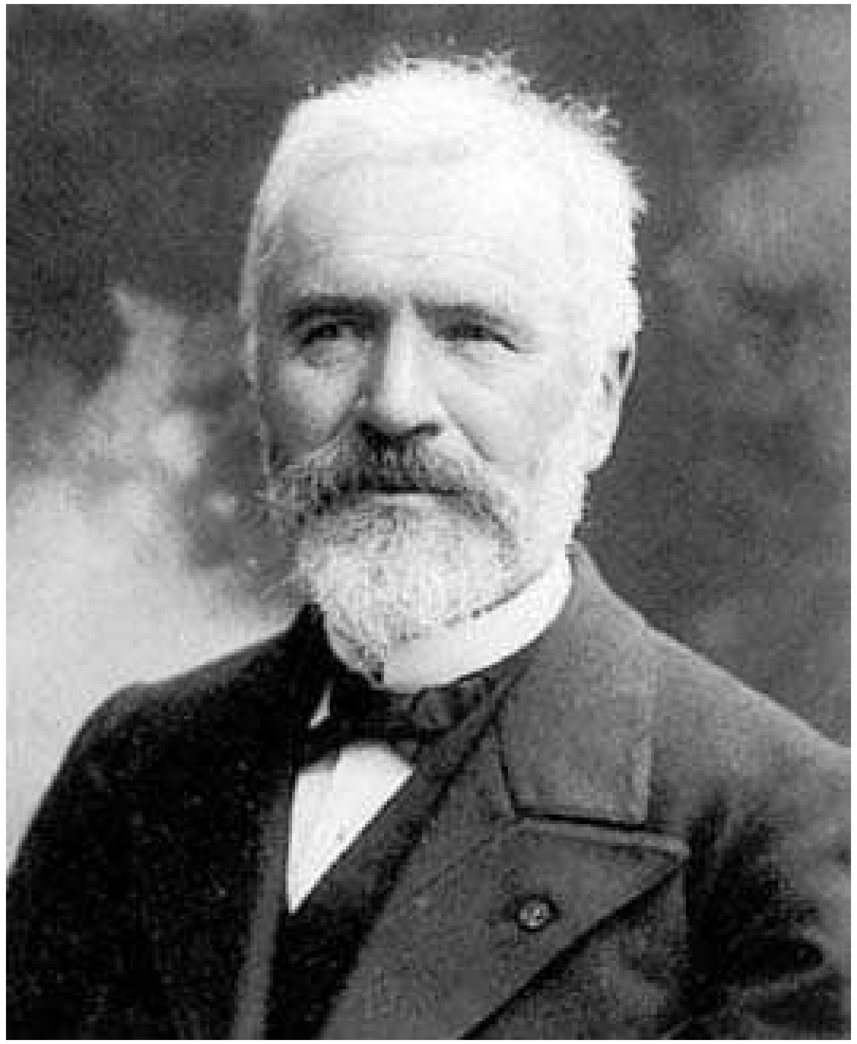
$$\begin{aligned} \mathrm{Log}\,2 &= \lim_{\substack{z \to 1 \\ |z|<1}} \mathrm{Log}(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right) \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \end{aligned}$$

■

Week 5: Line and Contour Integrals

13. Line and contour integrals, paths and curves
14. Properties of line integrals, path dependence
15. Cauchy-Goursat theorem and applications

Marie Ennemonde Camille Jordan (1838–1922) Edouard Jean-Baptiste Goursat (1858–1936)



Complex Integrals

Definition: Consider a complex-valued function of a real variable t

$$z: \ [a,b] \to \mathbb{C} \qquad z(t) = x(t) + iy(t)$$

where $x(t)$, $y(t)$ are continuous on $[a,b]$. Then we define the complex integral

$$\boxed{\int_a^b z(t)dt := \int_a^b (x(t) + iy(t))dt = \int_a^b x(t)dt + i\int_a^b y(t)dt}$$

where the real integrals giving the real and imaginary parts

$$\text{Re} \int_a^b z(t)dt = \int_a^b \text{Re } z(t)dt, \qquad \text{Im} \int_a^b z(t)dt = \int_a^b \text{Im } z(t)dt$$

exist by continuity as the limits of Riemann sums where $\{t_j\}$ is a partition of $[a,b]$ with $t_0 = a$, $t_n = b$ and $\Delta t_j = t_j - t_{j-1}$

$$\boxed{\int_a^b x(t)dt := \lim_{\substack{n\to\infty \\ \Delta t_j \to 0}} \sum_{j=1}^n x(t_j)\,\Delta t_j}$$

Complex Primitives

Theorem 40 (Complex Primitives)

If the complex-valued function $z(t)$ is continuous on $[a,b]$ and has a primitive $Z(t)$ such that $Z'(t) = z(t)$ then

$$\boxed{\int_a^b z(t)dt = [Z(t)]_a^b = Z(b) - Z(a)}$$

Example: Evaluate the integral $\int_0^1 (t^2 + it)\, dt$

$$\int_0^1 (t^2 + it)\, dt = \left[\frac{t^3}{3} + i\,\frac{t^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{2}\,i \qquad \blacksquare$$

Example: Evaluate the integral $\int_0^{2\pi} e^{it}\, dt$

$$\int_0^{2\pi} e^{it}\, dt = \int_0^{2\pi} (\cos t + i \sin t)\, dt = \Big[\sin t - i \cos t\Big]_0^{2\pi} = 0$$

Or better

$$\int_0^{2\pi} e^{it}\, dt = \left[\frac{1}{i}\, e^{it}\right]_0^{2\pi} = \frac{1}{i}(e^{2\pi i} - 1) = 0 \qquad \blacksquare$$

Primitives

Definition: If $f(z)$ and $F(z)$ are analytic in an open connected domain D and $F'(z) = f(z)$ then $F(z)$ is called a *primitive* (*antiderivative* or *indefinite integral*) of $f(z)$ and denoted

$$\boxed{F(z) = \int f(z)\, dz + \text{constant}}$$

Theorem 41 (Primitives) *If $F(z)$, $G(z)$ are both primitives of $f(z)$ in a connected open domain D so that $F'(z) = G'(z) = f(z)$ everywhere in D, then $F(z) - G(z)$ is constant in D.*

Proof: Let $H(z) = F(z) - G(z) = u + iv$. Since $H'(z) = 0$ it follows from the Cauchy-Riemann equations that in D

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Integrating gives

$$u = \text{const}, \quad v = \text{const} \quad \Rightarrow \quad H = u + iv = \text{const in } D \qquad \blacksquare$$

- Note that $f'(z) = 0$ in D implies $f(z)$ is constant in D.
- Note also that *connectedness* is essential since

$$f(z) = \begin{cases} 1, & |z| < 1 \\ 0, & |z| > 1 \end{cases}$$

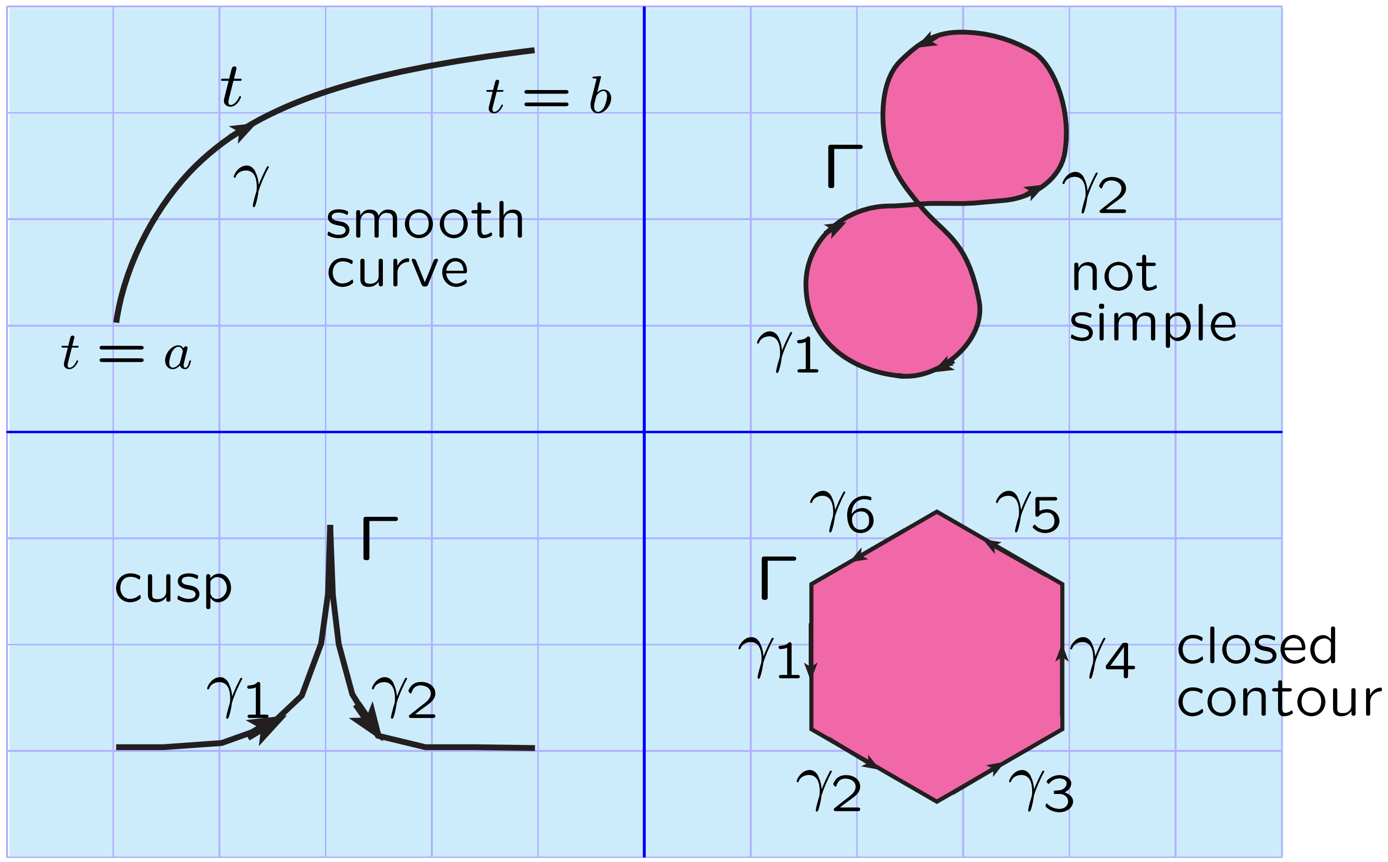
is analytic and $f'(z) = 0$ on its domain of definition D but f is not constant on the disconnected domain D.

Smooth Curves

We want to define contour integrals along *curves or contours* which generalize straight line segments and arcs of circles. To do this we define the notion of *smooth curves*.

Definition: A *smooth curve* γ is a function $z : [a, b] \subset \mathbb{R} \to \mathbb{C}$ such that $z(t) = x(t) + iy(t)$ where $x(t)$, $y(t)$ are C^1 and hence differentiable. The curve is *simple* (non-self-intersecting) if $z(t_1) \neq z(t_2)$ for $a \leq t_1 < t_2 < b$ and is *closed* if $z(a) = z(b)$. A smooth curve is *directed or oriented* with initial point $z(a)$ and endpoint $z(b)$.

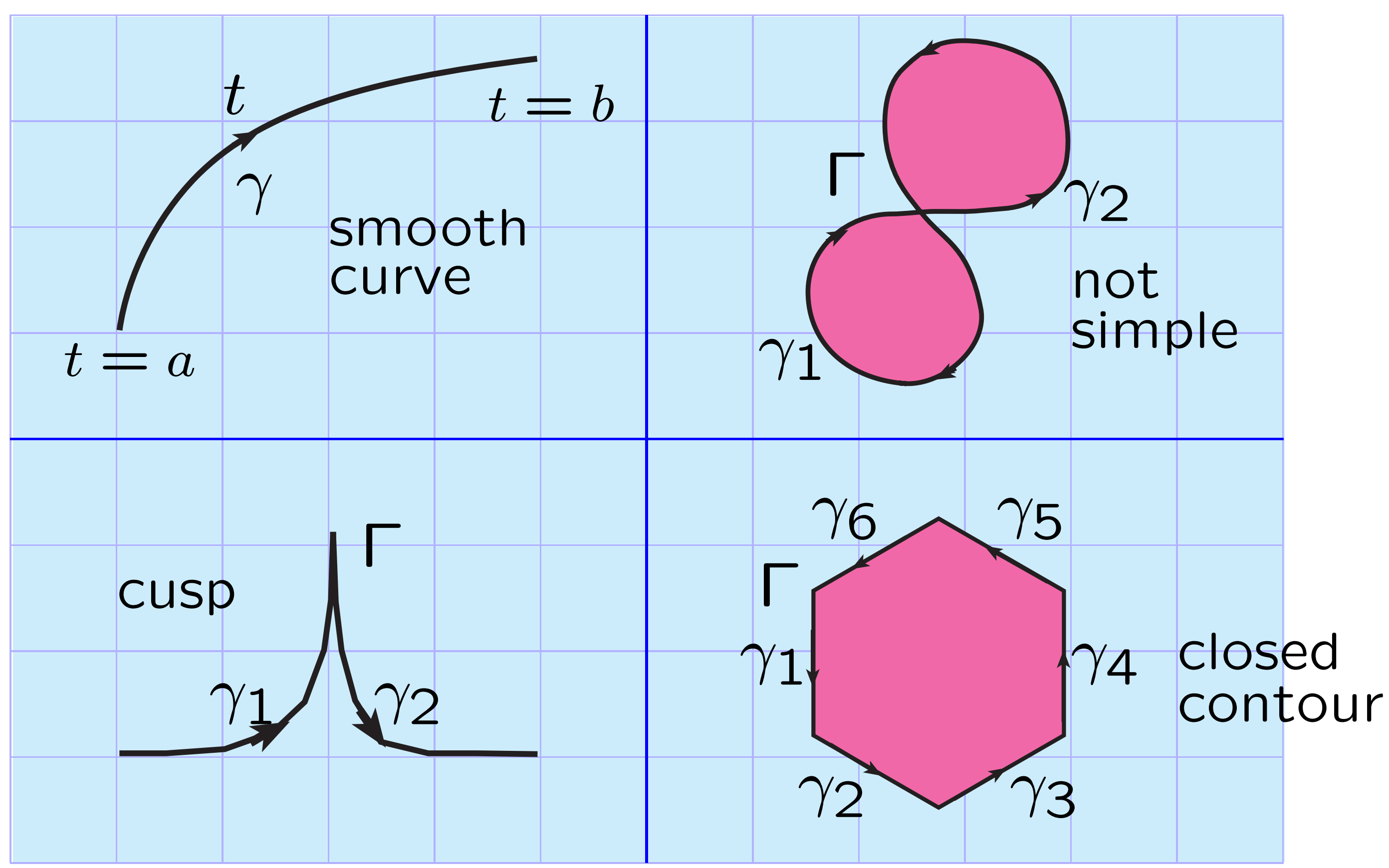
Definition: A *contour* $\Gamma = \sum_{j=1}^{n} \gamma_j$ consists of a finite sequence of smooth curves γ_j, $j = 1, 2, \ldots, n$ with endpoints joined to initial points. A contour is thus piecewise smooth (differentiable) and allows for self-intersections, corners and cusps (corners with common tangents).



Jordan Theorem

Theorem 42 (Jordan Theorem) *A simple closed curve Γ divides the plane $\mathbb{C}$ into two open regions having the curve as their common boundary. The bounded region is the interior of Γ and the unbounded region the exterior. The curve Γ is positively oriented if the interior always lies to the left.*

Proof: Very difficult — see textbook.



Contour Integrals

Definition: Let $f(z) = u(x,y) + iv(x,y)$ be continuous in an open region containing the smooth curve γ. Then we define the *contour integral* of $f(z)$ along γ by

$$\int_\gamma f(z)dz := \int_a^b f(z(t))\,\frac{dz}{dt}\,dt = \int_a^b (u+iv)(x'(t)+iy'(t))dt$$

$$= \int_a^b \Big[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\Big]dt$$

$$+\, i\int_a^b \Big[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\Big]dt$$

The real Riemann integrals exist because the integrands are continuous. In terms of line integrals in vector analysis

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$$

A contour integral over $\Gamma = \sum_{j=1}^n \gamma_j$ is defined by

$$\boxed{\int_\Gamma f(z)dz := \sum_{j=1}^n \int_{\gamma_j} f(z)dz}$$

A contour integral over a simple closed contour in the positive (counter-clockwise) sense is denoted

$$\boxed{\oint_\Gamma f(z)dz = \text{closed contour integral}}$$

Properties of Line Integrals

Theorem 43 (Properties of Line Integrals)
If A, B are constants, $f(z)$ and $g(z)$ are continuous in an open region containing Γ, and $-\Gamma$ denotes the contour Γ traversed in the opposite direction then

1. **Linear:** $$\int_\Gamma \Big(Af(z) + Bg(z)\Big)dz = A\int_\Gamma f(z)dz + B\int_\Gamma g(z)dz$$
2. **Orientation:** $$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$$
3. **Additivity:** $$\int_{\Gamma_1+\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$
4. **Bound:** $$\left|\int_\gamma f(z)dz\right| \leq \int_a^b |f(z)||z'(t)|dt, \quad \gamma\colon z = z(t), \ t \in [a,b]$$
5. **Change of Variables:** $$\int_\Gamma f(z)dz = \int_{\Gamma'} f(g(w))g'(w)dw$$

where $z = g(w)$, Γ is the image of Γ' under the change of variables and $g(w)$ is analytic in a region containing Γ'.

Example Line Integrals

Example: Evaluate the line integral $\int_\gamma z dz$ where γ is the straight line segment from 0 to $1+i$:

Solution: We parametrize the curve by $z = x + iy = (1+i)t = t + it$ so that $x(t) = t$ and $y(t) = t$ with $0 \le t \le 1$. Hence $x'(t) = 1$, $y'(t) = 1$, $z'(t) = 1+i$ and

$$\begin{aligned}\int_\gamma z dz &= \int_0^1 z \frac{dz}{dt} dt = \int_0^1 (1+i)t(1+i)dt \\ &= 2i \int_0^1 t dt = 2i \left[\frac{t^2}{2}\right]_0^1 = i \qquad \blacksquare\end{aligned}$$

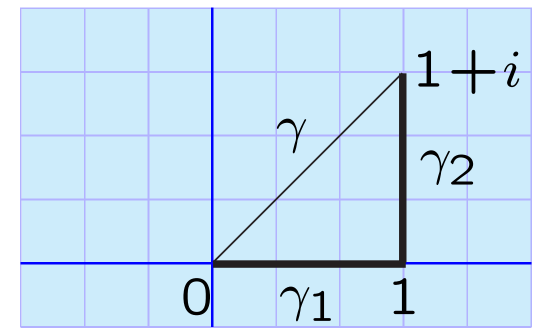
Example: Evaluate the closed contour integral $\oint_\Gamma \frac{dz}{z}$ where Γ is the circle $|z| = 1$ traversed twice in the counter-clockwise direction:

Solution: We parametrize the circle $|z| = 1$ as $z(t) = e^{it}$ with $t \in [0, 2\pi]$ so that $z(0) = z(2\pi) = 1$ and $z'(t) = ie^{it}$

$$\begin{aligned}\oint_\Gamma \frac{dz}{z} &= \oint_{|z|=1} \frac{dz}{z} + \oint_{|z|=1} \frac{dz}{z} = 2 \oint_{|z|=1} \frac{dz}{z} \\ &= 2 \int_0^{2\pi} z^{-1} \frac{dz}{dt} dt = 2 \int_0^{2\pi} e^{-it}(ie^{it})dt \\ &= 2i \int_0^{2\pi} dt = 4\pi i \qquad \blacksquare\end{aligned}$$

Path Dependence

Example: Evaluate the line integral $\int_\Gamma \overline{z} dz$ along the curve Γ where (a) $\Gamma = \gamma$ is the straight line segment from $z = 0$ to $z = 1 + i$, (b) $\Gamma = \gamma_1 + \gamma_2$ is the sum of two straight line segments from 0 to 1 and 1 to $1 + i$ and (c) where $\Gamma = \gamma_1 + \gamma_2 - \gamma$ is a closed contour.



(a) We parametrize the straight line segment by $z = (1 + i)t$ with $z'(t) = 1 + i$, $\overline{z} = (1 - i)t$ and $t \in [0, 1]$

$$\int_\gamma \overline{z}\, dz = \int_0^1 \overline{z} \frac{dz}{dt} dt = \int_0^1 (1 - i)t(1 + i)dt = 2\int_0^1 tdt = 2\left[\frac{t^2}{2}\right]_0^1 = 1$$

(b) We parametrize (i) $\gamma_1 : \ z = t$ with $z'(t) = 1$ and $t \in [0, 1]$ and (ii) $\gamma_2 : \ z = 1 + it$ with $z'(t) = i$ and $t \in [0, 1]$

$$\int_{\gamma_1} \overline{z}\, dz = \int_0^1 \overline{z} \frac{dz}{dt} dt = \int_0^1 tdt = \left[\frac{t^2}{2}\right]_0^1 = \tfrac{1}{2}$$

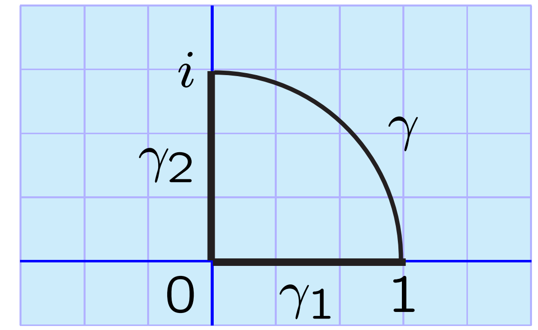
$$\int_{\gamma_2} \overline{z}\, dz = \int_0^1 \overline{z} \frac{dz}{dt} dt = \int_0^1 (1 - it)\, i\, dt = \int_0^1 (t + i)dt = \tfrac{1}{2} + i$$

$$\int_{\gamma_1+\gamma_2} \overline{z}\, dz = \int_{\gamma_1} + \int_{\gamma_2} \overline{z}\, dz = \tfrac{1}{2} + (\tfrac{1}{2} + i) = 1 + i$$

(c) By additivity, the closed contour integral is

$$\oint_\Gamma \overline{z}\, dz = \int_{\gamma_1} + \int_{\gamma_2} - \int_\gamma \overline{z}\, dz = \tfrac{1}{2} + (\tfrac{1}{2} + i) - 1 = i \qquad \blacksquare$$

Path Independence



Example: Evaluate the line integral $\int_\Gamma z^2 dz$ along the curve Γ where (a) $\Gamma = \gamma$ is the arc of the unit circle from $z = 1$ to $z = i$, (b) $\Gamma = \gamma_1 + \gamma_2$ is the sum of two straight line segments from 1 to 0 and 0 to i and (c) where $\Gamma = \gamma - \gamma_1 - \gamma_2$ is a closed contour.

(a) Parametrize the quarter circle $|z| = 1$ by $z = e^{it}$ with $z'(t) = ie^{it}$ and $t \in [0, \pi/2]$

$$\int_\gamma z^2\, dz = \int_0^{\pi/2} z^2 \frac{dz}{dt}\, dt = \int_0^{\pi/2} e^{2it} i e^{it} dt = i \int_0^{\pi/2} e^{3it} dt$$

$$= i\left[\frac{1}{3i} e^{3it}\right]_0^{\pi/2} = \frac{1}{3}(e^{3\pi i/2} - 1) = -\frac{1}{3}(1+i)$$

(b) We parametrize (i) $-\gamma_1 : \;\; z = t$ with $z'(t) = 1$ and $t \in [0, 1]$ and (ii) $\gamma_2 : \;\; z = it$ with $z'(t) = i$ and $t \in [0, 1]$

$$-\int_{\gamma_1} z^2\, dz = \int_{-\gamma_1} z^2\, dz = \int_0^1 z^2 \frac{dz}{dt}\, dt = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_{\gamma_2} z^2\, dz = \int_0^1 z^2 \frac{dz}{dt}\, dt = \int_0^1 (it)^2\, i\, dt = -i \int_0^1 t^2 dt = -\frac{i}{3}$$

$$\int_{\gamma_1+\gamma_2} z^2\, dz = \int_{\gamma_2} - \int_{-\gamma_1} z^2\, dz = -\frac{i}{3} - \frac{1}{3} = -\frac{1}{3}(1+i)$$

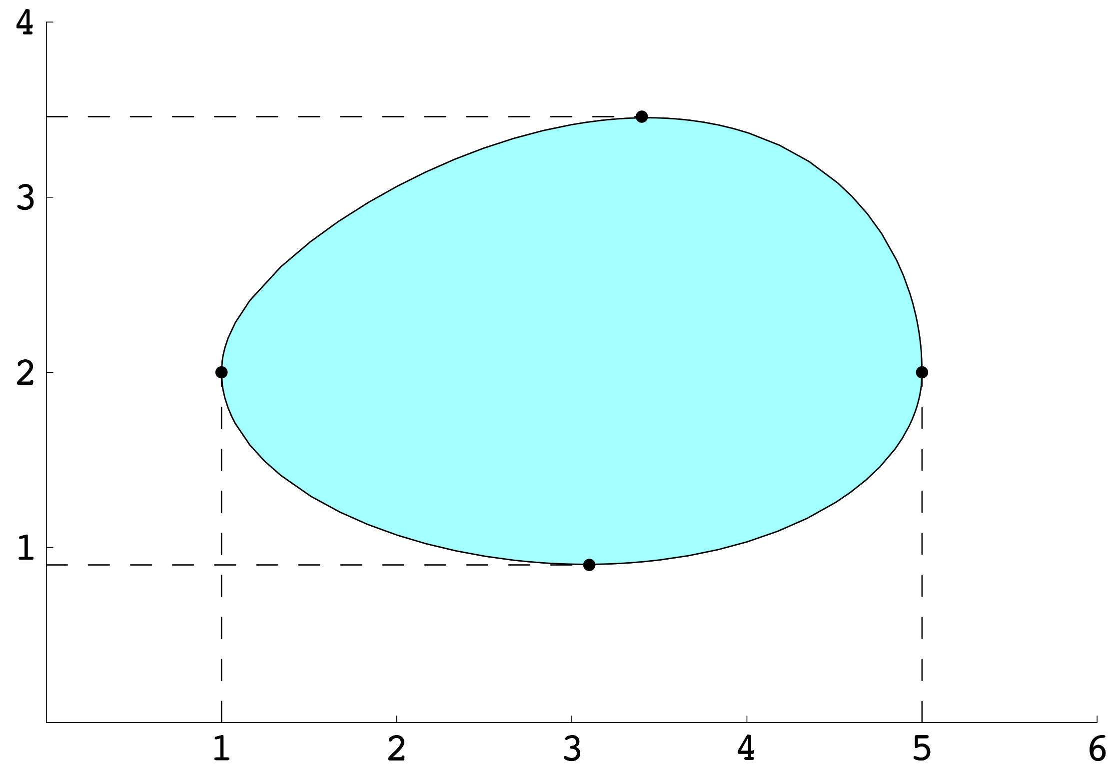
(c) By additivity, the closed contour integral is

$$\oint_\Gamma z^2\, dz = \int_\gamma - \int_{\gamma_1} - \int_{\gamma_2} z^2\, dz = -\frac{1}{3}(1+i) + \frac{1}{3}(1+i) = 0 \qquad \blacksquare$$

Green's Theorem

Theorem 44 (Green's Theorem) *If R is a closed region of the xy plane bounded by a simple closed curve Γ and if M and N are C^1 in R then*

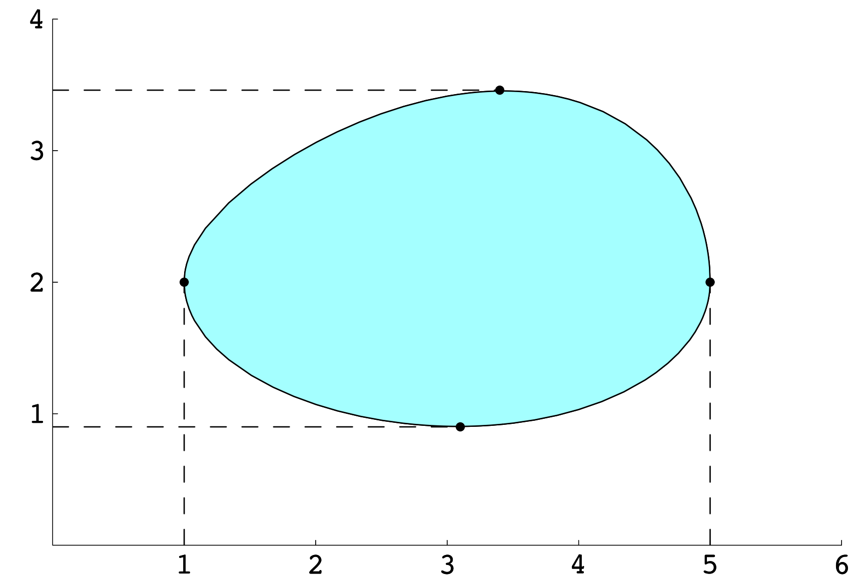
$$\boxed{\oint_{\Gamma=\partial R} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$



- Generally, by Green's theorem, if Γ is a simple closed curve

$$\oint_\Gamma \overline{z}\,dz = \oint_\Gamma x\,dx + y\,dy + i\oint_\Gamma x\,dy - y\,dx = 2i\{\text{area enclosed by } \Gamma\}$$

Proof of Green's Theorem



Proof: Assume first that R is convex (so that any straight lines parallel to coordinate axes cut R in at most two points) with lower (AEB), upper (AFB), left (EAF) and right (EBF) curves $y = Y_1(x)$, $y = Y_2(x)$, $x = X_1(y)$ and $x = X_2(y)$.

$$\begin{aligned}
\oint_\Gamma M\,dx &= \int_a^b M(x, Y_1(x))dx + \int_b^a M(x, Y_2(x))dx \\
&= -\int_a^b \left[M(x, Y_2) - M(x, Y_1)\right] dx \\
&= -\int_a^b \left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\partial M(x,y)}{\partial y}\,dy\right] dx = -\iint_R \frac{\partial M}{\partial y}\,dxdy \\
\oint_\Gamma N\,dy &= \int_e^f N(X_2, y)dy + \int_f^e N(X_1, y)dy \\
&= \int_e^f \left[\int_{x=X_1(y)}^{x=X_2(y)} \frac{\partial N(x,y)}{\partial x}\,dx\right] dy = \iint_R \frac{\partial N}{\partial x}\,dxdy
\end{aligned}$$

Adding gives the required result. If R is not convex (or is multi-connected), we can subdivide R into two or more convex regions by cuts and use additivity. ■

Cauchy's Theorem

Theorem 45 (Cauchy's Theorem) *If $f(z)$ is analytic in a simply-connected open domain D and $f'(z)$ is continuous in D then for any simple closed curve Γ in D*

$$\boxed{\oint_\Gamma f(z)dz = 0}$$

Proof: Let R be the union of Γ and its interior so that $\Gamma = \partial R$. The result then follows from Green's theorem (which requires Γ simple and $f'(z)$ continuous)

$$\begin{aligned}\oint_\Gamma f(z)dz &= \oint_\Gamma (u+iv)(dx+idy) = \oint_\Gamma udx - vdy + i\oint_\Gamma vdx + udy \\ &= \iint\limits_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \iint\limits_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0\end{aligned}$$

since by Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \blacksquare$$

Theorem 46 (Cauchy-Goursat Theorem)
If $f(z)$ is analytic in a simply-connected open domain D then for any closed contour Γ in D

$$\boxed{\oint_\Gamma f(z)dz = 0}$$

Proof: Difficult — see text. Removes requirements that Γ is *simple* and $f'(z)$ is *continuous* in R. Note $\Gamma = \partial R$ must be *positively* oriented so that R is always on the left. Also the region R need not be simply-connected provided *all* of the boundary of R is included in $\Gamma = \partial R$. $\blacksquare$

Examples of Cauchy's Theorem

Example: Verify Cauchy's theorem for $\oint_\Gamma \cos z\,dz$ where the closed contour traverses counter-clockwise the square with vertices at $z = 0, 1, 1+i, i$.

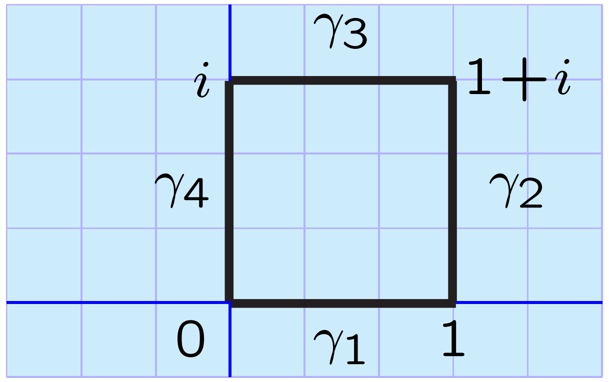
Solution: Since $f(z)$ is entire, all closed contour integrals must vanish. We parametrize the four edges by (i) γ_1: $z = t$, $t \in [0,1]$, (ii) γ_2: $z = 1+it$, $t \in [0,1]$, (iii) $-\gamma_3$: $z = i+t$, $t \in [0,1]$, (iv) $-\gamma_4$: $z = it$, $t \in [0,1]$:

$$\int_{\gamma_1} \cos z\,dz = \int_0^1 \cos t\,dt = \Big[\sin t\Big]_0^1 = \sin 1$$

$$\int_{\gamma_2} \cos z\,dz = i\int_0^1 \cos(1+it)dt = \Big[\sin(1+it)\Big]_0^1 = \sin(1+i) - \sin 1$$

$$\int_{-\gamma_3} \cos z\,dz = \int_0^1 \cos(i+t)\,dt = \Big[\sin(i+t)\Big]_0^1 = \sin(1+i) - \sin i$$

$$\int_{-\gamma_4} \cos z\,dz = i\int_0^1 \cos(it)\,dt = \Big[\sin(it)\Big]_0^1 = \sin i$$



$$\oint_\Gamma \cos z\,dz = \int_{\gamma_1} + \int_{\gamma_2} - \int_{-\gamma_3} - \int_{-\gamma_4} \cos z\,dz$$

$$= \sin 1 + [\sin(1+i) - \sin 1] - [\sin(1+i) - \sin i] - \sin i = 0 \qquad \blacksquare$$

Example: Evaluate the contour integral $\displaystyle\oint_{|z|=2} \frac{e^z\,dz}{(z^2-9)}$:

The integrand is analytic except at the poles $z = \pm 3$, so it is analytic in and on the circle $|z| = 2$. The closed contour integral therefore vanishes by Cauchy's theorem. $\blacksquare$

Conservative Vector Fields

Theorem 47 (Conservative Vector Fields)
If $f(z) = u(x, y) + iv(x, y)$ is analytic in a simply-connected open domain D, then the vector fields $\boldsymbol{A}_1 = (u, -v) = \nabla\varphi_1$ and $\boldsymbol{A}_2 = (v, u) = \nabla\varphi_2$ are conservative in $D \subset \mathbb{R}^2$ and $\int f(z)\,dz$ is independent of path.

Proof: Recall that, in vector analysis, a C^1 vector field $\boldsymbol{A}$ on a simply-connected open domain D is conservative and can be written as $\boldsymbol{A} = \nabla\varphi$ if and only its curl, $\nabla \times \boldsymbol{A}$, vanishes. So let $\boldsymbol{A}_1 = (u, -v, 0)$, $\boldsymbol{A}_2 = (v, u, 0)$ and use the Cauchy-Riemann equations to show that in $\mathbb{R}^3$

$$\nabla \times \boldsymbol{A}_1 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & -v & 0 \end{vmatrix} = -\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\boldsymbol{k} = \boldsymbol{0}$$

$$\nabla \times \boldsymbol{A}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v & u & 0 \end{vmatrix} = \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)\boldsymbol{k} = \boldsymbol{0}$$

Then restrict to $\mathbb{R}^2$ since u, v are functions of x, y only. ■

- Note that if $d\boldsymbol{r} = (dx, dy, dz)$ then since $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are conservative

$$\begin{aligned} \int f(z)dz &= \int (u + iv)(dx + idy) = \int u\,dx - v\,dy + i\int v\,dx + u\,dy \\ &= \int (u, -v, 0)\cdot d\boldsymbol{r} + i\int (v, u, 0)\cdot d\boldsymbol{r} = \int \boldsymbol{A}_1 \cdot d\boldsymbol{r} + i\int \boldsymbol{A}_2 \cdot d\boldsymbol{r} \\ &= \text{path independent integral} \end{aligned}$$

Week 6: Cauchy's Integral Formula

16. Fundamental theorem of calculus, path independence
17. Deformation of contours about simple poles
18. General Cauchy integral formula

Fundamental Theorem of Calculus

Theorem 48 (Fundamental Theorem of Calculus) *Suppose $f(z)$ is analytic in a simply-connected open domain D then*

$$\boxed{\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)}$$

is independent of the path in D joining z_1 and z_2. Here

$$\boxed{F(z) = \int_{z_0}^{z} f(w)dw, \qquad z_0 \in D}$$

is a primitive of $f(z)$ and is thus analytic in D

$$\boxed{F'(z) = f(z), \qquad z \in D}$$

Proof of Fundamental Theorem of Calculus

Proof: Suppose Γ_1 and Γ_2 are two contours joining z_1 and z_2 in D. Then $\Gamma = \Gamma_1 - \Gamma_2$ is a closed contour and path independence follows from Cauchy's theorem

$$0 = \oint_\Gamma f(z)dz = \int_{\Gamma_1 - \Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz$$

Since $f(z)$ must be analytic *everywhere* between Γ_1 and Γ_2, this region must be simply connected so that Γ_1 can be *continuously deformed* into Γ_2. By path independence, $F(z)$ is well-defined in D with derivative

$$\begin{aligned}
F'(z) &= \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\Big[\int_{z_0}^{z+\Delta z} - \int_{z_0}^{z} f(w)dw\Big] \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(w)dw \;=\; f(z)
\end{aligned}$$

This last limit follows, since $f(z)$ is continuous, by the following Lemma. ■

Estimating Integrals

Lemma 49 (Continuity Lemma) *If $f(z)$ is continuous in an open domain containing z and $z+\Delta z$ then*

$$\boxed{\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(w)dw \;=\; f(z)}$$

Proof: Since $f(z)$ is continuous we have

$$|f(w)-f(z)|<\epsilon \quad \text{whenever} \quad |w-z|<\delta(\epsilon)$$

Choose $|\Delta z|$ sufficiently small so that $|w-z|\le|\Delta z|<\delta(\epsilon)$ then

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(w)dw - f(z)\right| \;=\; \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(w)-f(z)]dw\right|$$

$$\le\; \frac{1}{|\Delta z|}\int_{t}^{t+\Delta t}\Big|f(w)-f(z)\Big|\left|\frac{dw}{dt}\right|dt \;\le\; \frac{\epsilon}{|\Delta z|}\int_{t}^{t+\Delta t}\left|\frac{dw}{dt}\right|dt \;=\; \frac{\epsilon}{|\Delta z|}\text{Length}(\Gamma) \;=\; \epsilon$$

Here we used the path independence of the contour integral to choose Γ to be the straight line segment between z and $z+\Delta z$ so that Length$(\Gamma)=|\Delta z|$. ■

Theorem 50 (Integral Estimate) *If $f(z)$ is continuous in an open domain containing Γ and $|f(z)|\le M$ on Γ, then*

$$\boxed{\left|\int_{\Gamma} f(z)dz\right| \le \int_{t_1}^{t_2}\Big|f(z)\Big|\left|\frac{dz}{dt}\right|dt \le M\,\text{Length}(\Gamma)}$$

Proof: The proof of this useful estimate is left as an exercise. ■

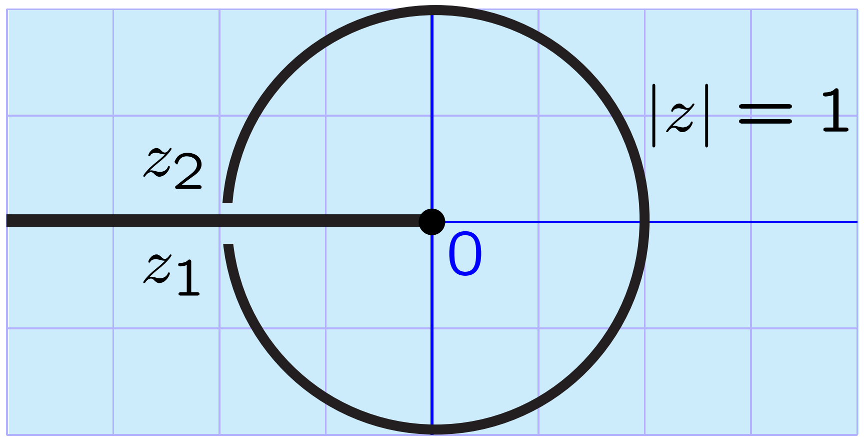
Examples of Fundamental Theorem

Example: Evaluate the line integrals $\int_{z_1}^{z_2} z^n dz$, $n \in \mathbb{Z}$ around the unit circle from $z_1 = e^{-i(\pi-\epsilon)}$ to $z_2 = e^{i(\pi-\epsilon)}$, $0 < \epsilon < \pi$ and hence obtain $\oint_{|z|=1} z^n dz$:

For $n \geq 0$, $f(z) = z^n$ is entire whereas, for $n \leq -1$, $f(z) = z^n$ is analytic in the punctured plane $\mathbb{C}\backslash\{0\}$. Since this is not simply-connected we introduce a cut and work in $\mathbb{C}\backslash(-\infty, 0]$:

$$\int_{z_1}^{z_2} z^n dz = \begin{cases} \left[\frac{z^{n+1}}{n+1}\right]_{z_1}^{z_2}, & n \neq -1 \\ \left[\operatorname{Log} z\right]_{z_1}^{z_2}, & n = -1 \end{cases}$$

$$\oint_{|z|=1} z^n dz = \lim_{\epsilon\to 0} \int_{z_1}^{z_2} z^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases} \qquad \blacksquare$$



Exercise: Verify this directly by evaluating the line integrals.

Example: Evaluate the closed contour integral $\oint_{|z+2|=1} \frac{dz}{z}$:

$F(z) = \operatorname{Log} z$ is *not a primitive* since it is not analytic at $z = -1, -3$ on the contour. However, the branch $\operatorname{Log}(-z)$ with $0 \leq \arg z < 2\pi$ is a primitive and is analytic in $\mathbb{C}\backslash[0, \infty)$. Now it follows as in the previous example that the closed contour integral vanishes

$$\oint_{|z+2|=1} \frac{dz}{z} = \lim_{z_2\to z_1} \left[\operatorname{Log}(-z)\right]_{z_1}^{z_2} = 0 \qquad \blacksquare$$

Trigonometric Integrals

Example: Show that

$$\boxed{\int_0^{2\pi} \cos^{2n} t\, dt = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}\, 2\pi, \qquad n = 1, 2, 3, \ldots}$$

Solution: Let $z = e^{it}$ so that $dz = ie^{it}dt = izdt$ and $dt = dz/iz$. Then $\cos t = \frac{1}{2}(z + z^{-1})$ and by the binomial theorem

$$\begin{aligned}
\int_0^{2\pi} \cos^{2n} t\, dt &= \oint_{|z|=1} \left[\tfrac{1}{2}(z+z^{-1})\right]^{2n} \frac{dz}{iz} = \frac{1}{2^{2n} i} \oint_{|z|=1} \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k} \frac{dz}{z} \\
&= \frac{1}{2^{2n}\, i} \binom{2n}{n} \oint_{|z|=1} \frac{dz}{z} = \frac{1}{2^{2n}\, i} \binom{2n}{n}\, 2\pi i \\
&= \frac{1}{2^{2n}} \frac{(2n)!}{n!\, n!} 2\pi = \frac{(2n)(2n-1)(2n-2)\cdots 1}{(2^n n!)(2^n n!)} 2\pi \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)(2n-2)\cdots 4\cdot 2} 2\pi
\end{aligned}$$

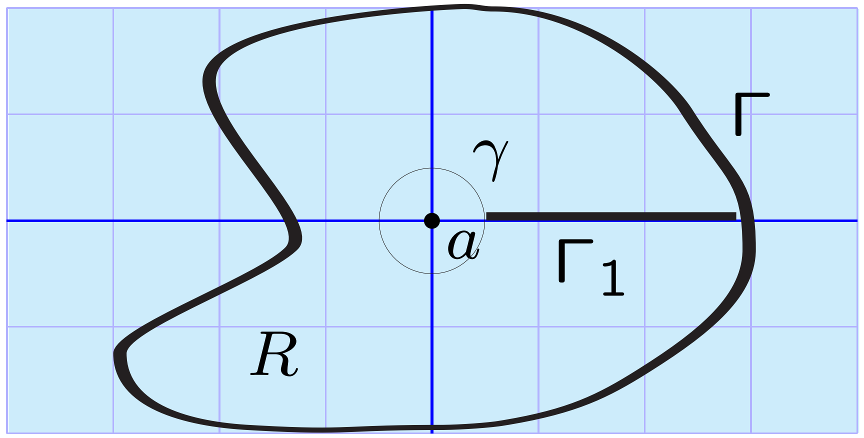
where we used the previous result

$$\oint_{|z|=1} z^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases} \qquad \blacksquare$$

Exercise: Establish the previous cosine integrals using the methods of real analysis, that is, integration by parts and recursion relations. $\blacksquare$

Closed Contours About Simple Poles

Theorem 51 (Closed Contours About Simple Poles)



If Γ is any closed contour not passing through $z = a$

$$\boxed{\oint_\Gamma \frac{dz}{z-a} = \begin{cases} 0, & a \text{ outside } \Gamma \\ 2\pi i, & a \text{ inside } \Gamma \end{cases}}$$

Proof: The integrand $f(z) = \dfrac{1}{z-a}$ is analytic in $\mathbb{C}\backslash\{a\}$. Hence if a lies outside Γ the integral vanishes by Cauchy's theorem. If a lies inside Γ, the contour can be *deformed* into a small circle γ of radius $\epsilon > 0$ centered on a with $z - a = \epsilon e^{it}$, $z'(t) = i\epsilon e^{it}$ and $0 \le t \le 2\pi$ so that

$$\oint_\Gamma \frac{dz}{z-a} = \oint_{|z-a|=\epsilon} \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\,\epsilon\, e^{it}}{\epsilon\, e^{it}}\, dt = i\int_0^{2\pi} dt = 2\pi i$$

Note that the analyticity domain of $f(z)$ between Γ and γ is not simply-connected but that introducing a cut Γ_1 gives a simply-connected domain R. Then Cauchy's theorem implies

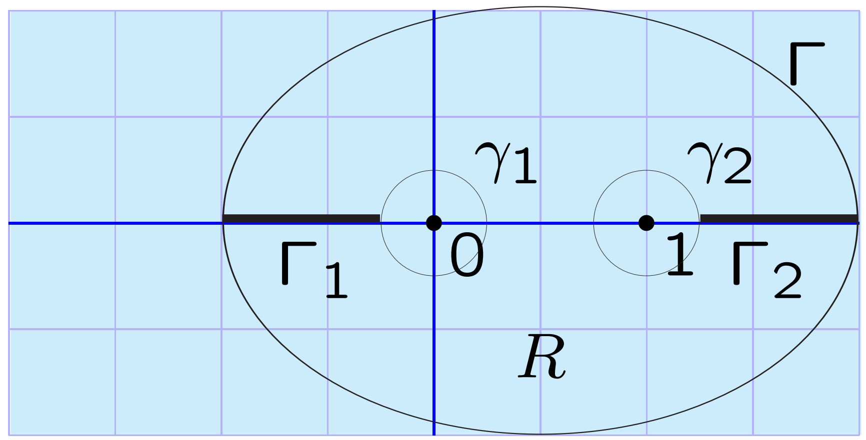
$$\oint_{\partial R} f(z)dz = \oint_\Gamma + \int_{-\Gamma_1} + \oint_{-\gamma} + \int_{\Gamma_1} f(z)dz = 0$$

and hence the *deformation* is valid

$$\boxed{\oint_\Gamma f(z)dz = \oint_\gamma f(z)dz} \qquad \blacksquare$$

Deforming Closed Contour Integrals

Example: Evaluate $\oint_\Gamma \frac{3z-2}{z^2-z}\,dz$ where the closed contour Γ is the ellipse $|z|+|z-1|=3$:



Solution: By partial fractions, $f(z)=\frac{3z-2}{z^2-z}=\frac{2}{z}+\frac{1}{z-1}$ is analytic in $\mathbb{C}\backslash\{0,1\}$.

We introduce small circles centered at $z=0,1$ and two cuts Γ_1,Γ_2 so that the region R between Γ and the circles is simply-connected. Cauchy's theorem then tells us that we can deform the contour

$$\oint_{\partial R} f(z)dz = \oint_\Gamma + \int_{\Gamma_1} + \oint_{-\gamma_1} + \int_{-\Gamma_1} + \int_{-\Gamma_2} + \oint_{-\gamma_2} + \int_{\Gamma_2} f(z)dz = 0$$

$$\Rightarrow \quad \oint_\Gamma f(z)dz \;=\; \oint_{\gamma_1} + \oint_{\gamma_2} f(z)dz = \oint_{\gamma_1} + \oint_{\gamma_2}\left[\frac{2}{z}+\frac{1}{z-1}\right]dz$$

$$= \; 2(2\pi i)+0+2\cdot 0+2\pi i = 6\pi i$$

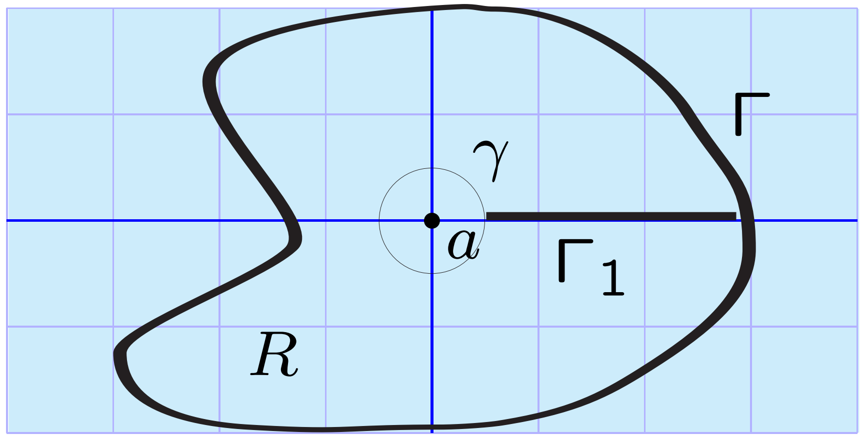
where each of the four integrals is evaluated using the previous theorem. ■

- This method easily generalizes to evaluate closed contour integrals of any rational function of z with only simple poles.

Cauchy Integral Formula

Theorem 52 (Cauchy Integral Formula) *Let Γ be a positively oriented simple closed contour. If $f(z)$ is analytic in a simply-connected open domain D containing Γ then*

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-a}\,dz, \qquad a \text{ inside } \Gamma}$$



Proof: The integrand is analytic in $D\backslash\{a\}$. We introduce a small circle γ of radius $\epsilon > 0$ centered on $z = a$ and a cut Γ_1 so that the region R between Γ and γ is simply-connected. We parametrize γ by $z - a = \epsilon e^{it}$ with $z'(t) = i\epsilon e^{it}$. By deforming the contour we find

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = \oint_\gamma \frac{f(z)}{z-a}\,dz$$

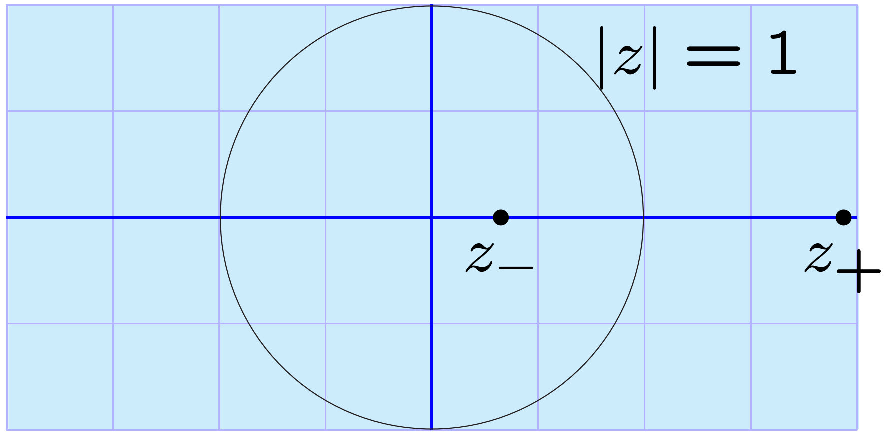
Since the LHS is independent of ϵ we can take $\epsilon \to 0$

$$\begin{aligned}\oint_\Gamma \frac{f(z)}{z-a}\,dz &= \lim_{\epsilon\to 0}\oint_\gamma \frac{f(z)}{z-a}\,dz = i\lim_{\epsilon\to 0}\int_0^{2\pi} f(a+\epsilon\, e^{it})dt \\ &= i\int_0^{2\pi}\lim_{\epsilon\to 0} f(a+\epsilon\, e^{it})dt = i\int_0^{2\pi} f(a)dt = 2\pi i f(a)\end{aligned}$$

where we have used the continuity of $f(z)$. ■

A Trigonometric Integral

Example: Evaluate the integral $I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta}$:



Let $z = e^{i\theta}$ so that $z'(\theta) = ie^{i\theta} = iz$ and $d\theta = -idz/z$. Then

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta} = -i \oint_{|z|=1} \frac{dz}{z(3 - z - z^{-1})} \\ &= i \oint_{|z|=1} \frac{dz}{z^2 - 3z + 1} = i \oint_{|z|=1} \frac{dz}{(z - z_+)(z - z_-)} \end{aligned}$$

where $z_\pm = (3 \pm \sqrt{5})/2 = 2.618.., 0.381..$
So setting

$$f(z) = (z - z_+)^{-1}$$

which is analytic inside $|z| = 1$, we have by the Cauchy integral formula

$$I = i \oint_{|z|=1} \frac{f(z)}{(z - z_-)} dz = (i)(2\pi i) f(z_-) = -\frac{2\pi}{z_- - z_+} = \frac{2\pi}{\sqrt{5}} \qquad \blacksquare$$

Exercise: Evaluate I using the methods of systematic integration for real integrals, that is, by using the substitution $t = \tan(\theta/2)$. $\blacksquare$

General Cauchy Integral Formula

Theorem 53 (General Cauchy Integral Formula) *Let Γ be a positively oriented simple closed contour. If $f(z)$ is analytic in a simply-connected open domain D containing Γ then*

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad a \text{ inside } \Gamma,\ n = 0, 1, 2, \ldots}$$

Proof: See text. This result is equivalent to

$$\boxed{\frac{d^n}{da^n} f(a) = \frac{d^n}{da^n}\left[\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-a}\, dz\right] = \frac{1}{2\pi i}\oint_\Gamma \frac{\partial^n}{\partial a^n}\frac{f(z)}{z-a}\, dz}$$

Example: Evaluate the integral $\displaystyle\oint_{|z|=4} \frac{e^z\, dz}{(z-2)^2}$:

Let $f(z) = e^z$ which is entire, then by the general Cauchy integral formula with $n = 1$

$$\oint_{|z|=4} \frac{e^z\, dz}{(z-2)^2} = 2\pi i f'(2) = 2\pi i e^2 \qquad \blacksquare$$

Corollary 54 (Analyticity of Derivatives) *If $f(z)$ is analytic in an open simply-connected domain D, then $f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$ are all analytic in D.*

Proof: Follows from the general Cauchy integral formula since, for a suitably chosen closed contour Γ, the derivatives exist and are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dw \qquad \blacksquare$$

Taylor's Theorem

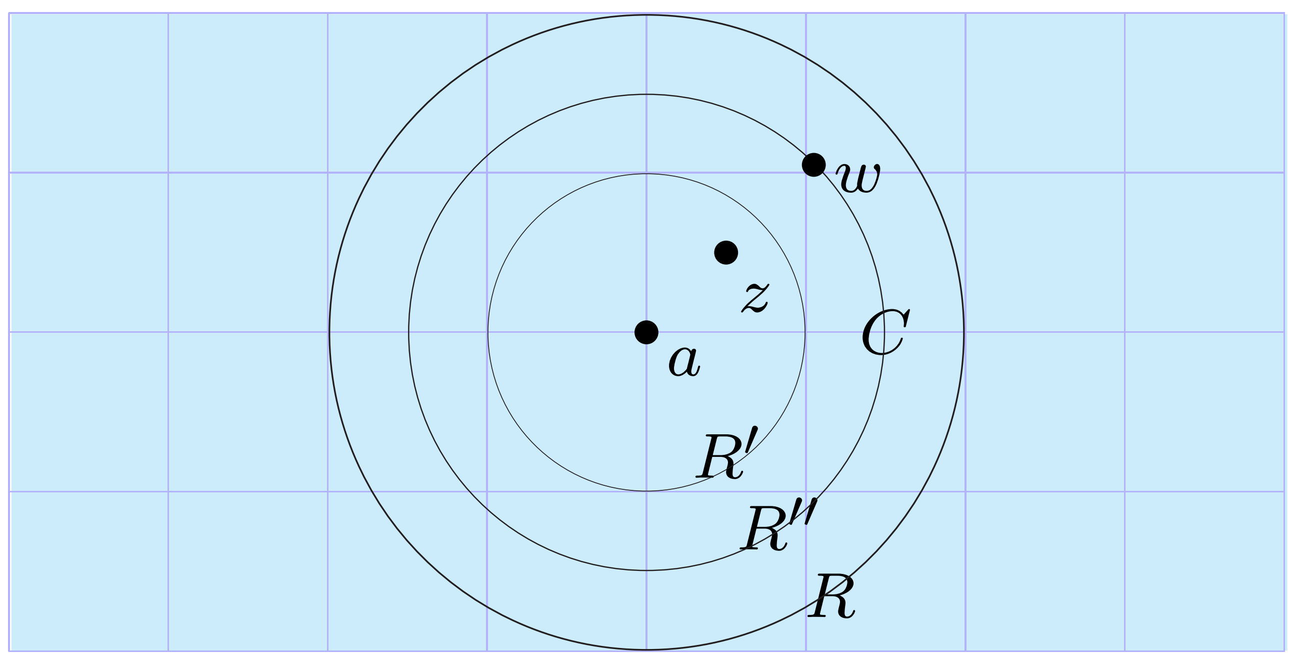
Theorem 55 (Taylor's Theorem)

If $f(z)$ is analytic in the disk $|z-a| < R$ then the Taylor series converges to $f(z)$ for all z in this disk. The convergence is uniform in any closed subdisk $|z-a| \leq R' < R$. Specifically,

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(z-a)^k + R_n(z)$$

where the remainder satisfies

$$\sup_{|z-a|\leq R'} |R_n(z)| \to 0 \quad \textit{as } n \to \infty$$



Proof of Taylor's Theorem

Proof: Let C be the circle $|z-a| = R''$ with $R' < R'' < R$. Then by Cauchy's integral formula with $n = 0, 1, 2, \ldots$ for z inside C

$$f(z) \;=\; \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw, \qquad \frac{f^{(n)}(a)}{n!} \;=\; \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)^{n+1}}\,dw$$

If we substitute

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\,\frac{1}{1-\frac{z-a}{w-a}} \\ &= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \cdots + \frac{(z-a)^n}{(w-a)^n} + \frac{\frac{(z-a)^{n+1}}{(w-a)^{n+1}}}{1-\frac{z-a}{w-a}}\right]\end{aligned}$$

into the first formula, integrate term-by-term and use the second formula we find

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(z-a)^k + R_n(z)$$

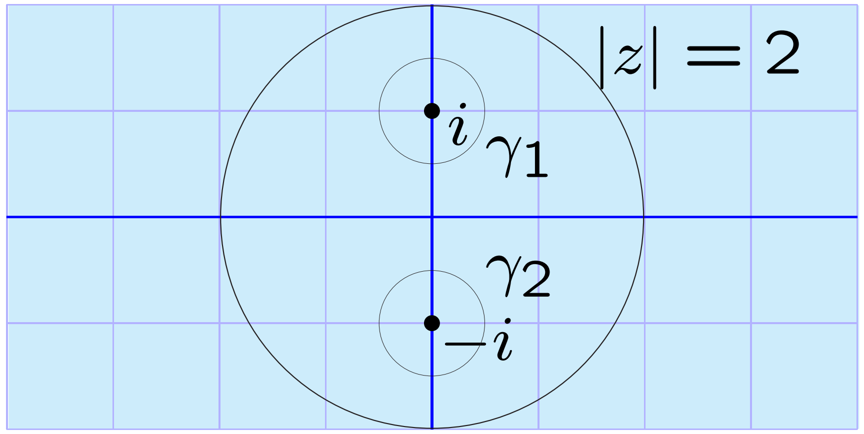
where the remainder $R_n(z)$ is uniformly bounded on $|z-a| \le R'$ by

$$\begin{aligned}|R_n(z)| \;&=\; \frac{1}{2\pi}\left|\int_C \frac{f(w)}{(w-z)}\,\frac{(z-a)^{n+1}}{(w-a)^{n+1}}\,dw\right| \\ &\le\; \frac{2\pi R''}{2\pi(R''-R')}\left(\frac{R'}{R''}\right)^{n+1} \max_{w\in C}|f(w)| \to 0 \quad \text{as} \quad n\to\infty \qquad \blacksquare\end{aligned}$$

Evaluation of Integrals

Example: Show that

$$\frac{1}{2\pi i}\oint_{|z|=2}\frac{e^{zt}dz}{z^2+1} \;=\; \sin t, \qquad t\in\mathbb{C}$$



Solution: The integrand is analytic except at the simple poles $z=\pm i$. We place circles γ_1 and γ_2 of radius ϵ around these poles so that by deformation of contours

$$\oint_{|z|=2}\frac{e^{zt}dz}{z^2+1} \;=\; \oint_{\gamma_1}+\oint_{\gamma_2}\frac{e^{zt}dz}{z^2+1} \;=\; \oint_{\gamma_1}\frac{f_1(z)dz}{z-i}+\oint_{\gamma_2}\frac{f_2(z)dz}{z+i}$$

where

$$f_1(z)=\frac{e^{zt}}{z+i}, \qquad f_2(z)=\frac{e^{zt}}{z-i}$$

Hence by the Cauchy integral formula

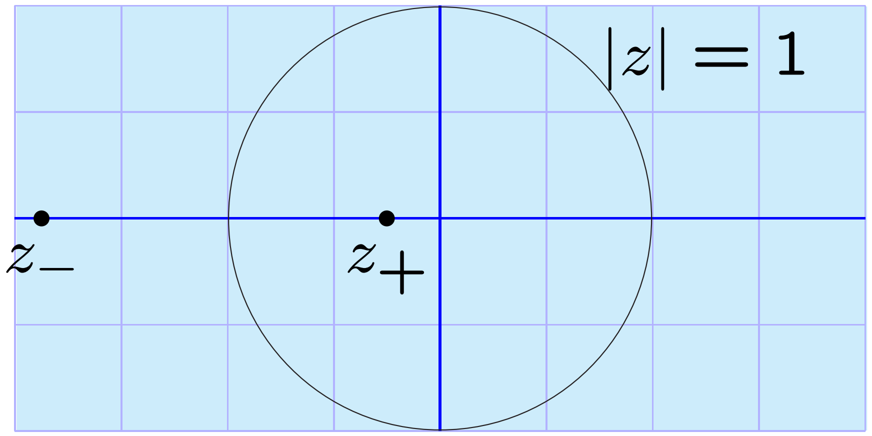
$$\frac{1}{2\pi i}\oint_{|z|=2}\frac{e^{zt}dz}{z^2+1} \;=\; f_1(i)+f_2(-i)=\frac{1}{2i}e^{it}-\frac{1}{2i}e^{-it} \;=\; \sin t \qquad \blacksquare$$

Another Trigonometric Integral

Example: Use the general Cauchy integral formula to evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^3}$$

Solution: Let $z = e^{i\theta}$ so that $z'(\theta) = ie^{i\theta} = iz$ and $d\theta = -idz/z$. Then



$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^3} = -i\oint_{|z|=1} \frac{8dz}{z(4+z+z^{-1})^3} \\ &= -i\oint_{|z|=1} \frac{8z^2\,dz}{(z^2+4z+1)^3} = -i\oint_{|z|=1} \frac{f(z)dz}{(z-z_+)^3} \end{aligned}$$

where $z_\pm = (-2 \pm \sqrt{3}) = -0.26.., -3.73..$ and

$$f(z) = \frac{8z^2}{(z-z_-)^3}$$

which is analytic inside $|z| = 1$. Note that

$$z_+ + z_- = -4, \qquad z_+ - z_- = 2\sqrt{3}, \qquad z_+z_- = 1$$

So, by the general Cauchy integral formula with $n = 2$

$$I = (-i)(2\pi i)f''(z_+)/2! = \pi\left[\frac{16[(z+z_-)^2+2zz_-]}{(z-z_-)^5}\right]_{z=z_+} = 16\pi\frac{16+2}{(2\sqrt{3})^5} = \frac{\pi}{\sqrt{3}} \qquad \blacksquare$$

Exercise: Evaluate I using the substitution $t = \tan(\theta/2)$. $\blacksquare$

Week 7: Singularities and Laurent Series

19. Isolated zeros and poles, removable and essential singularities
20. Laurent series
21. Residues

Charles Emile Picard (1856–1941)

Order m Zeros

Definition: A point $z = a$ is a *zero of order* m of $f(z)$ if $f(z)$ is analytic at $z = a$ and $f(z)$ and its first $m - 1$ derivatives vanish at $z = a$ but $f^{(m)}(a) \neq 0$. A zero of order 1 is a *simple zero*.

Lemma 56 (Order m Zero)
Let $f(z)$ be analytic at $z = a$. Then $f(z)$ has a zero of order m at $z = a$ if and only if

$$\boxed{f(z) = (z - a)^m g(z)}$$

where $g(z)$ is analytic at $z = a$ with $g(a) \neq 0$.

Proof: Using Taylor series

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n \quad \Rightarrow \quad f(z) = (z - a)^m g(z)$$

$$\text{with} \qquad g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(a)}{(n+m)!} (z - a)^n \quad \text{analytic at } z = a$$

Conversely,

$$g(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \quad \Rightarrow \quad f(z) = (z - a)^m \sum_{n=0}^{\infty} a_n (z - a)^n$$

and since $a_0 \neq 0$, $f(z)$ has a zero of order m at $z = a$. ■

Isolated Zeros

Theorem 57 (Isolated Zeros)

Zeros of an analytic function which is not identically zero are isolated, that is, if $z = a$ is a zero then there are no zeros in a punctured disk about $z = a$.

Proof: The Taylor series about $z = a$ with Taylor coefficients a_n converges to $f(z)$ on some open disk about $z = a$. If $a_n = 0$ for all n then $f(z) \equiv 0$. Otherwise, there is a smallest $m \geq 1$ such that $a_m \neq 0$ and $f(z) = (z - a)^m g(z)$ has an order m zero. Now $g(z)$ is analytic and therefore continuous at $z = a$ with $g(a) \neq 0$. Hence $g(z) \neq 0$ in an open disk about $z = a$ and the result follows. ■

Isolated Singularities

Definition: Let $f(z)$ have an isolated singularity at $z = a$ and Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \qquad 0 < |z-a| < R$$

(i) If $a_n = 0$ for all $n < 0$ then we say $z = a$ is a *removable singularity*.

(ii) If $a_{-m} \neq 0$ for some $m > 0$ but $a_n = 0$ for all $n < -m$, we say $z = a$ is a *pole of order* m.

(iii) If $a_{-n} \neq 0$ for an infinite number of $n > 0$, we say $z = a$ is an *essential singularity*.

Examples: From Laurent expansions we find

(i) $f(z) = \sin z/z$ has a removable singularity at $z = 0$.

(ii) $f(z) = e^z/z^3$ has a pole of order 3 at $z = 0$.

(iii) $f(z) = e^{1/z}$ has an essential singularity at $z = 0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots \qquad \blacksquare$$

Theorem 58 (Isolated Singularities) *If $f(z)$ has an isolated singularity at $z = a$ then*

(i) $z = a$ is a removable singularity $\Leftrightarrow$ $|f(z)|$ is bounded near $z = a$ $\Leftrightarrow$ $f(z)$ has a limit as $z \to a$ $\Leftrightarrow$ $f(z)$ can be redefined at $z = a$ so that $f(z)$ is analytic at $z = a$.

(ii) $z = a$ is a pole $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to a$ $\Leftrightarrow$ $f(z) = g(z)(z-a)^{-m}$ with $m > 0$ and $g(a) \neq 0$.

(iii) $z = a$ is an essential singularity $\Leftrightarrow$ $|f(z)|$ is neither bounded nor goes to infinity as $z \to a$ $\Leftrightarrow$ $f(z)$ assumes every complex value, with possibly one exception, in every neighbourhood of $z = a$.

Proof: See text: (i) is easy, (ii) is similar to previous theorem, (iii) is hard (Picard). $\blacksquare$

Laurent Series

Definition: A series of the form

$$\boxed{\sum_{n=-\infty}^{\infty} a_n(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}}$$

convergent in some open annulus $r < |z-a| < R$ is called a *Laurent series around* $z = a$.

- A Laurent series is the sum of two Taylor series, the first in positive powers of $w = z - a$ and the second in positive powers of $w = (z-a)^{-1}$, that is, in negative powers of $z - a$. The first series converges for $|z-a| < R$ and the second series converges for $|z-a| > r$ or $\left|\frac{1}{z-a}\right| < \frac{1}{r}$. The Laurent series therefore converges in the intersection of these two regions given by the open annulus or ring $r < |z-a| < R$.

- In practice, a Laurent series is usually obtained by combining suitable Taylor series. For example, the Laurent series for $f(z) = z^2 e^{1/z}$ about $z = 0$ is

$$z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right) = z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

- The inner and outer radii of convergence r, R are defined by the Cauchy-Hadamard formulas

$$\boxed{r = \limsup_{n\to\infty} |a_{-n}|^{1/n}, \qquad \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}}$$

In practice, however, r and R are usually determined by applying the ratio or root test.

Laurent Theorem

Theorem 59 (Laurent Theorem)

(i) Suppose $f(z)$ is analytic in an open annulus $r < |z-a| < R$ and let C be any circle with center at a lying in this annulus. Then the Laurent series with coefficients

$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)^{n+1}}\,dw, \quad n = 0, \pm 1, \pm 2, \ldots, \quad a \text{ is inside } C}$$

converges uniformly to $f(z)$ in any closed subannulus $r < \rho_1 \le |z-a| \le \rho_2 < R$.

(ii) Conversely, If $r < R$ and

$$\sum_{n=0}^{\infty} a_n(z-a)^n \text{ converges for } |z-a| < R, \qquad \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \text{ converges for } |z-a| > r$$

then there is a unique function $f(z)$ analytic in $r < |z-a| < R$ with the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$.

Proof: Similar to Taylor's theorem — see text. ■

- If $f(z)$ is analytic in $|z-a| < R$ then by Cauchy's theorem $a_n = 0$ for $n \le -1$ and the series reduces to the Taylor series. In practice, the coefficients are typically obtained by manipulating Taylor expansions about $z-a=0$ and $z-a=\infty$.

Example of Laurent Series

Example: Find the Laurent series of $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the annulus $1 < |z| < 2$:

Solution: Since the only singularities are at $z = 1, 2$, $f(z)$ is analytic in $1 < |z| < 2$. Using partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

since, by the geometric series, for $|z| < 2$ or $\left|\frac{z}{2}\right| < 1$

$$\frac{1}{(z-2)} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

and for $1 < |z|$ or $\left|\frac{1}{z}\right| < 1$

$$\frac{1}{(z-1)} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \qquad \blacksquare$$

Further Examples of Laurent Series

Example: Find the Laurent series of $f(z) = \dfrac{z^2 - 2z + 3}{z - 2}$ in the region $|z - 1| > 1$:

Solution: Since the region $|z-1| > 1$ excludes the singularity at $z = 2$, $f(z)$ is analytic in $|z - 1| > 1$. We now use the geometric series and expand for $\left|\frac{1}{z-1}\right| < 1$

$$\begin{aligned}
\frac{z^2 - 2z + 3}{z - 2} &= \frac{(z-1)^2 + 2}{(z-1) - 1} = \frac{(z-1)^2 + 2}{(z-1)} \frac{1}{1 - \frac{1}{(z-1)}} \\
&= \left[(z-1) + \frac{2}{(z-1)}\right]\left[1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots\right] \\
&= \left[(z-1) + 1 + \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \cdots\right] \\
&\qquad + \left[\frac{2}{(z-1)} + \frac{2}{(z-1)^2} + \cdots\right] \\
&= (z-1) + 1 + \sum_{n=1}^{\infty} \frac{3}{(z-1)^n} \qquad \blacksquare
\end{aligned}$$

Exercise: Find the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

in each of the regions (i) $0 < |z| < 1$, (ii) $|z| > 1$, (iii) $0 < |z - 1| < 1$ and (iv) $|z - 1| > 1$.

Residues

Definition: If $f(z)$ has an isolated singularity at the point $z = a$, so it is analytic in a punctured neighbourhood of a, then the coefficient a_{-1} of $(z-a)^{-1}$ in the Laurent series for $f(z)$ around a is called the *residue of* $f(z)$ *at* a and denoted $\text{Res}(f;a)$

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \Rightarrow \quad \text{Res}(a) = \text{Res}(f;a) = a_{-1}}$$

If $f(z)$ is analytic at the point $z = a$ then $f(z)$ has a Taylor expansion around a and

$$\text{Res}(a) = \text{Res}(f;a) = a_{-1} = 0$$

Lemma 60 (Order m Pole) *If $f(z)$ has a pole of order m at $z = a$ then*

$$\boxed{\mathit{Res}(f;a) = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]}$$

In particular, if $f(z)$ has a simple pole ($m = 1$) at $z = a$ then

$$\boxed{\mathit{Res}(f;a) = \lim_{z\to a} (z-a) f(z)}$$

Proof: Starting with the Laurent expansion

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + \cdots$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] = (m-1)!\, a_{-1} + m!\, a_0(z-a) + \cdots \to (m-1)!\, a_{-1} \quad \text{as} \quad z \to a \qquad \blacksquare$$

- If $f(z)$ has an essential singularity then the residue is obtained from the Laurent series of $f(z)$ in the punctured neighbourhood of a using $\text{Res}(f;a) = a_{-1}$.

Examples of Residues

Example: Find the residues of $f(z) = \dfrac{z^2}{(1-z)^2(2-z)}$:

Solution: There is a simple pole at $z = 2$ and a double pole at $z = 1$

$$\begin{aligned}
\text{Res}(2) &= \lim_{z\to 2} (z-2)\frac{z^2}{(1-z)^2(2-z)} = \lim_{z\to 2} \frac{-z^2}{(1-z)^2} = -4 \\
\text{Res}(1) &= \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(1-z)^2(2-z)}\right] = \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{(2-z)}\right] \\
&= \lim_{z\to 1} \left[\frac{2z}{2-z} + \frac{z^2}{(2-z)^2}\right] = 2 + 1 = 3 \qquad \blacksquare
\end{aligned}$$

Example: If $f(z) = P(z)/Q(z)$ is rational and the polynomial $Q(z)$ has a simple zero at $z = a$, use l'Hôpital's rule to show that

$$\boxed{\text{Res}(f;a) = \frac{P(a)}{Q'(a)}, \qquad Q'(a) \neq 0}$$

Solution: Using l'Hôpital's rule

$$\text{Res}(f;a) = \lim_{z\to a} \frac{(z-a)P(z)}{Q(z)} = \lim_{z\to a} \frac{P(z) + (z-a)P'(z)}{Q'(z)} = \frac{P(a)}{Q'(a)}, \qquad Q'(a) \neq 0 \qquad \blacksquare$$

More Residues

Example: Find the residue

$$\operatorname{Res}\Big(\frac{\sin z}{z^2 - z}; 1\Big)$$

Solution: Using the previous result

$$\operatorname{Res}\Big(\frac{\sin z}{z^2 - z}; 1\Big) = \frac{\sin z}{2z - 1}\Big|_{z=1} = \sin 1 \qquad \blacksquare$$

Example: Find the residue

$$\operatorname{Res}\Big(z^2 \sin\frac{1}{z}; 0\Big)$$

Solution: There is no pole, rather $z = 0$ is an essential singularity. The Laurent expansion is

$$z^2\Big(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\Big) = z - \frac{1}{3!z} + \frac{1}{5!z^3} - \cdots$$

Hence

$$\operatorname{Res}\Big(z^2 \sin\frac{1}{z}; 0\Big) = a_{-1} = -\frac{1}{6} \qquad \blacksquare$$

Residues as Integrals

Exercise: If C is any closed contour and $m \in \mathbb{Z}$, use deformation of contours and a change of integration variable to evaluate the contour integral to show that

$$\boxed{\frac{1}{2\pi i}\oint_C (z-a)^m dz \;=\; \text{Res}\Big((z-a)^m; a\Big) \;=\; \delta_{m,-1}, \qquad a \text{ inside } C}$$

The contour integral does not exist if $m < 0$ and the contour passes through a.

Example: If $f(z)$ is analytic at $z = a$ or has an isolated singularity at $z = a$, show that

$$\boxed{\text{Res}(f;a) \;=\; \lim_{\epsilon\to 0}\left[\frac{1}{2\pi i}\oint_{|z-a|=\epsilon} f(z)dz\right]}$$

This gives an alternative definition of residues.

Solution: Using the Laurent series gives

$$\begin{aligned}
\lim_{\epsilon\to 0}\left[\frac{1}{2\pi i}\oint_{|z-a|=\epsilon} f(z)dz\right] \;&=\; \lim_{\epsilon\to 0}\left[\frac{1}{2\pi i}\oint_{|z-a|=\epsilon}\sum_{n=-\infty}^{\infty} a_n(z-a)^n dz\right] \\
&=\; \lim_{\epsilon\to 0}\left[\sum_{n=-\infty}^{\infty}\frac{a_n}{2\pi i}\oint_{|z-a|=\epsilon}(z-a)^n dz\right] \;=\; a_{-1} \;=\; \text{Res}(f;a)
\end{aligned}$$

The term-by-term integration is justified because the Laurent series is absolutely and uniformly convergent in any closed sub-annulus within the open annulus of convergence. If $z = a$ is an isolated singularity, the limit $\epsilon \to 0$ ensures that there are no other singularities inside the circle $|z-a| = \epsilon$. If $f(z)$ is analytic at $z = a$, the integral vanishes for sufficiently small ϵ by Cauchy's theorem.

Week 8: Meromorphic Functions/Residues

22. Meromorphic functions, residue theorem
23. Improper integrals, evaluation of integrals involving rational functions
24. Meromorphic partial fractions

Meromorphic Functions

Definition: A function $f(z)$ is *meromorphic* in the domain D if at every point of D it is either analytic or has a pole.

- Clearly, a meromorphic function has zeros and (isolated) poles but no other singularities.

- We have seen that an *entire* (analytic everywhere) function can be expanded into an infinite Taylor series. In this sense an entire function is like an infinite polynomial — it has zeros but no poles. Typical entire functions are $\cos z$ and $\sin z$. A meromorphic function, such as $\cot z$, can always be written as the ratio of two entire functions — its zeros are the zeros of the numerator and its poles are the zeros of the denominator. Thus

$$\boxed{\cot z = \frac{\cos z}{\sin z}}$$

In this sense a meromorhic function is a generalization of a rational function allowing for infinite polynomials in the numerator and denominator and thus an infinite number of zeros and poles.

Exercise: Find all of the zeros and poles of the meromorphic functions $\cot z$ and $\tan z$.

Residue Theorem

Theorem 61 (Residue Theorem)
If Γ is a simple closed contour and $f(z)$ is analytic inside and on Γ except at the isolated points $z_1, z_2, \ldots, z_m$ inside Γ then

$$\boxed{\oint_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{m} \mathit{Res}(f; z_k)}$$

Proof: Let C_k be small circles about each isolated singularity $z = z_k$ so that on C_k we have the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(k)} (z - z_k)^n$$

Then by deformation of contours

$$\begin{aligned} \oint_\Gamma f(z)dz &= \sum_{k=1}^{m} \oint_{C_k} f(z)dz = \sum_{k=1}^{m} \oint_{C_k} \sum_{n=-\infty}^{\infty} a_n^{(k)} (z - z_k)^n dz \\ &= \sum_{k=1}^{m} \sum_{n=-\infty}^{\infty} a_n^{(k)} \oint_{C_k} (z - z_k)^n dz = 2\pi i \sum_{k=1}^{m} a_{-1}^{(k)} \\ &= 2\pi i \sum_{k=1}^{m} \text{Res}(f; z_k) \qquad \blacksquare \end{aligned}$$

The term-by-term integration is justified by the uniform convergence of the Laurent expansions. ■

Residue Theory and Integrals

Residue theory can be used to evaluate many types of integrals. For example,

$$\boxed{I = \int_0^{2\pi} U(\cos t, \sin t)\,dt}$$

where $U(x,y)$ is a continuous real rational function on $[0, 2\pi]$.

Example: Evaluate $\quad I = \int_0^{\pi} \frac{dt}{2-\cos t} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{2-\cos t}$

Solution: Let $z = e^{it}$ so that $dz = ie^{it}dt = izdt$ and $dt = dz/iz$. Then $\cos t = \frac{1}{2}(z+z^{-1})$ and

$$2I = \oint_{|z|=1} \frac{1}{2-\frac{1}{2}(z+z^{-1})}\frac{dz}{iz} = \frac{-2}{i}\oint_{|z|=1}\frac{dz}{z^2-4z+1}$$

The integrand has simple poles at $z_\pm = 2 \pm \sqrt{3}$ but only z_- lies inside the unit circle with residue

$$\text{Res}(z_-) = \lim_{z\to z_-}\frac{(z-z_-)}{(z-z_-)(z-z_+)} = \lim_{z\to z_-}\frac{1}{(z-z_+)} = -\frac{1}{2\sqrt{3}}$$

Hence

$$2I = \frac{-2}{i}2\pi i\Big(-\frac{1}{2\sqrt{3}}\Big) = \frac{2\pi}{\sqrt{3}} \quad \Rightarrow \quad I = \frac{\pi}{\sqrt{3}} \qquad \blacksquare$$

Improper Integrals

Definition: If $f(x)$ is continuous over $[a, \infty)$, its *improper integral* is defined by

$$\boxed{\int_a^\infty f(x)dx := \lim_{R\to\infty} \int_a^R f(x)dx}$$

provided this limit exists. If $f(x)$ is continuous on $(-\infty, \infty)$ we define the double improper integral

$$\boxed{\int_{-\infty}^\infty f(x)dx := \lim_{R\to\infty} \int_0^R f(x)dx + \lim_{R'\to\infty} \int_{-R'}^0 f(x)dx}$$

provided both limits exist. If these limits exist then

$$\int_{-\infty}^\infty f(x)dx := \lim_{R,R'\to\infty} \int_{-R'}^R f(x)dx$$

independent of how the limit is taken.

- The *(Cauchy) principle value* of the integral is defined by

$$\boxed{\mathrm{PV}\int_{-\infty}^\infty f(x)dx := \lim_{R\to\infty} \int_{-R}^R f(x)dx}$$

provided this limit exists. If the double improper integral exists it must equal its principal value, but the principal value integral can exist when the double integral does not exist.

Examples: Improper Integrals

Example: Evaluate the improper integral $\int_0^\infty e^{-2x} dx$:

$$\int_0^\infty e^{-2x} dx = \lim_{R\to\infty} \int_0^R e^{-2x} dx = \lim_{R\to\infty} \left[-\frac{e^{-2x}}{2} \right]_0^R$$
$$= \lim_{R\to\infty} \left[-\frac{e^{-2R}}{2} + \frac{1}{2} \right] = \tfrac{1}{2} \qquad \blacksquare$$

Example: Evaluate the principal value integral $\mathrm{PV}\int_{-\infty}^{\infty} x dx$:

$$\mathrm{PV}\int_{-\infty}^{\infty} x\, dx = \lim_{R\to\infty} \int_{-R}^{R} x\, dx = \lim_{R\to\infty} \left[\frac{x^2}{2} \right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0$$

even though the double integral $\int_{-\infty}^{\infty} x\, dx$ does not exist. $\blacksquare$

Improper Integrals and Residues

Lemma 62 *Let C_R be the semi-circular contour in the upper-half plane from $z = R$ to $z = -R$. If*

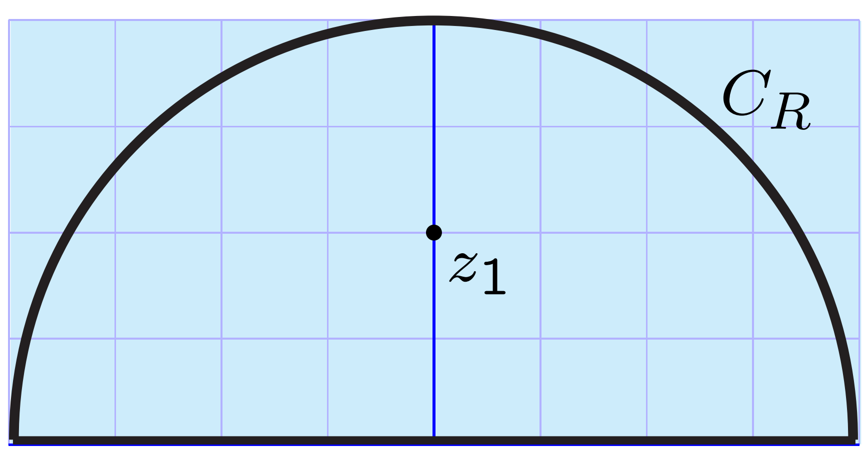
$$|f(z)| \le \frac{K}{|z|^2}, \qquad |z| \textit{ large}$$

then

$$\boxed{\lim_{R\to\infty} \left| \int_{C_R} f(z)dz \right| = 0}$$

Proof: We bound the integral

$$\left| \int_{C_R} f(z)dz \right| \le \frac{K}{R^2}\text{Length}(C_R) = \frac{K}{R^2}\pi R = \frac{K\pi}{R} \to 0 \quad \text{as} \quad R \to \infty \qquad \blacksquare$$



Principal Value Integrals

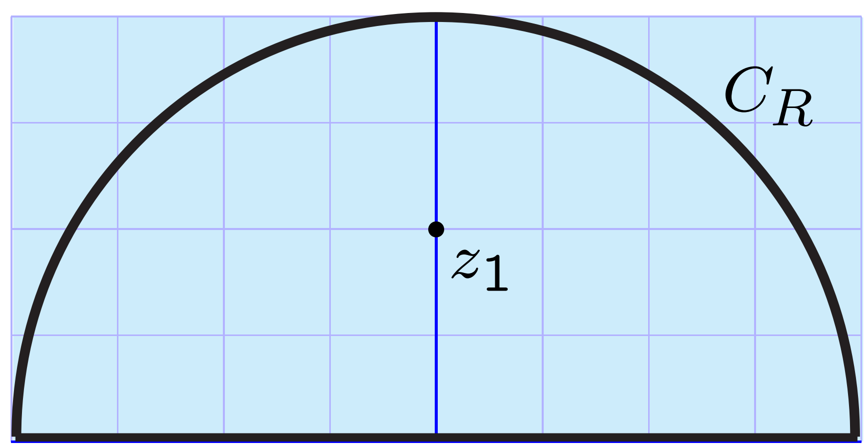
Theorem 63 (Principal Value Integrals)
Let $f(z) = P(z)/Q(z)$ be rational and analytic on the real axis so it is analytic in the upper half plane except at isolated poles. If in addition

$$\boxed{\text{degree } Q \geq 2 + \text{degree } P}$$

so that $f(z)$ satisfies the previous lemma, then residue theory can be used to evaluate the principal value integral

$$\begin{aligned} PV\int_{-\infty}^{\infty} f(x)dx &= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}\oint_{\Gamma_R} f(z)dz \\ &= 2\pi i \sum_{k=1}^{m} Res(f; z_k) \end{aligned}$$

by closing the contour in the upper half plane and summing over residues.



A Principal Value Integral

Example: Use the previous theorem to compute the principal value integral

$$\boxed{I = \text{PV}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2}}$$

Solution: The integrand $f(z) = P(z)/Q(z)$ is rational with degree $P = 2$, degree $Q = 4$ and double poles at $z = \pm i$. The previous theorem therefore applies. The residue at $z = i$ is

$$\text{Res}(f;i) = \lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)] = \lim_{z\to i}\frac{d}{dz}\Big[\frac{z^2}{(z+i)^2}\Big] = \lim_{z\to i}\Big[\frac{2z}{(z+i)^2} - \frac{2z^2}{(z+i)^3}\Big] = \frac{2i}{4i^2} + \frac{2}{8i^3} = \frac{1}{4i}$$

Therefore

$$\oint_{\Gamma_R} f(z)dz = 2\pi i\,\frac{1}{4i} = \frac{\pi}{2} \quad \text{for all } R > 1$$

and so

$$\lim_{R\to\infty}\oint_{\Gamma_R} f(z)dz = \lim_{R\to\infty}\Big[\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz\Big] = \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$$

The previous lemma applies since, for $|z| \geq \sqrt{2}$, we have using the triangle inequality $|z^2+1| \geq \Big||z|^2 - 1\Big| = |z|^2 - 1 \geq |z|^2 - \frac{1}{2}|z|^2 = \frac{1}{2}|z|^2$ and so

$$|f(z)| = \Big|\frac{z^2}{(z^2+1)^2}\Big| = \frac{|z|^2}{|z^2+1|^2} \leq \frac{|z|^2}{(\frac{1}{2}|z|^2)^2} \leq \frac{4}{|z|^2} \qquad \blacksquare$$

- Note that in these cases the double improper integral exists so that

$$\text{PV}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = 2\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+1)^2} = \frac{\pi}{2}$$

Meromorphic Partial Fractions

- It is often useful to expand rational functions into a finite number of partial fractions. For example

$$\frac{2z}{1-z^2} = \frac{1}{1-z} - \frac{1}{1+z}$$

Similarly, *meromorphic* functions can be expanded into an infinite number of partial fractions.

Exercise: Establish the partial fraction expansions of the following meromorphic functions

$$\begin{aligned}
\pi \cot \pi z &= \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \\
\pi \operatorname{cosec} \pi z &= \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2} \\
\pi \sec \pi z &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(\frac{2n-1}{2})^2 - z^2} \\
\pi \tan \pi z &= 2z \sum_{n=1}^{\infty} \frac{1}{(\frac{2n-1}{2})^2 - z^2} \\
\pi \tanh \pi z &= 2z \sum_{n=1}^{\infty} \frac{1}{(z^2 + \frac{2n-1}{2})^2} \\
\pi \coth \pi z &= \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}
\end{aligned}$$

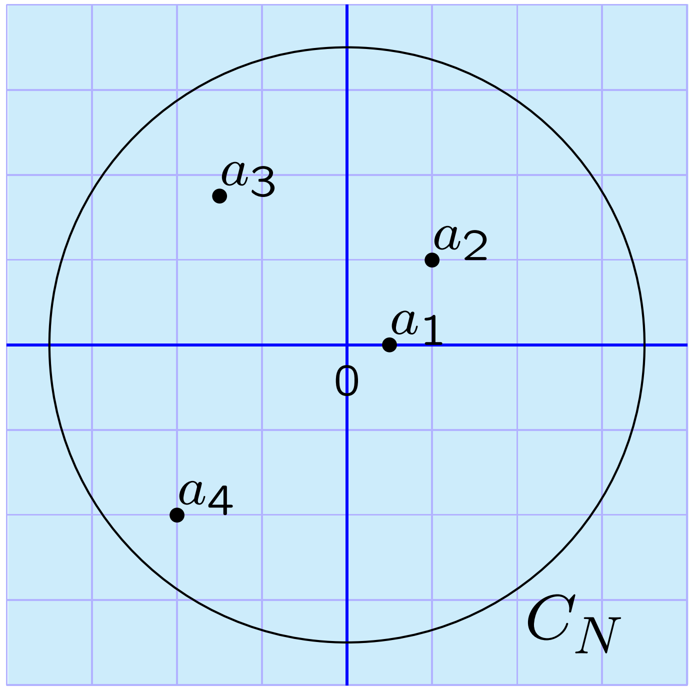
Partial Fraction Expansions

Theorem 64 (Partial Fraction Expansions)
Suppose $f(z)$ *is analytic at* $z = 0$ *and meromorphic in* $\mathbb{C}$ *with simple poles at* $z = a_1, a_2, a_3 \ldots$ *arranged in order of increasing modulus. Let* $b_1, b_2, b_3, \ldots$ *be the residues of* $f(z)$ *at* $z = a_1, a_2, a_3 \ldots$. *Suppose further that*

$$|f(z)| < M, \qquad \text{on circles } C_N: \ |z| = R_N \to \infty \text{ as } N \to \infty$$

where the circles do not pass through any poles and M *is independent of* N. *Then*

$$\boxed{f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)}$$



Proof of Partial Fraction Expansions

Proof: Consider $F(z) = \dfrac{f(z)}{z-w}$ with residues

$$\begin{aligned} \text{Res}(F; a_n) &= \lim_{z \to a_n} (z - a_n) \frac{f(z)}{z-w} = \frac{b_n}{a_n - w} \\ \text{Res}(F; w) &= \lim_{z \to w} (z - w) \frac{f(z)}{z-w} = f(w) \end{aligned}$$

So using the residue theorem and subtracting

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_N} \frac{f(z)dz}{z-w} &= f(w) + \sum_{\text{poles } a_n \text{ in } C_N} \frac{b_n}{a_n - w} \\ \frac{1}{2\pi i} \oint_{C_N} \frac{f(z)dz}{z} &= f(0) + \sum_{\text{poles } a_n \text{ in } C_N} \frac{b_n}{a_n} \end{aligned}$$

$$f(w) - f(0) + \sum_{\text{poles } a_n \text{ in } C_N} \left(\frac{b_n}{a_n - w} - \frac{b_n}{a_n} \right) = \frac{w}{2\pi i} \oint_{C_N} \frac{f(z)dz}{z(z-w)}$$

where $\quad \left| \oint_{C_N} \dfrac{f(z)dz}{z(z-w)} \right| \leq \dfrac{2\pi R_N M}{R_N(R_N - |w|)} \to 0 \quad \text{as} \quad N \to \infty$

so the result follows provided the series converges. ■

Partial Fraction Expansion of $\cot \pi z$

Example: Show that

$$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)}$$

Solution: The meromorphic function $f(z) = \pi \cot \pi z - \dfrac{1}{z}$ has simple poles at $z = a_n = n = \pm 1, \pm 2, \ldots$ with residues

$$b_n = \lim_{z \to n} (z-n) \left(\frac{\pi z \cos \pi z - \sin \pi z}{z \sin \pi z} \right) = \lim_{z \to n} \frac{(z-n)}{\sin \pi z} \frac{\pi z \cos \pi z - \sin \pi z}{z} = 1$$

By l'Hôpital, $f(z)$ has a removable singularity at $z = 0$

$$\begin{aligned}
\lim_{z \to 0} \left(\pi \cot \pi z - \frac{1}{z} \right) &= \lim_{z \to 0} \left(\frac{\pi z \cos \pi z - \sin \pi z}{z \sin \pi z} \right) \\
&= \lim_{z \to 0} \left(\frac{\pi \cos \pi z - \pi^2 z \sin \pi z - \pi \cos \pi z}{\sin \pi z + \pi z \cos \pi z} \right) = - \lim_{z \to 0} \left(\frac{\pi^2 z \sin \pi z}{\sin \pi z + \pi z \cos \pi z} \right) \\
&= - \lim_{z \to 0} \left(\frac{\pi^2 \sin \pi z}{\sin \pi z / z + \pi \cos \pi z} \right) = 0 = f(0)
\end{aligned}$$

The circles C_N of the previous theorem can be replaced with the squares Γ_N with vertices at $z = (N + \frac{1}{2})(\pm 1 \pm i)$ on which we can show

$$|\cot \pi z| \leq M = \coth(\pi/2), \qquad \text{independent of } N$$

The required result then follows as in the previous theorem. ■

Bound on Squares Γ_N

Let Γ_N be the square with vertices at $z = (N + \frac{1}{2})(\pm 1 \pm i)$.
Then on Γ_N

$$|\cot \pi z| \leq M = \coth(\pi/2), \qquad \text{independent of } N$$

(i) If $z = x + iy$ and $y > \frac{1}{2}$ or $y < -\frac{1}{2}$ respectively

$$|\cot \pi z| = \left|\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right| \leq \frac{e^{-2\pi y}+1}{|e^{-2\pi y}-1|} \leq \frac{1+e^{-\pi}}{1-e^{-\pi}} = \coth(\pi/2)$$

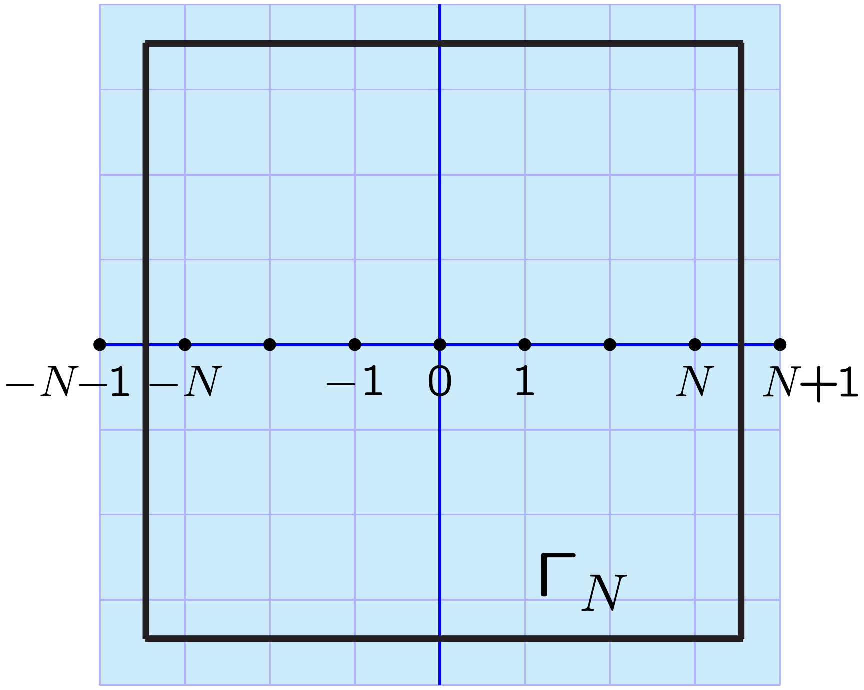
$$|\cot \pi z| = \left|\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right| \leq \frac{e^{-2\pi y}+1}{|e^{-2\pi y}-1|} = \frac{1+e^{2\pi y}}{1-e^{2\pi y}} \leq \frac{1+e^{-\pi}}{1-e^{-\pi}}$$

(ii) If $x = \pm(N + \frac{1}{2})$ and $-\frac{1}{2} \leq y \leq \frac{1}{2}$ then

$$|\cot \pi z| = |\cot(\pm\pi/2 + \pi i y)| = |\tanh \pi y| \leq \tanh(\pi/2) \leq \coth(\pi/2)$$

Hence

$$\left|\oint_{\Gamma_N} \frac{f(z)dz}{z(z-w)}\right| \leq \frac{4(2N+1)\,M}{(N+\frac{1}{2})(N+\frac{1}{2}-|w|)} \to 0, \qquad N \to \infty \qquad \blacksquare$$



The Contour Γ_N:
(showing poles of $\cot \pi z$)

Alternative Partial Fraction Expansion of $\cot \pi z$

Example: Establish the alternative form

$$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}}$$

Solution: The second form follows after rearranging the series

$$\begin{aligned}
\pi \cot \pi z &= \frac{1}{z} + \lim_{N \to \infty} \left[\sum_{n=-N}^{-1} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \sum_{n=1}^{N} \left(\frac{1}{z-n} + \frac{1}{n} \right) \right] \\
&= \frac{1}{z} + \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad \blacksquare
\end{aligned}$$

This is allowed because the double series is absolutely (and uniformly) convergent in $|z| \leq R$ by the Weierstrass M-test

$$\left| \frac{1}{z-n} + \frac{1}{n} \right| = \left| \frac{z}{n(n-z)} \right| \leq \frac{R}{n(n-R)} \leq \frac{2R}{n^2} = M_n, \quad n \geq 2R$$

since $\displaystyle\sum_{n \neq 0} \frac{1}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent harmonic series.

Infinite Product Form of $\sin \pi z$

Example: Integrate the partial fraction expansion of $\pi \cot \pi z$ term-by-term to obtain the infinite product

$$\boxed{\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}$$

Solution: Since the partial fraction expansion converges absolutely and uniformly we can integrate term-by-term

$$\pi \cot \pi z - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

$$\Rightarrow \quad \mathrm{Log}\, \frac{\sin \pi z}{\pi z} = \sum_{n=1}^{\infty} \mathrm{Log}(1 - \frac{z^2}{n^2}) = \mathrm{Log} \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

where the constant of integration vanishes. The result follows by taking exponentials. ■

- Putting $z = \frac{1}{2}$ in the infinite product gives the Wallis product

$$\boxed{\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots}$$

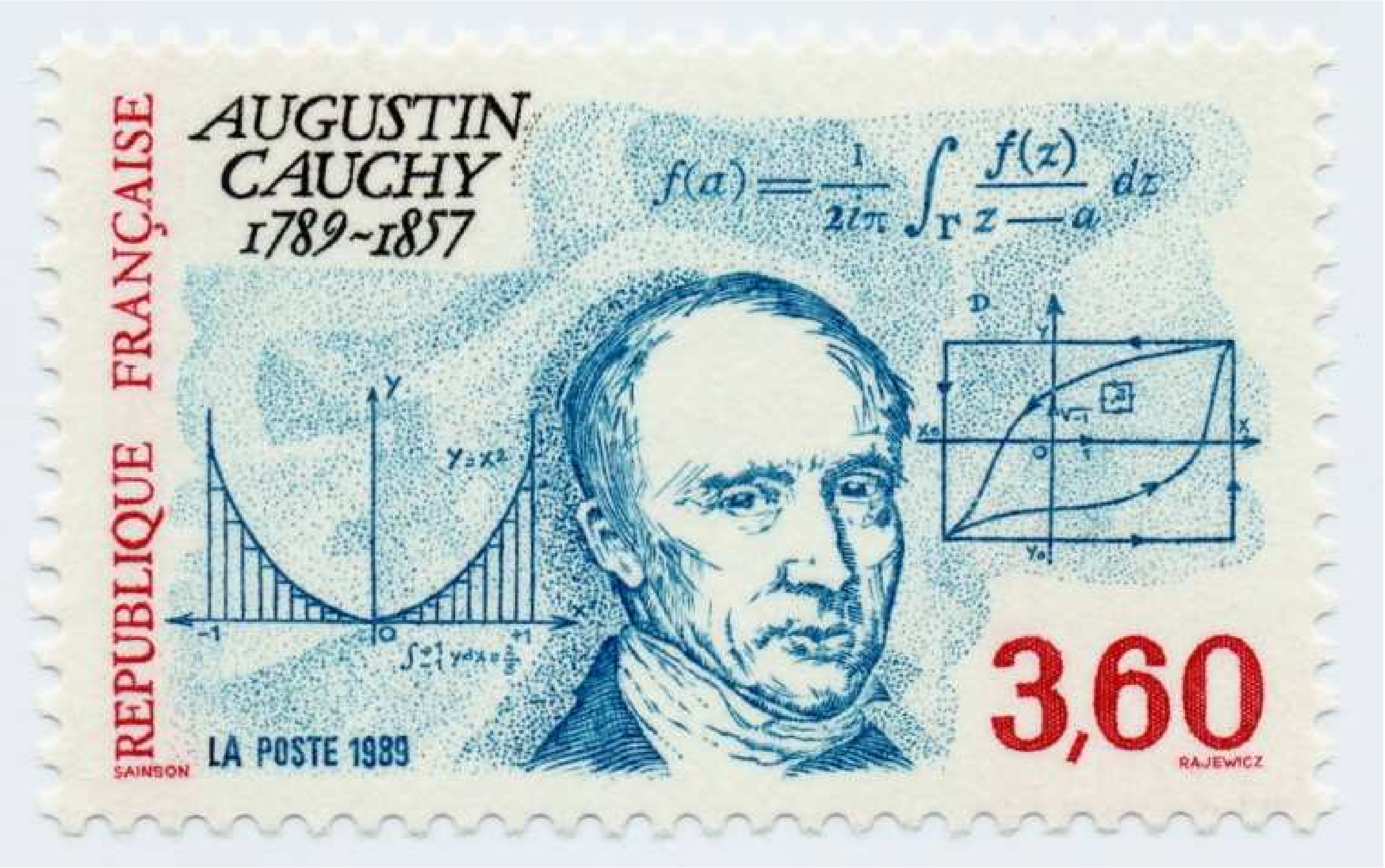
- The corresponding infinite product for cosine is

$$\boxed{\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)}$$

Week 9: Residue Calculus

25. Evaluation of integrals involving trigonometric functions
26. Evaluation of integrals using indented contours
27. Summation of series using the residue calculus

Augustin Louis Cauchy (1789–1857)



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Residues and Trigonometric Integrals

Residue theory can be used to evaluate many types of integrals. For example,

$$\boxed{I = \int_0^{2\pi} U(\cos t, \sin t)\,dt}$$

where $U(x, y)$ is a continuous real rational function of x, y on $[-1, 1] \times [-1, 1]$.

Example: Evaluate $$I = \int_0^{\pi} \frac{dt}{2 - \cos t} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{2 - \cos t}$$

Solution: Let $z = e^{it}$ so that $dz = ie^{it}dt = iz\,dt$ and $dt = dz/iz$. Then $\cos t = \frac{1}{2}(z + z^{-1})$ and

$$2I = \oint_{|z|=1} \frac{1}{2 - \frac{1}{2}(z + z^{-1})} \frac{dz}{iz} = \frac{-2}{i} \oint_{|z|=1} \frac{dz}{z^2 - 4z + 1}$$

The integrand has simple poles at $z_{\pm} = 2 \pm \sqrt{3}$ but only z_- lies inside the unit circle with residue

$$\text{Res}(z_-) = \lim_{z \to z_-} \frac{(z - z_-)}{(z - z_-)(z - z_+)} = \lim_{z \to z_-} \frac{1}{(z - z_+)} = -\frac{1}{2\sqrt{3}}$$

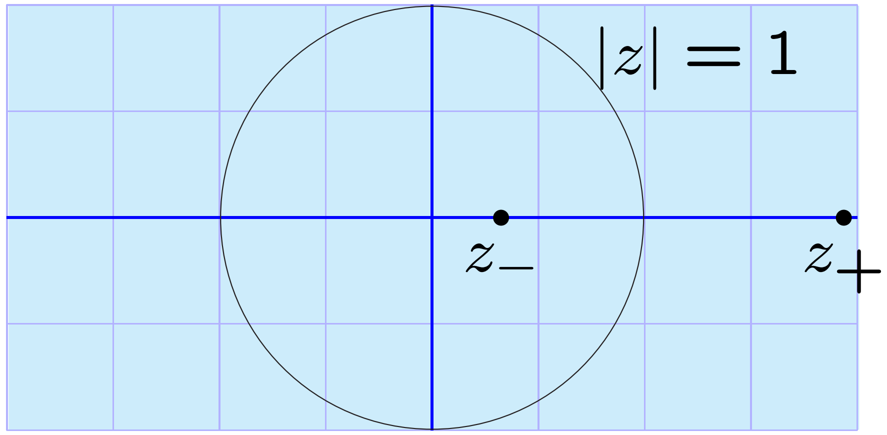
Hence

$$2I = \frac{-2}{i} 2\pi i \Big(-\frac{1}{2\sqrt{3}} \Big) = \frac{2\pi}{\sqrt{3}} \quad \Rightarrow \quad I = \frac{\pi}{\sqrt{3}} \qquad \blacksquare$$

A Trigonometric Integral by Residues

Example: Evaluate the integral $I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta}$:

This integral was evaluated previously using the Cauchy integral formula.



Let $z = e^{i\theta}$ so that $z'(\theta) = ie^{i\theta} = iz$ and $d\theta = -idz/z$. Then

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta} = -i \oint_{|z|=1} \frac{dz}{z(3 - z - z^{-1})} \\ &= i \oint_{|z|=1} \frac{dz}{z^2 - 3z + 1} = i \oint_{|z|=1} \frac{dz}{(z - z_+)(z - z_-)} \end{aligned}$$

where $z_\pm = (3 \pm \sqrt{5})/2 = 2.618.., 0.381..$ So setting

$$f(z) = (z - z_+)^{-1}$$

which is analytic inside $|z| = 1$, we have by the Cauchy integral formula

$$I = i \oint_{|z|=1} \frac{f(z)dz}{(z - z_-)} = (i)(2\pi i)f(z_-) = -\frac{2\pi}{z_- - z_+} = \frac{2\pi}{\sqrt{5}}$$

We obtain the same result using residue calculus and summing over the poles inside $|z| = 1$

$$I = (i)(2\pi i)\sum_k \text{Res}\Big(\frac{f(z)}{(z - z_-)}; z_k\Big) = (i)(2\pi i)f(z_-) = \frac{2\pi}{\sqrt{5}} \qquad \blacksquare$$

Another Trigonometric Integral

Exercise: For real $p \neq \pm 1$, evaluate the real definite trigonometric integral

$$\boxed{I = \int_0^{2\pi} \frac{dt}{1 - 2p\cos t + p^2}}$$

Solution: Let $z = e^{it}$ so that $dz = ie^{it}dt = izdt$ and $dt = dz/iz$. Then $\cos t = \frac{1}{2}(z + z^{-1})$ and

$$I = \oint_{|z|=1} \frac{dz}{iz[1 + p^2 - p(z + \frac{1}{z})]} = \frac{i}{p} \oint_{|z|=1} \frac{dz}{(z-p)(z-\frac{1}{p})}$$

There are now two situations to consider:

(i) $|p| > 1$: In this case there is a simple pole inside $|z| = 1$ at $z = 1/p$. So by the residue theorem

$$I = \frac{i}{p} 2\pi i \,\mathrm{Res}\left(\frac{1}{p}\right) = -\frac{2\pi}{p} \frac{1}{\frac{1}{p} - p} = \frac{2\pi}{p^2 - 1}$$

(ii) $|p| < 1$: In this case there is a simple pole at $z = p$ so

$$I = \frac{i}{p} 2\pi i \,\mathrm{Res}(p) = -\frac{2\pi}{p} \frac{1}{p - \frac{1}{p}} = \frac{2\pi}{1 - p^2}$$

(iii) If $|p| = 1$, that is, $p = \pm 1$, the integral does not exist since there is a pole on the contour of integration.

- The above results can be combined into the single formula

$$\int_0^{2\pi} \frac{dt}{1 - 2p\cos t + p^2} = \frac{2\pi}{|1 - p^2|} \quad \text{when} \quad p \neq \pm 1 \qquad \blacksquare$$

Uniformity on an Arc

Definition: (Uniformity on an Arc) If along a circular arc of radius R, $|f(z)| \le M_R$ where M_R does not depend on (the polar angle) θ, and $M_R \to 0$ as $R \to \infty$ (or $R \to 0$) we say that $f(z)$ tends uniformly to zero on $C_R = \{z : |z| = R\}$ as $R \to \infty$ (or $R \to 0$).

Example: Consider the function

$$f(z) = \frac{z}{z^2+1} \quad \text{on} \quad C_R = \{z : |z| = R\}$$

We deduce that

$$|f(z)| \le \begin{cases} \dfrac{R}{R^2-1}, & R > 1 \\ \dfrac{R}{1-R^2}, & R < 1 \end{cases}$$

It follows that $f(z)$ tends uniformly to zero when either $R \to \infty$ or $R \to 0$. ■

- In general any rational function whose denominator is of higher degree than the numerator tends uniformly to zero as $R \to \infty$.

- The polar angle can be restricted to a closed interval $\theta_0 \le \theta \le \theta_0 + \alpha$. We then require

$$\boxed{M_R = \max_{\substack{z=Re^{i\theta} \\ \theta_0 \le \theta \le \theta_0+\alpha}} |f(z)| \to 0 \quad \text{as} \quad R \to \infty}$$

Limiting Contours I

Theorem 65 (Limiting Contours I) *If on C_R, $zf(z)$ tends uniformly to zero as $R \to \infty$ then*

$$\boxed{\lim_{R\to\infty} \int_{C_R} f(z)\, dz \;=\; 0}$$

on the circular arc of radius R subtending an angle α at the origin

$$C_R = \{z : |z| = R, \quad \theta_0 \le \theta \le \theta_0 + \alpha\}$$

Proof: We have

$$|zf(z)| = R|f(z)| \le M_R$$

where M_R is independent of θ and $M_R \to 0$ as $R \to \infty$. It follows that

$$0 \;\le\; \left|\int_{C_R} f(z)\, dz\right| \;\le\; \int_{C_R} |f(z)|\,|dz| \;\le\; \frac{M_R}{R}\int_{C_R} |dz| \;=\; \alpha M_R \to 0 \quad \text{as} \quad R \to \infty$$

since on C_R, $z = Re^{i\theta}$, $dz = Rie^{i\theta} d\theta$, and hence

$$\int_{C_R} |dz| = \int_{\theta_0}^{\theta_0+\alpha} R\, d\theta = \alpha R \qquad \blacksquare$$

Limiting Contours II — Jordan's Lemma

Theorem 66 (Jordan's Lemma) *If $f(z)$ tends uniformly to zero on $C_R = \{z : z = Re^{i\theta},$ $0 \le \theta_0 \le \theta \le \theta_1 \le \pi\}$ as $R \to \infty$ then for $k > 0$:*

$$
\begin{aligned}
&\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z)\, dz = 0 &\qquad& C_R \text{ in 1st, 2nd quadrants}\\
&\lim_{R\to\infty} \int_{C_R} e^{-ikz} f(z)\, dz = 0 && C_R \text{ in 3rd, 4th quadrants}\\
&\lim_{R\to\infty} \int_{C_R} e^{kz} f(z)\, dz = 0 && C_R \text{ in 2nd, 3rd quadrants}\\
&\lim_{R\to\infty} \int_{C_R} e^{-kz} f(z)\, dz = 0 && C_R \text{ in 1st, 4th quadrants}
\end{aligned}
$$

Proof: We prove the first case. The other cases are similar. Note that on C_R

$$|dz| = R\, d\theta \qquad \text{and} \qquad |f(z)| \le M_R$$

It follows that

$$
\begin{aligned}
0 \;\le\; \left| \int_{C_R} e^{ikz} f(z)\, dz \right| \;&\le\; \int_{C_R} |e^{ikz}||f(z)||dz| \;\le\; RM_R \int_{\theta_0}^{\theta_1} e^{-kR\sin\theta}\, d\theta \\
&\le\; RM_R \int_0^{\pi} e^{-kR\sin\theta}\, d\theta \;=\; 2RM_R \int_0^{\pi/2} e^{-kR\sin\theta}\, d\theta \\
&\le\; 2RM_R \int_0^{\pi/2} e^{-2kR\theta/\pi}\, d\theta \;=\; \frac{\pi M_R}{k}(1 - e^{-kR}) \;\to\; 0 \quad \text{as} \quad R \to \infty
\end{aligned}
$$

Here we used $\sin(\frac{\pi}{2} - \theta) = \sin(\frac{\pi}{2} + \theta)$ and the inequality $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$. ■

Limiting Contours III

Theorem 67 (Limiting Contours III) *If on a circular arc C_r of radius r and centre a, $|(z-a)f(z)|$ tends uniformly to zero as $r \to 0$ then*

$$\boxed{\lim_{r\to 0} \int_{C_r} f(z)\,dz \;=\; 0}$$

Proof: On $C_r = \{z : z - a = re^{i\theta},\, \theta_0 \le \theta < \theta_0 + \alpha\}$ we have

$$|(z-a)f(z)| \;=\; r|f(z)| \;\le\; M_r$$

where $M_r \to 0$ as $r \to 0$. It follows that

$$\left|\int_{C_r} f(z)\,dz\right| \;\le\; \frac{M_r}{r}\int_{C_r} |dz| \;=\; \alpha M_r$$

which gives the required result. ∎

Limiting Contours IV

Theorem 68 (Limiting Contours IV) *If $f(z)$ has a simple pole at $z = a$ with residue Res(a) and if C_r is a circular arc of radius r and centre a subtending an angle α at $z = a$ then*

$$\boxed{\lim_{r\to 0}\int_{C_r} f(z)\,dz = i\alpha\,\mathit{Res}(a)}$$

Proof: The Laurent series for $f(z)$ can be expressed as

$$f(z) = \frac{\text{Res}(a)}{z-a} + \phi(z)$$

where $\phi(z)$ is analytic at $z = a$. We then have

$$\int_{C_r} f(z)\,dz = \int_{C_r} \frac{\text{Res}(a)}{z-a}\,dz + \int_{C_r} \phi(z)\,dz$$

The second integral vanishes as $r \to 0$ from previous theorem (since $\phi(z)$ is bounded at $z = a$). Furthermore, on C_r, we can write

$$z = a + re^{i\theta} \qquad \text{where} \qquad \theta_0 \le \theta \le \theta_0 + \alpha$$

It then follows that

$$\int_{C_r} \frac{\text{Res}(a)}{z-a}\,dz = \text{Res}(a)\int_{\theta_0}^{\theta_0+\alpha} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = i\alpha\,\text{Res}(a) \qquad \blacksquare$$

- Warning: This theorem only applies to the case of *simple* poles!

Fourier Example

Exercise: Evaluate the Fourier integral

$$\boxed{I = \int_{-\infty}^{\infty} e^{ikx}(a^2 + x^2)^{-1}\, dx, \qquad a > 0, \quad k \in \mathbb{R}}$$

Solution: The function $f(z) = (a^2 + z^2)^{-1}$ has simple poles at $z = \pm ia$.
(i) $k \geq 0$: Close the contour in the upper half plane. Provided $R > a$, we have

$$\begin{aligned}\left|\int_{C_R} e^{ikz} f(z)\, dz\right| \;&\leq\; \int_{C_R} |a^2 + z^2|^{-1}\, |dz| \;\leq\; (R^2 - a^2)^{-1} \int_{C_R} |dz| \\ &=\; \pi R(R^2 - a^2)^{-1} \to 0 \quad \text{as } R \to \infty\end{aligned}$$

since $|e^{ikz}| = |e^{ik(x+iy)}| = e^{-ky} \leq 1$ in the upper half plane. It follows that

$$\int_{-\infty}^{\infty} e^{ikx}(a^2 + x^2)^{-1}\, dx \;=\; 2\pi i\, \mathrm{Res}(e^{ikz} f(z); ia) \;=\; 2\pi i \left.\frac{e^{ikz}}{z + ia}\right|_{z=ia} \;=\; \frac{\pi}{a}\, e^{-ka}$$

(ii) $k \leq 0$: Closing the contour in the lower half plane gives

$$\int_{-\infty}^{\infty} e^{ikx}(a^2 + x^2)^{-1}\, dx \;=\; -2\pi i\, \mathrm{Res}(e^{ikz} f(z); -ia) \;=\; -2\pi i \left.\frac{e^{ikz}}{z - ia}\right|_{z=-ia} \;=\; \frac{\pi}{a}\, e^{ka}$$

(iii) The two results combine into the single result

$$\int_{-\infty}^{\infty} e^{ikx}(a^2 + x^2)^{-1}\, dx \;=\; \frac{\pi}{a}\, e^{-|k|a} \qquad \blacksquare$$

- Notice that taking real and imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{a^2 + x^2}\, dx = \frac{\pi}{a}\, e^{-|k|a}, \qquad \int_{-\infty}^{\infty} \frac{\sin(kx)}{a^2 + x^2}\, dx = 0$$

Indented Contours I

Example: Consider the contour integrals

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \int_{-\infty}^{\infty} \text{Im}\left[\frac{e^{ix}}{x}\right] dx, \qquad I' = \oint_C \frac{e^{iz}}{z}\, dz$$

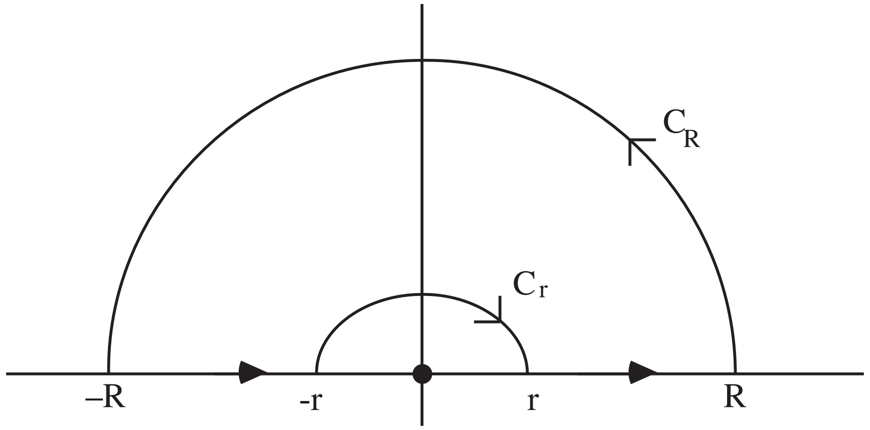
where C is the standard upper-half-plane contour indented by a small semi-circle C_r around the singularity at $z = 0$. Since $f(z) = e^{iz}/z$ is analytic inside C, Cauchy's theorem gives

$$0 = \oint_C \frac{e^{iz}}{z}\, dz = \int_{-R}^{-r} \frac{e^{ix}}{x}\, dx + \int_{C_r} \frac{e^{iz}}{z}\, dz + \int_r^R \frac{e^{ix}}{x}\, dx + \int_{C_R} \frac{e^{iz}}{z}\, dz$$

By Jordan's Lemma $\quad \lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz = 0$

Similarly, from the previous theorem

$$\lim_{r\to 0} \int_{C_r} \frac{e^{iz}}{z}\, dz = -i\pi\, \text{Res}(0) = -i\pi$$



where the negative sign comes from the fact that C_r is traversed clockwise. It follows that

$$\lim_{r\to 0,\, R\to\infty} \left\{ \int_{-R}^{-r} \frac{e^{ix}}{x}\, dx + \int_r^R \frac{e^{ix}}{x}\, dx \right\} \equiv \wp\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = i\pi$$

where the limit defines the *Cauchy principal value*. Equating real and imaginary parts gives

$$\wp\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0, \qquad \wp\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

The integral I is a conditionally (not absolutely) convergent improper integral. ■

Conditionally Convergent Improper Integrals

Theorem 69 (Conditionally Convergent Improper Integrals)
If $f(x)$ has a bounded primitive

$$|F(x)| \leq K < \infty, \qquad x \geq a > 0$$

then

$$\boxed{\int_a^\infty \frac{f(x)}{x^p}\,dx \quad \textit{converges} \quad \textit{for} \;\; p > 0}$$

Proof: *Integration by parts gives*

$$\lim_{R\to\infty}\int_a^R \frac{f(x)}{x^p}\,dx \;=\; \lim_{R\to\infty}\left.\frac{F(x)}{x^p}\right|_a^R + \lim_{R\to\infty} p\int_a^R \frac{F(x)}{x^{p+1}}\,dx \;=\; -\frac{F(a)}{a^p} + p\int_a^\infty \frac{F(x)}{x^{p+1}}\,dx$$

where the last improper integral is absolutely convergent by comparison with $\displaystyle\int_a^\infty \frac{dx}{x^{p+1}}$.

Example: The integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx$ is a convergent improper integral.

$$F(x) \;=\; \int f(x)\,dx \;=\; \int \sin x\,dx \;=\; -\cos x, \qquad |F(x)| \leq 1, \quad x \geq a > 0$$

So

$$\int_0^\infty \frac{\sin x}{x}\,dx \;=\; \int_0^a + \int_a^\infty \frac{\sin x}{x}\,dx$$

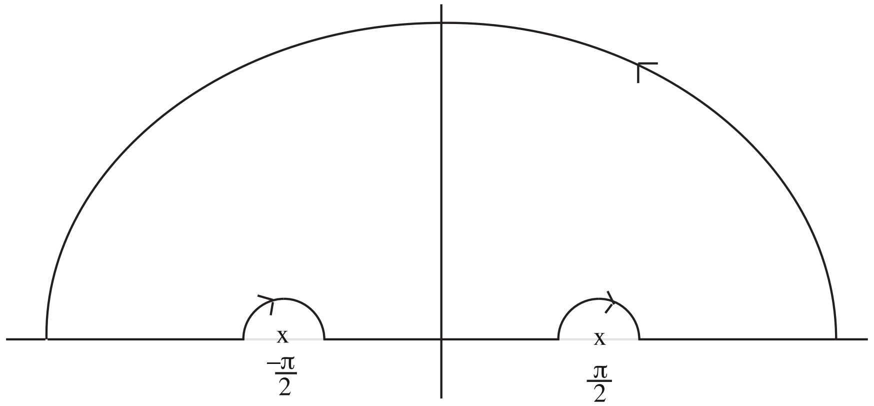
The first integral exists as a Riemann integral and the second is a convergent improper integral by the theorem. ■

Indented Contours II

Example: Evaluate the contour integral

$$\boxed{I = \oint_C \frac{e^{iz}\,dz}{\pi^2 - 4z^2}}$$

where C is the standard upper-half-plane contour indented by small semi-circles around the singularities at $z = \pm\pi/2$.



Solution: Following essentially the same steps as in the previous example

$$0 = \wp\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{\pi^2 - 4x^2} - i\pi\left[\mathrm{Res}\left(-\frac{\pi}{2}\right) + \mathrm{Res}\left(\frac{\pi}{2}\right)\right] = \wp\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{\pi^2 - 4x^2} - i\pi\left[-\frac{i}{4\pi} - \frac{i}{4\pi}\right]$$

Equating real parts gives

$$\wp\int_{-\infty}^{\infty} \frac{\cos x}{\pi^2 - 4x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{\pi^2 - 4x^2}\,dx = \frac{1}{2}$$

The Cauchy principal value of $\wp\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{\pi^2 - 4x^2}$ must be retained since otherwise the integral fails to exist due to divergences at $x = \pm\pi/2$. ■

- The technique of indented contours only works for *simple* poles on the real axis! For higher order poles the Cauchy principal values do not exist.

Series and Residues

Theorem 70 (Summing Series)

Residue theory can be used to sum the following types of series:

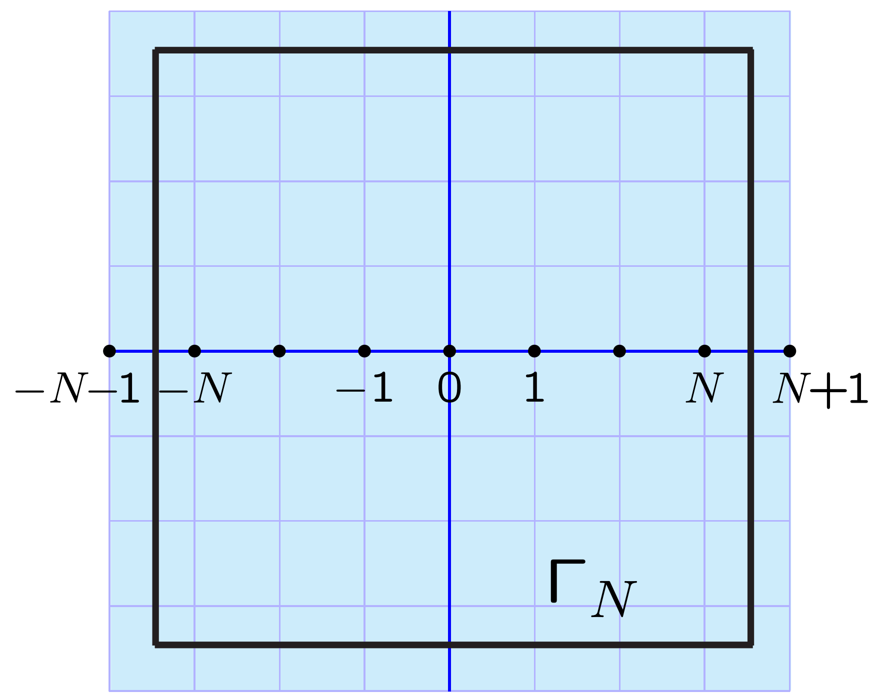
$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{\text{poles } z_j \text{ of } f(z)} Res(\pi \cot \pi z\, f(z); z_j)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{\text{poles } z_j \text{ of } f(z)} Res(\pi \csc \pi z\, f(z); z_j)$$

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) = \sum_{\text{poles } z_j \text{ of } f(z)} Res(\pi \tan \pi z\, f(z); z_j)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = \sum_{\text{poles } z_j \text{ of } f(z)} Res(\pi \sec \pi z\, f(z); z_j)$$

where the residues are summed only over the poles z_j of $f(z)$. The methods apply provided $f(z)$ is analytic except at isolated poles ($z_j \notin \mathbb{Z}$) and decays sufficiently rapidly as $|z| \to \infty$, for example, $|f(z)| \le K/|z|^2$.



The Contour Γ_N:
(showing poles of
$\cot \pi z$)

Summing Series: Proof

Proof: We consider just the first type. The function $F(z) = \pi \cot \pi z \, f(z)$ has simple poles at $z = n \in \mathbb{Z}$ with residues

$$\begin{aligned} \text{Res}(F; n) &= \lim_{z \to n} (z - n)\pi \cot \pi z \, f(z) \\ &= \lim_{z \to n} \frac{(z-n)}{\sin \pi z} \pi \cos \pi z \, f(z) = f(n) \end{aligned}$$

Let Γ_N be the square with vertices at $z = (N + \frac{1}{2})(\pm 1 \pm i)$. Then on Γ_N

$$|\cot \pi z| \leq M = \coth(\pi/2), \qquad \text{independent of } N$$

(i) If $z = x + iy$ and $y > \frac{1}{2}$ or $y < -\frac{1}{2}$ respectively

$$|\cot \pi z| = \left| \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right| \leq \frac{e^{-2\pi y} + 1}{|e^{-2\pi y} - 1|} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = \coth(\pi/2)$$

$$|\cot \pi z| = \left| \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right| \leq \frac{e^{-2\pi y} + 1}{|e^{-2\pi y} - 1|} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}}$$

(ii) If $x = \pm(N + \frac{1}{2})$ and $-\frac{1}{2} \leq y \leq \frac{1}{2}$ then

$$|\cot \pi z| = |\cot(\pm \pi/2 + \pi i y)| = |\tanh \pi y| \leq \tanh(\pi/2) \leq \coth(\pi/2)$$

Hence, since $|z| \geq N$ on Γ_N

$$\left| \oint_{\Gamma_N} \pi \cot \pi z \, f(z) dz \right| \leq \frac{\pi M K}{N^2} (8N + 4) \to 0 \quad \text{as} \quad N \to \infty$$

Summing Series: Proof (Continued)

Therefore by the residue theorem

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \oint_{\Gamma_N} \pi \cot \pi z\, f(z) dz = \sum_{\text{all poles of } F(z)} \text{Res}(\pi \cot \pi z\, f(z)) \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} f(n) + \sum_{\text{poles } z_j \text{ of } f(z)} \text{Res}(\pi \cot \pi z\, f(z); z_j) \\
&= \sum_{n=-\infty}^{\infty} f(n) + \sum_{\text{poles } z_j \text{ of } f(z)} \text{Res}(\pi \cot \pi z\, f(z); z_j)
\end{aligned}
$$

provided the double series converges. $\blacksquare$

Summing Example Series

Example: Show that for $a > 0$, $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth \pi a$:

The series is absolutely convergent for $a > 0$. Let $f(z) = \dfrac{1}{z^2+a^2}$ with simple poles at $z = \pm ai$. The residues of

$$F(z) = \pi \cot \pi z \, f(z) = \frac{\pi \cot \pi z}{z^2+a^2}$$

at $z = \pm ai$ are then

$$\operatorname{Res}(F; ai) = \lim_{z\to ai} \frac{(z-ai)\pi \cot \pi z}{(z+ai)(z-ai)} = \frac{\pi \cot \pi a i}{2ai} = -\frac{\pi}{2a} \coth \pi a$$

$$\operatorname{Res}(F; -ai) = \lim_{z\to -ai} \frac{(z+ai)\pi \cot \pi z}{(z+ai)(z-ai)} = \frac{\pi \cot \pi a i}{2ai} = -\frac{\pi}{2a} \coth \pi a$$

Hence, by the previous theorem,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\sum_{\text{poles } z_j \text{ of } f(z)} \operatorname{Res}(\pi \cot \pi z \, f(z); z_j) \; = \; \frac{\pi}{a} \coth \pi a \qquad \blacksquare$$

Example: Sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ as the limit $a \to 0$ of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$:

The second series is absolutely and uniformly convergent for $0 \le a \le R < \infty$ by the Weierstrass M-test. Using uniform convergence, continuity and l'Hôpital's rule

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{a\to 0} \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \lim_{a\to 0} \tfrac{1}{2}\Big[\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} - \frac{1}{a^2}\Big] = \lim_{a\to 0} \frac{\pi a \coth \pi a - 1}{2a^2} = \frac{\pi^2}{6} \qquad \blacksquare$$

Week 10: Applications/Cauchy Theorems

28. Gauss mean value theorem, maximum modulus principle, applications to harmonic functions
29. Liouville's theorem, the fundamental theorem of algebra
30. The identity theorem with a brief discussion of analytic continuation

Carl Friedrich Gauss (1777–1855)

Joseph Liouville (1809–1882)

Maximum Modulus Principle

Lemma 71 (Gauss Mean Value Theorem) *If $f(z)$ is analytic inside and on the circle C given by $|z-a|=r$, then the mean of $f(z)$ on C is $f(a)$*

$$\boxed{f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt}$$

Proof: On C, $z=a+re^{it}$, $z'(t)=ire^{it}$. So by Cauchy's integral formula

$$f(a) \;=\; \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})ire^{it}}{re^{it}}dt \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt \qquad \blacksquare$$

Theorem 72 (Maximum Modulus Principle)

If $f(z)$ is analytic in and on a simple closed curve Γ, then the maximum value of $|f(z)|$ occurs on Γ unless $f(z)$ is constant.

Proof: By the Gauss mean value theorem on $|z-a|=r$

$$|f(a)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|dt \qquad (*)$$

Proceed by contradiction. Suppose maximum is $|f(a)|$ for a inside Γ so that, on some circle C about a and inside Γ, $|f(a+re^{it})| \le |f(a)|$ and (if $f(z)$ is not constant)

$$|f(a+re^{it})| < |f(a)|, \quad \text{for some } t$$

Then, by continuity, this holds in some interval $t_1 < t < t_2$ and so the mean value satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|dt < |f(a)|$$

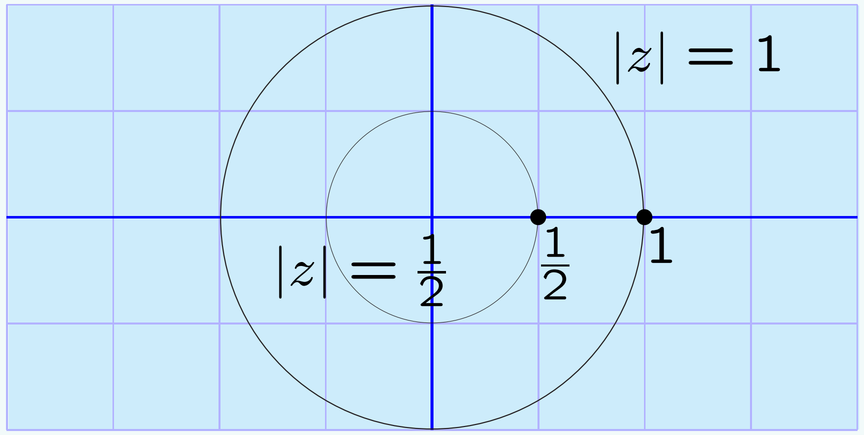
which contradicts $(*)$. Hence either $f(z)$ is constant or the maximum occurs on Γ. $\blacksquare$

Application of Maximum Modulus

Example: Let $f(z) = e^z/z$. Find the point where $|f(z)|$ takes its maximum on the annulus $\frac{1}{2} \leq |z| \leq 1$ and find its value.

Solution: By the maximum modulus principle, the maximium must occur on the boundary of the annulus which consists of the inner and outer circles:

$$|z| = \tfrac{1}{2}, \qquad |z| = 1$$



For $z = re^{i\theta}$ on this boundary we have

$$|f(z)| = \frac{e^{r\cos\theta}}{r}, \qquad r = \tfrac{1}{2}, 1$$

The maximum thus occurs when $\cos\theta = 1$ or $\theta = 0$. But now

$$|f(z)| = \begin{cases} 2\sqrt{e} \approx 3.3, & r = \frac{1}{2},\ \theta = 0 \\ e \approx 2.7, & r = 1,\ \theta = 0 \end{cases}$$

so the maximum value $|f(z)| = 2\sqrt{e}$ occurs at $z = \frac{1}{2}$. ■

Exercise: If $f(z) = e^z/z$, find the minimum of $|f(z)|$ on the annulus $\frac{1}{2} \leq |z| \leq 1$. (Hint: Find the maximum of $|g(z)|$ with $g(z) = 1/f(z) = ze^{-z}$.) ■

Exercise: Let $f(z) = 1/z$. Find the point or points where $|f(z)|$ takes its maximum and minimum values on the annulus $1 \leq |z| \leq 2$. ■

Liouville's Theorem

Lemma 73 (Cauchy's Inequality)
If $f(z)$ is analytic inside and on the circle C given by $|z-a|=r$ and $|f(z)|\le M$ on C, then

$$\boxed{|f^{(n)}(a)|\le \frac{M\,n!}{r^n},\qquad n=0,1,2,\ldots}$$

Proof: On C we have $z=a+re^{it}$ and $z'(t)=ire^{it}$. So using Cauchy's integral formula

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{(z-a)^{n+1}}\,z'(t)\right|dt$$

$$= \frac{n!}{2\pi}\int_0^{2\pi}\left|\frac{f(a+re^{it})\,ire^{it}}{r^{n+1}e^{(n+1)it}}\right|dt \le \frac{M\,n!}{r^n} \qquad \blacksquare$$

Theorem 74 (Liouville's Theorem) *If $f(z)$ is entire and bounded $|f(z)|\le M$ in $\mathbb{C}$, then $f(z)$ is constant.*

Proof: By Cauchy's inequality with $n=1$

$$|f'(z)|\le\frac{M}{r}$$

Letting $r\to\infty$, we find $f'(z)=0$ and so $f(z)$ is constant since $\mathbb{C}$ is connected. $\blacksquare$

- Note that for this theorem to apply $f(z)$ must be bounded as $|z|\to\infty$.

Application of Liouville's Theorem

Example: Prove the identity

$$\begin{aligned} & \sin(z+u)\sin(z-u)\sin(v+w)\sin(v-w) - \sin(z+w)\sin(z-w)\sin(v+u)\sin(v-u) \\ = \; & \sin(z+v)\sin(z-v)\sin(u+w)\sin(u-w), \qquad z,u,v,w \in \mathbb{C} \end{aligned}$$

Solution: View the LHS and RHS as functions of z and show

$$\frac{\text{LHS}}{\text{RHS}} = \{\text{entire and bounded}\}, \qquad u \neq \pm w + k\pi, \quad k \in \mathbb{Z}$$

The RHS vanishes when $z = \pm v + k\pi$, $k \in \mathbb{Z}$. Setting $z = \pm v + k\pi$ in the LHS gives

$$\sin(\pm v+u)\sin(\pm v-u)\sin(v+w)\sin(v-w) - \sin(\pm v+w)\sin(\pm v-w)\sin(v+u)\sin(v-u) = 0$$

It follows that LHS/RHS is an entire function of z. But LHS/RHS is continuous, periodic in the real direction and bounded in the imaginary direction since

$$f(z) = \frac{\sin(z+u)}{\sin(z+v)} = \frac{e^{2iz+iu} - e^{-iu}}{e^{2iz+iv} - e^{-iv}} = \text{bounded as } z \to \pm i\infty$$

$$\lim_{z \to \pm i\infty} |f(z)| = \left| \frac{e^{\mp iu}}{e^{\mp iv}} \right| \leq e^{|u|+|v|} = M$$

It follows that LHS/RHS is bounded. Since LHS/RHS is entire and bounded, it follows by Liouville's theorem that it is constant. Setting $z = w$ we find

$$\frac{\text{LHS}}{\text{RHS}} = 1 \qquad \blacksquare$$

Exercise: Prove this identity algebraically. $\blacksquare$

Exercise: Use Liouville's theorem to prove the identity

$$\sin v \, \sin(w-v) - \sin u \, \sin(w-u) = \sin(v-u)\sin(w-u-v) \qquad \blacksquare$$

Fundamental Theorem of Algebra

Theorem 75 (Fundamental Theorem of Algebra)
Every polynomial equation of degree $n \geq 1$

$$\boxed{P_n(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0, \quad a_0, a_1, \ldots, a_n \in \mathbb{C}}$$

with $a_n \neq 0$ *has exactly* n *roots (solutions) counted according to multiplicity.*

Proof: (i) First we show that there exists at least one root. Proceed by contradiction. Suppose $P_n(z) = 0$ has no root, then $f(z) = \dfrac{1}{P_n(z)}$ is entire. Moreover, since

$$\lim_{|z| \to \infty} |f(z)| = 0$$

$f(z)$ is bounded on $\mathbb{C}$. So, by Liouville's theorem, $f(z)$ and $P_n(z)$ are constant. But this is a contradiction. We conclude that $P_n(z) = 0$ must have at least one root.

(ii) Suppose $z = z_1$ is a root of $P_n(z) = 0$. Then we apply polynomial division

$$\begin{aligned} P_n(z) - P_n(z_1) &= (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) - (a_n z_1^n + a_{n-1} z_1^{n-1} + \cdots + a_1 z_1 + a_0) \\ &= a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \cdots + a_1(z - z_1) \\ &= (z - z_1) Q_{n-1}(z) = P_n(z) \end{aligned}$$

where $Q_{n-1}(z)$ is a polynomial of degree $n-1$. By iterating, $P_n(z)$ has exactly n roots. ■

- Note that the n roots need not be distinct. For example, the polynomial $P_2(z) = (z-1)^2$ has a double root at $z = 1$.

Factorization of Polynomials

Corollary 76 (Factorization of Polynomials)
Every polynomial of degree $n \geq 1$

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

with $a_n \neq 0$ *factorizes into* n *complex linear factors*

$$\boxed{P_n(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n)}$$

where the n *roots* $z_1, z_2, \ldots, z_n$ *of* $P_n(z)$ *satisfy*

$$\sum_{j=1}^{n} z_j = -\frac{a_{n-1}}{a_n}, \qquad \sum_{i<j} z_i z_j = \frac{a_{n-2}}{a_n}, \qquad \ldots \qquad z_1 z_2 \ldots z_n = (-1)^n \frac{a_0}{a_n}$$

Proof: Iterating the fundamental theorem of algebra gives

$$P_n(z) = (z - z_1) Q_{n-1}(z) = (z - z_1)(z - z_2) R_{n-2}(z) = \cdots = (z - z_1)(z - z_2) \cdots (z - z_n)\, S_0$$

where the polynomial S_0 of degree zero is a constant. Expanding this polynomial gives

$$\begin{aligned} P_n(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 \\ &= S_0 \Big[z^n - (z_1 + z_2 + \cdots + z_n) z^{n-1} + \cdots + (-1)^n z_1 z_2 \ldots z_n \Big] \end{aligned}$$

So equating coefficients we find $a_n = S_0$ and

$$\begin{aligned} a_{n-1} &= -S_0\, (z_1 + z_2 + \cdots + z_n) \\ a_{n-2} &= S_0\, (z_1 z_2 + z_1 z_3 + \cdots + z_{n-1} z_n) \\ \ldots & \quad \ldots \\ a_0 &= (-1)^n S_0\, z_1 z_2 \ldots z_n \qquad \blacksquare \end{aligned}$$

Roots and Polynomials

Example: If $z^4 = 1$ and $z \neq 1$ show that $1 + z + z^2 + z^3 = 0$.

Solution: The roots of $z^4 - 1 = (z^2 - 1)(z^2 + 1) = 0$ are $z = \pm 1,\ \pm i$. Hence

$$z^4 - 1 = (z-1)(z+1)(z-i)(z+i) = 0$$

But $z \neq 1$ so

$$(z+1)(z-i)(z+i) = 0 \quad \Rightarrow \quad (z+1)(z^2+1) = 0 \quad \Rightarrow \quad z^3 + z^2 + z + 1 = 0 \qquad \blacksquare$$

Corollary 77 (Roots in $\mathbb{Z}$)
Suppose $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $P_n(z)$ factors as

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = (z - z_1)(b_{n-1} z^{n-1} + \cdots + b_1 z + b_0)$$

where $z_1 \in \mathbb{Z}$ and $b_0, b_1, \ldots, b_{n-1} \in \mathbb{Z}$. Then the integer root z_1 must be an integer factor of $a_0 = -z_1 b_0$.

- This Corollary may yield integer roots of polynomials with integer coefficients:

Example: Find the 3 roots of the cubic

$$P_3(z) = z^3 - z^2 - z - 2 = 0$$

Solution: The factors of $a_0 = -2$ are $\pm 1, \pm 2$ so try $z = \pm 1, \pm 2$

$$P_3(2) = 8 - 4 - 2 - 2 = 0 \ \Rightarrow \ (z-2) \text{ is a factor}$$

$$\Rightarrow \ P_3(z) = (z-2)(z^2 + z + 1) \quad \text{by polynomial division}$$

$$\Rightarrow \ z = 2,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \qquad \blacksquare$$

Conjugate Roots and Quadratic Factors

Corollary 78 (Conjugate Roots) *Suppose* $a_0, a_1, \ldots, a_n \in \mathbb{R}$ *and*

$$P_n(z) \;=\; a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

where $z = z_1 \in \mathbb{C}$ *is a root* $P_n(z_1) = 0$. *Then* $z = \overline{z}_1$ *is also a root, that is,* $P_n(\overline{z}_1) = 0$.

Proof: Taking the complex conjugate of $P_n(z_1) = 0$ gives

$$\overline{P_n(z_1)} = \overline{a_n z_1^n + a_{n-1} z_1^{n-1} + \cdots + a_2 z_1^2 + a_1 z_1 + a_0} = \overline{0}$$
$$\Rightarrow \;\; a_n \overline{z}_1^n + a_{n-1} \overline{z}_1^{n-1} + \cdots + a_2 \overline{z}_1^2 + a_1 \overline{z}_1 + a_0 = P_n(\overline{z}_1) = 0 \qquad \blacksquare$$

Example: Expand into linear and quadratic factors the degree six polynomial

$$P_6(z) = z^6 - z^5 - 3z^4 + z^3 + 3z^2 - z - 2 = 0$$

Solution: Try $z = \pm 1, \pm 2$ as roots:

$$P_6(-1) = P_6(2) = 0 \;\Rightarrow\; (z+1), (z-2) \text{ are factors}$$

Using polynomial division

$$P_6(z) \;=\; (z+1)(z^5 - 2z^4 - z^3 + 2z^2 + z - 2) \;=\; (z+1)(z-2)(z^4 - z^2 + 1) = 0$$

Solving the quadratic in $w = z^2$ gives the other roots

$$z^2 = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm \pi i/3} \;\Rightarrow\; z = \pm e^{\pm \pi i/6}$$

Hence, since $e^{\frac{\pi i}{6}} + e^{-\frac{\pi i}{6}} = 2\cos\frac{\pi}{6} = \sqrt{3}$,

$$\begin{aligned} P_6(z) &= (z+1)(z-2)(z + e^{\frac{\pi i}{6}})(z + e^{-\frac{\pi i}{6}})(z - e^{\frac{\pi i}{6}})(z - e^{-\frac{\pi i}{6}}) \\ &= (z+1)(z-2)(z^2 + \sqrt{3}\,z + 1)(z^2 - \sqrt{3}\,z + 1) \qquad \blacksquare \end{aligned}$$

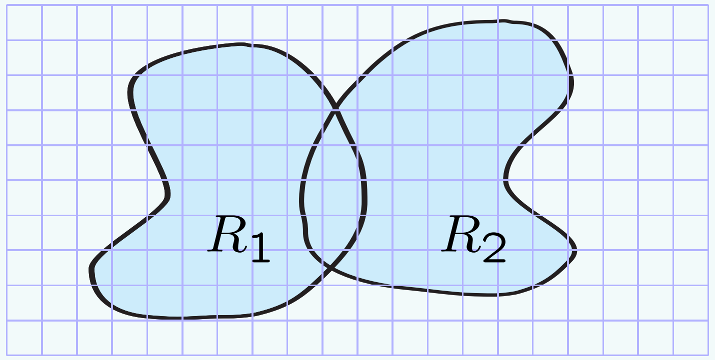
Analytic Continuation

Theorem 79 (Identity Theorem)
If $f_1(z)$ and $f_2(z)$ are analytic in an open connected region R and $f_1(z) = f_2(z)$ on some open neighbourhood of z, then $f_1(z) = f_2(z)$ on R.

- An analytic function in a connected region R is thus determined by its values on an arbitrarily small open neighbourhood in R. The proof derives from complex Taylor series and the result does not apply to real differentiable functions.

Definition: Let $f_1(z)$ be analytic in the open connected region R_1. Suppose $f_2(z)$ is analytic in the open connected region R_2 and $f_1(z) = f_2(z)$ in $R_1 \cap R_2$. Then we say that $f(z)$ is *an analytic continuation of* $f_1(z)$ and analytic in the combined region $R = R_1 \cup R_2$.

$$\boxed{f(z) = \begin{cases} f_1(z), & z \in R_1 \\ f_2(z), & z \in R_2 \end{cases}}$$



It is sometimes impossible to extend a function analytically beyond the boundary of a region. This boundary is then called a *natural boundary*.

Exercise: Show that $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ has a singularity at each root of unity satisfying $z^{2^n} = 1$. Deduce that $|z| = 1$ is a *natural boundary*. ■

Non-Uniqueness of Analytic Continuation

Theorem 80 (Non-Uniqueness of Analytic Continuation)
If an analytic function $f_1(z)$ in R_1 is extended by two different paths to an open connected region R_n, then the two analytic continuations are identical if no singularity lies between the two paths. If the two analytic continuations are different then a branch point lies between the two paths.

- If the function $f_1(z) = \operatorname{Log} z$, analytic in $|z-1| < 1$, is extended using Taylor series to the negative real axis by analytic continuation clockwise and anti-clockwise around the branch point at $z = 0$, then the two analytic continuations differ by $2\pi i$.

Exercise: Verify this using $R_1 : |z-1| < 1$, $R_2 : |z-i| < 1$, $R_3 : |z+1| < 1$ for the clockwise path and $R_1' : |z-1| < 1$, $R_2' : |z+i| < 1$, $R_3' : |z+1| < 1$ for the anti-clockwise path. ■

- There is no problem analytically continuing around a singularity in the form of a pole. Indeed, this is given by the Laurent expansion. Analytic continuation around poles is sometimes called meromorphic continuation.

- We can now give a more precise definition of a branch point:

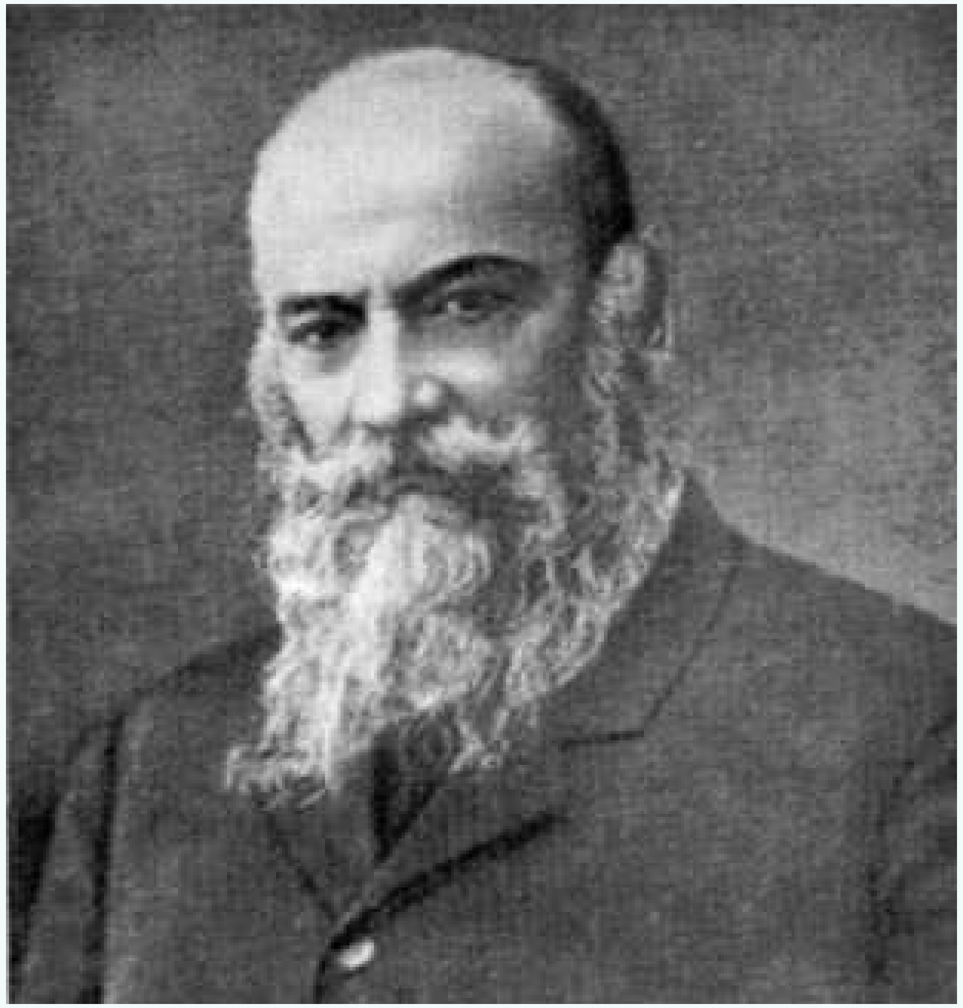
Definition: If analytic continuation of the function $f(z)$ full circle around a point $z = z_0$ brings you back to a different branch, then $z = z_0$ is a *branch point* of the multi-valued function $f(z)$. This is called non-trivial *monodromy*. Note that analytic continuation around a point on a branch cut always produces trivial monodromy.

Week 11: Conformal Transformations

31. Analytic functions as conformal mappings
32. Möbius transformations and basic properties
33. Conformal transformations from Möbius transformations

August Ferdinand Möbius (1790–1868)

Nikolai Egorovich Joukowsky (1847–1921)



Conformal Maps

Definition: A conformal map $f : U \to V$ is a function which preserves angles. More specifically, f is conformal at a point if the angle between any two C^1 curves through the point is preserved under the mapping.

Theorem 81 (Analytic Maps are Conformal) *A function $f(z)$ analytic in an open neighbourhood of a with $f'(a) \neq 0$ is conformal at $z = a$. Since, by continuity, $f'(z) \neq 0$ in an open neighbourhood of a, it follows that $f(z)$ is conformal in a neighbourhood of a.*

Proof: If $\gamma : [0, 1] \to \mathbb{C}$ is a C^1 curve and $f(\gamma(t))$ its image then the tangent slopes are

$$\arg(\gamma'(t)), \quad \gamma'(t) \neq 0; \quad \arg(f(\gamma(t))'), \quad f(\gamma(t))' = f'(\gamma(t))\gamma'(t) \neq 0 \text{ if } f'(z) \neq 0 \text{ and } \gamma'(t) \neq 0$$

Let $\gamma_1 : [0, 1] \to \mathbb{C}$ and $\gamma_2 : [0, 1] \to \mathbb{C}$ be C^1 curves through the point $z = a$ with

$$\gamma_1(t_1) = \gamma_2(t_2) = a$$

The tangents to the curves at $z = a$ are $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ and the angle between them is

$$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)), \qquad \gamma_1'(t_1) \neq 0, \quad \gamma_2'(t_2) \neq 0$$

Assuming $f'(a) \neq 0$ and applying the chain rule gives

$$\frac{f(\gamma_2(t_2))'}{f(\gamma_1(t_1))'} = \frac{f'(\gamma_2(t_2))\gamma_2'(t_2)}{f'(\gamma_1(t_1))\gamma_1'(t_1)} = \frac{f'(a)\gamma_2'(t_2)}{f'(a)\gamma_1'(t_1)} = \frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}$$

The result follows by taking the argument on the left and right since

$$\arg\left(\frac{z_2}{z_1}\right) = \arg(z_2) - \arg(z_1) \qquad \blacksquare$$

- It follows that the group of conformal maps in two dimensions is infinite dimensional. By contrast, this group is finite dimensional in three or higher dimensions.

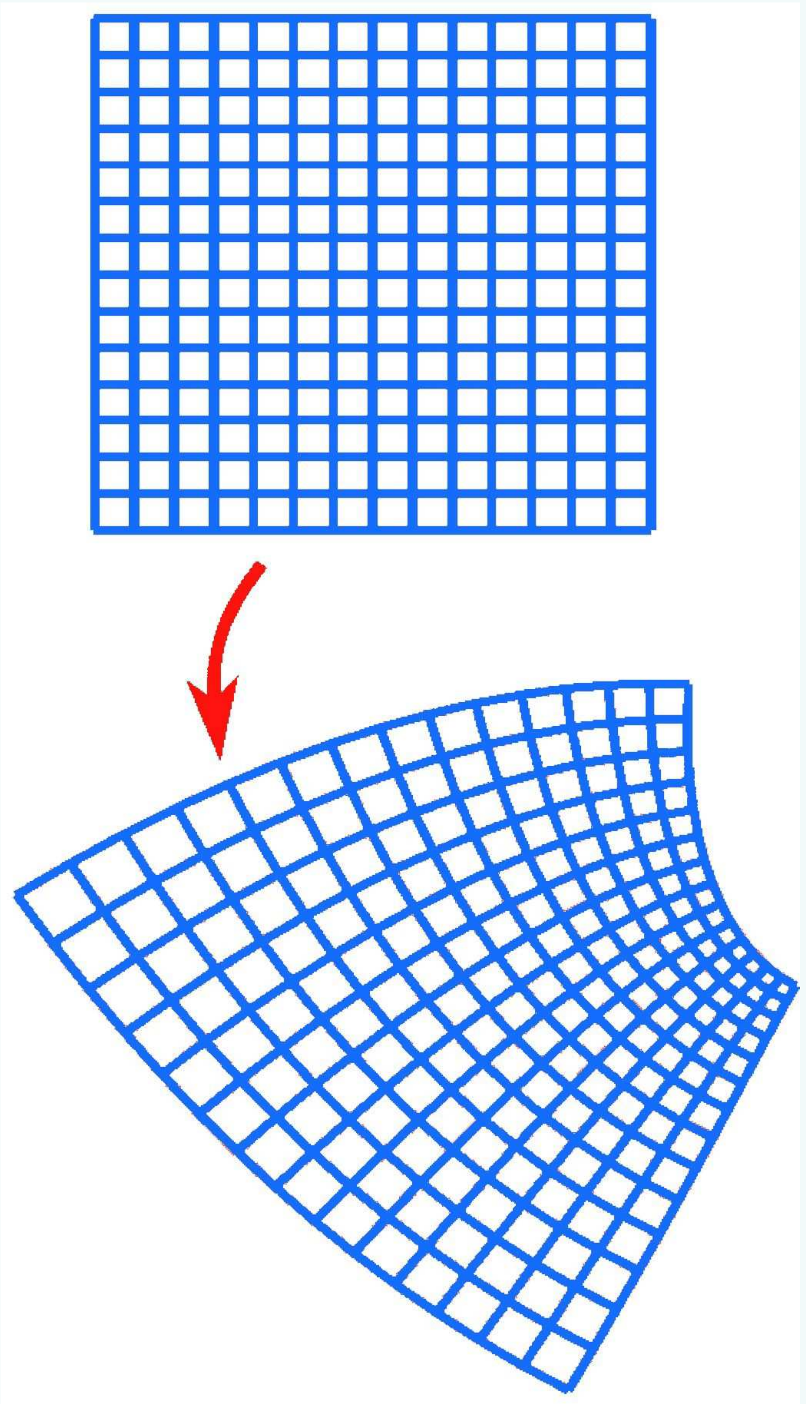
Riemann Mapping Theorem

- A non-constant analytic function maps open connected sets to open connected sets.

Theorem 82 (Riemann Mapping Theorem)
Let $D \neq \mathbb{C}$ be an open simply-connected domain. Then there is a one-to-one analytic function that maps D onto the interior of the unit circle. Moreover, one can prescribe an arbitrary point of D and a direction through that point which are mapped to the origin and the direction of the positive real axis, respectively. Under such restrictions the mapping is unique.

Proof: See textbook.

- Since a one-to-one analytic map is invertible, it follows that any open simply-connected domain can be mapped onto any other open simply-connected domain provided neither is $\mathbb{C}$.



Möbius Transformations

Definition: A *Möbius transformation* (linear fractional or bilinear transformation) is any non-constant function on $\hat{\mathbb{C}}$ of the form

$$\boxed{w = f(z) = \frac{az+b}{cz+d}, \qquad ad \neq bc, \qquad a,b,c,d \in \mathbb{C}}$$

$$f\Big(-\frac{d}{c}\Big) = \infty, \quad f(\infty) = \frac{a}{c} \quad (c \neq 0); \qquad f(\infty) = \infty \quad (c = 0)$$

- A Möbius transformation is conformal at every point except at its pole $z = -d/c$ since

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

- Möbius transformations form a group under compositions with the identity $I(z) = z$.

Exercise: The matrix associated with the Möbius transformation $w = f(z)$ is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

More precisely, $w = f(z)$ is associated with the set λA with $\lambda \neq 0$. Show that the matrix associated with the composition $f_1(f_2(z))$ of Möbius transformations is

$$A_1 A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \qquad \det A_1 A_2 = \det A_1 \det A_2 \neq 0$$

Also show that the inverse function f^{-1} satisfying $f(f^{-1}(z)) = f^{-1}(f(z)) = I(z)$ is given by

$$z = f^{-1}(w) = \frac{dw-b}{-cw+a}, \qquad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Elementary Möbius Maps

Möbius transformations include translations, rescalings, rotations and inversions.

- If we superimpose the z- and w-planes, we can view a conformal or Möbius map as a map from the complex plane onto itself $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

- Let $z_0 = x_0 + iy_0 \in \hat{\mathbb{C}}$ and $\lambda, \theta_0 \in \mathbb{R}$ with $\lambda > 0$. Then the elementary Möbius maps are:

1. Translation: $w = z + z_0$

This is translation in the Argand plane by the vector $(x_0, y_0) \in \mathbb{R}^2$.

2. Rescaling: $w = \lambda z$

This is a contraction if $0 < \lambda < 1$ and a magnification if $\lambda > 1$.

3. Rotation: $w = e^{i\theta_0} z$

This is a rotation of the complex plane about $z = 0$ by an angle θ_0. The rotation is anticlockwise of $\theta_0 > 0$ and clockwise if $\theta_0 < 0$.

4. Inversion: $w = \dfrac{1}{z}$

This is point-by-point inversion of the plane through the point at the origin $z = 0$ $(z \mapsto \dfrac{1}{\bar{z}})$ followed by reflection in the x-axis (complex conjugation).

Exercise: A linear transformation $w = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$ is a composition of a translation, rescaling and rotation.

Möbius Compositions

Theorem 83 (Möbius Compositions)

If $f(z)$ is a Möbius transformation then:

(i) $f(z)$ is the composition of a finite sequence of elementry maps in the form of translations, rescalings, rotations and inversions.

(ii) $f(z)$ maps $\hat{\mathbb{C}}$ one-to-one onto itself.

(iii) $f(z)$ maps the class of circles and lines to itself. Note that a circle is uniquely determined by 3 distinct noncollinear points. A line is uniquely determined by 2 distinct points to which we can add the point at ∞ on the Riemann sphere. A line is just a (great) circle which passes through ∞.

Proof: (i) We have

$$f(z) = \frac{az+c}{cz+d} = f_4(f_3(f_2(f_1(z))))$$

where

$$f_1(z) = z + \frac{d}{c}, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = -\frac{ad-bc}{c^2}z, \quad f_4(z) = z + \frac{a}{c}$$

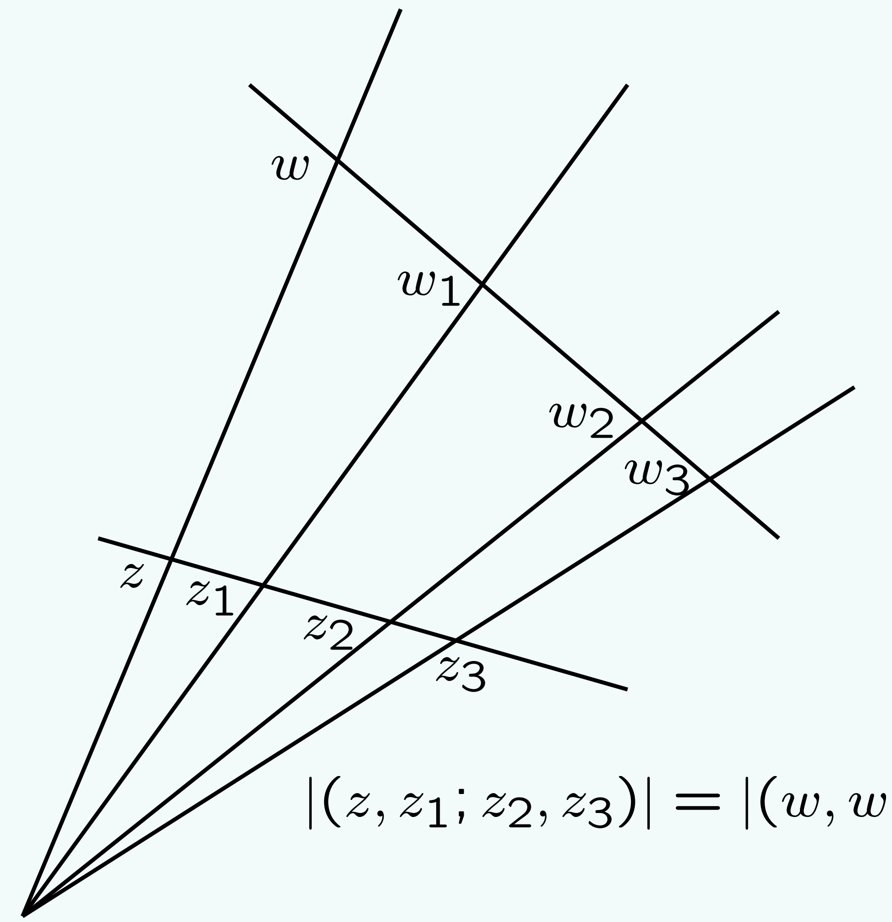
(ii)-(iii) See textbook. ■

Crossratio

- The crossratio is defined by

$$\boxed{(z, z_1; z_2, z_3) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} = \text{cross-ratio}}$$

- The crossratio is fundamental in perspective drawings and projective geometry.



$$|(z, z_1; z_2, z_3)| = |(w, w_1; w_2, w_3)|$$

Exercise: Show that

$$(z, z_1; z_2, z_3) + (z, z_2; z_1, z_3) = 1$$

$$(z, z_1; z_2, z_3) + (z_3, z_1; z_2, z) = 1$$

$$(z, z_1; z_2, z_3)(z, z_3; z_2, z_1) = 1$$

$$(z, z_1; z_2, z_3)(z_2, z_1; z, z_3) = 1$$

$$(z, z_1; z_2, z_3) = (z_1, z; z_3, z_2) = (z_2, z_3; z, z_1) = (z_3, z_2; z_1, z)$$

Exercise: Show that the crossratio is invariant under the following simultaneous transformations of z, z_1, z_2, z_3: (i) translations, (ii) rescalings, (iii) rotations (about the origin) and (iv) inversions (through the origin) and hence invariant under arbitrary Möbius transformations.

Constructing Möbius Transformations

Example: Find a Möbius transformation that maps the unit disk $|z| < 1$ onto the right half-plane $\mathrm{Re}\, w > 0$ such that $f(1) = \infty$ and $f(-1) = 0$.

Solution: A Möbius transformation takes the form

$$\boxed{w = f(z) = \frac{az + b}{cz + d}}$$

Since $f(1) = \infty$ and $f(-1) = 0$, we see that $d = -c$ to ensure the divergence at $z = 1$ and $b = a$ to ensure the vanishing at $z = -1$. Hence

$$f(z) = -\frac{a}{c}\frac{1+z}{1-z} = \lambda\frac{1+z}{1-z}$$

To ensure $f(\pm i)$ lie on the imaginary w-axis we find

$$f(\pm i) = \pm i\lambda, \qquad \lambda = -\frac{a}{c} \in \mathbb{R}$$

From the properties of Möbius transformations, we conclude that the unit circle is mapped into the line $\mathrm{Re}\, w = 0$. The image $f(0)$ of the point $z = 0$ at the centre of the disk

$$f(0) = -\frac{a}{c} = \lambda > 0$$

for this image to be in the right half-plane $\mathrm{Re}\, w > 0$. Choosing the positive rescaling factor $\lambda = 1$ gives the Möbius transformation

$$w = f(z) = \frac{1+z}{1-z}$$

- The answer is not unique since a rescaling by $\lambda \neq 1$ leaves the right half plane invariant.

Three-Point Uniqueness

- The general Möbius transformation

$$w = f(z) = \frac{az + b}{cz + d}$$

appears to involve four complex parameters a, b, c, d. However, since $ad \neq bc$, either $a \neq 0$ or $c \neq 0$ or both a and c are non-zero. We can therefore express the transformation with three unkown coefficients

$$w = f(z) = \frac{z + \frac{b}{a}}{\frac{c}{a} z + \frac{d}{a}} \qquad \text{or} \qquad w = f(z) = \frac{\frac{a}{c} z + \frac{b}{c}}{z + \frac{d}{c}}$$

- It follows that there exists a unique Möbius transformation $w = f(z)$ that maps the three distinct points z_1, z_2, z_3 onto the three points w_1, w_2, w_3 respectively. An implicit formula for the mapping $w = f(z)$ is given by

$$\boxed{\frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1}}$$

If the line or circles are oriented by the order of z_1, z_2, z_3 and w_1, w_2, w_3 respectively, then the Möbius transformation maps the region to the left (left-region) of z_1, z_2, z_3 onto the region to the left (left-region) of w_1, w_2, w_3.

Exercise: Show, using cross-ratios, that the unique Möbius transformation $w = f(z)$ such that $f(-1) = 0$, $f(0) = a$, $f(1) = \infty$ is

$$f(z) = a \frac{1 + z}{1 - z}$$

Circles to Circles

Example: Find a Möbius transformation that maps the circle through the distinct noncollinear points z_1, z_2, z_3 to the circle through the distinct noncollinear points w_1, w_2, w_3.

Solution: Suppose the Möbius transformation $w = T(z)$ is such that $T(z_1) = 0$, $T(z_2) = 1$ and $T(z_3) = \infty$, that is, $T(z)$ maps the circle onto the real axis $\operatorname{Im} w = 0$. Following the previous example, we find

$$T(z) = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} = (z, z_1; z_2, z_3) = \text{cross-ratio}$$

Similarly, suppose the Möbius transformation $z = S(w)$ is such that $S(w_1) = 0$, $S(w_2) = 1$ and $S(w_3) = \infty$, that is, $S(w)$ maps the circle onto the real axis $\operatorname{Im} z = 0$. We find

$$S(w) = \frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1} = (w, w_1; w_2, w_3) = \text{cross-ratio}$$

The required Möbius transformation is thus given by

$$w = f(z) = S^{-1}(T(z)) \quad \Leftrightarrow \quad S(w) = T(z) \quad \Leftrightarrow \quad (w, w_1; w_2, w_3) = (z, z_1; z_2, z_3)$$

because

$$\boxed{\begin{aligned} f(z_1) &= S^{-1}(T(z_1)) = S^{-1}(0) = w_1 \\ f(z_2) &= S^{-1}(T(z_2)) = S^{-1}(1) = w_2 \\ f(z_3) &= S^{-1}(T(z_3)) = S^{-1}(\infty) = w_3 \end{aligned}}$$

- The order of the points is important. Specifically, if the circles are oriented such that the points are traversed in the given order, then the left-region is mapped onto the left region where each left-region is an interior or exterior of the associated circle.

Conformal Mappings of Bounded Regions

Theorem 84 (Parametric Boundaries)
Suppose that a C^1 curve C, which may be closed or open, has parametric equations

$$x = F(t), \qquad y = G(t), \qquad t \in [a, b]$$

Then, assuming $f'(z) \neq 0$, the conformal map

$$z = f(w) = F(w) + iG(w), \qquad F, G \text{ analytic}$$

maps the real axis of the w-plane onto the curve C in the z-plane.

Proof: See textbook. For closed curves, this often gives map between bounded regions.

Exercise: Find a conformal transformation that maps the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}, \qquad a, b > 0$$

onto the interval $[0, 2\pi]$ along the real axis of the w-plane and the interior of the ellipse onto a suitable rectangle in the strip $0 < \operatorname{Re} w < 2\pi$ within the upper-half w-plane.

- Conformal transformations map simply-connected open sets to simply-connected open sets. To apply boundary conditions we need to include the boundary and extend these maps to the closed sets.

Theorem 85 (Caratheodory)
Suppose that U,V are a pair of simply-connected open sets whose boundaries $\partial U, \partial V$ are simple continuous closed (Jordan) curves. Then any conformal map of U one-to-one onto V extends to a continuous map of $U \cup \partial U$ one-to-one onto $V \cup \partial V$.

Proof: See textbook.

Conformal Mapping of Laplace's Equation

Theorem 86 (Conformal Mapping of Laplace's Equation)
(i) Suppose that an analytic function

$$w = f(z) = u(x, y) + iv(x, y)$$

maps a domain D_z in the z-plane onto a domain D_w in the w-plane. If $h(u, v)$ is a harmonic function of u, v on D_w, that is it satisfyies Laplace's equation $\nabla^2 h(u, v) = 0$, then

$$H(x, y) = h(u(x, y), v(x, y)) = \textit{harmonic function of } x, y \textit{ in } D_z$$

(ii) Suppose that $C = \partial D_z$ and $\Gamma = f(C) = \partial D_w$ are C^1. Then Dirichlet or Neumann boundary conditions are preserved. Explicitly, if along Γ

$$h(u, v) = h_0 \in \mathbb{R} \qquad \textit{or} \qquad \frac{dh}{dn} = \textit{normal derivative to } \Gamma = 0$$

then along C

$$H(x, y) = h_0 \in \mathbb{R} \qquad \textit{or} \qquad \frac{dH}{dN} = \textit{normal derivative to } C = 0$$

Proof: (i) Let $w = f(z) = u + iv$ and let $k = k(u, v)$ be the harmonic conjugate of $h = h(u, v)$. Then $g(w) = h + ik$ is analytic. So

$$H(x, y) = h(u(x, y), v(x, y)) = \mathrm{Re}\, g(w) = \mathrm{Re}\, g(f(z)) = \text{harmonic function of } x, y$$

because the composite function $g(f(z))$ is analytic.

- It follows that a solution $H(x, y)$ of Laplace's equation $\nabla^2 H(x, y) = 0$ in a complicated domain D_z in the xy-plane can be obtained by using a conformal mapping $f(z)$ of the domain in the xy-plane onto a simpler domain D_w in the uv-plane and solving Laplace's equation there.

Applications in Potential Theory

Consider Laplace's equation for $\phi = \phi(x, y)$ in a domain D with specified boundary conditions on $\Gamma = \partial D$

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$$

- **Heat Flow:**

In heat flow, Laplace's equation governs the temperature distribution $\phi(x, y)$ and the curves $\phi =$ constant are *isotherms*. Typically, the temperature is fixed on parts of the boundary $\Gamma = \partial D$. The rest of the boundary is assumed to be insulating so that the normal derivative to Γ vanishes $\dfrac{\partial\phi}{\partial n} = 0$.

- **Electrostatics:**

In electrostatics, $\phi(x, y)$ is the electric potential with electric field $\boldsymbol{E} = \nabla\phi$ and $\nabla \cdot \boldsymbol{E} = 0$. The curves $\phi =$ constant are *equipotentials*. Typically, one specifies either the potential or the normal component of $\boldsymbol{E}$ on the boundary $\Gamma = \partial D$.

- **Fluid Flow:**

In fluid flow, $\phi(x, y)$ is the stream function and the curves $\phi =$ constant are *streamlines*. For flow around a nonporous body, the perimeter must be part of a streamline.

Cylindrical Capacitor

Example: Find the electrostatic potential $\phi(z)$ between two long cylindrical conductors such that $\phi = 0$ on $\left|z - \frac{3}{10}\right| = \frac{3}{10}$ and $\phi = 1$ on $|z| = 1$.

Solution: Since $\operatorname{Log} w = \operatorname{Log}|w| + i\operatorname{Arg}(w)$, $w \neq 0$ is analytic in an annulus (with a cut), $\operatorname{Log}|w|$ is harmonic. So the radially symmetric solution we seek for two concentric cylinders of radii r, R with $0 < r < R$ is

$$\psi(w) \;=\; \frac{\operatorname{Log}(|w|/r)}{\operatorname{Log}(R/r)}, \qquad r < |w| < R$$

We need to find a conformal mapping to relate the two geometries. The required Möbius transformation and its inverse are

$$w = f(z) = -\frac{z - \frac{1}{3}}{z - 3}, \qquad z = f^{-1}(w) = \frac{3w + \frac{1}{3}}{w + 1}$$

It is verified that the inner and outer circles in the z-plane are mapped onto concentric circles in the w-plane

$$|w| = \left|f\Big(\frac{3}{10}(1 + e^{i\theta})\Big)\right| = \frac{1}{9} = r, \qquad |w| = |f(e^{i\theta})| = \frac{1}{3} = R, \qquad \theta \in [0, 2\pi]$$

The solution in the w-plane is thus

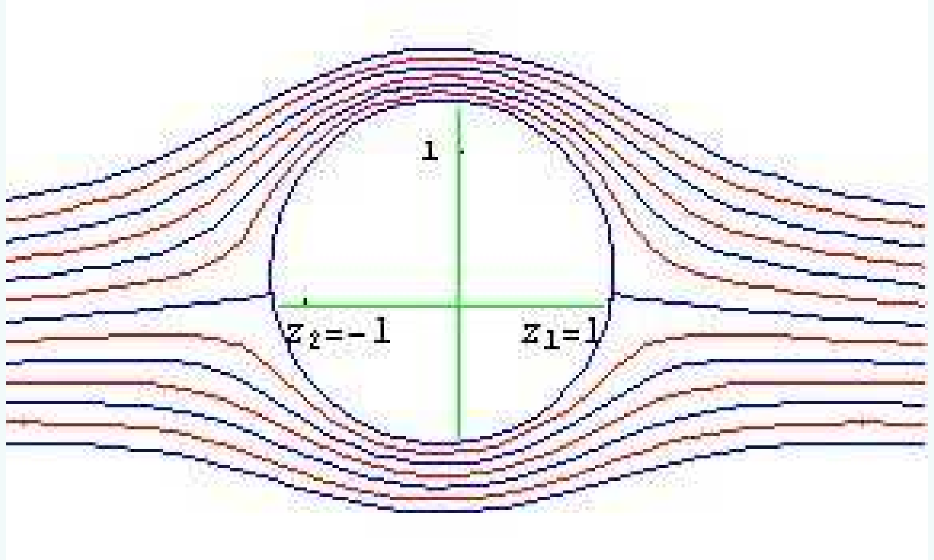
$$\psi(w) = \frac{\operatorname{Log} 9|w|}{\operatorname{Log} 3}, \qquad \frac{1}{9} < |w| < \frac{1}{3}$$

Transforming back, we find

$$\phi(z) = \psi\Big(\frac{z - \frac{1}{3}}{z - 3}\Big) = \frac{\operatorname{Log}(|9z - 3|/|z - 3|)}{\operatorname{Log} 3} \qquad \blacksquare$$

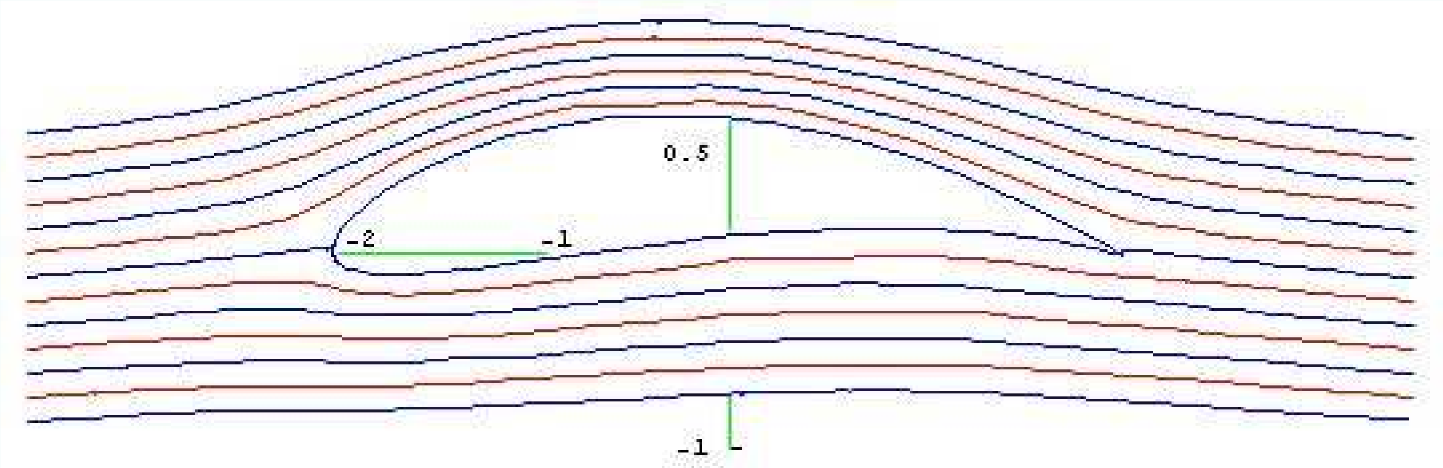
Joukowsky Airfoil

- In 1908, the mathematician Joukowsky considered the flow around an off-centre cylinder



- Miraculously, this cylinder is mapped onto the Joukowsky airfoil under the conformal Joukowsky mapping

$$\boxed{w = J(z) = z + \frac{1}{z}}$$



- It is therefore possible to obtain the airflow around the Joukowsky airfoil by studying the airflow around a cylinder. This technique is of major importance in aerodynamics!

Week 12: Gamma and Zeta Functions

34. The Gamma function
35. General discussion of the Zeta function
36. Revision

Leonhard Euler
(1707–1783)

Georg Friedrich Bernhard Riemann
(1826–1866)

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Gamma Function

Definition: The *gamma function* is defined by

$$\boxed{\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx, \qquad \mathrm{Re}\, p > 0}$$

From this integral expression, $\Gamma(p)$ has no branch points and is analytic for $\mathrm{Re}\, p > 0$.

- Integrating by parts gives

$$\begin{aligned}\Gamma(p) &= -\int_0^\infty x^{p-1} \frac{d}{dx}(e^{-x})\, dx = -\Big[x^{p-1}e^{-x}\Big]_0^\infty + (p-1)\int_0^\infty x^{p-2} e^{-x} dx \\ &= (p-1)\,\Gamma(p-1), \quad \text{provided} \quad \mathrm{Re}\, p > 1\end{aligned}$$

This recursion determines $\Gamma(p)$ for $p \in \mathbb{N}$. Since $\Gamma(1) = 1$, iteration gives

$$\Gamma(p) = (p-1)(p-2)(p-3)\ldots(3)(2)\Gamma(1) = (p-1)!$$

Similarly, if p is half an odd integer, iteration shows that $\Gamma(p)$ is a multiple of

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x}\, dx = 2\int_0^\infty e^{-y^2}\, dy = \sqrt{\pi}, \qquad x = y^2$$

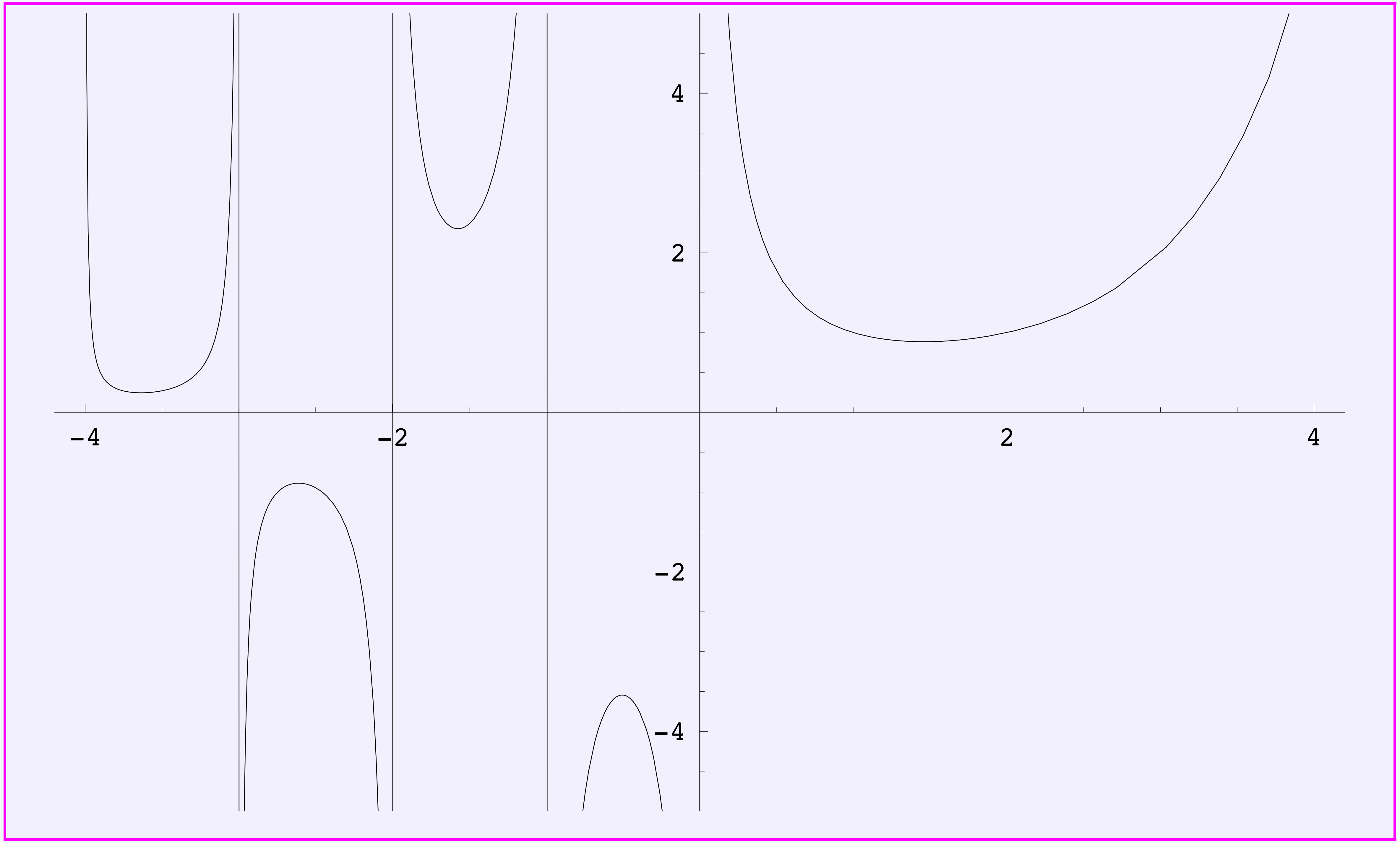
- Iterating the reverse recursion relation is used to analytically continue $\Gamma(p)$ to $\mathrm{Re}\, p \le 0$

$$\Gamma(p) = \Gamma(p+1)/p \quad \Rightarrow \quad \Gamma(p) = \frac{\Gamma(p+n+1)}{p(p+1)(p+2)\cdots(p+n)}, \qquad p \ne 0, -1, -2, -3, \ldots$$

It follows that $\Gamma(p)$ is analytic everywhere in $\mathbb{C}$ except for simple poles at $p = 0, -1, -2, \ldots$.

Graph of the Gamma Function

- A graph of $\Gamma(p)$ for real p:



Beta Function

Definition: A related function is the *beta function* $B(r,s)$ defined by

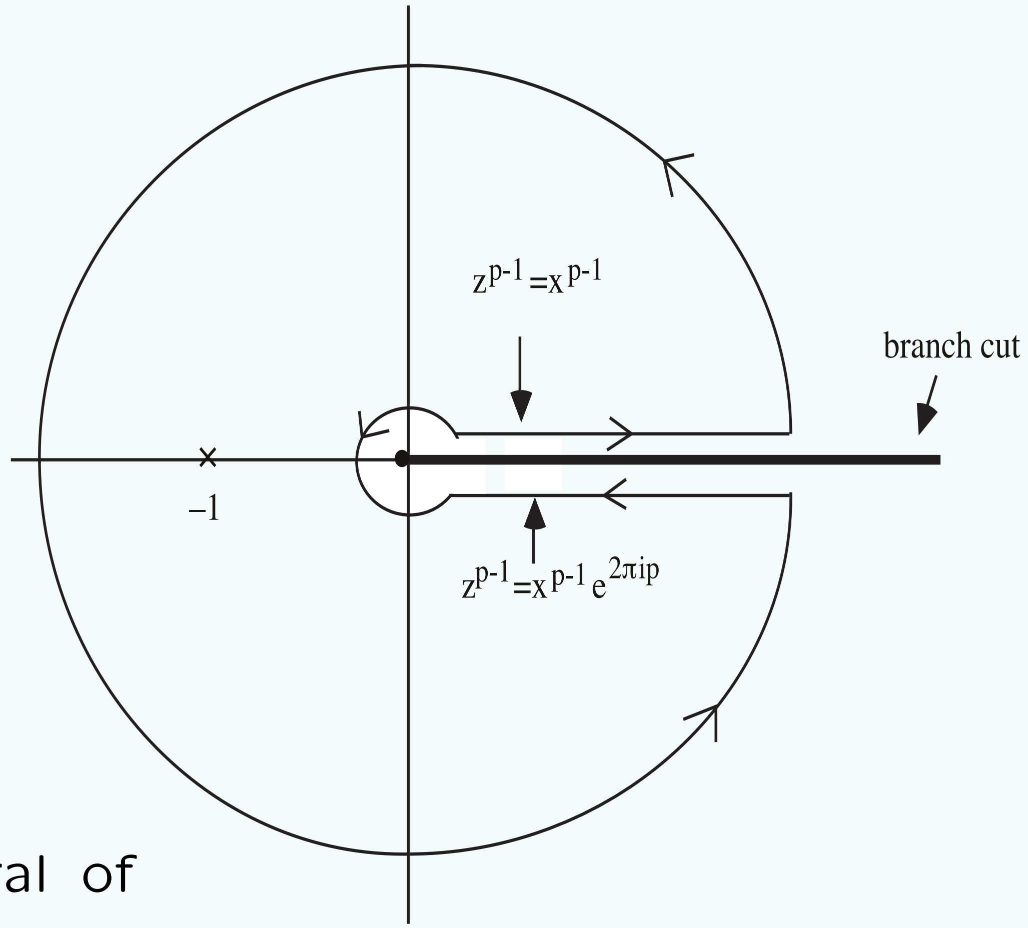
$$\boxed{B(r,s) = \int_0^1 u^{r-1}(1-u)^{s-1}\,du}$$

Consider the product

$$\Gamma(r)\Gamma(s) = \int_0^\infty x^{r-1}e^{-x}\,dx \int_0^\infty y^{s-1}e^{-y}\,dy$$

as a double integral over the first quadrant in the x-y plane. Substituting $x+y=u$ we find

$$\begin{aligned}\Gamma(r)\Gamma(s) &= \int_0^\infty e^{-u}\left(\int_0^u x^{r-1}(u-x)^{s-1}\,dx\right)du \\ &= \int_0^\infty u^{r+s-1}e^{-u}\,du \int_0^1 t^{r-1}(1-t)^{s-1}\,dt \\ &= \Gamma(r+s)B(r,s)\end{aligned}$$



In the second step we substituted $x = ut$, $dx = u\,dt$.
In particular, since $\Gamma(1) = 1$

$$\Gamma(p)\Gamma(1-p) = B(p,1-p) = \int_0^1 u^{p-1}(1-u)^{-p}\,du$$

Or, after substituting $u = x(1+x)^{-1}$,

$$\Gamma(p)\Gamma(1-p) = \int_0^\infty x^{p-1}(1+x)^{-1}\,dx$$

This integral can be evaluated by considering the integral of $z^{p-1}(1+z)^{-1}$ $(0 < \operatorname{Re} p < 1)$ around the contour C.

Reflection Formula

Using residues and limiting contours theorems I/III (Uniformity on an Arc) we find

$$\begin{aligned}
\oint_C z^{p-1}(1+z)^{-1}\,dz &= 2\pi i\,\mathrm{Res}(-1) = 2\pi i\,e^{i\pi(p-1)} = -2\pi i\,e^{i\pi p} \\
&= \int_0^\infty x^{p-1}(1+x)^{-1}\,dx + \int_\infty^0 x^{p-1}e^{2\pi i p}(1+x)^{-1}\,dx \\
&= -(e^{2\pi i p}-1)\int_0^\infty x^{p-1}(1+x)^{-1}\,dx
\end{aligned}$$

After rearranging we find

$$\int_0^\infty x^{p-1}(1+x)^{-1}\,dx = \frac{\pi}{\sin \pi p}$$

The final result is called the *reflection formula*

$$\boxed{\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}, \qquad 0 < \mathrm{Re}\,p < 1}$$

- Although we derived the reflection formula for $0 < \mathrm{Re}\,p < 1$, it extends straightforwardly to all $p \notin \mathbb{Z}$ using the recursion relation for the Γ function.
- Using the fact that $\pi/\sin(\pi z)$ has simple poles at $z = \pm n$, $n = 0, 1, 2\ldots$ with residues $(-1)^n$, the reflection formula shows that $\Gamma(p)$ has simple poles at $p = -n$, $n = 0, 1, 2, \ldots$ with residues

$$\frac{(-1)^n}{\Gamma(1+n)} = \frac{(-1)^n}{n!}$$

Gamma Function Summary

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx, \qquad \mathrm{Re}\, p > 0$$

Special Values	$\Gamma(1) = 1$ $\Gamma(1/2) = \sqrt{\pi}$
Recurrence Relation	$\Gamma(p) = (p-1)\Gamma(p-1)$
Reflection Formula	$\Gamma(p)\Gamma(1-p) = \pi / \sin(\pi p)$
Singularities	Simple poles at $p = 0, -1, -2, \cdots$
Residues	$\mathrm{Res}(-n) = (-1)^n / n! \qquad n = 0, 1, 2, \cdots$

Zeta Function

- The Riemann zeta function is defined by

$$\boxed{\zeta(p) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad \mathrm{Re}\, p > 1}$$

It is analytically (more precisely meromorphically) continued to $\mathrm{Re}\, p > 0$ through the alternating Riemann zeta function $\zeta^*(p)$/Dirichlet eta function $\eta(p)$

$$\boxed{\zeta^*(p) \;=\; \frac{\eta(p)}{1-2^{1-p}}, \qquad \eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, \qquad \mathrm{Re}\, p > 0, \quad p \neq 1}$$

with $\zeta^*(p) = \zeta(p)$ for $\mathrm{Re}\, p > 1$. The eta function $\eta(p)$ is absolutely convergent for $\mathrm{Re}\, p > 1$ and conditionally convergent for $0 < \mathrm{Re}\, p < 1$ by a convergence test for Dirichlet series

$$\Big|\sum_{k=1}^{n} a_k\Big| \text{ bounded for large } n \quad \Rightarrow \quad \Big|\sum_{n=1}^{\infty} \frac{a_n}{n^p}\Big| < \infty, \qquad a_n \in \mathbb{C}, \quad \mathrm{Re}\, p > 0;$$

- The Riemann zeta function is then analytically continued to $\mathrm{Re}\, p \leq 0$ by a reflection formula in the form of the *Riemann relation*

$$\boxed{2\Gamma(p)\,\zeta(p)\cos(\pi p/2) \;=\; (2\pi)^p \zeta(1-p)}$$

This formula is not obvious and requires an integral representation of the eta function

$$\eta(p) = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{x^{p-1}}{e^x + 1}\, dx, \qquad \mathrm{Re}\, p > 0$$

- It follows that $\zeta(p)$ has no branch points and is an analytic function in the complex p-plane except for a simple pole at $p = 1$ with residue equal to unity.

Zeta Function Summary

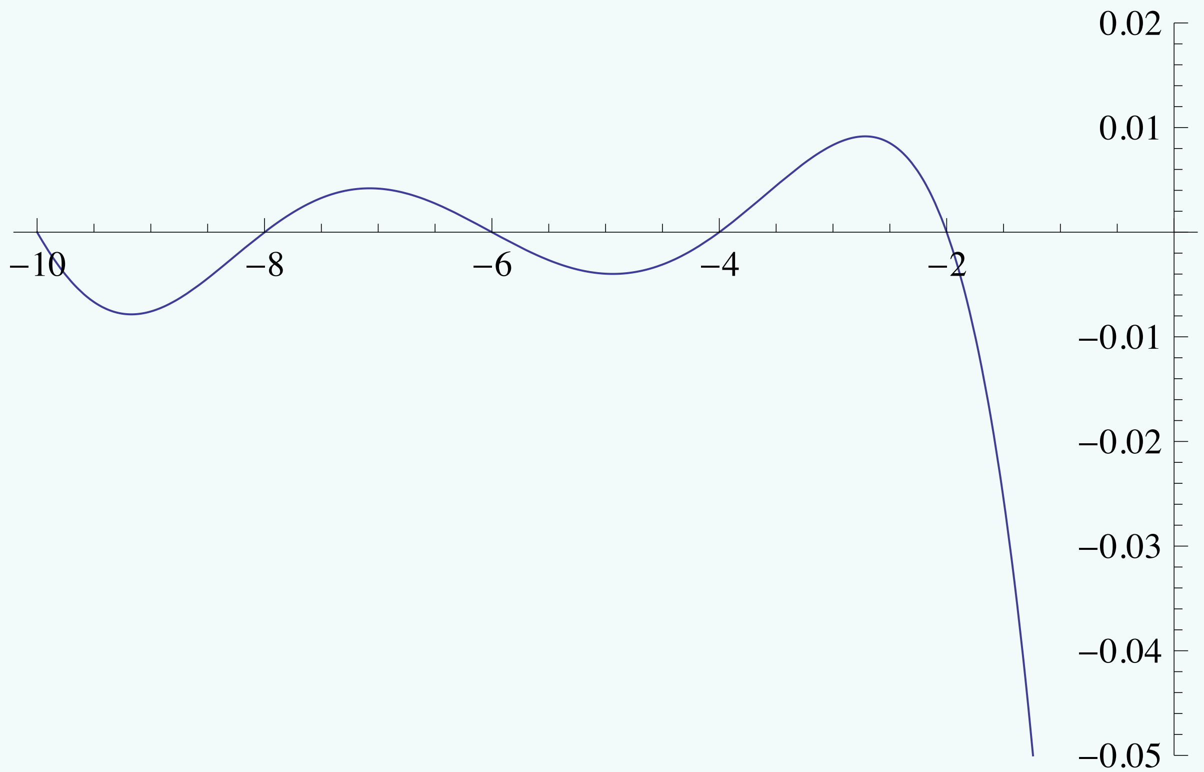
- Riemann/Euler Formulas

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} = \prod_{n \text{ prime}} \frac{1}{1-n^{-p}}, \qquad \mathrm{Re}\, p > 1$$

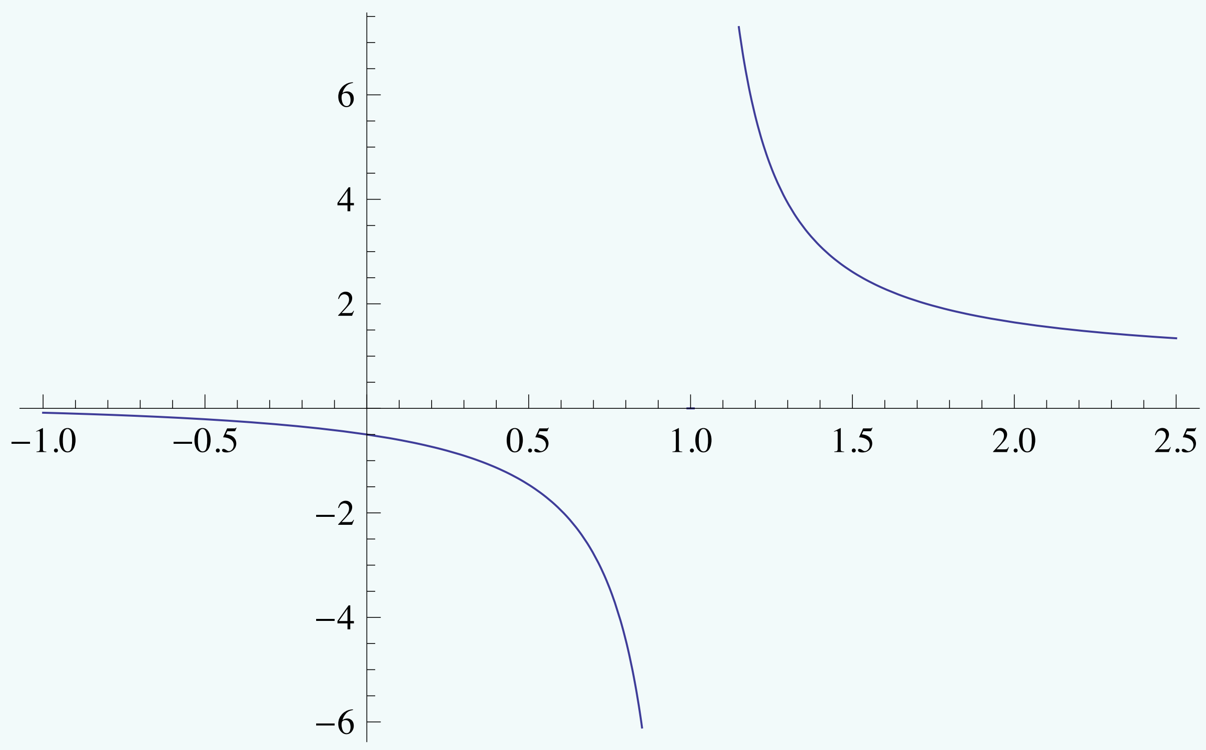
Special values	$\zeta(-1) = -1/12$ $\zeta(0) = -1/2$ $\zeta(2) = \pi^2/6$ $\zeta(4) = \pi^4/90$
Riemann Relation	$2\Gamma(p)\zeta(p)\cos(p\pi/2) = (2\pi)^p\zeta(1-p)$
Singularity	Simple pole at $p = 1$, $\mathrm{Res}(1) = 1$
Zeros	Trivial zeros at $p = -2, -4, -6, -8, \ldots$ Non-trivial zeros on the line $\mathrm{Re}\, p = \frac{1}{2}$

Zeta Function on Real Axis

Trivial Zeros on Negative Real Axis

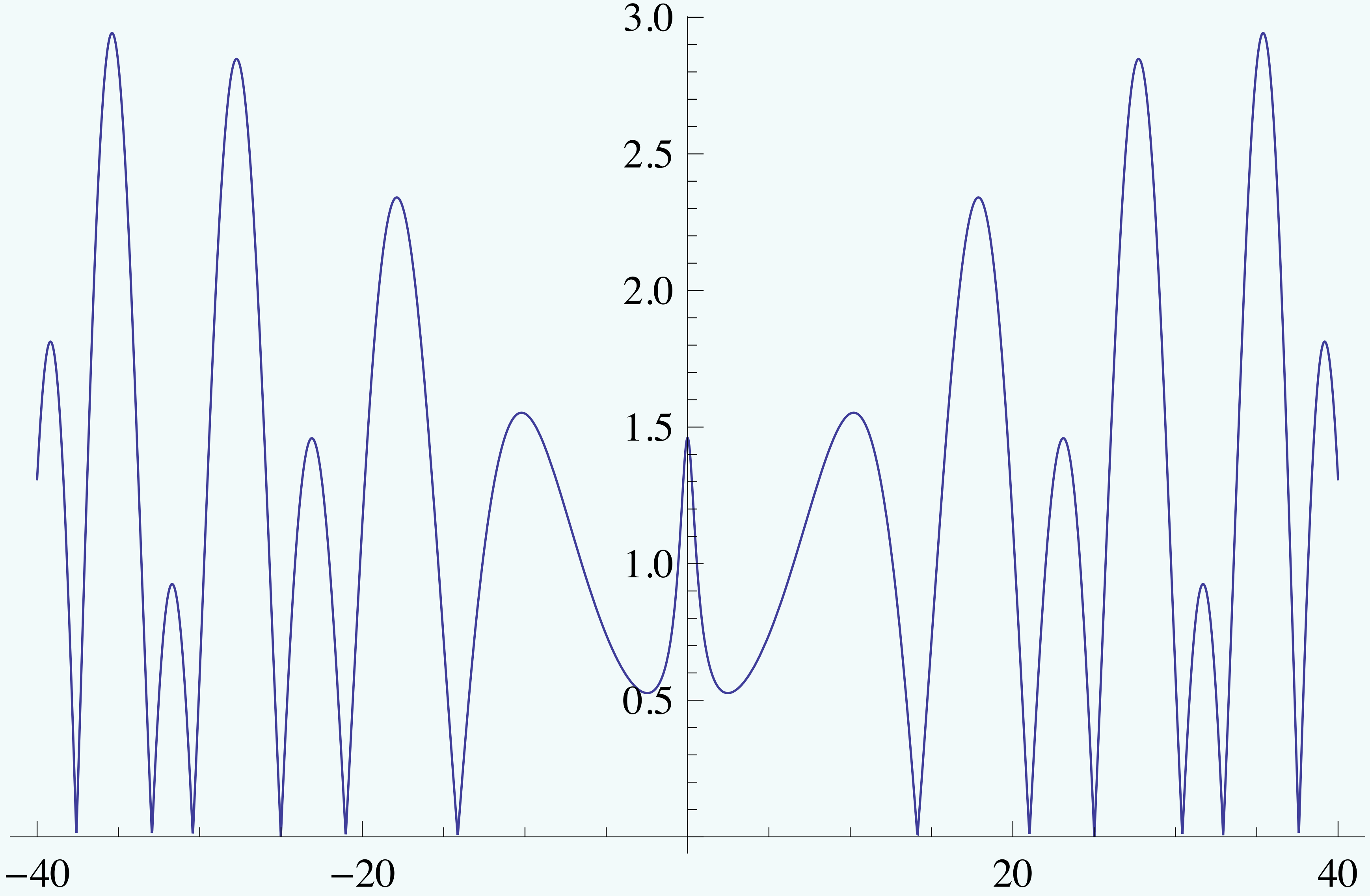


Pole at $z = 1$ on Positive Real Axis



Zeta Function on Imaginary Axis

Plot of $|\zeta(\frac{1}{2} + iy)|$ Showing Zeros on Imaginary Axis



- **Riemann Hypothesis.** Perhaps the most famous unproved mathematical conjecture is the *Riemann hypothesis* which states that all of the (non-trivial) zeros of $\zeta(p)$ lie exactly on the *critical line* $\operatorname{Re} p = 1/2$.