The Relative Chain Framework for Modular Termination

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1 Introduction

2 Preliminaries

2.1 Term Rewrite Systems

We consider the setting of term rewriting systems over a **signature** $T(\Sigma, V)$. A signature consists of a finite set of **function symbols** Σ and an infinite set of **variables** V. Σ is presumed to be partitioned into a set of **defined functions** Σ_{def} and **constructors** Σ_{con}. Each $f \in \Sigma$ has an associated **arity**, and $\Sigma^{(i)}$ denotes all the function symbols in Σ with arity $i \in \mathbb{N}$.

$T(\Sigma, V)$ is defined inductively as follows:

1. $\Sigma^{(0)} \subseteq T(\Sigma, V)$ and $V \subseteq T(\Sigma, V)$.
2. If $f \in \Sigma^{(n)}$, and $M_i \in T(\Sigma, V)$ for each $i \in [n]$, then $f(M_1, \cdots, M_n) \in T(\Sigma, V)$.

The recursive structure of $T(\Sigma, V)$ prompts us to think of terms in a TRS as **syntax trees**. We define $Pos(M) \subseteq \mathbb{N}^*$ for each $M \in T(\Sigma, V)$ to be the set of positions in M's syntax tree. The root of the tree has position ϵ; and the ith child of a node at position p has position pi. $M|_p$ denotes M's subterm located at position p, and $M[N]_p$ denotes M with its term at position p replaced by $N \in T(\Sigma, V)$.

A **(one-hole) context** $C[\cdot] : T(\Sigma, V) \to T(\Sigma, V)$ is a term with a hole $\circ$ at some position. $C[M]$ denotes the context $C[\cdot]$ with its hole replaced by the term M. A **substitution** $\theta : X \to T(\Sigma, V)$ is a function mapping variables to terms, for which the set $dom(\theta) := \{x \mid \theta(x) \neq x\}$ is finite. $M\theta$ denotes the term M with each variable replaced by its θ-image.

A reduction system $(R, \to)$ consists of a set R equipped with a binary relation $\to \subseteq R \times R$. A reduction is **terminating** iff there exist no sequences $x_1 x_2 \cdots \in R^\omega$ with $x_1 \to x_2 \to \cdots$. A reduction system over $T(\Sigma, V)$ is a **term rewrite syste (TRS)** iff it is closed under contexts and substitutions (i.e. if $M \to N$, then $C[M\theta] \to C[N\theta]$ for all contexts $C[\cdot]$ and substitutions θ).

Here, we will consider TRSs defined by a set of oriented equations $R \subseteq T(\Sigma, V) \times T(\Sigma, V)$. For an equation $M \approx N \in R$, we have that $C[M\theta] \to_R C[N\theta]$ for all contexts $C[\cdot]$ and substitutions θ. We say that an equation $M \approx N$ is **stable** iff M is headed by a defined function symbol. A TRS is stable iff all of its equations are stable.

2.2 The Dependency Pair Framework

Theorem 2.1 ([1]). *A TRS is terminating iff it does not induce an infinite chain.*

3 Relative Chains

Definition 3.1. *Consider two TRSs R and S, and assume that R comes equipped with dependency pairs $\langle \cdot, \cdot \rangle_R$. A sequence of these pairs $\langle s_1, t_1 \rangle_R \ \langle s_2, t_2 \rangle_R \cdots$ forms an R/S-chain iff there exists some sequence of substitutions $\theta_1 \theta_2 \cdots$ satisfying $t_i \theta_i \to_S^* s_{i+1} \theta_{i+1}$ for each i.*

Definition 3.2. *Consider some TRS $S := R_1 \cup R_2$, which is the union of two other TRSs. An S-chain is **spanning** iff it is of the form $\cdots p_1 p_2 \cdots$ where $p_1 \in DP(R_1)$ and $p_2 \in DP(R_2)$. In this case, we say that this chain spans **from** R_1 **to** R_2.*

Theorem 3.3. *Consider two TRSs R_1 and R_2. $S := R_1 \cup R_2$ is terminating if both of the following hold:*

1. *There are no infinite R_1/S- or R_2/S-chains.*
2. *Every spanning S-chain only spans from R_1 to R_2 (or vice versa).*

Proof. We proceed by contrapositive. Assume that S is nonterminating. Hence, by Theorem 2.1, $R_1 \cup R_2$ induces an infinite chain. We now demonstrate that any such chain violates one of our conditions. Observe that an infinite chain must repeat some finite sequence s of dependency pairs infinitely many times. This cycle must occur either entirely within $DP(R_1)$ or $DP(R_2)$, or must alternate between $DP(R_1)$ and $DP(R_2)$. More formally, we have two cases:

1. $s \in DP(R_1)^* \cup DP(R_2)^*$. Hence R_1 or R_2 induces an infinite chain, which violates condition (1).
2. $s = \cdots x_1 x_2 \cdots y_2 y_1 \cdots$, where $x_i, y_i \in DP(R_i)$ for $i \in \{1, 2\}$ (i.e. s spans from R_1 to R_2 and from R_2 to R_1). This directly violates condition (2).

□

4 Flat Termination and Stability

References

[1] Thomas Arts and Jürgen Giesl. "Termination of term rewriting using dependency pairs". In: *Theoretical Computer Science* 236.1 (2000), pp. 133–178. DOI: 10.1016/S0304-3975(99)00207-8.