P1 Calculus 3
Calculus of Vectors and Fields

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Course Page www.robots.ox.ac.uk/~dwm/Courses/1CA3

This course on calculus is concerned with the **calculus of vectors**, and the **calculus of scalar and vector fields**.

We start by applying calculus of single variables to individual vectors and vector relationships defined in fixed coordinate systems. We then consider how to describe curves in 3D space and introduce spatially moving coordinates, followed by two varieties of temporally moving systems.

We then move on to consider calculus applied to scalar and vector *fields*. A scalar field is such that a scalar quantity has a defined value at each location in some space. For example, one might define pressure $P(x, y, z, t) \equiv P(\mathbf{x}, t)$ at each location in the atmosphere. The wind velocities generated by this pressure field will have different magnitudes and directions at each location, and therefore comprise a vector field $\mathbf{v}(\mathbf{x}, t)$. The overall aim is to examine how calculus can be applied effectively in 3D without recourse to arbitrary coordinate systems.

Notation. We will use a bold font $\mathbf{a}, \mathbf{b}, \ldots, \boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$ to represent vectors and the corresponding non-bold font $a, b, \ldots, \alpha, \beta, \ldots$ for their magnitudes. Unit vectors will wear hats $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \ldots, \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \ldots$. The unit vector along Cartesian x, y, z axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Later we will require unit vectors in cylindrical and spherical coordinates. These will be $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$.

In your written work, **underline the vector symbol** and be meticulous doing so. Utter chaos ensues if you can't distinguish vectors from scalars in expressions like $r^2\mathbf{r}$. Remember too that if the left hand side of an equation is a vector, then so must the right hand side; and ditto for scalars.

Lecture Content

1. Differentiation of vectors and vector expressions. Space curves, the position vector and its differential. Arc-length. Tangents, normals, and binormials. Coordinate sytems that change — I: Frenet-Serret; II: vector derivatives in plane-polars; III rotation and Coriolis.
2. Introduction to scalar and vector fields. Scalar field and the gradient operator grad, examples and significance. Gradients, fields and potentials. An introduction to line integrals using vectors.
3. Vectors fields and the divergence operator div: examples and significance of divergence. Flux density and surface integrals. Vectors fields and the curl operator: examples and significance. Combining vector operators.
4. Vector calculus in other coordinate systems. The position vector and its differential in curvilinear systems, and in the polars. Metric scale parameters, and their link to multiple integration. Some examples of the geometry, involving simple line and surface integrals.

Tutorial Sheets

The tutorial sheet associated with this course is

- 1P1J: Calculus of Vectors and Fields

Reading

- James, G. (2004) **Advanced Modern Engineering Mathematics**, Prentice-Hall, 3rd Ed., ISBN: 0-13-045425-7 (paperback).
- Kreyszig, E. (1999) **Advanced Engineering Mathematics**, John Wiley & Sons, 8th Ed., ISBN: 0-471-15496-2 (paperback).
- Sokolnikoff, I.S. and Redheffer, R.M. (1966) **Mathematics and Physics of Modern Engineering**, McGraw-Hill 2nd ed. (A classic.)

Course WWW Pages

Pdf copies of these notes (in colour), copies of the lecture slides, the tutorial sheets, corrections, answers to FAQs etc, will be accessible from

www.robots.ox.ac.uk/∼dwm/Courses/1CA3 ,

and

www.canvas.ox.ac.uk

Lecture 1

Differentiating Vector Functions of a Single Variable

Part A: Fixed coordinate systems

1.1 Differentiation of a vector with respect to a scalar

It should be no great surprise that we often wish differentiate vector functions. For example, suppose you were driving along a wiggly road. Your position vector (the vector that points from the origin to where you are *now*) is $\mathbf{r}(t)$ at time t. Differentiating $\mathbf{r}(t)$ with respect to time will yield your velocity $\mathbf{v}(t)$, and differentiating $\mathbf{v}(t)$ will yield acceleration $\mathbf{a}(t)$. (Differentiating an acceleration gives a jerk $\mathbf{j}(t)$; and we don't really want to know what happens if you differentiate a jerk.)

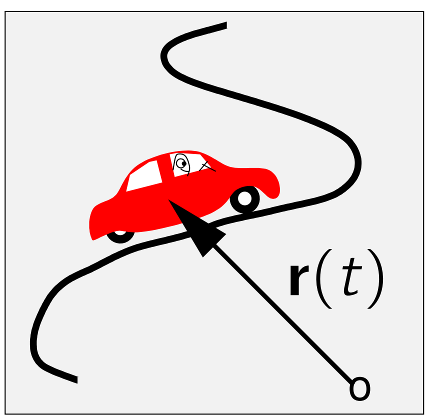


Figure 1.1: A position vector $\mathbf{r}(t)$ varying with time t.

A vector expression in n dimensions on the left and right hand sides of an equation is entirely equivalent to writing n scalar equations in the components of the vector. This indicates that the fundamental definition of the derivative must extend effortlessly to vectors. So, if we have a vector function $\mathbf{f}(p)$ of some single parameter p, it must be that

$$\frac{d\mathbf{f}}{dp} = \lim_{\delta p \to 0} \frac{\mathbf{f}(p + \delta p) - \mathbf{f}(p)}{\delta p} \ . \tag{1.1}$$

If we write $\mathbf{f}$ in terms of components relative to a FIXED coordinate system (for example, $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$ are fixed)

$$\mathbf{f}(p) = f_x(p)\hat{\mathbf{\imath}} + f_y(p)\hat{\mathbf{\jmath}} + f_z(p)\hat{\mathbf{k}} \quad \Rightarrow \frac{d\mathbf{f}}{dp} = \frac{df_x}{dp}\hat{\mathbf{\imath}} + \frac{df_y}{dp}\hat{\mathbf{\jmath}} + \frac{df_z}{dp}\hat{\mathbf{k}} \ . \tag{1.2}$$

That is, in order to differentiate a vector defined with respect to fixed axes, one simply differentiates each component separately.

For example, suppose $\mathbf{r}(t)$ is the position vector of a moving object:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}} \ . \tag{1.3}$$

Then its instantaneous velocity and acceleration are

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{\imath}} + \frac{dy}{dt}\hat{\mathbf{\jmath}} + \frac{dz}{dt}\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\hat{\mathbf{\imath}} + \frac{d^2y}{dt^2}\hat{\mathbf{\jmath}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} \ . \tag{1.4}$$

Returning to the general problem, it is obvious that $d\mathbf{f}/dp$ has a different magnitude from $\mathbf{f}$, but note that in general it also has a different direction. Because we are just dividing the vector $d\mathbf{f}$ by a scalar dp, it must be $d\mathbf{f}/dp$ has the same direction as $d\mathbf{f}$ but, not surprisingly, the change $d\mathbf{f}$ has a different direction to $\mathbf{f}$.

1.1.1 Differentiating vector relationships

It also follows that all the familiar rules of differentiation apply to addition, subtraction and multiplication — both scalar products and vector products. Thus, for the two product rules:

$$\frac{d}{dp}(\mathbf{f} \cdot \mathbf{g}) = \frac{d\mathbf{f}}{dp} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{d\mathbf{g}}{dp} \quad \text{and} \quad \frac{d}{dp}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dp} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dp} \ . \tag{1.5}$$

If you need convincing, a scalar or vector product will involve a summation with a particular pattern over terms involving products $f_i g_j$, where the subscripts indicate components. Differentation gives a summation *with the same pattern* over terms involving $(df_i/dp)g_j + f_i(dg_j/dp)$. And if you still need convincing, try writing it out fully ...

1.1.2 The chain rule

You'll also realize that the chain rule applies. If $\mathbf{f} = \mathbf{f}(u)$ and $u = u(t)$, then

$$\frac{d\mathbf{f}}{dt} = \frac{d\mathbf{f}}{du}\frac{du}{dt} \ . \tag{1.6}$$

♣ Example #1

Q: A 3D vector $\mathbf{f}$ of constant magnitude is varying over time. What can you say about the direction of $d\mathbf{f}/dt$?

A: Using intuition: if only its direction is changing, then the vector must be tracing out paths on the surface of a sphere. We would intuit that the derivative $d\mathbf{f}/dt$ is orthogonal to $\mathbf{f}$.

To prove this write

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{f}) = \mathbf{f}\cdot\frac{d\mathbf{f}}{dt} + \frac{d\mathbf{f}}{dt}\cdot\mathbf{f} = 2\mathbf{f}\cdot\frac{d\mathbf{f}}{dt} \quad . \tag{1.7}$$

But $(\mathbf{f}\cdot\mathbf{f}) = f^2$ which we are told is constant.

$$\Rightarrow \frac{d}{dt}(\mathbf{f}\cdot\mathbf{f}) = 0 \qquad \Rightarrow 2\mathbf{f}\cdot\frac{d\mathbf{f}}{dt} = 0 \tag{1.8}$$

and hence $\mathbf{f}$ and $d\mathbf{f}/dt$ must be perpendicular when $|\mathbf{f}|$ is constant.

♣ Example #2

Q: The position of a vehicle is $\mathbf{r}(u)$ where u is the amount of fuel consumed by some time t. Work out an expression for the acceleration.

A: The velocity and acceleration are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt} \qquad \mathbf{a} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left(\frac{d\mathbf{r}}{du}\frac{du}{dt}\right) = \frac{d^2\mathbf{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2} \; . \tag{1.9}$$

1.2 Integration of a vector function

As with scalars, integration of a vector function of a single scalar variable is the reverse of differentiation. That is,

$$\int_{p_1}^{p_2}\left[\frac{d\mathbf{f}(p)}{dp}\right]dp = \mathbf{f}(p_2) - \mathbf{f}(p_1) \; . \tag{1.10}$$

However, more interesting types of integral are possible, especially when the vector is a function of more than one variable. This requires the introduction of the concepts of scalar and vector fields. See later.

1.3 Differentiation a vector wrt a vector [An aside]

This is possible, but we don't need to bother with it here as it is just a notational device used in linear algebra. Imagine differentiating each component of the vector "on top" with respect to each component of the vector "on the bottom". You end up with a matrix of partial derivatives, not a vector.

1.4 The position vector, space curves, and derivatives

It is impossible to overstate the importance of the the position vector $\mathbf{r}$ in vector calculus. It plays a starring rôle. We first consider its use in defining space curves.

A **space curve** is — as it says on the can — a curve in 3D space. We will assume that each point on the curve has a different position vector $\mathbf{r}$. Referring to Fig. 1.2(a), suppose $\mathbf{r}$ is parameterized by some scalar p, so that by varying p we trace out the complete curve $\mathbf{r}(p)$. We can write $\mathbf{r}(p + \delta p) = \mathbf{r}(p) + \delta\mathbf{r}$. The small vector $\delta\mathbf{r}$ is obviously a secant to the curve, and $\delta\mathbf{r}/\delta p$ points in the same direction. (It must do! We are just dividing a little vector by a little scalar.)

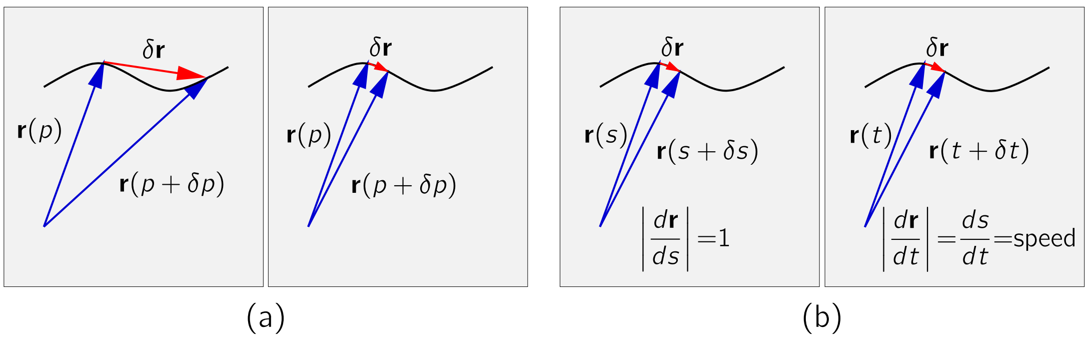


Figure 1.2: (a) $\delta\mathbf{r}$ is a secant to the curve but, in the limit as $\delta p \to 0$, becomes a tangent. (b) Two special parameters, arc-length s and time t.

But in the limit as δp tends to zero

$$\lim_{\delta p \to 0} \frac{\delta\mathbf{r}}{\delta p} \to \frac{d\mathbf{r}}{dp} \tag{1.11}$$

a quantity which, in the limit, must be a **tangent to the space curve**. Note though that using a general parameter p there is nothing special about the magnitude of the tangent. (For utter clarity, one might call it a non-unit tangent.)

Fig. 1.3 shows three of the infinity of ways of parametrizing a curve. However, there is one very special parametrization when p measures **arc-length**, denoted by s.

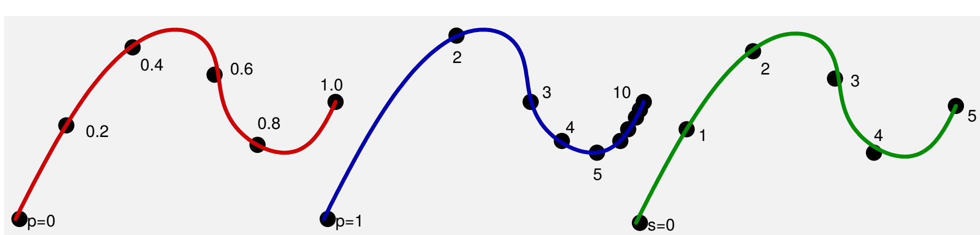


Figure 1.3: Different parametrizations describe the same curve. Arc-length s is special as it measures actual distance along the curve.

The difference in arc-length between two points on the curve is the actual distance travelled along the curve. But, for infinitesimally small movements, it must be that $ds = |d\mathbf{r}|$, so that $d\mathbf{r}/ds$ must be of unit length.

> If a curve $\mathbf{r}(s)$ is parametrized by the arc length s
> $$\frac{d\mathbf{r}}{ds} \text{ is everywhere a } \textbf{UNIT tangent} \text{ to the curve.}$$

In general, parameter p will not be arc-length. But the chain rule tells us that

$$\frac{d\mathbf{r}}{dp} = \frac{d\mathbf{r}}{ds}\frac{ds}{dp}\ . \tag{1.12}$$

So, the direction of the derivative $d\mathbf{r}/dp$ is that of a tangent to the curve, and its magnitude is $|ds/dp|$, the rate of change of arc length w.r.t the parameter.

An immediately interesting case (see Fig. 1.2b) is when p is time t. Then

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \tag{1.13}$$

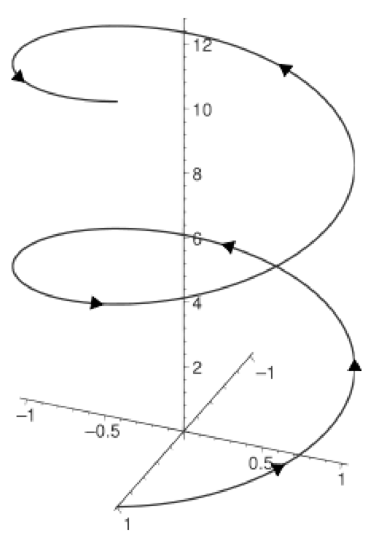
This indicates that when moving along a curve, the vector velocity is the unit tangent ($d\mathbf{r}/ds$) multiplied by the scalar speed (ds/dt). This makes sense!

1.4.1 ♣ Example: application to a helix

Q1: In the following equation parameter α can take any value. By considering its value in the range 0 to 2π, sketch the 3D space curve

$$\mathbf{r} = a\cos(\alpha)\hat{\mathbf{\imath}} \quad + \quad a\sin(\alpha)\hat{\mathbf{\jmath}} \quad + \quad \frac{\alpha P}{2\pi}\hat{\mathbf{k}}\ . \tag{1.14}$$

A1: The $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}$ components just trace out a circle. The $\hat{\mathbf{k}}$ component increases linearly from 0 and will be P when $\alpha = 2\pi$. It should be obvious that this is a helix with radius a and pitch P. The sketch is



Q2: Determine the **unit tangent** to the helix, and hence determine the relationship between α and arc-length s given $s = 0$ when $\alpha = 0$.

A2: The parameter α is (probably) not special. The **non-unit tangent** is

$$\frac{d\mathbf{r}}{d\alpha} = -a\sin(\alpha)\hat{\mathbf{\imath}} \quad + \quad a\cos(\alpha)\hat{\mathbf{\jmath}} \quad + \quad \frac{P}{2\pi}\hat{\mathbf{k}}\ . \tag{1.15}$$

Its magnitude is

$$\left|\frac{d\mathbf{r}}{d\alpha}\right| = \frac{ds}{d\alpha} = \sqrt{a^2 + (P/2\pi)^2}\ . \tag{1.16}$$

Hence the **unit tangent** is

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\alpha}\frac{d\alpha}{ds} = \frac{1}{\sqrt{a^2 + (P/2\pi)^2}}\left[-a\sin(\alpha)\hat{\mathbf{\imath}} \quad + \quad a\cos(\alpha)\hat{\mathbf{\jmath}} \quad + \quad \frac{P}{2\pi}\hat{\mathbf{k}}\right]\ . \tag{1.17}$$

Integrating $ds/d\alpha$ and using the given limit we find

$$\alpha = \frac{s}{\sqrt{a^2 + (P/2\pi)^2}}\ . \tag{1.18}$$

Q3. Show that the unit tangent to the curve has a constant elevation angle w.r.t the xy-plane, and determine the angle's magnitude.

A3. The unit tangent's projections onto the xy plane and the $\hat{\mathbf{k}}$ axis have magnitudes

$$\frac{a}{\sqrt{a^2 + (P/2\pi)^2}} \quad \text{and} \quad \frac{(P/2\pi)}{\sqrt{a^2 + (P/2\pi)^2}}. \tag{1.19}$$

Hence the elevation angle is $e = \tan^{-1}(P/(2\pi a))$, which is constant.

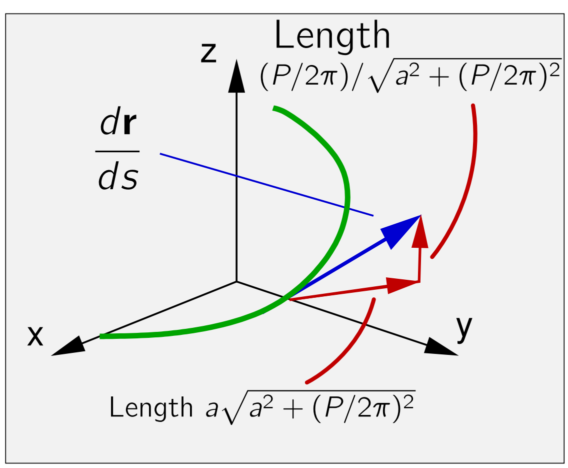


Figure 1.4: Working out the elevation angle between the tangent and the xy-plane.

1.5 More on r, its change dr, & parameters s, p

It is worth stressing that the position vector $\mathbf{r}$ in Cartesian coordinates is **always**

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \text{or using the parameter} \quad \mathbf{r}(p) = x(p)\hat{\mathbf{i}} + y(p)\hat{\mathbf{j}} + z(p)\hat{\mathbf{k}}\ . \tag{1.20}$$

It follows immediately that the differential

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\ . \tag{1.21}$$

But we have already noted that $ds = |d\mathbf{r}|$ and hence, as sketched in Fig. 1.5,

$$ds^2 = dx^2 + dy^2 + dz^2\ . \tag{1.22}$$

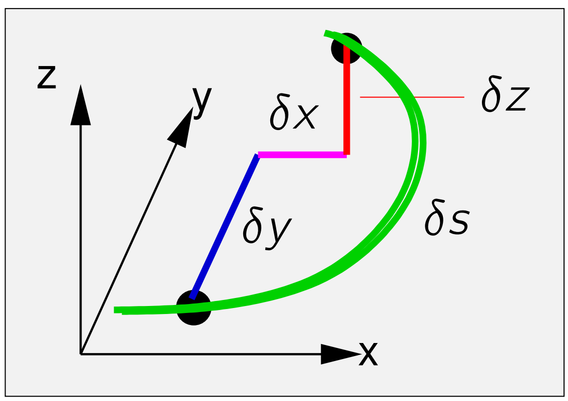


Figure 1.5: The curvature is WILDLY exaggerated! In the limit, δs is straight and $ds^2 = dx^2 + dy^2 + dz^2$.

So if a curve is parameterized in terms of p, and given as $x(p)$, $y(p)$ and $z(p)$,

$$\frac{ds}{dp} = \left[\frac{dx}{dp}^2 + \frac{dy}{dp}^2 + \frac{dz}{dp}^2 \right]^{1/2}\ . \tag{1.23}$$

This will be unity if and only if $p = s$.

From ds/dp one can easily find the relationship between s and p by integration.

Where next?

So far we have referred our vectors to a coordinate system that is fixed in space.

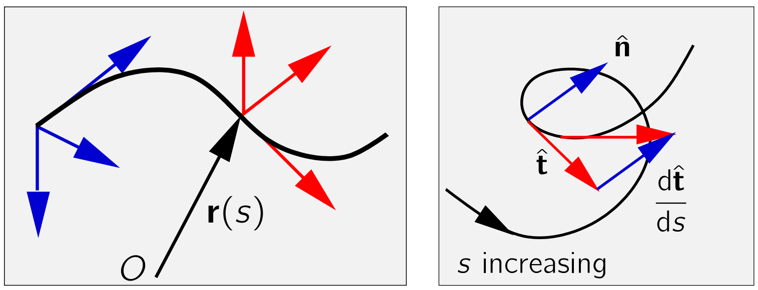
We now come to consider *three areas* where the coordinate systems change ...

Our brains seem better wired for fixed coordinates, so this can be confusing at times!

Part B: Coordinate systems that change

1.6 Coordinate systems that change (1): Frenet-Serret relations

We'll start with the idea of two French engineers and mathematicians, F-J. Frenet and J. A. Serret. They defined a local orthogonal coordinate system which is *intrinsic* to the curve — the system moves as one moves along the curve as illustrated in Fig. 1.6(a). The orientation of the coordinate system depends on the value of the arc-length parameter s at each point. It is helpful to note that we can always refer this changing intrinsic system to an external fixed system $Oxyz$.



(a) (b)

Figure 1.6: (a) A coordinate frame intrinsic to the curve. (b) The unit tangent $\hat{\mathbf{t}}$ changing as s changes.

The local system is defined by three mutually perpendicular directions. These are:

1. **Tangent $\hat{\mathbf{t}}$.** For the first intrinsic direction defined at at the point $\mathbf{r}(s)$ we can use the **unit tangent $\hat{\mathbf{t}}$**. We know already that

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} \; . \tag{1.24}$$

2. **Principal Normal $\hat{\mathbf{n}}$.** Recall our earlier proof that if $\mathbf{f}$ was a vector with variable direction but fixed magnitude then $d\mathbf{f}/dt$ and $\mathbf{f}$ were orthogonal. But $\hat{\mathbf{t}}$ has constant magnitude and varies over s, so that $d\hat{\mathbf{t}}/ds$ must be perpendicular to $\hat{\mathbf{t}}$. Hence for the second direction we use *the principal normal* $\hat{\mathbf{n}}$ defined by

$$\kappa\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} : \text{ where } \kappa \geq 0 \text{ by convention.} \tag{1.25}$$

   κ is the **curvature**, and $\kappa = 0$ for a straight line.

3. **The Binormal $\hat{\mathbf{b}}$**

   The local coordinate frame is completed by defining **the binormal** as

$$\hat{\mathbf{b}}(s) = \hat{\mathbf{t}}(s) \times \hat{\mathbf{n}}(s) \; . \tag{1.26}$$

With $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ defined, the **Frenet-Serret relationships** emerge from differentiating dot products between them.

First, since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$,

$$\frac{d}{ds}\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} \;=\; \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \;=\; \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} \;=\; 0\ . \tag{1.27}$$

But $\hat{\mathbf{b}} \cdot \hat{\mathbf{n}} = 0$, so

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} = 0. \tag{1.28}$$

This says $d\hat{\mathbf{b}}/ds$ is perpendicular to $\hat{\mathbf{t}}$; but was also know that because $\hat{\mathbf{b}}$ has fixed magnitude it is perpendicular to $\hat{\mathbf{b}}$. Hence it must be along the direction of $\hat{\mathbf{n}}$:

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau(s)\hat{\mathbf{n}}(s) \tag{1.29}$$

where by definition τ is the curve's **torsion**. (The negative sign is a matter of convention.)

Differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$, we find

$$\frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot (\kappa\hat{\mathbf{n}}) = 0 \qquad \Rightarrow \frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{t}} = -\kappa \tag{1.30}$$

$$\frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{b}}}{ds} = \frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot (-\tau\hat{\mathbf{n}}) = 0 \qquad \Rightarrow \frac{d\hat{\mathbf{n}}}{ds} \cdot \hat{\mathbf{b}} = \tau\ . \tag{1.31}$$

Together these give

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s). \tag{1.32}$$

The twists and turns in a space curve are therefore described by the local curvature and torsion, and we now have all three of

The Frenet-Serret relationships

$$d\hat{\mathbf{t}}/ds \;=\; \kappa\hat{\mathbf{n}} \tag{1.33}$$
$$d\hat{\mathbf{n}}/ds \;=\; -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s) \tag{1.34}$$
$$d\hat{\mathbf{b}}/ds \;=\; -\tau(s)\hat{\mathbf{n}}(s) \tag{1.35}$$

1.6.1 ♣ Example: F-S applied to that helix

Q Derive the curvature $\kappa(s)$ and torsion $\tau(s)$ of the helix

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\beta}\right)\hat{\mathbf{\imath}} + a\sin\left(\frac{s}{\beta}\right)\hat{\mathbf{\jmath}} + h\left(\frac{s}{\beta}\right)\hat{\mathbf{k}} \tag{1.36}$$

where for compactness we've written $h = P/2\pi$ and $\beta = \sqrt{a^2 + h^2}$.

A The unit tangent in $[\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}]$ components is

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \left[-\frac{a}{\beta}\sin\left(\frac{s}{\beta}\right), \quad \frac{a}{\beta}\cos\left(\frac{s}{\beta}\right), \quad \frac{h}{\beta}\right] \ . \tag{1.37}$$

Differentiation gives

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}} = \left[-\frac{a}{\beta^2}\cos\left(\frac{s}{\beta}\right), \quad -\frac{a}{\beta^2}\sin\left(\frac{s}{\beta}\right), \quad 0\right] \tag{1.38}$$

The curvature is always positive, so

$$\kappa = \frac{a}{\beta^2} = \frac{a}{(a^2 + (P/2\pi)^2)} \quad \text{and} \quad \hat{\mathbf{n}} = \left[-\cos\left(\frac{s}{\beta}\right), \quad -\sin\left(\frac{s}{\beta}\right), \quad 0\right] \ . \tag{1.39}$$

So κ is a constant (it doesn't depend on s), and $\hat{\mathbf{n}}$ is parallel to the xy-plane (it has no $\hat{\mathbf{k}}$ component). It points to the central axis of the helix.
Now find the binormal: (here C, S is shorthand for $\cos, \sin(s/\beta)$)

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ (-a/\beta)S & (a/\beta)C & (h/\beta) \\ -C & -S & 0 \end{vmatrix} = \left[\frac{h}{\beta}\sin\left(\frac{s}{\beta}\right), \ -\frac{h}{\beta}\cos\left(\frac{s}{\beta}\right), \ \frac{a}{\beta}\right] \tag{1.40}$$

and differentiate $\hat{\mathbf{b}}$ to find an expression for the torsion

$$\frac{d\hat{\mathbf{b}}}{ds} = \left[\frac{h}{\beta^2}\cos\left(\frac{s}{\beta}\right), \quad \frac{h}{\beta^2}\sin\left(\frac{s}{\beta}\right), \quad 0\right] = \frac{-h}{\beta^2}\hat{\mathbf{n}} \tag{1.41}$$

so the torsion is

$$\tau = \frac{h}{\beta^2} = \frac{(P/2\pi)}{(a^2 + (P/2\pi)^2)} \ , \tag{1.42}$$

again a constant.

Both the curvature and torsion being constants, ie. independent of s, seems entirely sensible. The helix never changes it shape and, if you were to move along it, no one point would be distinguished from any other.

1.7 Coordinate systems that move (2): derivatives in plane polars

Another type of moving coordinate system is, unlike the previous example, referred to the original fixed coordinate system. To keep this simple we will consider motion in a plane, and consider plane polar coordinates. Note that we'll continue to denote the plane polar angle by ϕ to avoid confusion with the spherical polar angle.

In plane polar coordinates, the position vector of any point P is given by

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \\ &= r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} \\ &= r\hat{\mathbf{r}} \end{aligned} \tag{1.43}$$

where we have introduced the unit radial vector

$$\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \quad . \tag{1.44}$$

The other basic unit vector in plane polars is orthogonal to $\hat{\mathbf{r}}$ and is

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \tag{1.45}$$

so that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, $\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$, and $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$.

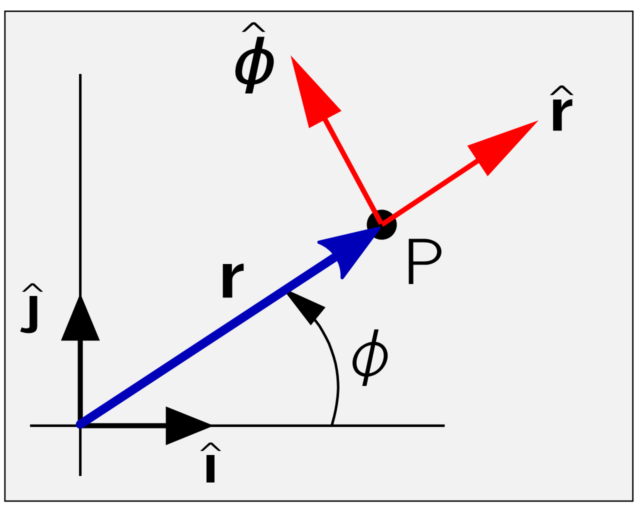


Figure 1.7:

Now suppose point P is moving.

Then $\mathbf{r}$ is a function of time t, as of course are r, ϕ, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

The velocity of P is

$$\begin{aligned}\dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{r}}) &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\phi}{dt}(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \\ &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} \quad = \text{radial} + \text{tangential}\end{aligned} \tag{1.46}$$

The radial and tangential components of velocity of P are therefore dr/dt and $rd\phi/dt$, respectively.

Differentiating again gives the acceleration of P

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} + \frac{dr}{dt}\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} + r\frac{d^2\phi}{dt^2}\hat{\boldsymbol{\phi}} - r\frac{d\phi}{dt}\frac{d\phi}{dt}\hat{\mathbf{r}} \\ &= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2}\right]\hat{\boldsymbol{\phi}}\end{aligned} \tag{1.47}$$

One might guess this system is convenient when there is motion related to a circle. Three obvious cases are:

1. Motion along a radial spoke, $\Rightarrow\phi =$constant:

$$\ddot{\mathbf{r}} = \frac{d^2r}{dt^2}\hat{\mathbf{r}}\ . \tag{1.48}$$

2. Motion in a circle with varying angular speed, $\Rightarrow r =$constant:

$$\ddot{\mathbf{r}} = -r\left(\frac{d\phi}{dt}\right)^2\hat{\mathbf{r}} + r\frac{d^2\phi}{dt^2}\hat{\boldsymbol{\phi}}\ . \tag{1.49}$$

3. Motion in a circle with fixed angular speed, $\Rightarrow r =$constant and $d\phi/dt$=constant:

$$\ddot{\mathbf{r}} = -r\left(\frac{d\phi}{dt}\right)^2\hat{\mathbf{r}} \tag{1.50}$$

   which is good old centripetal acceleration.

Extending to 3D. The arugments presented could readily be extended to 3D using, say, spherical polar coordinates where the basis vectors are $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}$.

Another note. It is probably futile to try to remember these relationships. Instead learn the basics — the definition of the position vector $\mathbf{r}$ and $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$, and then be confident about differentiation.

1.8 Coordinate frames that move (3): rotation & Coriolis

Suppose you are sitting in a fixed coordinate system $Oxyz$, observing a point P on a body that rotates with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through an origin defined in the body, as shown in Fig. 1.8. The angular velocity vector $\boldsymbol{\omega}$ points in the same direction as the axis of rotation. It is therefore fixed both with respect to the rotating frame *and* the fixed frame.

We can break the position vector of point P into two parts $\mathbf{r}_\mathrm{P} = \mathbf{r}_0 + \mathbf{r}$. But $\mathbf{r}_0$ is a constant and plays no part in the motion of P, and for further analysis we translate the fixed coordinate system to the body origin.

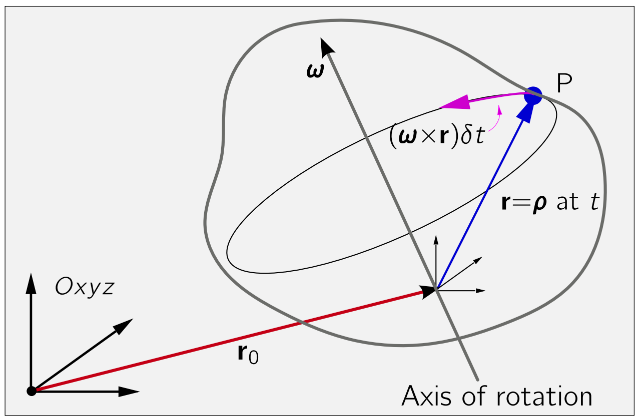


Figure 1.8: You sit in a fixed frame and observe the movement of a point P on a body which rotated.

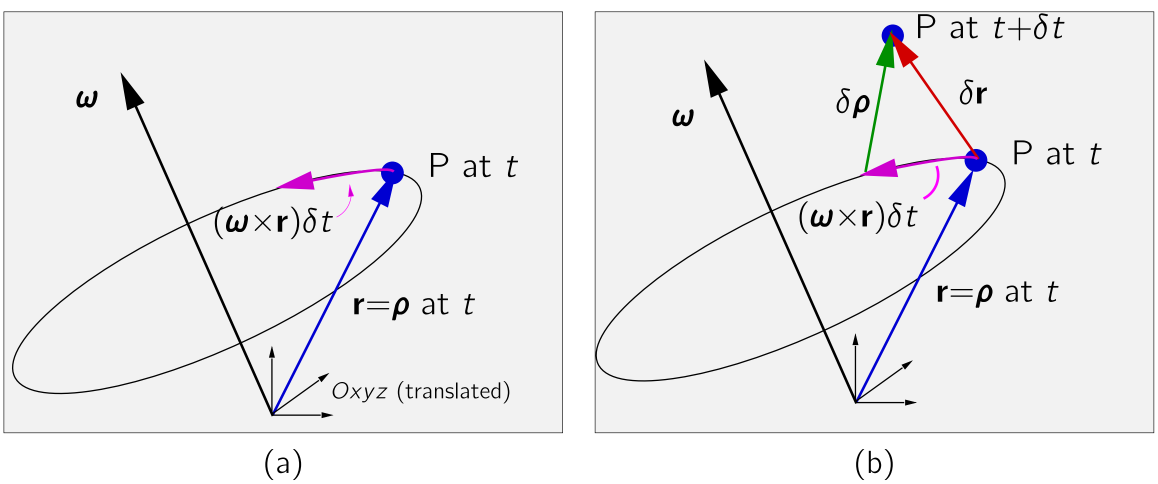


Figure 1.9: Vectors in the rotating and fixed frames when (a) P is stationary in the rotating frame, and (b) when it is moving.

Let $\boldsymbol{\rho}$ be the position, as described in the rotating frame, of the point P The point's position in the *translated* fixed frame is just $\mathbf{r}$. Now, **at some instant** t, let the rotating coordinate frame be be aligned with the fixed coord system. So at instant t, and only at instant t, $\mathbf{r} = \boldsymbol{\rho}$.

First suppose that point P is stationary in the rotating frame (Fig 1.9(a)). Its velocity is therefore zero in the rotating frame, but at any instant in the fixed frame P's velocity is

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \boldsymbol{\rho} \ . \tag{1.51}$$

Although the frames are aligned, this is true whether or not the origins of coordinates coincide. Notice that $d\mathbf{r}/dt$ will have fixed magnitude, and will always be perpendicular to the axis of rotation.

Now let the point P move in the rotating frame. There will be two contributions to its motion when described in the fixed frame — one due to its motion within the rotating frame, and one due to the rotation itself. Over an interval δt the movement is

$$\delta\mathbf{r} \ = \ \delta\boldsymbol{\rho} \ + \ (\boldsymbol{\omega} \times \boldsymbol{\rho})\delta t \ . \tag{1.52}$$

Hence the **instantaneous velocity** in the fixed frame is the sum of the body defined motion and the rotational motion:

$$\frac{d\mathbf{r}}{dt} \ = \ \frac{D\boldsymbol{\rho}}{Dt} \ + \ \boldsymbol{\omega} \times \boldsymbol{\rho} \ \equiv \ \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\boldsymbol{\rho}. \tag{1.53}$$

Here, the capital D is used to indicate differentiation in the rotating frame. We've also introduced an operator notation. Note that we could replace $\boldsymbol{\rho}$ on the RHS with $\mathbf{r}$. At the instant under consideration, they are the same.

But, again because the frames are aligned at instant t, this equation applies to *any* vector. In particular, the instantaneous acceleration is

$$\frac{d^2\mathbf{r}}{dt^2} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\frac{d\mathbf{r}}{dt} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\mathbf{r} \ . \tag{1.54}$$

The **instantaneous acceleration** observed in the fixed frame is therefore

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{D^2\boldsymbol{\rho}}{Dt^2} \ + \ 2\boldsymbol{\omega} \times \frac{D\boldsymbol{\rho}}{Dt} \ + \ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{1.55}$$

- The first term is the acceleration of the point P in the rotating frame measured in the rotating frame.

- The last term is the centripetal acceleration to due to the rotation. (Yes! Its magnitude is $\omega^2 r$ and its direction is that of $-\mathbf{r}$. Check it out.)

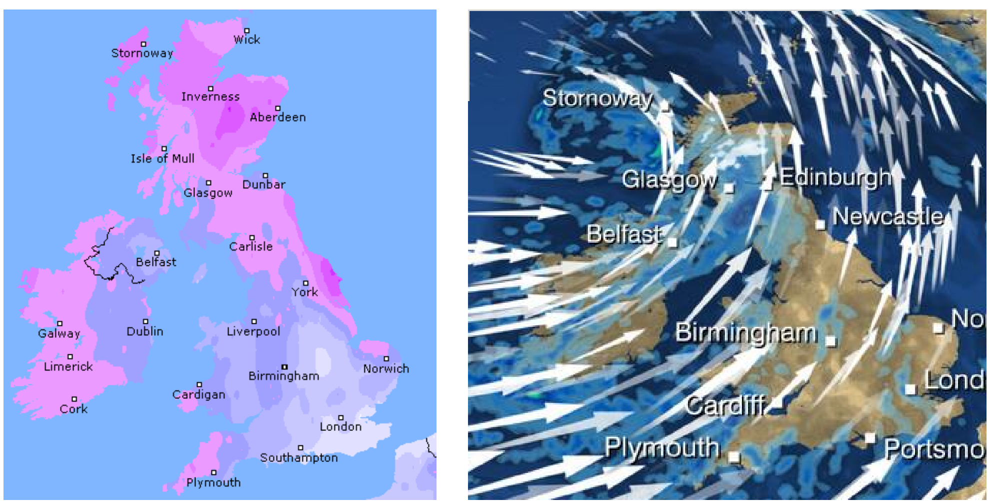
- The middle term is an extra term which arises because of the velocity of P in the rotating frame. It is known as the **Coriolis acceleration**, named after the French engineer who first identified it.

Because of the rotation of the earth, the Coriolis acceleration is of great importance in meteorology and accounts for the occurrence of high pressure anti-cyclones and low pressure cyclones in the northern hemisphere, in which the Coriolis acceleration is produced by a pressure gradient. It is also a very important component of the acceleration (hence the force exerted) by a rapidly moving robot arm, whose links whirl rapidly about rotary joints. You'll learn more about this in the 2nd year A3 Dynamics of Machines course.

Lecture 2

Scalar fields and the grad operator.

Our focus now turns away from individual vectors and towards scalar and vector quantities which are defined over regions in space. When a scalar function $U(\mathbf{r})$ is determined or defined at each position $\mathbf{r}$ in some region, we say that U is a **scalar field** in that region. Similarly, if a vector function $\mathbf{v}(\mathbf{r})$ is defined at each point, then $\mathbf{v}$ is a **vector field** in that region. Familiar examples of 2D fields are shown in Fig. 2.1.



(a) (b)

Figure 2.1: Examples of (a) a scalar field (temperature); (b) a vector field (wind velocity)

As you would guess, there are strong links between fields of each type. In heat transfer, a scalar temperature field will give rise to vector field of heat flow, in fluid mechanics a scalar pressure field will give rise to a vector field of velocity flow, and in electrostatics an scalar potential field will give rise to a vector electric field.

In each case there is a sense of "flow" from a higher "potential to do something" to a lower one — one would therefore guess that gradients, but 3D gradients, will be involved.

This leads us to define the *gradient* of a scalar field using a differential vector operator. Acting on a scalar field it delivers a vector field of gradients. (An early warning! We've been talking about high to low potential, but the gradient vector field will, of course, point from low to high values of the scalar field.)

2.1 The gradient of a scalar field

Consider a scalar field, such as temperature, defined throughout some region of space. One might ask how the value would vary as we moved off in an arbitrary direction in 3D. Here we find out how.

If $U(\mathbf{r}) = U(x, y, z)$ is a scalar field in 3 dimensions then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\text{grad}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \ . \tag{2.1}$$

It is usual to define the **vector operator** $\boldsymbol{\nabla}$, called "del" or "nabla" [1]. It is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z} \ . \tag{2.2}$$

and then

$$\text{grad}U \equiv \boldsymbol{\nabla} U \ .$$

Note that $\boldsymbol{\nabla}$ is *vector operator* which operates on a *scalar field* and returns a *vector field*.

Also note, without thinking too carefully about it, that the gradient of a scalar field tends to point in the 3D direction of greatest change of the field. Later we will be more precise.

2.2 ♣ Worked examples of gradient evaluation

1: $U = x^2$, a 1D example

$$\Rightarrow \boldsymbol{\nabla} U = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) x^2 = 2x\hat{\mathbf{i}} \ . \tag{2.3}$$

2: $U = r^2 = x^2 + y^2 + z^2$

$$\Rightarrow \boldsymbol{\nabla} U = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) (x^2+y^2+z^2) = 2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}} \ = \ 2\,\mathbf{r} \ . \tag{2.4}$$

[1] del because (I guess) it's an inverted delta, and nabla because (according to Wikipedia) nabla is Greek for a Phoenician harp — but, heck, you knew that ...

3: $U = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \boldsymbol{\nabla} U = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) (c_1 x + c_2 y + c_3 z) = c_1 \hat{\mathbf{i}} + c_2 \hat{\mathbf{j}} + c_3 \hat{\mathbf{k}} = \mathbf{c} \ . \quad (2.5)$$

4: $U = U(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$.

U is a function of r alone so dU/dr exists. As $U = U(x, y, z)$ as well,

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial U}{\partial y} = \frac{dU}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial U}{\partial z} = \frac{dU}{dr}\frac{\partial r}{\partial z} \ . \quad (2.6)$$

$$\Rightarrow \boldsymbol{\nabla} U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \quad = \frac{dU}{dr}\left(\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}} \right) \quad (2.7)$$

But $r = (x^2 + y^2 + z^2)^{1/2}$, so

$$\frac{\partial r}{\partial x} = 2x \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \ , \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \ .$$

$$\Rightarrow \boldsymbol{\nabla} U = \frac{dU}{dr}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r} \right) = \frac{dU}{dr}\left(\frac{\mathbf{r}}{r} \right) \quad = \frac{dU}{dr}\hat{\mathbf{r}} \ . \quad (2.8)$$

Note that it makes sense for a radially symmetric scalar field to have a gradient which is a radially symmetric vector field.

5: From paper P2 you will recognize $\Phi(r)$ as the electric potential a distance r from a point charge Q in vacuum.

$$\Phi(r) = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{r} \right) \ .$$

We can use Example 4 to find the gradient

$$\boldsymbol{\nabla}\Phi = -\frac{Q}{4\pi\epsilon_o}\frac{\hat{\mathbf{r}}}{r^2} \ . \quad \text{or we could write} \ -\boldsymbol{\nabla}\Phi = \frac{Q}{4\pi\epsilon_o}\frac{\hat{\mathbf{r}}}{r^2} \ . \quad (2.9)$$

You will recognize the RHS of the second expression as the electric field $\mathbf{E}$ around a point charge. You already knew that in one dimension $E = -d\Phi/dx$, and now we've found out that

> In 3D we write the vector electric field as the -ve gradient of the potential:
>
> $$\mathbf{E} = -\boldsymbol{\nabla}\Phi \ .$$

Remember too that the potential Φ is a scalar, and the electric field is a vector.

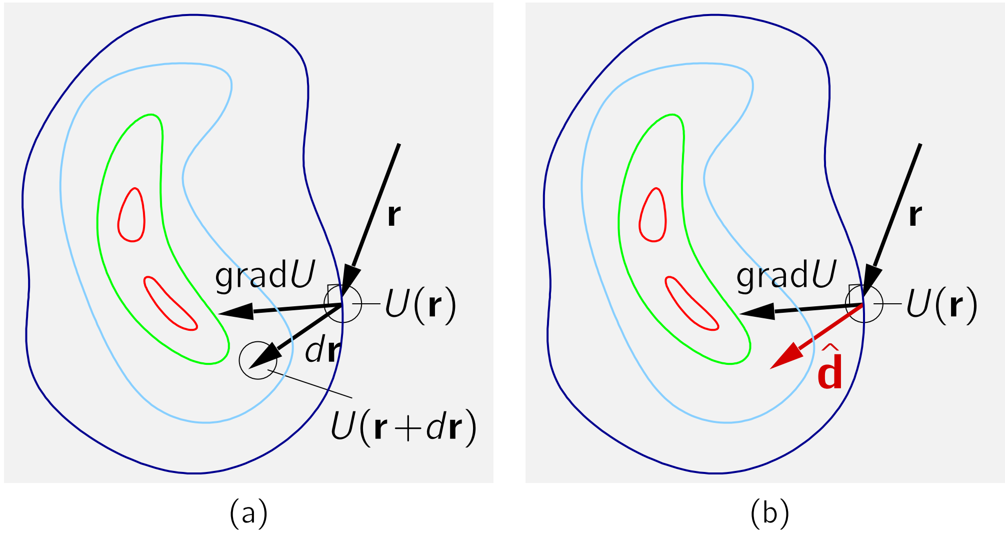


Figure 2.2: The directional derivative: The rate of change of U with respect to distance in direction $\hat{\mathbf{d}}$ is $\nabla U \cdot \hat{\mathbf{d}}$.

2.3 The significance of grad

Referring to Fig. 2.2(a), if our current position is $\mathbf{r} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$ in some scalar field $U(\mathbf{r}) = U(x, y, z)$, and we move an infinitesimal distance $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$, the value of U changes to $U(x + dx, y + dy, z + dz)$. From Calculus 2.1 we know that the change dU is the total differential:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ . \tag{2.10}$$

But $\nabla U = (\hat{\mathbf{i}}\partial U/\partial x + \hat{\mathbf{j}}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in U can also be written as the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ . \tag{2.11}$$

Now divide both sides by ds (allowed because these are total differentials):

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ . \tag{2.12}$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a *unit vector* in the direction of $d\mathbf{r}$.

This result can be paraphrased (Fig. 2.2(b)) as:

Statement #1: grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (ie, the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

Now imagine sitting at one point and changing the direction of $\hat{\mathbf{d}}$. Looking at Fig. 2.2(b) and/or Eq. (2.12) it is evident that

Statement #2: At any position $\mathbf{r}$ in a scalar field U, grad$U$ points in the direction of greatest change of U at P, and has magnitude equal to the rate of change of U with respect to distance in that direction.

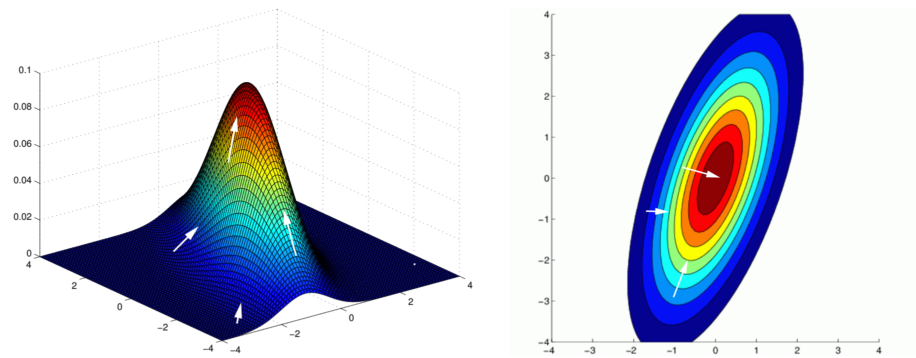


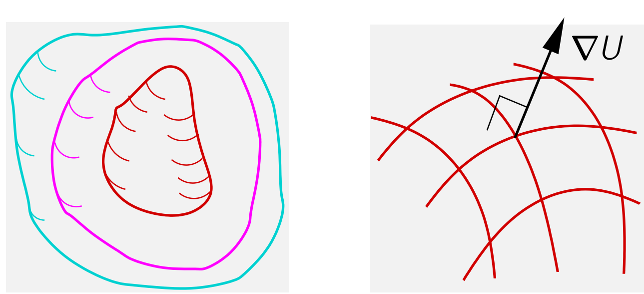
Figure 2.3: ∇U is in the direction of greatest (positive!) change of U wrt distance. (Positive ⇒"uphill".)

Another nice property emerges if we think of a surface of constant U – that is the locus (x, y, z) for $U(x, y, z) =$ constant. If we move a tiny amount within that iso-U surface, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in that U=constant surface

$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 . \qquad (2.13)$$

But $d\mathbf{r}/ds$ is a tangent to the surface, so this result shows that (Fig. 2.4)

Statement #3: grad$U$ is everywhere NORMAL to a surface of constant U.



Surfaces of constant U *Level surfaces* Surface of constant U

Figure 2.4: grad$U$ is everywhere NORMAL to a surface of constant U.

2.4 Grad, fields and potential

You were reminded earlier of electric field **E**, and that in one dimension (say along the x-direction) the increase $d\Phi$ in electric potential Φ when moving a distance dx in a field E is $d\Phi = -E dx$. Equivalently the field in the x-direction is given by $E = -\frac{d\Phi}{dx}$, the negative potential gradient in the x-direction.

If we were interested in the change of potential ϕ when we moved by dx in one dimension we would write

$$d\Phi = -E(x)dx \; . \tag{2.14}$$

The equivalent in 3D is

$$d\Phi = \left(\frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \right) = \boldsymbol{\nabla} \Phi \cdot d\mathbf{r} = -\mathbf{E} \cdot d\mathbf{r} \tag{2.15}$$

Suppose we asked what the change in potential was between A and B. In 1D we would write

$$\Phi_B - \Phi_A = \int_{\Phi_A}^{\Phi_B} d\Phi = -\int_{x_A}^{x_B} E(x)dx \; . \tag{2.16}$$

But in 3D

$$\Phi_B - \Phi_A = \int_{\Phi_A}^{\Phi_B} d\Phi = -\int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} \; . \tag{2.17}$$

$\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E} \cdot d\mathbf{r}$ is an example of a **line integral**. We discuss these now.

2.5 What is a line integral?

Line integrals are concerned with measuring the integrated interaction with a field as you move through it on some defined path.

Many texts describe line integrals without using vector calculus, which rather hides their physical importance. We shall review both approaches: you are likely to find the vector approach more satisfying.

Suppose we have a scalar field $F(\mathbf{r}) \equiv F(x, y, z)$ defined in space. Now consider moving throught the field along a space curve which you have chopped into elemental arc-lengths δs_i. Each element is associated with a position (x_i, y_i, z_i) and a function value $F(x_i, y_i, z_i)$. The line integral is defined as

$$\int_C F(x, y, z) ds = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} F(x_i, y_i, z_i) \delta s_i \tag{2.18}$$

over the path C from s_{start} to s_{end}.

In practice, the method of solution depends on how the space curve is parameterized.

1: The simplest form is when we just happen to know $F(s)$ – so F need not be known everywhere, but only be known on the particular path taken. Then the line integral is simply

$$I = \int_{s_{\mathrm{start}}}^{s_{\mathrm{end}}} F(s)\, ds \; . \tag{2.19}$$

♣ **Example:** Suppose $F(s)$ is fuel consumption (say in mpg, or its SI equivalent) as a function of distance along the path. The line integral would give the total fuel consumed.

2: The more general form is when $F(x, y, z)$ defines a field, and the path taken through the field is defined in terms of a parameter p. That is, the path is defined by the space curve

$$\mathbf{r}(p) = [x(p), y(p), z(p)] \; . \tag{2.20}$$

We'll assume that the start and end values of the parameter $p_{\mathrm{start,end}}$ are given or easily found.

Finding the integral involves determining $F(p)$ by replacing x, y, z with their correspon- ing functions of p $x(p), y(p), z(p)$, and then writing

$$I = \int F ds = \int_{p_{\mathrm{start}}}^{p_{\mathrm{end}}} F(p) \frac{ds}{dp}\, dp \quad . \tag{2.21}$$

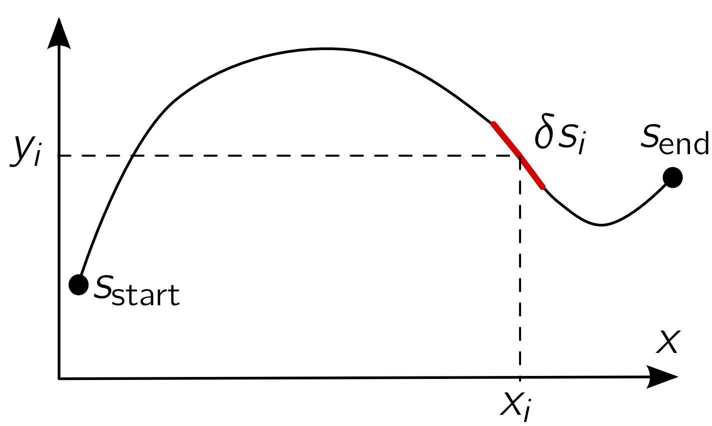


Figure 2.5: The line integral in 2D.

But because $ds^2 = dx^2 + dy^2 + dz^2$, we can replace ds/dp with

$$\frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2\right]^{1/2} , \tag{2.22}$$

and the integral is now one entirely in p.

♣ Example.

Q: Find the line integral from points $[xyz] = [000]$ to $[422]$ when $F(x, y, z) = xy/z^2$ and the path is $[x, y, z] = [p, p^{1/2}, p^{1/2}]$.

A:

(i) Notice that $p_{\text{start}} = 0$ and $p_{\text{end}} = 4$.

(ii) Use $x{=}p$, $y{=}p^{1/2}$, $z{=}p^{1/2}$ into $F = xy/z^2$ to find $F(p) = p^{1/2}$.

(iii) Then work out

$$\frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2\right]^{1/2} = \left[1^2 + \left(\frac{1}{2\sqrt{p}}\right)^2 + \left(\frac{1}{2\sqrt{p}}\right)^2\right]^{1/2} = \left[1 + \frac{1}{2p}\right]^{1/2} \tag{2.23}$$

(iv) So the line integral ends up as

$$I = \int_{p=0}^{p=4} F(p) ds = \int_{p=0}^{p=4} F(p)\frac{ds}{dp} dp = \int_{p=0}^{p=4} \left[p + \frac{1}{2}\right]^{1/2} dp = \boxed{\text{DIY}} = \frac{26}{3\sqrt{2}} . \tag{2.24}$$

♣ Example

Q: Derive the line integral $\int_L (x - y^2) ds$ where s is arc length and the path L is that segment of the straight line $y = 2x$ between $x = 0$ to $x = 1$.

A: We want to turn this integral into $\int_{x=0}^{x=1} F(x) dx$.

As the path lies in the x, y-plane any $dz = 0$, so that

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ which here } = \sqrt{1 + 2^2} dx = \sqrt{5} dx \tag{2.25}$$

Thus

$$I = \int_L (x - y^2) ds = \int_{x=0}^{1} (x - 4x^2)\sqrt{5} dx \tag{2.26}$$

$$= \sqrt{5}\left[\frac{x^2}{2} - \frac{4}{3}x^3\right|_0^1 = -\frac{5\sqrt{5}}{6} . \tag{2.27}$$

2.6 Line integrals using vectors

With vectors, problems are usually formulated using $\mathbf{r}$ as the position instead of (x, y, z) and $d\mathbf{r}$ instead of ds. You will immediately realize that this is no great change, and so the calculations involved are essentially identical.

The most common form of line integral is when the integrand $\mathbf{F}(\mathbf{r})$ is a vector field (because $\mathbf{F}()$ is a vector function) dotted with $d\mathbf{r}$ giving a scalar integral:

$$I = \int_L \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \quad = \quad \lim_{\delta \mathbf{r}_i \to 0} \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i .$$

For example, in Fig. 2.6 the total work done by a force $\mathbf{F}$ as it moves a point from A to B along a given path L is given by a line integral of this form. If the force $\mathbf{F}$ at $\mathbf{r}$ moves by $d\mathbf{r}$, then the element of work done is $dW = \mathbf{F} \cdot d\mathbf{r}$, and the total work done traversing the space curve is

$$W = \int_L \mathbf{F} \cdot d\mathbf{r} \ . \tag{2.28}$$

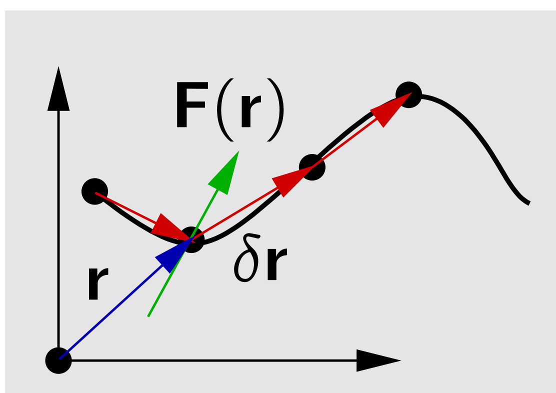
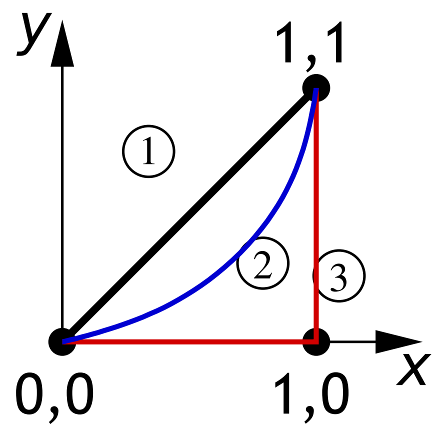


Figure 2.6: Line integral. $\mathbf{F}(\mathbf{r})$ is a force.

2.6.1 ♣ A trio of examples

Q1: A body moves through a force field $\mathbf{F} = x^2y\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}}$ at it moves between $[0,0]$ and $[1,1]$. Determine the work done on the body when the path is

1. along the line $y = x$.
2. along the curve $y = x^n$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$.



A1: The problem involves the x-y plane. The position vector is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, and so $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$. Then in general the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2y\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}) = \int_C (x^2y\,dx + xy^2\,dy) \ . \tag{2.29}$$

Path 1: $y{=}x$ gives $dy{=}dx$. It seems easiest to convert all y references to x.

$$\int_{[0,0]}^{[1,1]} (x^2y\,dx + xy^2\,dy) = \int_{x=0}^{x=1} (x^2x\,dx + xx^2\,dx) = \int_{x=0}^{x=1} 2x^3\,dx = \left[x^4/2\right|_{x=0}^{x=1} = 1/2 \ . \tag{2.30}$$

Path 2: $y{=}x^n$ gives $dy{=}nx^{n-1}dx$. Again convert all y references to x.

$$\int_{[0,0]}^{[1,1]} (x^2y\,dx + xy^2\,dy) = \int_{x=0}^{x=1} (x^{n+2}dx + nx^{n-1}.x.x^{2n}dx) \tag{2.31}$$

$$= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{3n}dx) \tag{2.32}$$

$$= \frac{1}{n+3} + \frac{n}{3n+1} \tag{2.33}$$

Path 3: is not smooth, so we must break it into two. Along the first section, $y = 0$ and $dy = 0$, and on the second $x = 1$ and $dx = 0$, so

$$\int_A^B (x^2y\,dx + xy^2\,dy) = \int_{x=0}^{x=1} (x^2 0\,dx) + \int_{y=0}^{y=1} 1.y^2\,dy = 0 + \left[y^3/3\right|_{y=0}^{y=1} = 1/3 \ . \tag{2.34}$$

Conclude: In general, line integrals depend not only the start and end points, but on the path taken between the start and end points.

By the way, here is a neat check on those results. Notice that that path (2) morphs into path (1) when $n = 1$, and that path (2) morphs into path (3) when $n \to \infty$. Our line integral result morphs too!

$$\frac{1}{n+3} + \frac{n}{3n+1} = 1/2 \text{ when } n = 1 \qquad \frac{1}{n+3} + \frac{n}{3n+1} = 1/3 \text{ when } n \to \infty\ . \quad (2.35)$$

♣ Another example

Q2: Now repeat path (2) from Q1, but using the force $\mathbf{F} = xy^2\hat{\mathbf{\imath}} + x^2y\hat{\mathbf{\jmath}}$.

A2: For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so the line integral is

$$\int_{[0,0]}^{[1,1]} (y^2\, x\, dx + y\, x^2\, dy) = \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{n-1}.x^2.x^n dx) \quad (2.36)$$

$$= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{2n+1}dx) \quad (2.37)$$

$$= \frac{1}{2n+2} + \frac{n}{2n+2} \quad (2.38)$$

$$= \frac{1}{2} \quad (2.39)$$

This result is **independent** of n.

In fact, for the last example field $\mathbf{F} = xy^2\hat{\mathbf{\imath}} + x^2y\hat{\mathbf{\jmath}}$ you could try **any path** between the same two points and the result would always be 1/2.

If you tried another pair of points, the line integral would (in general) not be 1/2, but whatever result you derived would again be independent of the path.

This happens when the field $\mathbf{F}$ being moved through is a **conservative field**. Line integrals through a conservative field depend only on the the start and end positions, not on the path.

2.7 Conservative fields

To summarize that result, and to introduce an obvious corollary,

If $\mathbf{F}$ is a conservative field

- The line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of path between A and B.
- The line integral around a closed path $\oint \mathbf{F} \cdot d\mathbf{r}$ is zero.

But how can we recognize $\mathbf{F}$ is conservative without checking every path?

Consider the 2D scalar field $U(x, y) = x^2y^2/2$. Recall the definition of the perfect or total differential

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \tag{2.40}$$

which in this case is

$$dU = y^2xdx + yx^2dy \ . \tag{2.41}$$

So our line integral is actually the integral of a total differential, and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B (y^2xdx + yx^2dy) = \int_A^B dU \quad = U_B - U_A \ , \tag{2.42}$$

which depends only on the difference in U between start and end points, not on the path between them.

In other words

If vector field $\mathbf{F}$ is the gradient ∇U of a scalar field U then $\mathbf{F}$ is conservative.

The scalar field U is then a **potential field.**

All scalar potential fields U have an associated vector field ∇U, but not every vector field $\mathbf{F}$ is the gradient of a scalar field.

Think for a moment about any electric field and any gravitational field. Are they conservative?

Lecture 3

Vector fields and the Div and Curl operators

In Lecture 2 we introduced the del operator which when applied to a *scalar field* $U(\mathbf{r})$ generate the *vector field* of gradients, ∇U.

In this lecture we delve further into the properties of fields. We again use the del operator, but now apply it to *vector fields*, to obtain the

- the **divergence** of a vector field, and
- the **curl** of a vector field.

The divergence of vector field is a *scalar* field, while the curl of the vector field is a *vector* field. As with grad, you will need to know how to apply the operator, and its underlying physical meaning and use. Although it might not be immediately obvious to you, together these operators provide a powerful method of describing physics in 3D without involving arbitrary coordinate systems.

3.1 The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

If $\mathbf{a}(x, y, z)$ is a vector function of position in 3 dimensions, that is $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$, then its divergence at any point is defined in Cartesian co-ordinates by

$$\operatorname{div}\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \ . \tag{3.1}$$

To clarify the notation used here, a_x denotes the x-component of $\mathbf{a}$, and a_x is a function of (x, y, z); and similarly for the other components.

But we can write the divergence as a scalar product between the ∇ operator and the vector field:

$$\operatorname{div} \mathbf{a} \equiv \boldsymbol{\nabla} \cdot \mathbf{a} = \left(\hat{\mathbf{\imath}} \frac{\partial}{\partial x} + \hat{\mathbf{\jmath}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \cdot \mathbf{a} \ . \tag{3.2}$$

This reminds you that the divergence of a vector field is a scalar field. You *cannot* compute the divergence of scalar field.

3.2 Some examples of divergence evaluation

You should work through these: here we'll look at examples (1) and (3).

	$\mathbf{a}$	$\operatorname{div} \mathbf{a}$
1)	$\mathbf{c}$	0
2)	$x\hat{\mathbf{\imath}}$	1
3)	$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$	3
4)	$\mathbf{r}/r^3$	0
5)	$r\mathbf{c}$, for $\mathbf{c}$ constant	$(\mathbf{r} \cdot \mathbf{c})/r$

♣ Example (1)

As $\mathbf{c} = c_x\hat{\mathbf{\imath}} + c_y\hat{\mathbf{\jmath}} + c_z\hat{\mathbf{k}}$ and all the $c_{x,y,z}$ are constant it is obvious on differentiation that the divergence is zero. A field with zero divergence is described as **solenoidal**. In P2 you will learn that the magnetic field strength[1] $\mathbf{H}$ inside a long thin straight solenoid with its axis aligned, say, with $\hat{\mathbf{k}}$ is uniform with a value of

$$\mathbf{H} = n\mathtt{I}\hat{\mathbf{k}} \tag{3.3}$$

where n is the number of turns per unit length and $\mathtt{I}$ is the current flowing. The lines of $\mathbf{H}$ are all parallel — they do not diverge.

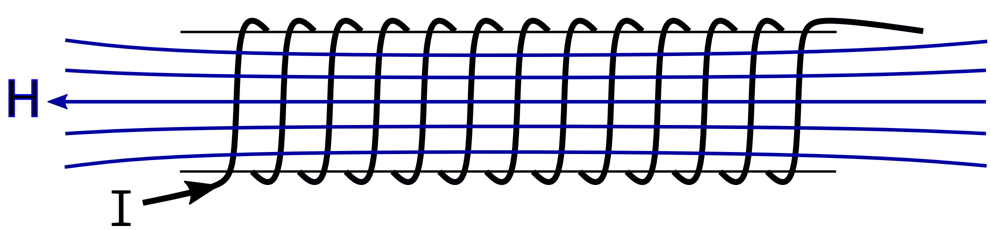


Figure 3.1: The **H** (magnetic field strength) inside a solenoid fed with dc I is constant.

Note that although the solenoid is a convenient way of remembering the description, we will now see that other non-constant fields can also have zero divergence.

[1] Note the distinction between magnetic field strength **H** (units: $\mathrm{A\,m^{-1}}$) and flux density **B** (units: Tesla or $\mathrm{Wb\,m^{-2}}$).

♣ Example (3)

Consider the x component of $\mathbf{a} = \mathbf{r}/r^3$

$$
\begin{aligned}
a_x &= x\,(x^2+y^2+z^2)^{-3/2}\ . && (3.4)\\
\Rightarrow \frac{\partial a_x}{\partial x} &= \frac{\partial}{\partial x}\left\{x\,(x^2+y^2+z^2)^{-3/2}\right\} && (3.5)\\
&= 1\,(x^2+y^2+z^2)^{-3/2} + x\frac{-3}{2}(x^2+y^2+z^2)^{-5/2}\,2x && (3.6)\\
&= r^{-3}\left(1-3x^2r^{-2}\right)\ . && (3.7)
\end{aligned}
$$

The terms in y and z are similar (with $3x^2$ replaced by $3y^2$ etc), so that

$$
\begin{aligned}
\mathrm{div}\,(\mathbf{r}/r^3) &= r^{-3}\left(1+1+1-3(x^2+y^2+z^2)r^{-2}\right) = r^{-3}\,(3-3) && (3.8)\\
&= 0\ .
\end{aligned}
$$

Note that $\mathbf{r}/r^3 = \hat{\mathbf{r}}/r^2$. So a radial vector field with a $1/r^2$ magnitude dependence has zero divergence.

3.3 The significance of the divergence

Imagine a flow of water, spraying out of a hosepipe, say. The flow, just like the wind in Fig. 2.1(b), is a vetor field. Let's denote it by $\mathbf{a}(\mathbf{r})$. To be precise, this vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{a}$ per unit time. Now we take an infinitesimal volume element dV and figure out the balance of the flow of $\mathbf{a}$ in to, and out of, dV.

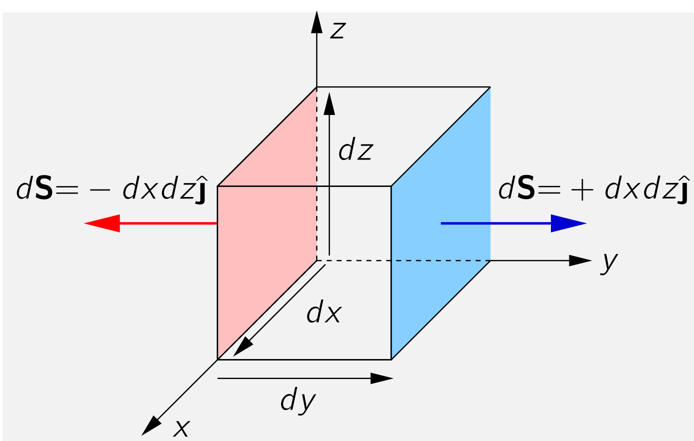


Figure 3.2: Elemental volume for calculating divergence.

In Cartesian co-ordinates the volume element $dV = dxdydz$. Think first about the face of area $dxdz$ perpendicular to the y axis and facing outwards in the negative y direction. That is, the one with surface area $d\mathbf{S} = -dxdz\hat{\mathbf{j}}$ (coloured pink in Fig. 3.2).

The component of the vector $\mathbf{a}$ normal to this face is $\mathbf{a}\cdot\hat{\mathbf{j}} = a_y$, and is pointing inwards, and so the its contribution to the OUTWARD flux from this surface is

$$\mathbf{a}\cdot d\mathbf{S} \quad = \quad -\,a_y(x,y,z)\,dx\,dz\ . \tag{3.9}$$

To be clear, $a_y(x,y,z)$ is the y-component of $\mathbf{a}$ measured at position (x,y,z). Remember that $\mathbf{a}$ is a vector field and has, in general, a different value at each point (x,y,z) in that field.

A similar contribution, but of opposite sign, will arise from the opposite face (coloured blue in Fig. 3.2), but we must remember that we have moved along y by an amount dy. Hence the OUTWARD flux is

$$a_y(x,y{+}dy,z)\,dx\,dz = \left(a_y + \frac{\partial a_y}{\partial y}dy\right)\,dx\,dz\ . \tag{3.10}$$

Adding both gives the total outward flux from these two faces as

$$\left(-\,a_y + a_y + \frac{\partial a_y}{\partial y}dy\right)dx\,dz\ = \frac{\partial a_y}{\partial y}\,dy\,dx\,dz = \frac{\partial a_y}{\partial y}\,dV\ . \tag{3.11}$$

Summing the other faces gives a total outward flux from the volume

$$\text{Total OUTWARD flux (efflux)} = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)dV = \boldsymbol{\nabla}\cdot\mathbf{a}\ \ dV \tag{3.12}$$

So we see that

The divergence of a vector field is a measure of *net efflux per unit volume* at each point in the field.

An equivalent statement is:

The divergence of a vector field is the *net flux generation per unit volume* at each point in the field.

Note that flux generation must always gives rise to efflux, so there is no ambiguity here.

3.4 Flux density and flux

From your work in P2 you are familiar with magnetic flux density $\mathbf{B}$ and electric flux density $\mathbf{D}$, and have no doubt made sketches of these which represent the flux density by the number of field lines per unit area. As we've already discussed, flux implies a flow of something, and other quantities can be similarly represented as a flux and flux density — the flow of a fluid, heat, radiation, and so on.

The "per unit area" in the definition of a flux density refers to an area which cuts the flow locally at 90° — that is, an area whose normal lies along the direction of flow.

But to evaluate a local element of flux, dF, flowing through an *actual* surface we need to multiply the flux density by an element of area on the actual surface — that is, taking care of vector orientation. The effective area intersecting the flow depends on the cosine of the angle between the flow and the surface normal. Thus, following Figure 3.3, the elements of flux through each $d\mathbf{S}$ is

$$dF = \mathbf{B} \cdot d\mathbf{S} \ . \tag{3.13}$$

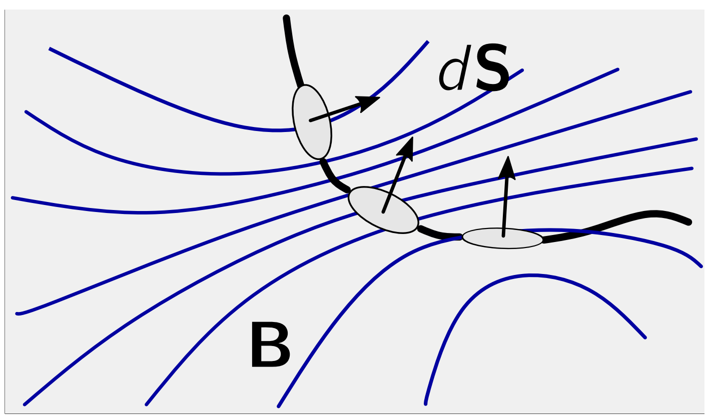


Figure 3.3: A $d\mathbf{S}$ with direction lying along the field direction captures all the available flux, while when perpendicular it captures none.

If we want to find the total flux through some extended surface R we have to perform a **surface integral**

$$F = \int_S \mathbf{B} \cdot d\mathbf{S} \ . \tag{3.14}$$

We have to be aware that both $\mathbf{B}$ and $d\mathbf{S}$ will change as we moved over the surface — $\mathbf{B}$ in magnitude and direction, and $d\mathbf{S}$ just in direction.

We will come back to deal with simple curved surfaces in Lecture 4, but here is a simpler example where the vector surface is in a fixed direction.

3.4.1 ♣ Example

Q: Evaluate the flux $\int \mathbf{B} \cdot d\mathbf{S}$ through the side of a unit cube at $x = 1$ in the direction of increasing x when $\mathbf{B}=y\hat{\mathbf{\imath}} + z\hat{\mathbf{\jmath}} + x\hat{\mathbf{k}}$.

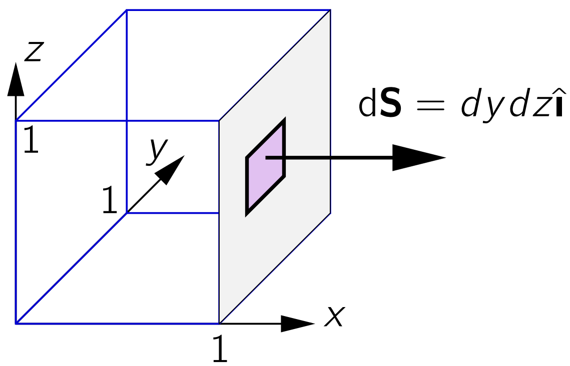


Figure 3.4:

A: Refer to Fig. 3.4. $d\mathbf{S}$ is perpendicular to the surface. Its $\pm$ direction actually depends on the nature of the problem, but we are told here to use the $+\hat{\mathbf{\imath}}$ direction.

Hence the element of surface area is

$$d\mathbf{S} = dy\, dz\, \hat{\mathbf{\imath}} \ . \tag{3.15}$$

Thus, through the face of the cube at $x = 1$,

$$\begin{aligned}
\mathbf{B} \cdot d\mathbf{S} &= (y\hat{\mathbf{\imath}} + z\hat{\mathbf{\jmath}} + x\hat{\mathbf{k}}) \cdot (dy\, dz\, \hat{\mathbf{\imath}}) && (3.16)\\
&= (y\hat{\mathbf{\imath}} + z\hat{\mathbf{\jmath}} + \underbrace{1}_{x=1}\hat{\mathbf{k}}) \cdot (dy\, dz\, \hat{\mathbf{\imath}}) && (3.17)\\
&= y\, dy\, dz \ . && (3.18)
\end{aligned}$$

Now just perform a double integration over the face ...

$$\begin{aligned}
\Rightarrow \int_{\text{Face}} \mathbf{B} \cdot d\mathbf{S} &= \int_{z=0}^{1} \int_{y=0}^{1} y\, dy\, dz && (3.19)\\
&= \frac{1}{2} y^2\Big|_0^1 z\Big|_0^1 \quad = \quad \frac{1}{2} \ . && (3.20)
\end{aligned}$$

3.5 The curl of a vector field

You have applied the del operator $\boldsymbol{\nabla}$ to a scalar field $\boldsymbol{\nabla} U$ to obtain the gradient, and dotted it with a vector field $\boldsymbol{\nabla}\cdot\mathbf{a}$ to obtain the divergence. You are now overwhelmed by the urge to cross it with a vector field.

This delivers the **curl of a vector field**:

$$\mathrm{curl}(\mathbf{a}) \equiv \boldsymbol{\nabla}\times\mathbf{a}\ . \tag{3.21}$$

The best way to work it out is to follow the pseudo-determinant recipe for vector products, so that

$$\boldsymbol{\nabla}\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \tag{3.22}$$

$$= \left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\hat{\mathbf{\imath}}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\hat{\mathbf{\jmath}}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}} \tag{3.23}$$

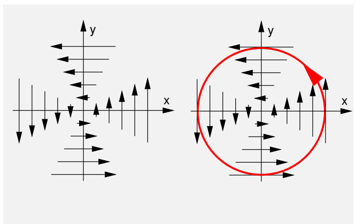
where, as before, a_x is the x-component of $\mathbf{a}$, and so on.

3.6 ♣ Examples of curl evaluation

Here are two straightforward examples, with a sketch of the the field $\mathbf{a}$ for the first case.

	$\mathbf{a}$	$\boldsymbol{\nabla}\times\mathbf{a}$
1)	$-y\hat{\mathbf{\imath}}+x\hat{\mathbf{\jmath}}$	$2\hat{\mathbf{k}}$
2)	$x^2y^2\hat{\mathbf{k}}$	$2x^2y\hat{\mathbf{\imath}}-2xy^2\hat{\mathbf{\jmath}}$

Examples



Sketch of field $\mathbf{a}$ in Example (1)

Working through the first,

$$\boldsymbol{\nabla}\times\mathbf{a} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0-0)\hat{\mathbf{\imath}}-(0-0)\hat{\mathbf{\jmath}}+(1-(-1))\hat{\mathbf{k}} = 2\hat{\mathbf{k}}\ . \tag{3.24}$$

3.7 The signficance of curl

Example 1) provides a clue. The field $\mathbf{a} = -y\hat{\mathbf{\imath}} + x\hat{\mathbf{\jmath}}$ is sketched above, and looks like the velocity field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega} = [0, 0, 1]$.) This field has a curl of $2\hat{\mathbf{k}}$, which is in the right-hand screw sense out of the page. You can also see that a field like this must give a non-zero value to the line integral around the complete loop, denoted $\oint_C \mathbf{a} \cdot d\mathbf{r}$.

That line integral, $\oint_C \mathbf{a} \cdot d\mathbf{r}$, is called the **circulation** of the vector field **a** around path C, and ...

The **curl** of the vector field **a** is the **circulation per unit area** (or **vorticity**) of the field, measured in the local plane of the chosen area.

It is no surprise then that a vector field with zero curl is said to be **irrotational**.

3.7.1 Mathematical derivation

One can understand curl's mathematical definition by constructing the rectangular element dx by dy shown in Figure 3.5.

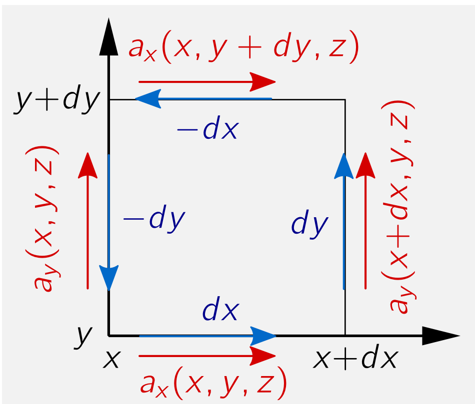


Figure 3.5: A small element used to calculate curl.

Consider the circulation round the perimeter of a rectangular element. The fields in the x direction at the bottom and top are

$$a_x(x, y, z) \qquad \text{and} \qquad a_x(x, y+dy, z) = a_x(x, y, z) + \frac{\partial a_x}{\partial y} dy, \tag{3.25}$$

and the fields in the y direction at the left and right are

$$a_y(x, y, z) \qquad \text{and} \qquad a_y(x+dx, y, z) = a_y(x, y, z) + \frac{\partial a_y}{\partial x} dx \tag{3.26}$$

where, as in the case of div, we have to be careful to calculate components at the correct positions.

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows, where the minus signs take account of the path being opposed to the field:

$$\begin{aligned} dC &= +[a_x\ dx] + [a_y(x{+}dx, y, z)\ dy] - [a_x(x, y{+}dy, z)\ dx] - [a_y\ dy] \qquad (3.27)\\ &= +[a_x\ dx] + \left[\left(a_y + \frac{\partial a_y}{\partial x}dx\right) dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y}dy\right) dx\right] - [a_y\ dy]\\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy \qquad = (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \qquad (3.28)\end{aligned}$$

where $d\mathbf{S} = dxdy\hat{\mathbf{k}}$. This satisfies the definition above.

3.8 Combining vector operators

The operators grad, div and curl provide a powerful method of describing and manipulating scalar and vector fields. It is possible also to combine operators, with some combinations providing short cuts during analysis.

HLT contains many of them, and here we will state a few, but then use just two. The important thing to remember is that the partial derivatives involved are "live", so that the product rule is often needed. (It is of course possible to derive these using brute force, but the more elegant way is using the Levi-Civita symbol, the Kronecker delta, and Einstein summation.)

3.8.1 div (grad$U$) or ∇^2 of a scalar field

Recall that ∇U of *any* scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute $\nabla \cdot (\nabla U)$.

$$\begin{aligned} \nabla \cdot (\nabla U) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) U\right) \qquad (3.29)\\ &= \left[\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right] U \qquad (3.30)\\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U \qquad (3.31)\\ \text{or} \quad \nabla^2 U &= \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) \qquad (3.32)\end{aligned}$$

∇^2 is called the Laplacian, and it occurs frequently when modelling diffusion for heat and fluid flow, electric potentials, and wave propagation in 3D.

3.8.2 curl grad $U = \mathbf{0}$

$$\nabla \times \nabla U = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \quad (3.33)$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) + \hat{\mathbf{j}}\,() + \hat{\mathbf{k}}\,() \qquad = \mathbf{0} \quad , \quad (3.34)$$

Note that the output is a zero *vector*. This indicates, for example, that no electric field $\mathbf{E}$ possesses circulation. Can you demonstrate why?

3.8.3 div curl $\mathbf{a} = 0$

$$\nabla \cdot (\nabla \times \mathbf{a}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \quad (3.35)$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \qquad = 0 \quad (3.36)$$

3.8.4 div and curl of $U\mathbf{a}$

Suppose $U(\mathbf{r})$ is a scalar field and that $\mathbf{a}(\mathbf{r})$ is a vector field. The product $U\mathbf{a}$ is a vector field, so we can compute its divergence and curl. (For example the density $\rho(\mathbf{r})$ of a fluid is a scalar field, and the instantaneous velocity of the fluid $\mathbf{v}(\mathbf{r})$ is a vector field. The mass flow rate is $\rho(\mathbf{r})\mathbf{v}(\mathbf{r})$.)

The divergence (a scalar) of the product $U\mathbf{a}$ is given by:

$$\nabla \cdot (U\mathbf{a}) = U(\nabla \cdot \mathbf{a}) + (\nabla U) \cdot \mathbf{a} \quad (3.37)$$

Similarly the curl of the vector field $U\mathbf{a}$, and the result should be a vector field:

$$\nabla \times (U\mathbf{a}) = U\nabla \times \mathbf{a} + (\nabla U) \times \mathbf{a} \quad . \quad (3.38)$$

3.8.5 curl curl a

The following important identity is stated, and left as an exercise:

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \quad (3.39)$$

where

$$\nabla^2 \mathbf{a} = \nabla^2 a_x \hat{\mathbf{i}} + \nabla^2 a_y \hat{\mathbf{j}} + \nabla^2 a_z \hat{\mathbf{k}} \quad . \quad (3.40)$$

3.8.6 div($\mathbf{a} \times \mathbf{b}$)

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \ . \tag{3.41}$$

3.8.7 [**An Aside**] Combining identities in physical problems

There is no better example of the description power of vector operators than their use in showing that Maxwell's four equations give rise to electromagnetic wave propagation. Denoting the volume density of free charge in a medium by ρ, its conductivity by σ, and permittivities and permeabilities by $\epsilon_r\epsilon_o$ and $\mu_r\mu_o$, and the current density by $\mathbf{J}$, the four equations are

$$\underbrace{\nabla \cdot \mathbf{D} = \rho}_{\text{Gauss}} \qquad \underbrace{\nabla \cdot \mathbf{B} = 0}_{\text{No free magnetic monopoles}} \qquad \underbrace{\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}}_{\text{Faraday-Lenz}} \qquad \underbrace{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}}_{\text{Ampère}}$$

In addition, we can assume the following, which should all be familiar to you:

$$\underbrace{\mathbf{J} = \sigma\mathbf{E}}_{\text{Ohm's Law in vectors}} \qquad \mathbf{B} = \mu_r\mu_0\mathbf{H} \qquad \mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$$

We'll assume that our medium contains no free charge and has zero conductivity. Then

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \epsilon_r\epsilon_0\nabla \cdot \mathbf{E} = \rho = 0 && \Rightarrow \nabla \cdot \mathbf{E} = 0. \\
\nabla \cdot \mathbf{B} &= \mu_r\mu_0\nabla \cdot \mathbf{H} = 0 && \Rightarrow \nabla \cdot \mathbf{H} = 0 \\
\nabla \times \mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \\
\nabla \times \mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t)
\end{aligned} \tag{3.42}$$

Above we found that "curl curl = grad div minus delsquared" — so here

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \qquad = \nabla \times (-\mu_r\mu_0(\partial\mathbf{H}/\partial t)) \tag{3.43}$$

But we know $\nabla \cdot \mathbf{E} = 0$. Also we can swap the order of partial differentiation on the rhs (remember that from Calculus II.1 ...) so we arrive at

$$-\nabla^2\mathbf{E} = -\mu_r\mu_0\frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu_r\mu_0\frac{\partial}{\partial t}\left(\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \tag{3.44}$$

$$\Rightarrow \nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{3.45}$$

A similar equation for $\mathbf{H}$ can be found (by taking $\nabla \times (\nabla \times \mathbf{H})$). These are 3D wave equations — the solutions for both the electric and magnetic fields are waves with phase velocity $v = 1/\sqrt{\mu_r\mu_0\epsilon_r\epsilon_0}$. In a vacuum, this is $c = 1/\sqrt{\mu_0\epsilon_0} = 3 \times 10^8 \mathrm{m\,s^{-1}}$.

The important points here are: (i) that our knowledge of the grad, div and curl of fields have made a daunting task really straightforward; and (ii) that we never once had to worry about coordinate systems.

3.9 Summary

In this lecture we have

- Defined and explain the div and curl of a vector field.
- We saw how operators could be combined using the basic rules of differential calculus.
- We also introduced the idea of the surface integral so that we could consider divergence in terms of net efflux from a volume.

Lecture 4

Vector calculus in other coordinate systems

4.1 Introduction

We have defined the grad, div, and curl operators, and developed line and surface integrals using Cartesian coordinates. However, as you learnt in your Calculus 2.1 course on multiple integration, the symmetry of a particular problem might indicate that another coordinate system should be adopted (for example, plane, cylindrical, or spherical polars).

The grad, div, and curl operators can be transformed to give direct expressions in other systems. Indeed, their definitions are in HLT. However, *this is often unnecessary*. When the scalar or vector field is given in Cartesians, it is often easier to evaluate ∇U, etc, in Cartesians, and then transform. Furthermore, after finding grad, div or curl, one is usually interested in performing a line integral or surface integral, and one is best off *delaying transforming to new coordinates* until the last moment. Examples later will make this clear.

4.2 Curvilinear and orthogonal curvilinear coordinates

Before considering the common sets of polar coordinates, we ought first to consider a general curvilinear coordinate system, $Ouvw$. In this, the vectors spanning 3D space $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$, do not have to be a right angles to one another — they can the edges of any parallelogram.

An *orthogonal* curvilinear coordinate system is (unsuprisingly) one where they are orthonal, so that $\hat{\mathbf{u}}\times\hat{\mathbf{v}}=\hat{\mathbf{w}}$, as are its cyclic permutations $\hat{\mathbf{v}}\times\hat{\mathbf{w}}=\hat{\mathbf{u}}$, and $\hat{\mathbf{w}}\times\hat{\mathbf{u}}=\hat{\mathbf{v}}$.

In Calculus 2.1 you learnt that when changing variables from Cartesians $Oxyz$ to some general coordinate system $Ouvw$ we could not simply replace an area element $dxdy$ by $dudv$; or a volume element $dxdydz$ by $dudvdw$. The reason was that du and so on were not guaranteed to be proper lengths.

In multiple integration you saw that the correction factor to apply to the elements could be found as the *modulus of the Jacobian*. But you will recall that you reached that by considering the area of parallelograms and the volumes of parallelopipeds. We will repeat the analysis here, but your greater knowledge of vectors will provide greater insight.

4.2.1 Transforming the position vector and its differential

We have seen the central role that the position vector and its differential plays in vector calculus, and it makes sense to start here with them in Cartesians

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \\ d\mathbf{r} &= dx\hat{\mathbf{\imath}} + dy\hat{\mathbf{\jmath}} + dz\hat{\mathbf{k}} \\ |d\mathbf{r}| &= ds = \sqrt{dx^2 + dy^2 + dz^2}\ . \end{aligned} \tag{4.1}$$

In general curvilinear coordinates we are interested in writing the position vector $\mathbf{r}$ and its differential $d\mathbf{r}$ as a sum of terms involving the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$.

But the very first thing to stress is that

$$\left.\begin{aligned} \mathbf{r} &\neq u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}} \\ d\mathbf{r} &\neq du\hat{\mathbf{u}} + dv\hat{\mathbf{v}} + dw\hat{\mathbf{w}} \\ |d\mathbf{r}| = ds &\neq \sqrt{du^2 + dv^2 + dw^2} \end{aligned}\right\} \textbf{THESE ARE BAD} \tag{4.2}$$

The badness arises because length or 'metric' scales have been lost.

Instead we must write

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} + h_v dv\hat{\mathbf{v}} + h_w dw\hat{\mathbf{w}} \tag{4.3}$$

where the $h_{u,v,w}$ are metric scale coefficients which turn du etc into proper lengths.

To find expressions for the scale parameters, remember that the position vector must be some function of u, v, w — $\mathbf{r} = \mathbf{r}(u, v, w)$ — and so the total or perfect differential can be written without thought as

$$d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial u}\right) du + \left(\frac{\partial \mathbf{r}}{\partial v}\right) dv + \left(\frac{\partial \mathbf{r}}{\partial w}\right) dw\ . \tag{4.4}$$

Now compare Eq. (4.4) that with Eq. (4.3): because u, v, and w, and hence du, dv, and dw are *independent*, we can match coefficients of du etc, and write that

$$h_u\hat{\mathbf{u}} = \left(\frac{\partial \mathbf{r}}{\partial u}\right) \text{ and hence } h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \text{ where } \left(\frac{\partial \mathbf{r}}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}}\right)\ . \tag{4.5}$$

and similarly for v and w. This looks complicated — but for any transformation we will know $x = x(u, v, w)$, etc, so can work out $\partial\mathbf{r}/\partial u$, etc. A grind, but not difficult.

To summarize so far:

If we are considering coordinates u, v, w

$$\mathbf{r} = x(u, v, w)\hat{\mathbf{\imath}} + y(u, v, w)\hat{\mathbf{\jmath}} + z(u, v, w)\hat{\mathbf{k}} \tag{4.6}$$

and

$$d\mathbf{r} = h_u du \hat{\mathbf{u}} \ + \ h_v dv \hat{\mathbf{v}} \ + \ h_w dw \hat{\mathbf{w}} \tag{4.7}$$

We know

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv + \frac{\partial \mathbf{r}}{\partial w} dw \ . \tag{4.8}$$

so that

$$h_u = \left| \frac{\partial \mathbf{r}}{\partial u} \right| \qquad h_v = \left| \frac{\partial \mathbf{r}}{\partial v} \right| \qquad h_w = \left| \frac{\partial \mathbf{r}}{\partial w} \right| \tag{4.9}$$

and

$$\left| \frac{\partial \mathbf{r}}{\partial u} \right| = \left[\left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 \right]^{1/2} \text{ and similiarly for } v, w. \tag{4.10}$$

Don't try to remember all this! The key to unlock all this is (i) writing down the perfect differential, and making the comparison between Eq. (4.7) and Eq. (4.8).

Fig. 4.1 is a 2D illustration that under transformation vector $d\mathbf{r}$ remains the same, but must be described differently in Cartesian and curvilinear coordinates. (The diagram is similar to that in Calculus 2.1.)

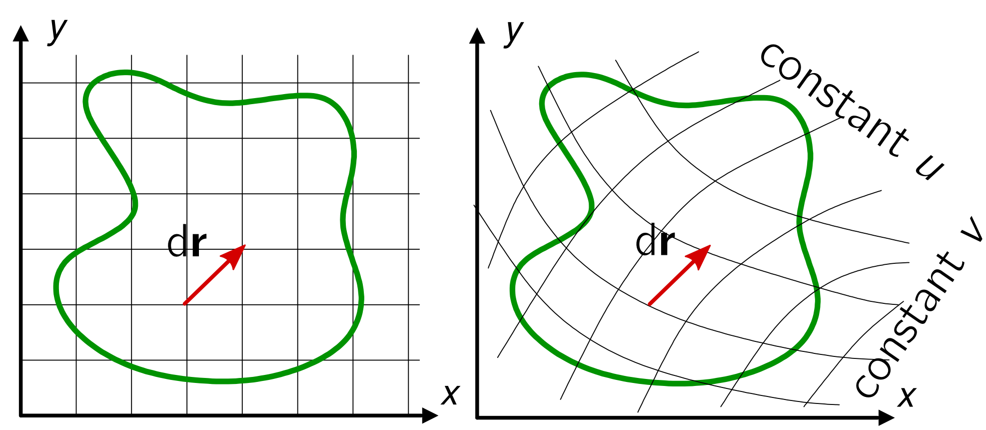


Figure 4.1: Lines of constant u and v appear as curves on the xy-plane.

4.3 Line, surface, volume integrals, in curvi/orthog-curvi coords

Line integrals: If you want to perform a line integral in the $\hat{\mathbf{u}}$ direction, the vector line element you want is (and similarly for the other directions)

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} \; + \; h_v dv\hat{\mathbf{v}} \; + \; h_w dw\hat{\mathbf{w}} \tag{4.11}$$

Surface integrals: Consider the curvilinear trio $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$. If you want a surface element in the $\hat{\mathbf{w}}$ direction, use (and similarly for the other directions)

$$d\mathbf{S}_w \quad = h_u h_v du dv(\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \quad = \quad \underbrace{h_u \, h_v \, du \, dv \, \hat{\mathbf{w}}}_{\text{for an orthogonal curvilinear trio}} \tag{4.12}$$

Volume integrals: And the extension to volume elements (a scalar) is also obvious.

The volume element is a parallelopiped, given by

$$\begin{aligned} dV \;&=\; (h_u du\hat{\mathbf{u}}) \times (h_v du\hat{\mathbf{v}}) \cdot (h_w dw\hat{\mathbf{w}}) &(4.13)\\ &=\; h_u h_v h_w \;\; du \; dv \; dw \;\; (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \quad = \quad \underbrace{h_u h_v h_w \; du \; dv \; dw}_{\text{for an orthogonal curvilinear trio}} &(4.14)\end{aligned}$$

4.3.1 Two questions ...

1. What are Orthogonal curvilinear coords?

They are ones where the $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ vectors are mutually perpendicular. They form a right-handed set with

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} \tag{4.15}$$

and so on, cyclically.

2. Do Jacobians still work?

Yes — and yes for all curvilinear coordinate systems, irrespective of whether orthogonal or not.

Thinking about volume elements we know that

$$h_u h_v h_w (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \cdot \frac{\partial \mathbf{r}}{\partial w} \tag{4.16}$$

But

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}} \right) \tag{4.17}$$

and similarly for v and w. Putting the components of $\partial \mathbf{r}/\partial u$ on the top line and those for v and w on the second and third, the scalar triple product becomes

$$\text{mod } h_u h_v h_w (\hat{\mathbf{u}} \times \hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} = \quad \text{mod} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{4.18}$$

If we were always sure that the coordinates were a right-handed trio the mod would be unnecessary on both sides. The modulus is used as an insurance.

4.3.2 All the polars are orthogonal curvilinear coordinate systems

Let us now specialize the analysis to the three common examples of orthogonal curvilinear coordinates — plane, cylindrical, and spherical polars.

In each case let us following a standard plan to understand the geometry involved:

1. Define the unit basis vectors
2. Determine the position vector
3. Determine the h coefficients
4. Use these to define the differential of the position vector. The general line element is identical to the differential
5. Use the components of the line element to define the *vector* surface element(s)
6. Use the components of the line element to define the *scalar* volume element

4.4 Plane polars: key geometry

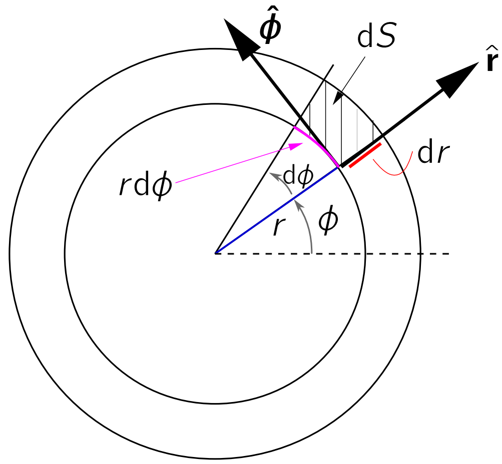


Figure 4.2:

1. **Unit basis vectors**

$$\hat{\mathbf{r}} = (\cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}}) \tag{4.19}$$
$$\hat{\boldsymbol{\phi}} = (-\sin\phi\hat{\mathbf{\imath}} + \cos\phi\hat{\mathbf{\jmath}}) \tag{4.20}$$

2. **Position vector**

$$\mathbf{r} = r\hat{\mathbf{r}} = r\cos\phi\hat{\mathbf{\imath}} + r\sin\phi\hat{\mathbf{\jmath}} \tag{4.21}$$

3. **Metric scale coefficients**

$$h_r\hat{\mathbf{r}} = \partial\mathbf{r}/\partial r = \cos\phi\hat{\mathbf{\imath}} + \sin\phi\hat{\mathbf{\jmath}} \qquad \Rightarrow h_r = 1$$
$$h_\phi\hat{\boldsymbol{\phi}} = \partial\mathbf{r}/\partial\phi = r(-\sin\phi\hat{\mathbf{\imath}} + \cos\phi\hat{\mathbf{\jmath}}) \qquad \Rightarrow h_\phi = r$$

4. **Differential**

$$d\mathbf{r} = h_r dr\hat{\mathbf{r}} + h_\phi d\phi\hat{\boldsymbol{\phi}} = dr\hat{\mathbf{r}} + r d\phi\hat{\boldsymbol{\phi}} \tag{4.22}$$

5. Surface element

The element is in the plane, but its normal sticks out of the plane — which is dodgy in plane polars. This indicates that the scalar value will suffice here!

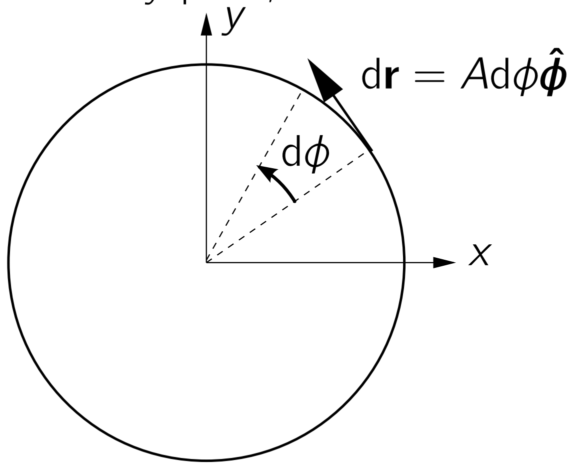
$$d\mathbf{S} = h_r h_\phi\, dr\, d\phi(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = r\, dr\, d\phi\, \hat{\mathbf{k}} \tag{4.23}$$
$$dS = r\, dr\, d\phi\ . \tag{4.24}$$

6. Volume element There isn't one! We are in flat world.

NB: We've (very properly) used the h coefficients to calculate values, but do notice that the results "make sense" from simple considerations. (Eg, swinging the line of length r around angle $d\phi$ traces a length $r\, d\phi$.

4.4.1 ♣ Example: Line integral in plane polars

Q: Evaluate the line integral $\oint_C \mathbf{f} \cdot d\mathbf{r}$, where $\mathbf{f} = -y^3\hat{\mathbf{i}} + x^3\hat{\mathbf{j}}$ and the path (or contour) C is the circle of radius A in the xy-plane, centred on the origin.



A: On the circle of interest $r = A$ is fixed, and we only need ϕ to define exactly where we are.

We know that

$$x = A\cos\phi \qquad \text{and} \qquad y = A\sin\phi \ , \tag{4.25}$$

so that

$$\mathbf{f} = -y^3\hat{\mathbf{i}} + x^3\hat{\mathbf{j}} \quad = A^3(-\sin^3\phi\hat{\mathbf{i}} + \cos^3\phi\hat{\mathbf{j}}). \tag{4.26}$$

In general,

$$d\mathbf{r} = dr\hat{\mathbf{r}} + rd\phi\hat{\boldsymbol{\phi}} \ , \tag{4.27}$$

but on the chosen path $r = A$ and so $dr = 0$. Hence

$$d\mathbf{r} = Ad\phi\hat{\boldsymbol{\phi}} = Ad\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \ . \tag{4.28}$$

Taking the dot products

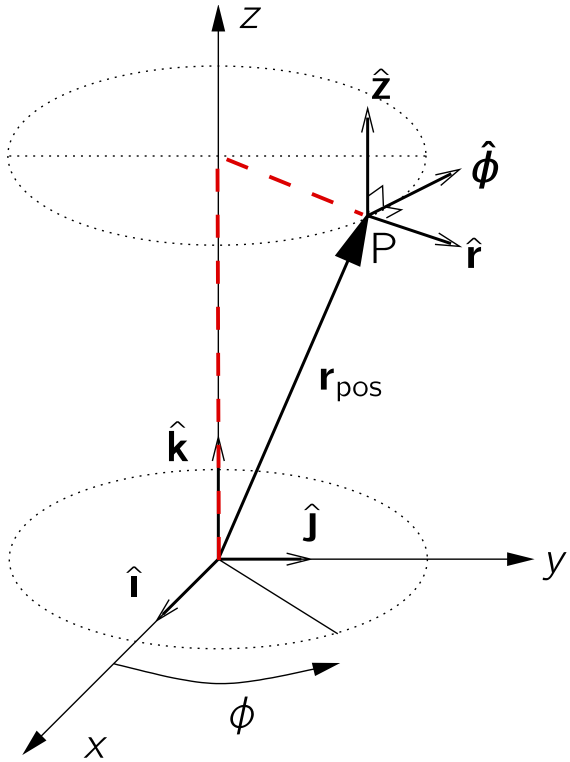
$$\oint_C \mathbf{f} \cdot d\mathbf{r} = A^4 \int_{\phi=0}^{2\pi} (\sin^4\phi + \cos^4\phi)d\phi = A^4\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right) \ = A^4\frac{3\pi}{2} \ . \tag{4.29}$$

Notice that we used a mixture of Cartesians for the vectors, and polars for the coefficients. This is perfectly permissible. We *could* have converted $\mathbf{f}$ to use $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$, etc, but the approach taken is easier.

Notice too that we could have found $d\mathbf{r}$ by writing

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} = Ad\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \tag{4.30}$$

4.5 Cylindrical polars: key geometry



1. Unit vectors

These are as in plane polars, but with the addition of a $\hat{\mathbf{z}}$ axis along the Cartesian $\hat{\mathbf{k}}$ direction.

$$\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \tag{4.31}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \tag{4.32}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{k}} \tag{4.33}$$

2. Pos vector

There a source of confusion here! The position vector does not point in the direction of the radial unit vector $\hat{\mathbf{r}}$, so that $r\,\hat{\mathbf{r}}$ is NOT the position vector! Instead we denote the position vector by $\mathbf{r}_{\text{pos}}$. Then

$$\mathbf{r}_{\text{pos}} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{4.34}$$
$$= r\hat{\mathbf{r}} + z\hat{\mathbf{z}} \tag{4.35}$$

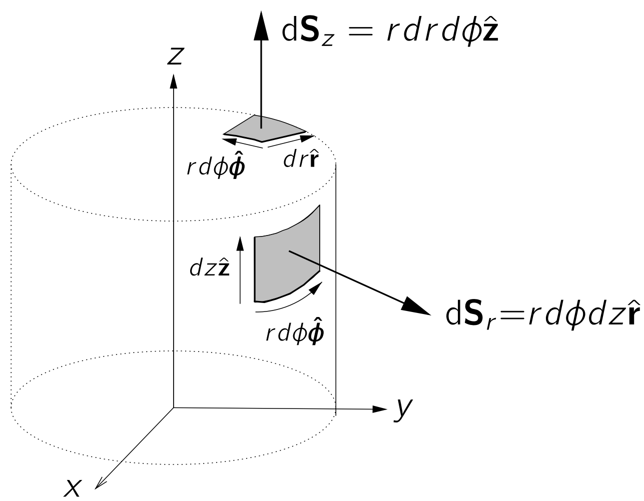


Figure 4.3:

3. Calculation of h using $\mathbf{r}_{\mathbf{pos}}$

Recall that $\mathbf{r}_{\text{pos}} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, so that ...

$$\partial\mathbf{r}_{\text{pos}}/\partial r = h_r\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \tag{4.36}$$

$$\partial\mathbf{r}_{\text{pos}}/\partial\phi = h_\phi\hat{\boldsymbol{\phi}} = r(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \tag{4.37}$$

$$\partial\mathbf{r}_{\text{pos}}/\partial z = h_z\hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{4.38}$$

$$\Rightarrow h_r = 1, \quad h_\phi = r, \quad h_z = 1\ . \tag{4.39}$$

4. The differential

$$d\mathbf{r}_{\text{pos}} = dr\ \hat{\mathbf{r}} + rd\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}} \tag{4.40}$$

5. Surface elements

There are three, but the last very rare! The first relates to the patch on the wallr; the second to the patch on the top (or, with a change in sign, on the base).

$$d\mathbf{S}_r = h_\phi\, h_z\, d\phi\, dz\, (\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = r\, d\phi\, dz\, \hat{\mathbf{r}} \tag{4.41}$$

$$d\mathbf{S}_z = \boxed{\text{DIY}} = rdrd\phi\hat{\mathbf{z}} \tag{4.42}$$

$$d\mathbf{S}_\phi = \boxed{\text{DIY}} = dr\, dz\hat{\boldsymbol{\phi}} \quad \text{Can you sketch this?} \tag{4.43}$$

6. Volume element

You know what to expect, but in full:

$$dV = h_r h_\phi h_z\ dr\ d\phi\ dz\ (\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) \cdot \hat{\mathbf{z}} \tag{4.44}$$

$$= 1\ \ r\ \ 1\ dr\ d\phi\ dz\ (\ 1\ ) \qquad = r\ dr\ d\phi\ dz\ . \tag{4.45}$$

Again, notice that the geometry makes sense, without resorting to the h coefficients.

4.5.1 ♣ Example. Surface integral using cylindrical polars.

Q: Find the surface integral of the field $\mathbf{f} = (x^2 + y^2)(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) + z^3\hat{\mathbf{k}}$ over the upper surface and side wall of a cylinder of radius A and height h aligned with z-axis and with its base sitting on the $z = 0$ plane.

A: Unlike the previous example, here it *is* convenient (but not essential) to express quantities using $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ as the vectors. So, we write

$$\mathbf{f} \quad = \quad r^2\mathbf{r} + z^3\hat{\mathbf{z}} \quad = \quad r^3\hat{\mathbf{r}} + z^3\hat{\mathbf{z}} \; . \tag{4.46}$$

Top: The surface element for the top is

$$d\mathbf{S}_z = r\,dr\,d\phi\,\hat{\mathbf{z}} \; , \tag{4.47}$$

Hence for the top, as $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = 0$, and then $z = h$,

$$\iint_{\text{top}} \mathbf{f} \cdot d\mathbf{S}_z = \int_{\phi=0}^{2\pi} \int_{r=0}^{A} (r^3\hat{\mathbf{r}} + z^3\hat{\mathbf{z}}) \cdot r\,dr\,d\phi\hat{\mathbf{z}} = \int_{\phi=0}^{2\pi} d\phi \int_{r=0}^{A} h^3 r\,dr = 2\pi h^3(A^2/2) \; . \tag{4.48}$$

Side: Remembering that the radius is A, the surface element for the side wall is

$$d\mathbf{S}_r = A d\phi dz \hat{\mathbf{r}} \; . \tag{4.49}$$

Around the side $r = A$, so

$$\iint_{\text{side}} \mathbf{f} \cdot d\mathbf{S}_r = \int_{\phi=0}^{2\pi} \int_{z=0}^{h} (r^3\hat{\mathbf{r}} + z^3\hat{\mathbf{z}}) \cdot dz A d\phi \hat{\mathbf{r}} = A^4 \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{h} dz = A^4 2\pi h \; . \tag{4.50}$$

Hence the total is

$$\iint_{\text{top+side}} \mathbf{f} \cdot d\mathbf{S} = 2\pi h A^2 \left(\frac{h^2}{2} + A^2 \right) \; . \tag{4.51}$$

4.6 Spherical polars: key geometry

1. Unit vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{4.52}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \tag{4.53}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \tag{4.54}$$
$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad \text{(and cyclic permutations)} \tag{4.55}$$

2. Position vector

$$\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}} \tag{4.56}$$
$$= r\hat{\mathbf{r}} \tag{4.57}$$

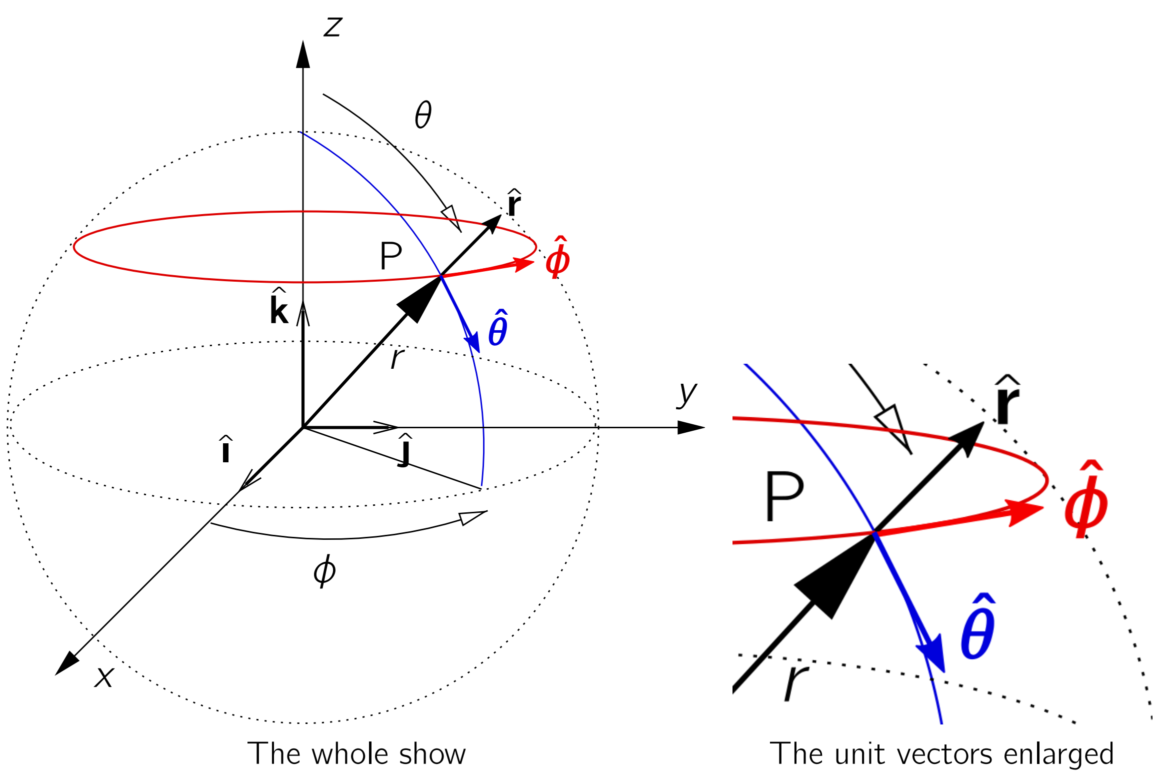
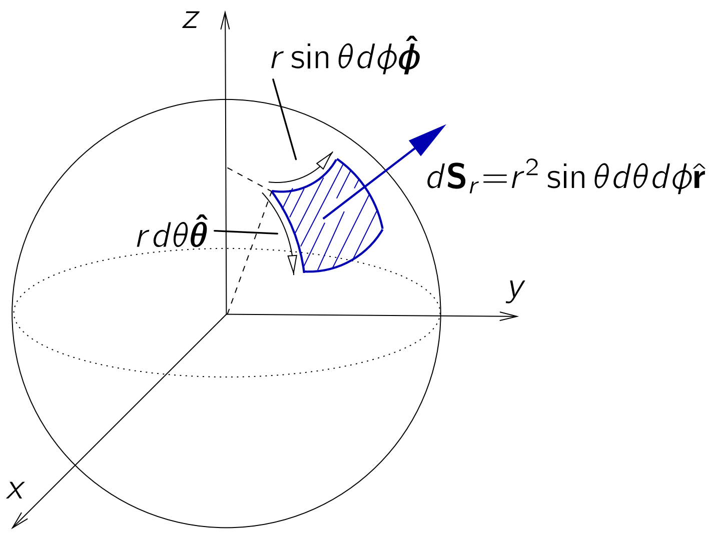
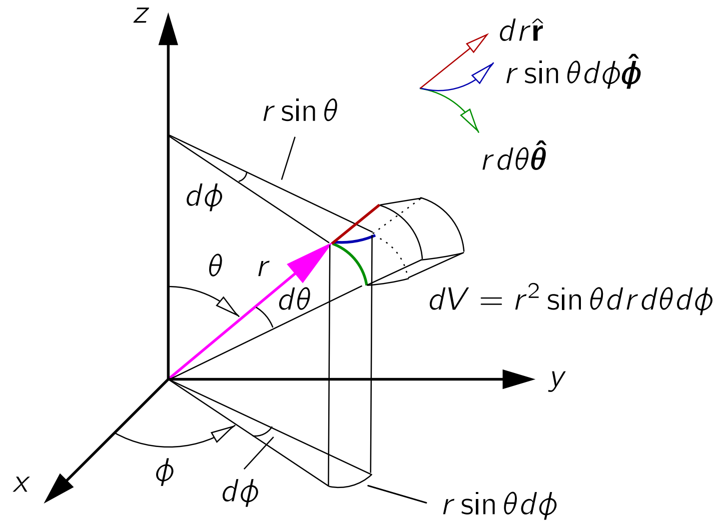


Figure 4.4:





3. h parameters

$$
\begin{aligned}
h_r &= \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right)^{1/2} = 1 && (4.58)\\
h_\theta &= \boxed{\text{grind}} = r && (4.59)\\
h_\phi &= \boxed{\text{grind}} = r\sin\theta && (4.60)
\end{aligned}
$$

4. Differential

$$
d\mathbf{r} = dr\ \hat{\mathbf{r}} + rd\theta\ \hat{\boldsymbol{\theta}} + r\sin\theta\ d\phi\ \hat{\boldsymbol{\phi}} \qquad (4.61)
$$

5. Surface elements:

$$
\begin{aligned}
d\mathbf{S}_r &= r^2\ \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}} && (4.62)\\
\Rightarrow d\mathbf{S}_r &= A^2\ \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}}\ \ \textbf{at fixed radius } A && (4.63)\\
d\mathbf{S}_\phi &= \boxed{\text{DIY}}\quad \text{but rarely seen} && (4.64)\\
d\mathbf{S}_\theta &= \boxed{\text{DIY}}\quad \text{even more rarely seen} && (4.65)
\end{aligned}
$$

6. Volume element

$$
\begin{aligned}
dV &= h_r\ \ h_\theta\ \ \ h_\phi\ \ \ \ dr\ d\theta\ d\phi\ \ (\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}})\cdot\hat{\boldsymbol{\phi}} && (4.66)\\
&= 1\ \ \ r\ \ \ r\sin\theta\ \ dr\ d\theta\ d\phi\ \ (1) && (4.67)\\
&= r^2\ \sin\theta\ dr\ d\theta\ d\phi\ (\text{as expected!}) && (4.68)
\end{aligned}
$$

4.6.1 ♣ Example: Surface integral in spherical polars

Q: Evaluate $\int_S \mathbf{f} \cdot d\mathbf{S}$, where $\mathbf{f} = z^3\hat{\mathbf{k}}$ and S is the sphere of radius A centred on the origin.

A: In spherical polars $z = r\cos\theta$ and $r = A$ on the surface of the sphere. Thus on the surface

$$\mathbf{f} = A^3 cos^3\theta\ \hat{\mathbf{k}} \quad \text{and} \quad d\mathbf{S} = A^2 \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}} \tag{4.69}$$

Hence

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A^3 cos^3\theta\ A^2 \sin\theta\ [\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}]\ d\theta d\phi\ . \tag{4.70}$$

But $\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} = \cos\theta$, so that

$$\begin{aligned} \int_S \mathbf{f} \cdot d\mathbf{S} &= A^5 \int_0^{2\pi} d\phi\ \int_0^{\pi} cos^3\theta \sin\theta\ [\cos\theta]\ d\theta \\ &= A^5 2\pi \int_0^{\pi} cos^4\theta\ \sin\theta\ \ d\theta \\ &= 2\pi A^5 \frac{1}{5} \left[-\cos^5\theta\right]_0^{\pi} \quad = \quad \frac{4\pi A^5}{5} \end{aligned} \tag{4.71}$$

4.7 Line, surface, & volume integrals involving div, grad, & curl.

♣ **Q(i):** The vector field $\mathbf{v}$ is

$$\mathbf{v} = \frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}} \ . \tag{4.72}$$

Find $\mathbf{v} \cdot d\mathbf{S}$ integrated over the surface of a sphere of radius A.

A(i): You notice that the problem involves spherical symmetry.

Thinking about the position vector $\mathbf{r}$ indicates that $\mathbf{v} = \dfrac{\mathbf{r}}{r} = \hat{\mathbf{r}}$.

But on the surface of a sphere of radius A

$$d\mathbf{S} = dS\hat{\mathbf{r}} = A^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}} \tag{4.73}$$

$$\Rightarrow \int \mathbf{v} \cdot d\mathbf{S} \quad = \quad A^2 \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi(\underbrace{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}}_{\text{unity}}) \quad = \quad 4\pi A^2 \tag{4.74}$$

Q(ii): Find the divergence of $\mathbf{v}$ and derive $\int \boldsymbol{\nabla} \cdot \mathbf{v} dV$ integrated over the volume of the same sphere.

A(ii): We have to find div $\mathbf{v}$. If you were wondering, there is a formula for this which use spherical polars directly. But we are to use only the Cartesian system.

Now think about each component of $\mathbf{v}$ in Cartesians ...

$$v_x = \frac{x}{r} = \frac{x}{(x^2+y^2+z^2)^{1/2}} \tag{4.75}$$

Differentiating wrt x gives

$$\frac{\partial v_x}{\partial x} = \frac{(x^2+y^2+z^2)^{1/2} - x^2(x^2+y^2+z^2)^{-1/2}}{(x^2+y^2+z^2)} = \frac{r - x^2/r}{r^2} \ . \tag{4.76}$$

Symmetry can be used to write the other contributions.

$$\Rightarrow \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{3r - (x^2+y^2+z^2)/r}{r^2} = \frac{2}{r} \ . \tag{4.77}$$

A(ii) ctd: To finish, we must integrate $2/r$ through the volume of a sphere of radius A. This is a straightforward multiple integral in spherical polars ...

We know

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{4.78}$$

and hence

$$\begin{aligned}
\int \nabla \cdot \mathbf{v} dV &= \int_{r=0}^{A} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{2}{r} \underbrace{r^2 \sin\theta dr \, d\theta \, d\phi}_{dV} \\
&= 2 \int_{r=0}^{A} r \, dr \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \\
&= 2 \left. \frac{r^2}{2} \right|_0^A \left. (-\cos\theta) \right|_0^{\pi} \left. \phi \right|_0^{2\pi} \\
&= 4\pi A^2
\end{aligned} \tag{4.79}$$

So we have the same result from both integrals!

This is an example of Gauss' Law which you will learn about next year. In short it says

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{v} dV \tag{4.80}$$

where V is a volume and S is the surface of that volume.

♣ **Q(i):** A vector field is $\mathbf{v} = 2y\hat{\mathbf{\imath}} - x\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$.

Find curl $\mathbf{v}$ and then evaluate $\int_S \mathrm{curl}\,\mathbf{v} \cdot d\mathbf{S}$ integrated over the surface of a hemisphere of radius A centred at the origin and for which $z \geq 0$.

A(i):

$$\text{(i) } \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & -x & z \end{vmatrix} = \hat{\mathbf{\imath}}(0) - \hat{\mathbf{\jmath}}(0) + \hat{\mathbf{k}}(-3) = -3\hat{\mathbf{k}}$$

The surface element on the hemisphere's surface is $d\mathbf{S} = A^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$.

Hence (note the $\theta = \pi/2$ limit for the hemisphere!)

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{S} = -3A^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta d\theta d\phi (\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \tag{4.81}$$

But $(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) = \cos\theta$, so

$$\begin{aligned} \int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{S} &= -3A^2 \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta d\theta \int_{\phi=0}^{2\pi} d\phi & (4.82) \\ &= -3A^2 \quad \frac{1}{2}\sin^2\theta \Big|_0^{\pi/2} \quad \phi \Big|_0^{2\pi} \\ &= -3A^2 \quad \frac{1}{2} \quad 2\pi \\ &= -3\pi A^2 \end{aligned}$$

Q(ii): Derive the line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$ where C is the closed circular contour $x^2 + y^2 = A^2$, $z = 0$.

A(ii): The circle has radius A, and $d\mathbf{r} = A\,d\phi\,\hat{\boldsymbol{\phi}} = A\,d\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}})$. Hence

$$\begin{aligned} \mathbf{v} \cdot d\mathbf{r} &= (2y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (A\,d\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}})) \\ &= A\,(-2y\sin\phi - x\cos\phi)\,d\phi \end{aligned} \tag{4.83}$$

But on the circle $x = A\cos\phi$ and $y = A\sin\phi$, so

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_{\phi=0}^{2\pi} A^2\,(-2\sin^2\phi - \cos^2\phi)\,d\phi \tag{4.84}$$

But $\int_0^{2\pi} \sin^2\phi d\phi = \int_0^{2\pi} \cos^2\phi d\phi = \pi$, so that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = A^2(-3\pi) \qquad = -3\pi A^2\ . \tag{4.85}$$

Again the results are the same!

This is an example of Stokes Law which in short tells us that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{v} \cdot d\mathbf{S} \tag{4.86}$$

when C is some bounding contour, and S is *any* "capping" surface bounded by C.

More next year!

4.8 Summary

Our analysis has provided a method of using the definition of the position vector $\mathbf{r}$ in a new coordinate system to find

- the metric coefficients or length scales $h_{u,v,w}$ associated with each key axis

$$h_u = |(\partial\mathbf{r}/\partial u)| \quad \text{etc;} \tag{4.87}$$

- the directions of the base coordinate vectors

$$\hat{\mathbf{u}} = (\partial\mathbf{r}/\partial u)/h_u \quad \text{etc;} \tag{4.88}$$

- the definition of the general line element

$$d\mathbf{r} = h_u du\hat{\mathbf{u}} + h_v dv\hat{\mathbf{v}} + h_w dw\hat{\mathbf{w}} \tag{4.89}$$

  which you can simplify if one or more of du, dv and dw are zero on your particular path;
- the definition, size and direction of the surface elements $d\mathbf{S}$; and
- the definition and size of volume element dV.

We also found that ...

- for our commonly used systems of polars, the results are remarkably intuitive.