Physics 155 HW #1

Geometry, limits, and series

1.) In one of the derivations in class, we used the fact that the angle between two intersecting chords of a circle that also subtend a diameter was 90° (a right angle). In pictures this means

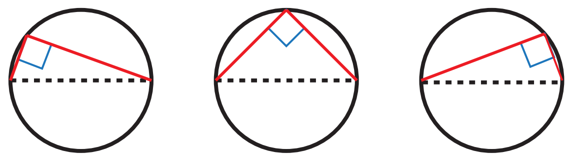


Figure 1: *Illustration of the theorem with three different triangles.*

You will prove this fact in this problem. Feel free to use your own technique if you like, but it *cannot* use trigonometry. You can only draw lines and use the Pythagorean theorem. I sketch a way to proceed below:

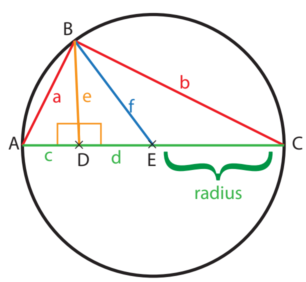


Figure 2: *One approach to the proof.*

Take any chord that subtends a diagonal of the circle and forms the triangle ABC. Drop a perpendicular from the diameter that intersects the circle at point B and denote the intersection point with the diameter as point D. Let E denote the center of the circle of radius r. The lengths of the line segments are noted as a, b, c, d, and e on the diagram. You can prove that the angle is 90° if you can show that $a^2 + b^2 = (2r)^2 = 4r^2$, since Pythagoras says this must be a right triangle. Find the appropriate relations that allow you to prove this fact.

2.) For the inscribed and circumscribed polygons that Archimedes used to estimate π, use trigonometry to show that $s_n = 2\sin\frac{\pi}{n}$ and $t_n = 2\tan\frac{\pi}{n}$.

Then show that s_n and s_{2n} satisfy

$$ {s_{2n}}^2 = \frac{{s_n}^2}{2 + \sqrt{4 - {s_n}^2}} $$

and

$$ \frac{2}{t_{2n}} = \frac{2}{t_n} + \sqrt{1 + \left(\frac{2}{t_n}\right)^2} $$

also hold by using the trig half-angle formulas. The following picture might be helpful.

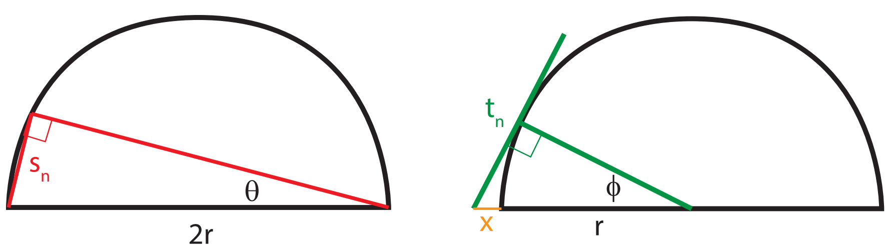


Figure 3: *Triangles needed for inscribed and circumscribed perimeters.*

You need to find the angles θ and ϕ to proceed.

3.) The Taylor series expansion for $\sin(x)$ is

$$ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... $$

Construct a polynomial approximation for $\sin(x)$ by truncating this series at some point, which has an error no larger than 1×10^{-4} for all three points $0, \frac{\pi}{4}, \frac{\pi}{2} = x$. Determine the cutoff n and explain how you determined the error was what it was.

Answer the following question. Is the polynomial of the order you found the *unique* polynomial that fits $\sin(x)$ to that accuracy or are there other ones? Note if you answer there are other ones you are *not* required to find an example. (If you can plot these results, it may help answer the question, but it is not required.)

4.) *Derive* the power series for e^x as a MacLaurin series (Taylor series expansion about $x = 0$).

5.) Assume x is small, take the power series for $\sin(x)$ and for $\cos(x)$ and compute, by expanding the denominator in a power series, the power series for $\tan(x)$ through seventh order in x. For example, the third-order contribution becomes

$$\tan(x) = \frac{x - \frac{1}{6}x^3 + ...}{1 - \frac{1}{2}x^2 + ...} = x(1 - \frac{1}{6}x^2)(1 + \frac{1}{2}x^2 + \frac{1}{4}x^4 + ...)$$

$$= x(1 + \frac{1}{3}x^2 + ...) = x + \frac{1}{3}x^3$$

Note that the expansion of the denominator gets tricky at higher order.

$$\frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ...} = 1 + (\frac{1}{2}x^2 - \frac{1}{24}x^4) + (\frac{1}{2}x^2 - \frac{1}{24}x^4)^2 + ...$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + ...$$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + ...$$

You will need this expansion through x^6 for this question.

Physics 155 HW # 2

1.) Using the methods developed in class, find the general formula for

$$\sum_{j=1}^{N} j^3 \tag{1}$$

2.) Evaluate $\int_0^\infty dx\, e^{-ax}$ for a real number $a > 0$. Use the following procedure:

a.) expand e^{-ax} in a MacLaurin series in x

b.) interchange the limit of integration and summing the series

c.) integrate each term in the series, but do not evaluate at the two limits yet

d.) sum the series to express the antiderivative as a function of e^{-ax}

e.) evaluate at the limits to get the final answer

Note that step b.) is only valid for an absolutely convergent series.

3.) a.) Repeat the above procedure for $\int_0^1 \sin(ax)\, dx$ for $a > 0$.

b.) Find a mathematical argument for why the above integral in a.) is less than or equal to $\dfrac{a}{2}$ by considering an appropiate bound for $\sin(ax)$.

4.) In lecture we stated, without proof, that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{2}$$

Provide a proof for this result and numerically evaluate for a range of different n values. How large does n need to be to get 4 digits of accuracy for e?

5.) Describe why a logarithm table to base 10 is more convenient to calculate with than a table to base e.

Physics 155 HW # 3

1.) The logarithmic derivative of a function $f(x)$ is given by:

$$\frac{\frac{df(x)}{dx}}{f(x)} = \frac{d}{dx} \ln f(x) \tag{1}$$

a.) Compute the logarithmic derivative of ax^n and $a(x-c)^n$.

b.) Prove that the logarithmic derivative of a product of two functions $W(x) = U(x)V(x)$ is the sum of the logarithmic derivatives of U and V.

c.) Find the logarithmic derivative of

$$(x-a_1)^{n_1}(x-a_2)^{n_2}(x-a_3)^{n_3}...(x-a_p)^{n_p}$$

for integers n_1, n_2,..., n_p.

d.) Find a function whose logarithmic derivative is 0, 1, or $2x$.

2.) Use the appropriate trig substitution to integrate (*i.e.* compute the antiderivative)

$$\int dx \sqrt{\frac{1+x}{1-x}} \tag{2}$$

Hint: manipulate the equation to get a $\sqrt{1-x^2}$ in the denominator before using a trig substitution.

3.) Use induction to show that

$$\begin{aligned} \frac{d^n}{dx^n}(fg) &= \frac{d^n f}{dx^n} g + \binom{n}{1} \frac{d^{n-1} f}{dx^{n-1}} \frac{dg}{dx} + \binom{n}{2} \frac{d^{n-2} f}{dx^{n-2}} \frac{d^2 g}{dx^2} + \ldots \\ &+ \binom{n}{n-1} \frac{df}{dx} \frac{d^{n-1} g}{dx^{n-1}} + f \frac{d^n g}{dx^n} \end{aligned} \tag{3}$$

where $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ is the binomial coefficient.

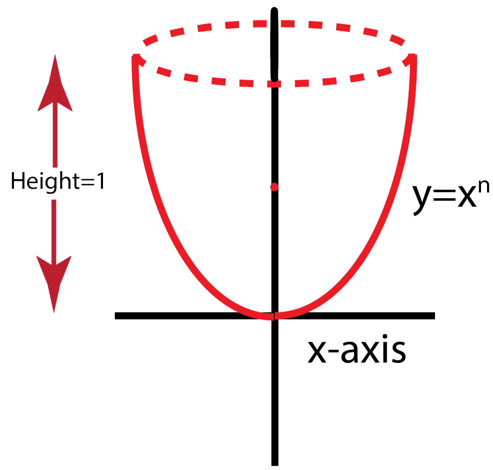
4.) Evaluate the following integrals analytically:

a.) $$\int_0^a \frac{dx}{x^2 + a^2} \tag{4}$$

b.) $$\int_{-1}^1 dx \, \sqrt{e^x} \tag{5}$$

c.) $$\int_0^1 dx \, \cos^{-1}\left(\sqrt{1 - x^2}\right) \tag{6}$$

5.) Evaluate the surface area and the volume of the surface of revolution given by a function $y = x^n$ with a height $h = 1$.



For the integrals, work them out to a one-dimensional integral. You won't be able to integrate the general case, so do only $n = 1$ and $n = 2$, which can be integrated. Also determine these values numerically (give a decimal answer).