Partie 2 : Problème Inverse.

Données :

- $T = T_0 + \sum_{k=1}^{n} \alpha_k \cdot \theta_k$.
- $A_k = \alpha_k \cdot A_0$
- $\theta_k = 0$ sur Γ_D.
- But : obtenir T_{opt} dans Ω_p (pare-choc)

$\hookrightarrow$ minimiser : $J(\vec{\alpha}) = \frac{1}{2} \cdot \int_{\Omega_p} \left(\left(T_0 + \sum \alpha_k \theta_k \right) - T_{opt} \right)^2 d\Omega + \frac{\beta}{2} \cdot \|\alpha\|^2$; $\beta \geqslant 0$.

Exercice 6 :

$$\left(\left(T_0 + \sum \alpha_k \theta_k \right) - T_{opt} \right)^2 = \left[\left(T_0 - T_{opt} \right) + \sum \alpha_k \theta_k \right]^2$$
$$= \left(T_0 - T_{opt} \right)^2 + 2 \cdot \left(T_0 - T_{opt} \right) \cdot \sum \alpha_k \theta_k + \left(\sum \alpha_k \theta_k \right)^2.$$

d'où $J(\vec{\alpha}) = \underbrace{\frac{1}{2} \int_{\Omega_p} \left(T_0 - T_{opt} \right)^2 d\Omega}_{= c} + \underbrace{\int_{\Omega_p} \left(T_0 - T_{opt} \right) \cdot \sum \alpha_k \theta_k \, d\Omega}_{} + \frac{1}{2} \int_{\Omega_p} \underbrace{\left(\sum \alpha_k \theta_k \right)^2}_{} \cdot d\Omega + \frac{\beta}{2} \cdot \underbrace{\|\alpha\|^2}_{= \int_{\Omega_p} \sum \alpha_k^2 \, d\Omega}$

$$= \sum \alpha_k \cdot \int_{\Omega_p} \left(T_0 - T_{opt} \right) \cdot \theta_k \, d\Omega.$$
$$= - \sum \alpha_k b_k.$$
$$= - \left(\alpha , b \right)_{\mathbb{R}^n}.$$

$= \left(\sum \alpha_k \theta_k \right) \left(\sum \alpha_{k'} \theta_{k'} \right)$

fubini

$= \sum_k \sum_j \alpha_k \alpha_j \theta_k \theta_j$

or, $\left(A\vec{\alpha}, \vec{\alpha} \right)_{\mathbb{R}^n} = \sum_{r=1}^{n} \sum_{r'=1}^{n} A_{rr'} \alpha_r \cdot \alpha_{r'}$

$$= \sum_{r=1}^{n} \sum_{r'=1}^{n} \left(\int_{\Omega_p} \theta_k \theta_{k'} \, d\Omega + \beta . \delta_{rr'} \right) \alpha_r \cdot \alpha_{r'}$$

$$= \int_{\Omega_p} \left(\sum_r \sum_{r'} \theta_r \theta_{r'} \right) \alpha_r \alpha_{r'} \, d\Omega + \underbrace{\sum_r \sum_{r'} \beta \delta_{rr'} \alpha_r \cdot \alpha_{r'}}_{= \beta \cdot \sum_r \alpha_r^2}.$$

Finalement : $\boxed{J(\vec{\alpha}) = c - (\vec{\alpha}, \vec{b}) + \frac{1}{2} \cdot (A\vec{\alpha}, \vec{\alpha})}$

Exercice 7

Q1) D'après le calcul de l'exercice 6 :

- $$(A\vec{\alpha}, \vec{\alpha})_{\mathbb{R}^n} = \int_{\Omega_p} \left(\sum_r \sum_{r'} \theta_r \theta_{r'}\right) \alpha_r \alpha_{r'} \, d\Omega + \beta \cdot \sum_r \alpha_r^2 .$$
$$= \int_{\Omega_p} \underbrace{\left(\sum \alpha_k \theta_r\right)^2}_{\geqslant 0} \cdot d\Omega + \underbrace{\beta \cdot \sum_r \alpha_r^2}_{> 0} . \quad (\vec{\alpha} \neq \vec{0})$$
$$> 0$$

- $A \in S_n^{++}(\mathbb{R}) \Rightarrow$ $A \in GL_n(\mathbb{R})$.

$A \in S_n(\mathbb{R})$ car $\forall (r, r') \in [\![1, n]\!]^2$, $A_{rr'} = A_{r'r}$

Q2) $\vec{\alpha} = \vec{\alpha}_m + \vec{\alpha}'$

- $\vec{\alpha}_m$ vérifiant $A\vec{\alpha}_m = \vec{b}$

- $$J(\vec{\alpha}) = J(\vec{\alpha}_m + \vec{\alpha}')$$
$$= c - (\vec{\alpha}_m + \vec{\alpha}', \vec{b}) + \frac{1}{2} \cdot (A\vec{\alpha}_m + A\vec{\alpha}', \vec{\alpha}_m + \vec{\alpha}')$$
$$= \underbrace{c - (\vec{\alpha}_m, \vec{b})}_{} - (\vec{\alpha}', \vec{b}) + \underbrace{\frac{1}{2} (A\vec{\alpha}_m, \vec{\alpha}_m)}_{} + \frac{1}{2} (A\vec{\alpha}', \vec{\alpha}_m)$$
$$+ \frac{1}{2} (\underbrace{A\vec{\alpha}_m}_{\vec{b}}, \vec{\alpha}') + \frac{1}{2} (A\vec{\alpha}', \vec{\alpha}')$$
$$= J(\vec{\alpha}_m) - (\vec{\alpha}', \vec{b}) + \frac{1}{2} (A\vec{\alpha}', \vec{\alpha}_m) + \frac{1}{2} (\vec{b}, \vec{\alpha}') + \frac{1}{2} (A\vec{\alpha}', \vec{\alpha}')$$

Or, $(A\vec{\alpha}', \vec{\alpha}'_m) = (\vec{\alpha}', A\vec{\alpha}'_m) = (\vec{\alpha}', \vec{b}')$

Finalement : $\boxed{J(\vec{\alpha}) = J(\vec{\alpha}'_m) + \frac{1}{2} \cdot (A\vec{\alpha}', \vec{\alpha}')}$

Q3). $\frac{1}{2} (A\vec{\alpha}', \vec{\alpha}') > 0$ (Q1)

- donc $J(\alpha_m)$ représente bien un minimum.

- En prenant 2 minimums α_m et α_n, on a : $\left.\begin{array}{l} A\vec{\alpha}'_m = \vec{b}' \\ A\vec{\alpha}'_n = \vec{b}' \end{array}\right\} \Rightarrow A(\vec{\alpha}'_m - \vec{\alpha}'_n) = \vec{0}'$

  Or, $A \in GL_n(\mathbb{R})$

  Donc $\alpha_m - \alpha_n = 0$

  Donc $\alpha_m = \alpha_n$.

Exercice 8

complexité : $\boxed{\Theta(\text{Nbtri} \times n^2)}$. avec n : nombre de résistances placée.