Domain Theory in Constructive and Predicative Univalent Foundations

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Abstract

We develop domain theory in constructive univalent foundations without Voevodsky's resizing axioms. In previous work in this direction, we constructed the Scott model of PCF and proved its computational adequacy, based on directed complete posets (dcpos). Here we further consider algebraic and continuous dcpos, and construct Scott's D_∞ model of the untyped λ-calculus. A common approach to deal with size issues in a predicative foundation is to work with *information systems* or *abstract bases* or *formal topologies* rather than dcpos, and *approximable relations* rather than Scott continuous functions. Here we instead accept that dcpos may be large and work with type universes to account for this. For instance, in the Scott model of PCF, the dcpos have carriers in the second universe $\mathcal{U}_1$ and suprema of directed families with indexing type in the first universe $\mathcal{U}_0$. Seeing a poset as a category in the usual way, we can say that these dcpos are large, but locally small, and have small filtered colimits. In the case of algebraic dcpos, in order to deal with size issues, we proceed mimicking the definition of accessible category. With such a definition, our construction of Scott's D_∞ again gives a large, locally small, algebraic dcpo with small directed suprema.

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1 Introduction

In domain theory [1] one considers posets with suitable completeness properties, possibly generated by certain elements called *compact*, or more generally generated by a certain *way below* relation, giving rise to algebraic and continuous domains. As is well known, domain theory has applications to programming language semantics [40, 38, 31], higher-type computability [25], topology, topological algebra and more [17, 16].

In this work we explore the development of domain theory from the univalent point of view [43, 46]. Additionally, we work constructively (we don't assume excluded middle or choice axioms) and predicatively (we don't assume Voevodky's resizing principles [44, 45, 46], and so, in particular, powersets are large). Most of the work presented here has been formalized in the proof assistant Agda [6, 15, 10] (see Section 7 for details). In our predicative setting, it is extremely important to check universe levels carefully, and the use of a proof assistant such as Agda has been invaluable for this purpose.

In previous work in this direction [8] (extended by Brendan Hart [18]), we constructed the Scott model of PCF and proved its computational adequacy, based on directed complete posets (dcpos). Here we further consider algebraic and continuous dcpos, and construct Scott's D_∞ model of the untyped λ-calculus [38].

A common approach to deal with size issues in a predicative foundation is to work with *information systems* [39], *abstract bases* [1] or *formal topologies* [36, 7] rather than dcpos, and *approximable relations* rather than (Scott) continuous functions. Here we instead accept that dcpos may be large and work with type universes to account for this. For instance, in our development of the Scott model of PCF [40, 31], the dcpos have carriers in the second universe $\mathcal{U}_1$ and suprema of directed families with indexing type in the first universe $\mathcal{U}_0$. Seeing a poset as a category in the usual way, we can say that these dcpos are large, but locally small, and have small filtered colimits. In the case of algebraic dcpos, in order to deal with size issues, we proceed mimicking the definition of accessible category [27]. With such a definition, our construction of Scott's D_∞ again gives a large, locally small, algebraic dcpo with small directed suprema.

Organization

Section 2: Foundations. *Section 3*: (Im)predicativity. *Section 4*: Basic domain theory, including directed complete posets, continuous functions, lifting, Ω-completeness, exponentials, powersets as dcpos. *Section 5*: Limit and colimits of dcpos, Scott's D_∞. *Section 6*: Way below relation, bases, compact element, continuous and algebraic dcpos, ideal completion, retracts, examples. *Section 7*: Conclusion and future work.

Related Work

Domain theory has been studied predicatively in the setting of *formal topology* [36, 7] in [37, 28, 29, 26] and the more recent categorical paper [22]. In this predicative setting, one avoids size issues by working with abstract bases or formal topologies rather than dcpos, and approximable relations rather than Scott continuous functions. Hedberg [19] presented these ideas in Martin-Löf Type Theory and formalized them in the proof assistant ALF. A modern formalization in Agda based on Hedberg's work was recently carried out in Lidell's master thesis [24].

Our development differs from the above line of work in that it studies dcpos directly and uses type universes to account for the fact that dcpos may be large. There are two Coq formalizations of domain theory in this direction [5, 11]. Both formalizations study ω-chain complete preorders, work with setoids, and make use of Coq's impredicative sort `Prop`. In our development we avoid the use of setoids thanks to the adoption of the univalent point of view. Moreover, we work predicatively and we work with directed sets rather than ω-chains, as we intend our theory to be also applicable to topology and algebra [17, 16].

There are also constructive accounts of domain theory aimed at program extraction [4, 30]. Both [4] and [30] study ω-chain complete posets (ω-cpos) and define notions of ω-continuity for them. Interestingly, Bauer and Kavkler [4] note that there can only be non-trivial examples of ω-continuous ω-cpos when Markov's Principle holds [4, Proposition 6.2]. This leads the authors of [30] to weaken the definition of ω-continuous ω-cpo by using the double negation of existential quantification in the definition of the way below relation [30, Remark 3.2]. In light of this, it is interesting to observe that when we study directed complete posets (dcpos) rather than ω-cpos, and continuous dcpos rather than ω-continuous ω-cpos, we can avoid Markov's Principle or a weakened notion of the way below relation to obtain non-trivial continuous dcpos (see for instance Examples 58, 59 and 82).

Another approach is the field of *synthetic domain theory* [35, 34, 20, 32, 33]. Although the work in this area is constructive, it is still impredicative, based on topos logic, but more importantly it has a focus different from that of regular domain theory: the aim is to isolate a few basic axioms and find models in (realizability) toposes where "every object is a domain and every morphism is continuous". These models often validate additional axioms, such as Markov's Principle and countable choice, and moreover falsify excluded middle. Our development has a different goal, namely to develop regular domain theory constructively and predicatively, but in a foundation compatible with excluded middle and choice, while not relying on them or Markov's Principle or countable choice.

2 Foundations

We work in intensional Martin-Löf Type Theory with type formers $+$ (binary sum), Π (dependent products), Σ (dependent sum), Id (identity type), and inductive types, including $\mathbf{0}$ (empty type), $\mathbf{1}$ (type with exactly one element $\star : \mathbf{1}$), $\mathbf{N}$ (natural numbers). Moreover, we have type universes (for which we typically write $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ or $\mathcal{T}$) with the following closure conditions. We assume a universe $\mathcal{U}_0$ and two operations: for every universe $\mathcal{U}$ a successor universe $\mathcal{U}^+$ with $\mathcal{U} : \mathcal{U}^+$, and for every two universes $\mathcal{U}$ and $\mathcal{V}$ another universe $\mathcal{U} \sqcup \mathcal{V}$ such that for any universe $\mathcal{U}$, we have $\mathcal{U}_0 \sqcup \mathcal{U} \equiv \mathcal{U}$ and $\mathcal{U} \sqcup \mathcal{U}^+ \equiv \mathcal{U}^+$. Moreover, $(-) \sqcup (-)$ is idempotent, commutative, associative, and $(-)^+$ distributes over $(-) \sqcup (-)$. We write $\mathcal{U}_1 :\equiv \mathcal{U}_0^+$, $\mathcal{U}_2 :\equiv \mathcal{U}_1^+, \ldots$ and so on. If $X : \mathcal{U}$ and $Y : \mathcal{V}$, then $X + Y : \mathcal{U} \sqcup \mathcal{V}$ and if $X : \mathcal{U}$ and $Y : X \to \mathcal{V}$, then the types $\Sigma_{x:X} Y(x)$ and $\Pi_{x:X} Y(x)$ live in the universe $\mathcal{U} \sqcup \mathcal{V}$; finally, if $X : \mathcal{U}$ and $x, y : X$, then $\mathsf{Id}_X(x, y) : \mathcal{U}$. The type of natural numbers $\mathbf{N}$ is assumed to be in $\mathcal{U}_0$ and we postulate that we have copies $\mathbf{0}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{U}}$ in every universe $\mathcal{U}$. All our examples go through with just two universes $\mathcal{U}_0$ and $\mathcal{U}_1$, but the theory is more easily developed in a general setting.

In general we adopt the same conventions of [43]. In particular, we simply write $x = y$ for the identity type $\mathsf{Id}_X(x, y)$ and use $\equiv$ for the judgemental equality, and for dependent functions $f, g : \Pi_{x:X} A(x)$, we write $f \sim g$ for the pointwise equality $\Pi_{x:X} f(x) = g(x)$.

Within this type theory, we adopt the univalent point of view [43]. A type X is a *proposition* (or *truth value* or *subsingleton*) if it has at most one element, i.e. the type $\mathsf{is\text{-}prop}(X) :\equiv \prod_{x,y:X} x = y$ is inhabited. A major difference between univalent foundations and other foundational systems is that we *prove* that types are propositions or properties. For instance, we can show (using function extensionality) that the axioms of directed complete poset form a proposition. A type X is a *set* if any two elements can be identified in at most one way, i.e. the type $\prod_{x,y:X} \mathsf{is\text{-}prop}(x = y)$ is inhabited.

We will assume two extensionality principles:

(i) *Propositional extensionality*: if P and Q are two propositions, then we postulate that $P = Q$ exactly when both $P \to Q$ and $Q \to P$ are inhabited.

(ii) *Function extensionality*: if $f, g : \prod_{x:X} A(x)$ are two (dependent) functions, then we postulate that $f = g$ exactly when $f \sim g$.

Function extensionality has the important consequence that the propositions form an exponential ideal, i.e. if X is a type and $Y : X \to \mathcal{U}$ is such that every $Y(x)$ is a proposition, then so is $\Pi_{x:X} Y(x)$. In light of this, universal quantification is given by Π-types in our type theory.

In Martin-Löf Type Theory, an element of $\prod_{x:X} \sum_{y:Y} \phi(x, y)$, by definition, gives us a function $f : X \to Y$ such that $\prod_{x:X} \phi(x, f(x))$. In some cases, we wish to express the weaker "for every $x : X$, there exists some $y : Y$ such that $\phi(x, y)$" without necessarily having an assignment of x's to y's. A good example of this is when we define directed families later (see Definition 7). This is achieved through the propositional truncation.

Given a type $X : \mathcal{U}$, we postulate that we have a proposition $\|X\| : \mathcal{U}$ with a function $|-| : X \to \|X\|$ such that for every proposition $P : \mathcal{V}$ in any universe $\mathcal{V}$, every function $f : X \to P$

factors (necessarily uniquely, by function extensionality) through $|-|$. Diagrammatically,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "|-|"'] & & P \\ & \|X\| \arrow[ur, dashed] & \end{tikzcd}$$

Existential quantification $\exists_{x:X} Y(x)$ is given by $\|\Sigma_{x:X} Y(x)\|$. One should note that if we have $\exists_{x:X} Y(x)$ and we are trying to prove some proposition P, then we may assume that we have $x : X$ and $y : Y(x)$ when constructing our inhabitant of P. Similarly, we can define disjunction as $P \vee Q :\equiv \|P + Q\|$.

3 Impredicativity

We now explain what we mean by (im)predicativity in univalent foundations.

▶ **Definition 1** (Has size, `has-size` in [14]). *A type $X : \mathcal{U}$ is said to* have size $\mathcal{V}$ *for some universe $\mathcal{V}$ when we have $Y : \mathcal{V}$ that is equivalent to X, i.e. X $\mathsf{has\text{-}size}$ $\mathcal{V} :\equiv \sum_{Y:\mathcal{V}} Y \simeq X$.*

Here, the symbol $\simeq$ refers to Voevodsky's notion of equivalence [14, 43]. The type X $\mathsf{has\text{-}size}$ $\mathcal{V}$ is a proposition if and only if the univalence axiom holds [14].

▶ **Definition 2** (Type of propositions $\Omega_{\mathcal{U}}$). *The type of propositions in a universe $\mathcal{U}$ is $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \mathsf{is\text{-}prop}(P) : \mathcal{U}^+$.*

Observe that $\Omega_{\mathcal{U}}$ itself lives in the successor universe $\mathcal{U}^+$. We often think of the types in some fixed universe $\mathcal{U}$ as *small* and accordingly we say that $\Omega_{\mathcal{U}}$ is *large*. Similarly, the powerset of a type $X : \mathcal{U}$ is large. Given our predicative setup, we must pay attention to universes when considering powersets.

▶ **Definition 3** ($\mathcal{V}$-powerset $\mathcal{P}_{\mathcal{V}}(X)$, $\mathcal{V}$-subsets). *Let $\mathcal{V}$ be a universe and $X : \mathcal{U}$ type. We define the* $\mathcal{V}$-powerset *$\mathcal{P}_{\mathcal{V}}(X)$ as $X \to \Omega_{\mathcal{V}} : \mathcal{V}^+ \sqcup \mathcal{U}$. Its elements are called* $\mathcal{V}$-subsets *of X.*

▶ **Definition 4** ($\in, \subseteq$). *Let x be an element of a type X and let A be an element of $\mathcal{P}_{\mathcal{V}}(X)$. We write $x \in A$ for the type $\mathsf{pr}_1(A(x))$. Given two $\mathcal{V}$-subsets A and B of X, we write $A \subseteq B$ for $\prod_{x:X}(x \in A \to x \in B)$.*

Functional and propositional extensionality imply that $A = B \iff A \subseteq B$ and $B \subseteq A$.

▶ **Definition 5** (Total type $\mathbb{T}(A)$). *Given a $\mathcal{V}$-subset A of a type X, we write $\mathbb{T}(A)$ for the* total type *$\sum_{x:X} x \in A$.*

One could ask for a *resizing axiom* asserting that $\Omega_{\mathcal{U}}$ has size $\mathcal{U}$, which we call *the propositional impredicativity of $\mathcal{U}$*. A closely related axiom is *propositional resizing*, which asserts that every proposition $P : \mathcal{U}^+$ has size $\mathcal{U}$. Without the addition of such resizing axioms, the type theory is said to be *predicative*. As an example of the use of impredicativity in mathematics, we mention that the powerset has unions of arbitrary subsets if and only if propositional resizing holds [14, `existence-of-unions-gives-PR`].

We mention that the resizing axioms are actually theorems when classical logic is assumed. This is because if $P \vee \neg P$ holds for every proposition in $P : \mathcal{U}$, then the only propositions (up to equivalence) are $\mathbf{0}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{U}}$, which have equivalent copies in $\mathcal{U}_0$, and $\Omega_{\mathcal{U}}$ is equivalent to a type $\mathbf{2}_{\mathcal{U}} : \mathcal{U}$ with exactly two elements. The existence of a computational interpretation of propositional impredicativity axioms for univalent foundations is an open problem, however [42, 41].

4 Basic Domain Theory

Our basic ingredient is the notion of *directed complete poset* (dcpo). In set-theoretic foundations, a dcpo can be defined to be a poset that has least upper bounds of all directed families. A naive translation of this to our foundation would be to proceed as follows. Define a poset in a universe $\mathcal{U}$ to be a type $P : \mathcal{U}$ with a reflexive, transitive and antisymmetric relation $- \sqsubseteq - : P \times P \to \mathcal{U}$. According to the univalent point of view, we also require that the type P is a *set* and the values $p \sqsubseteq q$ of the order relation are *subsingletons*. Then we could say that the poset $(P, \sqsubseteq)$ is *directed complete* if every directed family $I \to X$ with indexing type $I : \mathcal{U}$ has a least upper bound. The problem with this definition is that there are no interesting examples in our constructive and predicative setting. For instance, assume that the poset $\mathbf{2}$ with two elements $0 \sqsubseteq 1$ is directed complete, and consider a proposition $A : \mathcal{U}$ and the directed family $A + \mathbf{1} \to \mathbf{2}$ that maps the left component to 1 and the right component to 0. By case analysis on its hypothetical supremum, we conclude that the negation of A is decidable. This amounts to weak excluded middle, which is known to be equivalent to De Morgan's Law, and doesn't belong to the realm of constructive mathematics. To try to get an example, we may move to the poset Ω_0 of propositions in the universe $\mathcal{U}_0$, ordered by implication. This poset does have all suprema of families $I \to \Omega_0$ indexed by types I in the *first universe* $\mathcal{U}_0$, given by existential quantification. But if we consider a directed family $I \to \Omega_0$ with I in the *same universe* as Ω_0 lives, namely the *second universe* $\mathcal{U}_1$, existential quantification gives a proposition in the *third universe* $\mathcal{U}_2$ and so doesn't give its supremum. In this example, we get a poset such that

- **(i)** the carrier lives in the universe $\mathcal{U}_1$,
- **(ii)** the order has truth values in the universe $\mathcal{U}_0$, and
- **(iii)** suprema of directed families indexed by types in $\mathcal{U}_0$ exist.

Regarding a poset as a category in the usual way, we have a large, but locally small, category with small filtered colimits (suprema). This is typical of all the examples we have considered so far in practice, such as the dcpos in the Scott model of PCF and Scott's D_∞ model of the untyped λ-calculus. We may say that the predicativity restriction increases the universe usage by one. However, for the sake of generality, we formulate our definition of dcpo with the following universe conventions:

- **(i)** the carrier lives in a universe $\mathcal{U}$,
- **(ii)** the order has truth values in a universe $\mathcal{T}$, and
- **(iii)** suprema of directed families indexed by types in a universe $\mathcal{V}$ exist.

So our notion of dcpo has three universe parameters $\mathcal{U}, \mathcal{V}, \mathcal{T}$. We will say that the dcpo is *locally small* when $\mathcal{T}$ is not necessarily the same as $\mathcal{V}$, but the order has truth values of size $\mathcal{V}$. Most of the time we mention $\mathcal{V}$ explicitly and leave $\mathcal{U}$ and $\mathcal{T}$ to be understood from the context.

▶ **Definition 6** (Poset). *A* poset $(P, \sqsubseteq)$ *is a set* $P : \mathcal{U}$ *together with a proposition-valued binary relation* $\sqsubseteq : P \to P \to \mathcal{T}$ *satisfying:*

- **(i)** reflexivity*:* $\prod_{p:P} p \sqsubseteq p$*;*
- **(ii)** antisymmetry*:* $\prod_{p,q:P} p \sqsubseteq q \to q \sqsubseteq p \to p = q$*;*
- **(iii)** transitivity*:* $\prod_{p,q,r:P} p \sqsubseteq q \to q \sqsubseteq r \to p \sqsubseteq r$.

▶ **Definition 7** (Directed family). *Let* $(P, \sqsubseteq)$ *be a poset and* I *any type. A family* $\alpha : I \to P$ *is* directed *if it is inhabited (i.e.* $\|I\|$ *is pointed) and* $\Pi_{i,j:I} \exists_{k:I} \alpha_i \sqsubseteq \alpha_k \times \alpha_j \sqsubseteq \alpha_k$.

▶ **Definition 8** ($\mathcal{V}$-directed complete poset, $\mathcal{V}$-dcpo). *Let* $\mathcal{V}$ *be a type universe. A* $\mathcal{V}$-directed complete poset *(or* $\mathcal{V}$-dcpo*, for short) is a poset* $(P, \sqsubseteq)$ *such that every directed family* $I \to P$ *with* $I : \mathcal{V}$ *has a supremum in* P.

We will sometimes leave the universe $\mathcal{V}$ implicit, and simply speak of "a dcpo". On other occasions, we need to carefully keep track of universe levels. To this end, we make the following definition.

▶ **Definition 9** ($\mathcal{V}$-$\mathsf{DCPO}_{\mathcal{U},\mathcal{T}}$). *Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. We write $\mathcal{V}$-$\mathsf{DCPO}_{\mathcal{U},\mathcal{T}}$ for the type of $\mathcal{V}$-dcpos with carrier in $\mathcal{U}$ and order taking values in $\mathcal{T}$.*

▶ **Definition 10** (Pointed dcpo). *A dcpo D is* pointed *if it has a least element, which we will denote by $\bot_D$, or simply $\bot$.*

▶ **Definition 11** (Locally small). *A $\mathcal{V}$-dcpo D is* locally small *if we have $\sqsubseteq_{\mathsf{small}} : D \to D \to \mathcal{V}$ such that $\prod_{x,y:D}(x \sqsubseteq_{\mathsf{small}} y) \simeq (x \sqsubseteq_D y)$.*

▶ **Example 12** (Powersets as pointed dcpos). Powersets give examples of pointed dcpos. The subset inclusion $\subseteq$ makes $\mathcal{P}_{\mathcal{V}}(X)$ into a poset and given a (not necessarily directed) family $A_{(-)} : I \to \mathcal{P}_{\mathcal{V}}(X)$ with $I : \mathcal{V}$, we may consider its supremum in $\mathcal{P}_{\mathcal{V}}(X)$ as given by $\lambda x.\exists_{i:I}\, x \in A_i$. Note that $(\exists_{i:I}\, x \in A_i) : \mathcal{V}$ for every $x : X$, so this is well-defined. Finally, $\mathcal{P}_{\mathcal{V}}$ has a least element, the empty set: $\lambda x.\mathbf{0}_{\mathcal{V}}$. Thus, $\mathcal{P}_{\mathcal{V}}(X) : \mathcal{V}$-$\mathsf{DCPO}_{\mathcal{V}^+ \sqcup \mathcal{U}, \mathcal{V} \sqcup \mathcal{U}}$. When $\mathcal{V} \equiv \mathcal{U}$ (as in Example 59), we get the simpler, locally small $\mathcal{P}_{\mathcal{U}}(X) : \mathcal{U}$-$\mathsf{DCPO}_{\mathcal{U}^+,\mathcal{U}}$. ⌟

Fix two $\mathcal{V}$-dcpos D and E.

▶ **Definition 13** (Continuous function). *A function $f : D \to E$ is* (Scott) continuous *if it preserves directed suprema, i.e. if $I : \mathcal{V}$ and $\alpha : I \to D$ is directed, then $f(\bigsqcup \alpha)$ is the supremum in E of the family $f \circ \alpha$.*

▶ **Lemma 14.** *If $f : D \to E$ is continuous, then it is monotone, i.e. $x \sqsubseteq_D y$ implies $f(x) \sqsubseteq_E f(y)$.*

Proof. Given $x, y : D$ with $x \sqsubseteq y$, consider the directed family $\mathbf{1} + \mathbf{1} \to D$ defined as $\mathsf{inl}(\star) \mapsto x$ and $\mathsf{inr}(\star) \mapsto y$. Its supremum is y and f must preserve it, so $f(x) \sqsubseteq f(y)$. ◀

▶ **Lemma 15.** *If $f : D \to E$ is continuous and $\alpha : I \to D$ is directed, then so is $f \circ \alpha$.*

Proof. Using Lemma 14. ◀

▶ **Definition 16** (Strict function). *Suppose that D and E are pointed. A continuous function $f : D \to E$ is* strict *if $f(\bot_D) = \bot_E$.*

4.1 Lifting

▶ **Construction 17** ($\mathcal{L}_{\mathcal{V}}(X)$, η_X, cf. [8, 13]). Let $X : \mathcal{U}$ be a set. For any universe $\mathcal{V}$, we construct a pointed $\mathcal{V}$-dcpo $\mathcal{L}_{\mathcal{V}}(X) : \mathcal{V}$-$\mathsf{DCPO}_{\mathcal{V}^+ \sqcup \mathcal{U}, \mathcal{V}^+ \sqcup \mathcal{U}}$, known as the *lifting* of X. Its carrier is given by the type $\sum_{P:\mathcal{V}} \mathsf{is\text{-}prop}(P) \times (P \to X)$ of *partial elements* of X.

Given a partial element $(P, i, \varphi) : \mathcal{L}_{\mathcal{V}}(X)$, we write $(P, i, \varphi){\downarrow}$ for P and say that the partial element is defined if P holds. Moreover, we often leave the second component implicit, writing (P, φ) for (P, i, φ).

The order is given by $l \sqsubseteq_{\mathcal{L}_{\mathcal{V}}(X)} m :\equiv (l{\downarrow} \to l = m)$, and it has a least element given by $(\mathbf{0}, \mathbf{0}\text{-}\mathsf{is\text{-}prop}, \mathsf{unique\text{-}from\text{-}}\mathbf{0})$ where $\mathbf{0}\text{-}\mathsf{is\text{-}prop}$ is a witness that the empty type is a proposition and $\mathsf{unique\text{-}from\text{-}}\mathbf{0}$ is the unique map from the empty type.

Given a directed family $\left(Q_{(-)}, \varphi_{(-)}\right) : I \to \mathcal{L}_{\mathcal{V}}(X)$, its supremum is given by $(\exists_{i:I} Q_i, \psi)$, where ψ is such that

$$\begin{tikzcd}
\sum_{i:I} Q_i \arrow[rr, "(i,q)\mapsto\varphi_i(q)"] \arrow[dr, "|-|"'] & & D \\
& \exists_{i:I} Q_i \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes. (This is possible, because the top map is weakly constant (i.e. any of its values are equal) and D is a set [23, Theorem 5.4].)

Finally, we write $\eta_X : X \to \mathcal{L}_{\mathcal{V}}(X)$ for the embedding $x \mapsto (\mathbf{1}, \mathbf{1}\text{-is-prop}, \lambda u.x)$. ⌟

Note that we require X to be a set, so that $\mathcal{L}_{\mathcal{V}}(X)$ is a poset, rather than an ∞-category. In practice, we often have $\mathcal{V} \equiv \mathcal{U}$ (see for instance Example 58, Section 5.2, or the Scott model of PCF [8]), but we develop the theory for the more general case. We can describe the order on $\mathcal{L}_{\mathcal{V}}(X)$ more explicitly, as follows.

▶ **Lemma 18.** *If we have elements (P, φ) and (Q, ψ) of $\mathcal{L}_{\mathcal{V}}(X)$, then $(P, \varphi) \sqsubseteq (Q, \psi)$ holds if and only if we have $f : P \to Q$ such that $\prod_{p:P} \varphi(p) = \psi(f(p))$.*

Observe that this exhibits $\mathcal{L}_{\mathcal{V}}(X)$ as locally small. We will show that $\mathcal{L}_{\mathcal{V}}(X)$ is the *free* pointed $\mathcal{V}$-dcpo on a set X, but to do that, we first need a lemma.

▶ **Lemma 19.** *Let D be a pointed $\mathcal{V}$-dcpo. Then D has suprema of families indexed by propositions in $\mathcal{V}$, i.e. if $P : \mathcal{V}$ is a proposition, then any $\alpha : P \to D$ has a supremum $\bigvee \alpha$.*

Moreover, if E is a (not necessarily pointed) $\mathcal{V}$-dcpo and $f : D \to E$ is continuous, then $f(\bigvee \alpha)$ is the supremum of the family $f \circ \alpha$.

Proof. Let D be a pointed $\mathcal{V}$-dcpo, $P : \mathcal{V}$ a proposition and $\alpha : P \to D$ a function. Now define $\beta : \mathbf{1}_{\mathcal{V}} + P \to D$ by $\mathsf{inl}(\star) \mapsto \bot_D$ and $\mathsf{inr}(p) \mapsto \alpha(p)$. Then, β is easily seen to be directed and so it has a well-defined supremum in D, which is also the supremum of α. The second claim follows from the fact that β is directed, so continuous maps must preserve its supremum. ◀

▶ **Lemma 20.** *Let $X : \mathcal{U}$ be a set and let (P, φ) be an arbitrary element of $\mathcal{L}_{\mathcal{V}}(X)$. Then $(P, \varphi) = \bigvee_{p:P} \eta_X(\varphi(p))$.*

▶ **Theorem 21.** *The lifting $\mathcal{L}_{\mathcal{V}}(X)$ gives the free $\mathcal{V}$-dcpo on a set X. Put precisely, if $X : \mathcal{U}$ is a set, then for every $\mathcal{V}$-dcpo $D : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U}',\mathcal{T}'}$ and function $f : X \to D$, there is a unique continuous function $\overline{f} : \mathcal{L}_{\mathcal{V}}(X) \to D$ such that*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\eta_X"'] & & D \\
& \mathcal{L}_{\mathcal{V}}(X) \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

commutes.

Proof. We define $\overline{f} : \mathcal{L}_{\mathcal{V}}(X) \to D$ by $(P, \varphi) \mapsto \bigvee_{p:P} f(\varphi(p))$, which is well-defined by Lemma 19 and easily seen to be continuous. For uniqueness, suppose that we have $g : \mathcal{L}_{\mathcal{V}}(X) \to D$ continuous such that $g \circ \eta_X = f$. Let (P, φ) be an arbitrary element of $\mathcal{L}_{\mathcal{V}}(X)$. Using Lemma 20, we have:

$$
\begin{aligned}
g(P,\varphi) &= g\left(\bigvee_{p:P} \eta_X(\varphi(p))\right) \\
&= \bigvee_{p:P} g(\eta_X(\varphi(p))) && \text{(by Lemma 19 and continuity of } g\text{)} \\
&= \bigvee_{p:P} f(\phi(p)) && \text{(by assumption on } g\text{)} \\
&= \overline{f}(P,\varphi) && \text{(by definition),}
\end{aligned}
$$

as desired. ◀

There is yet another way in which the lifting is a free construction, cf. [8, Section 4.3]. What is noteworthy about this is that freely adding subsingleton suprema automatically gives all directed suprema.

▶ **Definition 22** ($\Omega_{\mathcal{V}}$-complete)**.** *A poset* $(P, \sqsubseteq)$ *is* $\Omega_{\mathcal{V}}$-complete *if it has suprema for all families indexed by a proposition in* $\mathcal{V}$*.*

▶ **Theorem 23.** *The lifting* $\mathcal{L}_{\mathcal{V}}(X)$ *gives the free* $\Omega_{\mathcal{V}}$*-complete poset on a set* X*. Put precisely, if* $X : \mathcal{U}$ *is a set, then for every* $\Omega_{\mathcal{V}}$*-complete poset* $(P, \sqsubseteq)$ *with* $P : \mathcal{U}'$ *and* $\sqsubseteq$ *taking values in* $\mathcal{T}'$ *and function* $f : X \to P$*, there exists a unique monotone* $\overline{f} : \mathcal{L}_{\mathcal{V}}(X) \to P$ *preserving all suprema indexed by propositions in* $\mathcal{V}$*, such that*

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\eta_X"'] & & P \\
& \mathcal{L}_{\mathcal{V}}(X) \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}
$$

commutes.

Proof. Similar to the proof of Theorem 21; also see [8, Proof of Theorem 4.16]. ◀

Finally, a variation of Construction 17 freely adds a least element to a dcpo.

▶ **Construction 24** ($\mathcal{L}'_{\mathcal{V}}(D)$)**.** Let $D : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U},\mathcal{T}}$ be a $\mathcal{V}$-dcpo. We construct a *pointed* $\mathcal{V}$-dcpo $\mathcal{L}'_{\mathcal{V}}(D) : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{V}^+ \sqcup \mathcal{U}, \mathcal{V} \sqcup \mathcal{T}}$. Its carrier is given by the type $\sum_{P:\mathcal{V}} \mathsf{is\text{-}prop}(P) \times (P \to D)$.

The order is given by $(P,\varphi) \sqsubseteq_{\mathcal{L}'_{\mathcal{V}}(D)} (Q,\psi) :\equiv \sum_{f:P\to Q} \prod_{p:P} \varphi(p) = \psi(f(p))$ and has a least element $(\mathbf{0}, \mathbf{0}\text{-}\mathsf{is\text{-}prop}, \mathsf{unique\text{-}from\text{-}}\mathbf{0})$.

Now let $\alpha :\equiv \left(Q_{(-)}, \varphi_{(-)}\right) : I \to \mathcal{L}'_{\mathcal{V}}(D)$ be a directed family. Consider $\Phi : \left(\sum_{i:I} Q_i\right) \to D$ given by $(i,q) \mapsto \varphi_i(q)$. The supremum of α is given by $(\exists_{i:I} Q_i, \psi)$, where ψ takes a witness that $\sum_{i:I} Q_i$ is inhabited to the directed (for which we needed $\exists_{i:I} Q_i$) supremum $\bigsqcup \Phi$ in D.

Finally, we write $\eta'_D : D \to \mathcal{L}'_{\mathcal{V}}(D)$ for the continuous map $x \mapsto (\mathbf{1}, \mathbf{1}\text{-}\mathsf{is\text{-}prop}, \lambda u.x)$. ⌟

▶ **Theorem 25.** *The construction* $\mathcal{L}'_{\mathcal{V}}(D)$ *gives the free pointed* $\mathcal{V}$*-dcpo on a* $\mathcal{V}$*-dcpo* D*. Put precisely, if* $D : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U},\mathcal{T}}$ *is a* $\mathcal{V}$*-dcpo, then for every pointed* $\mathcal{V}$*-dcpo* $E : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U}',\mathcal{T}'}$ *and continuous function* $f : D \to E$*, there is a unique strict continuous function* $\overline{f} : \mathcal{L}'_{\mathcal{V}}(D) \to E$ *such that*

$$
\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "\eta'_D"'] & & E \\
& \mathcal{L}'_{\mathcal{V}}(D) \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}
$$

commutes.

Proof. Similar to the proof of Theorem 21. ◀

4.2 Exponentials

▶ **Construction 26** (E^D)**.** Let D and E be two $\mathcal{V}$-dcpos. We construct another $\mathcal{V}$-dcpo E^D as follows. Its carrier is given by the type of continuous functions from D to E.

These functions are ordered pointwise, i.e. if $f, g : D \to E$, then

$$f \sqsubseteq_{E^D} g :\equiv \prod_{x:D} f(x) \sqsubseteq_E g(x).$$

Accordingly, directed suprema are also given pointwise. Explicitly, let $\alpha : I \to E^D$ be a directed family. For every $x : D$, we have the family $\alpha_x : I \to E$ given by $i \mapsto \alpha_i(x)$. This is a directed family in E and so we have a well-defined supremum $\bigsqcup \alpha_x : E$ for every $x : D$. The supremum of α is then given by $x \mapsto \bigsqcup \alpha_x$, where one should check that this assignment is indeed continuous.

Finally, if E is pointed, then so is E^D, because, in that case, the function $x \mapsto \bot_E$ is the least continuous function from D to E. ⌟

▶ Remark 27. In general, the universe levels of E^D can be quite large and complicated. For if $D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$ and $D : \mathcal{V}\text{-DCPO}_{\mathcal{U}',\mathcal{T}'}$, then $E^D : \mathcal{V}\text{-DCPO}_{\mathcal{V}^+ \sqcup \mathcal{U} \sqcup \mathcal{T} \sqcup \mathcal{U}' \sqcup \mathcal{T}', \mathcal{U} \sqcup \mathcal{T}'}$. Even if $\mathcal{V} = \mathcal{U} \equiv \mathcal{T} \equiv \mathcal{U}' \equiv \mathcal{T}'$, the carrier of E^D still lives in the "large" universe $\mathcal{V}^+$. (Actually, this scenario cannot happen non-trivially in a predicative setting, since non-trivial dcpos cannot be "small" [9].) Even so, as observed in [8], if we take $\mathcal{U} \equiv \mathcal{T} \equiv \mathcal{U}' \equiv \mathcal{T}' \equiv \mathcal{U}_1$ and $\mathcal{V} \equiv \mathcal{U}_0$, then D, E, E^D are all elements of $\mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$.

5 Scott's D_∞

We now construct, predicatively, Scott's famous pointed dcpo D_∞ which is isomorphic to its own function space $D_\infty^{D_\infty}$ (Theorem 39). We follow Scott's original paper [38] rather closely, but with two differences. Firstly, we explicitly keep track of the universe levels to make sure that our constructions go through predicatively. Secondly, [38] describes sequential (co)limits, while we study the more general directed (co)limits (Section 5.1) and then specialize to sequential (co)limits later (Section 5.2).

5.1 Limits and Colimits

▶ **Definition 28** (Deflation)**.** *Let D be a dcpo. An endofunction $f : D \to D$ is a* deflation *if $f(x) \sqsubseteq x$ for all $x : D$.*

▶ **Definition 29** (Embedding-projection pair)**.** *Let D and E be two dcpos. An* embedding-projection pair *from D to E consists of two continuous functions $\varepsilon : D \to E$ (the* embedding*) and $\pi : E \to D$ (the* projection*) such that:*

(i) *ε is a section of π;*
(ii) *$\varepsilon \circ \pi$ is a deflation.*

For the remainder of this section, fix the following setup. Let $\mathcal{V}$, $\mathcal{U}$, $\mathcal{T}$ and $\mathcal{W}$ be type universes. Let $(I, \sqsubseteq)$ be a directed preorder with $I : \mathcal{V}$ and $\sqsubseteq$ taken values in $\mathcal{W}$. Suppose that we have:

(i) for every $i : I$, a $\mathcal{V}$-dcpo $D_i : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$;
(ii) for every $i, j : I$ with $i \sqsubseteq j$, an embedding-projection pair $(\varepsilon_{i,j}, \pi_{i,j})$ from D_i to D_j;

such that

(i) for every $i : I$, we have $\varepsilon_{i,i} = \pi_{i,i} = \mathsf{id}$;

(ii) for every $i, j, k : I$ with $i \sqsubseteq j \sqsubseteq k$, we have $\varepsilon_{i,k} \sim \varepsilon_{j,k} \circ \varepsilon_{i,j}$ and $\pi_{i,k} \sim \pi_{i,j} \circ \pi_{j,k}$.

▶ **Construction 30** (D_∞)**.** Given the above inputs, we construct another $\mathcal{V}$-dcpo $D_\infty : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U}\sqcup\mathcal{V}\sqcup\mathcal{W},\mathcal{U}\sqcup\mathcal{T}}$ as follows. Its carrier is given by the type:

$$\sum_{\sigma:\prod_{i:I} D_i} \prod_{j:I, i\sqsubseteq j} \pi_{i,j}(\sigma_j) = \sigma_i.$$

These functions are ordered pointwise, i.e. if $\sigma, \tau : I \to D_i$, then

$$\sigma \sqsubseteq_{D_\infty} \tau :\equiv \prod_{i:I} \sigma_i \sqsubseteq_{D_i} \tau_i.$$

Accordingly, directed suprema are also given pointwise. Explicitly, let $\alpha : A \to D_\infty$ be a directed family. For every $i : I$, we have the family $A \to D_i$ given by $a \mapsto (\alpha(a))_i$, and denoted by α_i. One can show that α_i is directed and so we have a well-defined supremum $\bigsqcup \alpha_i : D_i$ for every $i : I$. The supremum of α is then given by the function $i : I \mapsto \bigsqcup \alpha_i$, where one should check that $\pi_{i,j}(\bigsqcup \alpha_j) = \bigsqcup \alpha_i$ holds whenever $i \sqsubseteq j$. ⌟

▶ Remark 31. We allow for general universe levels here, which is why D_∞ lives in the relatively complicated universe $\mathcal{U} \sqcup \mathcal{V} \sqcup \mathcal{W}$. In concrete examples, such as in Section 5.2, the situation simplifies to $\mathcal{V} = \mathcal{W} = \mathcal{U}_0$ and $\mathcal{U} = \mathcal{T} = \mathcal{U}_1$.

▶ **Construction 32** ($\pi_{i,\infty}$)**.** For every $i : I$, we have a continuous function $\pi_{i,\infty} : D_\infty \to D_i$, given by $\sigma \mapsto \sigma_i$. ⌟

▶ **Construction 33** ($\varepsilon_{i,\infty}$)**.** For every $i, j : I$, consider the function

$$\kappa : D_i \to \left(\sum_{k:I} i \sqsubseteq k \times j \sqsubseteq k\right) \to D_j$$
$$\kappa_x(k, l_i, l_j) = \pi_{i,j}(\varepsilon_{i,k}(x)).$$

Using directedness of $(I, \sqsubseteq)$, we can show that for every $x : D_i$ the map κ_x is weakly constant (i.e. all its values are equal). Therefore, we can apply [23, Theorem 5.4] and factor κ_x through $\exists_{k:I}(i \sqsubseteq k \times j \sqsubseteq k)$. But $(I, \sqsubseteq)$ is directed, so $\exists_{k:I}(i \sqsubseteq k \times j \sqsubseteq k)$ is a singleton. Thus, we obtain a function $\rho_{i,j} : D_i \to D_j$ such that: if we are given $k : I$ with $(l_i, l_j) : i \sqsubseteq k \times j \sqsubseteq k$, then $\rho_{i,j}(x) = \kappa_x(k, l_i, l_j)$.

Finally, this allows us to construct for every $i : I$, a continuous function $\varepsilon_{i,\infty} : D_i \to D_\infty$ by mapping $x : D_i$ to the function $\lambda(j : I).\rho_{i,j}(x)$. ⌟

▶ **Theorem 34.** *For every $i : I$, the pair $(\varepsilon_{i,\infty}, \pi_{i,\infty})$ is an embedding-projection pair.*

▶ **Lemma 35.** *Let $i, j : I$ such that $i \sqsubseteq j$. Then $\pi_{i,j} \circ \pi_{j,\infty} \sim \pi_i$, and $\varepsilon_{j,\infty} \circ \varepsilon_{i,j} \sim \varepsilon_{i,\infty}$.*

▶ **Theorem 36.** *The dcpo D_∞ with the maps $(\pi_{i,\infty})_{i:I}$ is the limit of $\Big((D_i)_{i:I}, (\pi_{i,j})_{i\sqsubseteq j}\Big)$. That is, given*

(i) *a $\mathcal{V}$-dcpo $E : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U}',\mathcal{T}'}$,*

(ii) *continuous functions $f_i : E \to D_i$ for every $i : I$,*

such that $\pi_{i,j} \circ f_j \sim f_i$ whenever $i \sqsubseteq j$, we have a continuous function $f_\infty : E \to D_\infty$ such that $\pi_{i,\infty} \circ f_\infty \sim f_i$ for every $i : I$. Moreover, f_∞ is the unique such continuous function.

The function f_∞ is given by mapping $y : E$ to the function $\lambda(i : I).f_i(y)$.

▶ **Theorem 37.** *The dcpo D_∞ with the maps $(\varepsilon_{i,\infty})_{i:I}$ is the colimit of $\left((D_i)_{i:I}, (\varepsilon_{i,j})_{i \sqsubseteq j}\right)$. That is, given*

(i) *a $\mathcal{V}$-dcpo $E : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U}',\mathcal{T}'}$,*

(ii) *continuous functions $g_i : D_i \to E$ for every $i : I$,*

such that $g_j \circ \varepsilon_{i,j} \sim g_i$ whenever $i \sqsubseteq j$, we have a continuous function $g_\infty : D_\infty \to E$ such that $g_\infty \circ \varepsilon_{i,\infty} \sim g_i$ for every $i : I$. Moreover, g_∞ is the unique such continuous function.

The function g_∞ is given by $\sigma \mapsto \bigsqcup_{i:I} g_i(\sigma_i)$, where one should check that the family $i \mapsto g_i(\sigma_i)$ is indeed directed.

Proof. For uniqueness, it is useful to know that an element $\sigma : D_\infty$ can be expressed as the directed supremum $\bigsqcup_{i:I} \varepsilon_{i,\infty}(\sigma_i)$. The rest can be checked directly. ◀

It should be noted that in both universal properties E can have its carrier in any universe $\mathcal{U}'$ and its order taking values in any universe $\mathcal{T}'$, even though we required all D_i to have their carriers and orders in two fixed universes $\mathcal{U}$ and $\mathcal{T}$, respectively.

5.2 Scott's Example Using Self-exponentiation

We now show that we can construct Scott's D_∞ [38] predicatively. Formulated precisely, we construct a pointed $D_\infty : \mathcal{U}_0\text{-}\mathsf{DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ such that D_∞ is isomorphic to its self-exponential $D_\infty^{D_\infty}$.

We employ the machinery from Section 5.1. Following [38, pp. 126–127], we inductively define pointed dcpos $D_n : \mathcal{U}_0\text{-}\mathsf{DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ for every natural number n:

(i) $D_0 :\equiv \mathcal{L}_{\mathcal{U}_0}(\mathbf{1}_{\mathcal{U}_0})$;

(ii) $D_{n+1} :\equiv D_n^{D_n}$.

Next, we inductively define embedding-projection pairs (ε_n, π_n) from D_n to D_{n+1}:

(i) $\varepsilon_0 : D_0 \to D_1$ is given by mapping $x : D_0$ to the continuous function that is constantly x;
$\pi_0 : D_1 \to D_0$ is given by evaluating a continuous function $f : D_0 \to D_0$ at $\perp$;

(ii) $\varepsilon_{n+1} : D_{n+1} \to D_{n+2}$ takes a continuous function $f : D_n \to D_n$ to the continuous composite $D_{n+1} \xrightarrow{\pi_n} D_n \xrightarrow{f} D_n \xrightarrow{\varepsilon_n} D_{n+1}$;
$\pi_{n+1} : D_{n+2} \to D_{n+1}$ takes a continuous function $f : D_{n+1} \to D_{n+1}$ to the continuous composite $D_n \xrightarrow{\varepsilon_n} D_{n+1} \xrightarrow{f} D_{n+1} \xrightarrow{\pi_n} D_n$.

In order to apply the machinery from Section 5.1, we will need embedding-projection pairs $(\varepsilon_{n,m}, \pi_{n,m})$ from D_n to D_m whenever $n \leq m$. Let n and m be natural numbers with $n \leq m$ and let k be the natural number $m - n$. We define the pairs by induction on k:

(i) if $k = 0$, then we set $\varepsilon_{n,n} = \pi_{n,n} = \mathsf{id}$;

(ii) if $k = l + 1$, then $\varepsilon_{n,m} = \varepsilon_{n+l} \circ \varepsilon_{n,n+l}$ and $\pi_{n,m} = \pi_{n,n+l} \circ \pi_{n+l}$.

So, Constructions 30, 32 and 33 give us $D_\infty : \mathcal{U}_0\text{-}\mathsf{DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ with embedding-projection pairs $(\varepsilon_{n,\infty}, \pi_{n,\infty})$ from D_n to D_∞ for every natural number n.

▶ **Lemma 38.** *Let n be a natural number. The function $\pi_n : D_{n+1} \to D_n$ is strict. Hence, so is $\pi_{n,m}$ whenever $n \leq m$.*

Proof. The first statement is proved by induction on n. The second by induction on k with $k :\equiv m - n$. ◀

▶ **Theorem 39.** *The dcpo D_∞ is pointed and isomorphic to $D_\infty^{D_\infty}$.*

Proof. Since every D_n is pointed, we can consider the function $\sigma : \prod_{n:\mathbf{N}} D_n$ given by $\sigma(n) :\equiv \bot_{D_n}$. Then σ is an element of D_∞ by Lemma 38 and it is the least, so D_∞ is indeed pointed.

We start by constructing a continuous function $\varepsilon'_\infty : D_\infty \to D_\infty^{D_\infty}$. By Theorem 37, it suffices to define continuous functions $\varepsilon'_n : D_n \to D_\infty^{D_\infty}$ for every natural number n such that $\varepsilon'_m \circ \varepsilon_{n,m} \sim \varepsilon'_n$ whenever $n \leq m$. We do so as follows:

(i) $\varepsilon'_{n+1} : D_{n+1} \equiv D_n^{D_n} \to D_\infty^{D_\infty}$ is given by mapping a continuous function $f : D_n \to D_n$ to the continuous composite $D_\infty \xrightarrow{\pi_{n,\infty}} D_n \xrightarrow{f} D_n \xrightarrow{\varepsilon_{n,\infty}} D_\infty$;

(ii) $\varepsilon'_0 : D_0 \to D_\infty^{D_\infty}$ is defined as the continuous composite $D_0 \xrightarrow{\varepsilon_0} D_1 \xrightarrow{\varepsilon'_1} D_\infty$.

Next, we construct a continuous function $\pi'_\infty : D_\infty^{D_\infty} \to D_\infty$. By Theorem 36, it suffices to define continuous functions $\pi'_n : D_n \to D_\infty^{D_\infty}$ for every natural number n such that $\pi_{n,m} \circ \pi'_m \sim \pi'_n$ whenever $n \leq m$. We do so as follows:

(i) $\pi'_{n+1} : D_\infty^{D_\infty} \to D_{n+1} :\equiv D_n^{D_n}$ is given by mapping a continuous function $f : D_\infty \to D_\infty$ to the continuous composite $D_n \xrightarrow{\varepsilon_{n,\infty}} D_\infty \xrightarrow{f} D_\infty \xrightarrow{\pi_{n,\infty}} D_n$;

(ii) $\pi'_0 : D_\infty^{D_\infty} \to D_0$ is defined as the continuous composite $D_\infty \xrightarrow{\pi'_1} D_1 \xrightarrow{\pi_0} D_0$.

It remains to prove that ε'_∞ and π'_∞ are inverses. To this end, it is convenient to have an alternative description of the maps ε'_∞ and π'_∞.

$$\text{For every } \sigma : D_\infty, \text{ we have } \varepsilon'_\infty(\sigma) = \bigsqcup_{n:\mathbf{N}} \varepsilon'_{n+1}(\sigma_{n+1}). \tag{1}$$

$$\text{For every continuous } f : D_\infty \to D_\infty, \text{ we have } \pi'_\infty(f) = \bigsqcup_{n:\mathbf{N}} \epsilon_{n+1,\infty}\big(\pi'_{n+1}(f)\big). \tag{2}$$

Using these equations we can prove that ε'_∞ and π'_∞ are inverses exactly as in [38, Proof of Theorem 4.4]. ◀

▶ Remark 40. Of course, Theorem 39 is only interesting in case $D_\infty \not\simeq \mathbf{1}$. Fortunately, D_∞ has (infinitely) many elements besides $\bot_{D_\infty}$. For instance, we can consider $x_0 :\equiv \eta(\star) : D_0$ and $\sigma_0 : \prod_{n:\mathbf{N}} D_n$ given by $\sigma_0(n) :\equiv \varepsilon_{0,n}(x_0)$. Then, σ_0 is an element of D_∞ not equal to $\bot_{D_\infty}$, because $x_0 \neq \bot_{D_0}$.

6 Continuous and Algebraic Dcpos

We next consider dcpos generated by certain elements called compact, or more generally generated by a certain way below relation, giving rise to algebraic and continuous domains.

6.1 The Way Below Relation

▶ **Definition 41** (Way below relation, $x \ll y$). *Let D be a $\mathcal{V}$-dcpo and $x, y : D$. We say that x is* way below *y, denoted by $x \ll y$, if for every $I : \mathcal{V}$ and directed family $\alpha : I \to D$, whenever we have $y \sqsubseteq \bigsqcup \alpha$, then there exists some element $i : I$ such that $x \sqsubseteq \alpha_i$ already. Symbolically,*

$$x \ll y :\equiv \prod_{I:\mathcal{V}} \prod_{\alpha:I\to D} \Big(\mathsf{is\text{-}directed}(\alpha) \to y \sqsubseteq \bigsqcup \alpha \to \exists_{i:I}\, x \sqsubseteq \alpha_i\Big).$$

▶ **Lemma 42.** *The way below relation enjoys the following properties.*

(i) *It is proposition-valued.*

(ii) *If $x \ll y$, then $x \sqsubseteq y$.*
(iii) *If $x \sqsubseteq y \ll v \sqsubseteq w$, then $x \ll w$.*
(iv) *It is antisymmetric.*
(v) *It is transitive.*

▶ **Lemma 43.** *Let D be a dcpo. Then $x \sqsubseteq y$ implies $\prod_{z:D}(z \ll x \to z \ll y)$.*

Proof. By Lemma 42(iii). ◀

▶ **Definition 44** (Compact)**.** *Let D be a dcpo. An element $x : D$ is called* compact *if $x \ll x$.*

▶ **Example 45.** The least element of a pointed dcpo is always compact.

▶ **Example 46** (Compact elements in $\mathcal{L}_{\mathcal{V}}(X)$)**.** Let $X : \mathcal{U}$ be a set. An element $(P, \varphi) : \mathcal{L}_{\mathcal{V}}(X)$ is compact if and only if P is decidable.

▶ **Definition 47** (Kuratowski finite)**.** *A type X is* Kuratowski finite *if there exists some natural number $n : \mathbf{N}$ and a surjection $e : \mathsf{Fin}(n) \twoheadrightarrow X$.*

That is, X is *Kuratowski finite* if its elements can be finitely enumerated, possibly with repetitions.

▶ **Example 48** (Compact elements in $\mathcal{P}_{\mathcal{U}}(X)$)**.** Let $X : \mathcal{U}$ be a set. An element $A : \mathcal{P}_{\mathcal{U}}(X)$ is compact if and only if its total type $\mathbb{T}A$ is Kuratowski finite.

Proof. Write $\iota : \mathsf{List}(X) \to \mathcal{P}_{\mathcal{U}}(X)$ for the map that regards a list on X as a subset of X. The inductively generated type $\mathsf{List}(X)$ of lists on X lives in the same universe $\mathcal{U}$ as X.

Suppose that A is compact. We must show that $\mathbb{T}(A)$ is Kuratowski finite. Consider the map $\alpha : \mathsf{List}(\mathbb{T}(A)) \to \mathcal{P}_{\mathcal{U}(X)}$ which takes a list $[(x_0, p_0), \ldots, (x_{n-1}, p_{n-1})]$ to $\iota([x_0, \ldots, x_{n-1}])$. Since the empty list is an element of $\mathsf{List}(\mathbb{T}(A))$ and because we can concatenate lists, α is directed. Moreover, $\mathsf{List}(\mathbb{T}(A)) : \mathcal{U}$ and $A = \bigsqcup \alpha$ holds. Hence, by compactness, there exists some $l :\equiv [(x_0, p_0), \ldots, (x_{n-1}, p_{n-1})] : \mathsf{List}(\mathbb{T}(A))$ such that $A \subseteq \alpha(l)$ already. Hence, the map $m : \mathsf{Fin}(n) \mapsto (x_m, p_m) : \mathbb{T}(A)$ is a surjection, so $\mathbb{T}(A)$ is Kuratowski finite.

Conversely, suppose that A is a subset such that $\mathbb{T}(A)$ is Kuratowski finite. We must prove that it is compact. Let $B_{(-)} : I \to \mathcal{P}_{\mathcal{U}}(X)$ be directed such that $A \subseteq \bigsqcup_{i:I} B_i$. Since $\exists_{i:I} A \subseteq B_i$ is a proposition, we can use Kuratowski finiteness of $\mathbb{T}(A)$ to obtain a natural number n and a surjection $e : \mathsf{Fin}(n) \twoheadrightarrow \mathbb{T}(A)$. For each $m : \mathsf{Fin}(n)$, find i_m such that $e_m \in B_{i_m}$. By directedness of I, there exists $k : I$ such that $e_m \in B_k$ for every $m : \mathsf{Fin}(n)$. Hence, $\exists_{k:I}\, A \subseteq B_k$, as desired. ◁

6.2 Continuous Dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it [3]. Predicatively, we must be careful, because if x is an element of a dcpo D, then $\sum_{y:D} y \ll x$ is typically large, so its directed join need not exist for size reasons. Our solution is to use a predicative version of bases [1] that accounts for size issues. For the special case of algebraic dcpos, our situation is the poset analogue of accessible categories [2]. Indeed, in category theory requiring smallness is common, even in impredicative settings, see for instance [21], where continuous dcpos are generalized to continuous categories.

▶ **Definition 49** (Basis, approximating family)**.** *A* basis *for $\mathcal{V}$-dcpo D is a function $\beta : B \to D$ with $B : \mathcal{V}$ such that for every $x : D$ there exists some $\alpha : I \to B$ with $I : \mathcal{V}$ such that*
(i) *$\beta \circ \alpha$ is directed and its supremum is x;*

(ii) *$\beta(\alpha_i) \ll x$ for every $i : I$.*

We summarise these requirements by saying that α is an approximating family *for x.*

Moreover, we require that $\ll$ is small when restricted to the basis. That is, we have $\ll^B : B \to B \to \mathcal{V}$ such that $(\beta(b) \ll \beta(b')) \simeq \left(b \ll^B b'\right)$ for every $b, b' : B$.

▶ **Definition 50** (Continuous dcpo)**.** *A dcpo D is* continuous *if there exists some basis for it.*

We postpone giving examples of continuous dcpos until we have developed the theory further, but the interested reader may look ahead to Examples 58, 59 and 82.

A useful property of bases is that it allows us to express the order fully in terms of the way below-relation, giving a converse to Lemma 43.

▶ **Lemma 51.** *Let D be a dcpo with basis $\beta : B \to D$. Then $x \sqsubseteq y$ holds if and only if $\prod_{b:B}(\beta(b) \ll x \to \beta(b) \ll y)$.*

Proof. The left-to-right implication holds by Lemma 43. For the converse, suppose that we have $x, y : D$ such that for every $\prod_{b:B}(\beta(b) \ll x \to \beta(b) \ll y)$. Since $x \sqsubseteq y$ is a proposition, we can obtain $\alpha : I \to B$ such that $\beta \circ \alpha$ is directed and $\bigsqcup \beta \circ \alpha = x$ and $\beta(\alpha_i) \ll x$ for every $i : I$. It then suffices to show that $\bigsqcup \beta \circ \alpha \sqsubseteq y$. Since $\bigsqcup$ gives the *least* upper bound, it is enough to prove that $\beta(\alpha_i) \sqsubseteq y$ for every $i : I$, but this holds by our hypothesis, our assumption that $\beta(\alpha_i) \ll x$ for every $i : I$, and Lemma 42(ii). ◀

▶ **Lemma 52.** *Let D be a $\mathcal{V}$-dcpo with a basis $\beta : B \to D$. Then $\sqsubseteq$ is small when restricted to the basis, i.e. $\beta(b_1) \sqsubseteq \beta(b_2)$ has size $\mathcal{V}$ for every two elements $b_1, b_2 : B$. Hence, we have $\sqsubseteq^B : B \to B \to \mathcal{V}$ such that $\prod_{b_1,b_2:B}\left(b_1 \sqsubseteq^B b_2\right) \simeq (\beta(b_1) \sqsubseteq \beta(b_2))$.*

Proof. Let $b_1, b_2 : B$ and note that we have the following equivalences:

$$\begin{aligned}
(\beta(b_1) \sqsubseteq \beta(b_2)) &\simeq \prod_{b:B}(\beta(b) \ll \beta(b_1) \to \beta(b) \ll \beta(b_2)) && \text{(by Lemma 51)} \\
&\simeq \prod_{b:B}\left(b \ll^B b_1 \to b \ll^B b_2\right) && \text{(by definition of a basis),}
\end{aligned}$$

but the latter is a type in $\mathcal{V}$. ◀

The most significant properties of a basis are the interpolation properties. We consider nullary, unary and binary versions here. The binary interpolation property actually follows fairy easily from the unary one, but we still record it, because we wish to show that bases are examples of the abstract bases that we define later (cf. Example 62).

▶ **Lemma 53** (Nullary interpolation)**.** *Let D be a dcpo with a basis $\beta : B \to D$. For every $x : D$, there exists some $b : B$ such that $\beta(b) \ll x$.*

Proof. Immediate from the definitions of a basis and a directed family. ◀

▶ **Lemma 54** (Unary interpolation)**.** *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$ and let $x, y : D$. If $x \ll y$, then there exists some $b : B$ such that $x \ll \beta(b) \ll y$.*

Proof. Our proof is a predicative version of [12]. Let $x, y : D$ with $x \ll y$. Since β is a basis, there exists an approximating family $\alpha : I \to D$ for y. Consider the family

$$\left(K :\equiv \sum_{b:B}\sum_{i:I} b \ll^B \alpha_i : \mathcal{V}\right) \xrightarrow{\mathsf{pr}_1} B \xrightarrow{\beta} D. \tag{†}$$

▷ Claim. The family (†) is directed.

Proof. By directedness of α and nullary interpolation, the type K is inhabited.

Now suppose that we have $b_1, b_2 : B$ and $i_1, i_2 : I$ with $b_1 \ll^B \alpha_{i_1}$ and $b_2 \ll^B \alpha_{i_2}$. By directedness of α, there exists $k : I$ with $\alpha_{i_1}, \alpha_{i_2} \sqsubseteq^B \alpha_k$. Since β is a basis for D, there exists an approximating family $\gamma : J \to B$ for $\beta(\alpha_k)$. From $b_1 \ll^B \alpha_{i_1}$ we obtain $b_1 \ll^B \alpha_k$ and similarly, $b_2 \ll^B \alpha_k$. Hence, there exist $j_1, j_2 : J$ such that $b_1 \sqsubseteq^B \gamma_{j_1}$ and $b_2 \sqsubseteq^B \gamma_{j_2}$. By directedness of J, there exists $m : J$ with $\gamma_{j_1}, \gamma_{j_2} \sqsubseteq^B \gamma_m$. Thus, putting this all together, we see that: $b_1, b_2 \sqsubseteq^B \gamma_{j_m} \ll^B \alpha_k$. Hence, (†) is directed. ◁

Thus, (†) has a supremum s in D.

▷ Claim. We have $y \sqsubseteq s$.

Proof. Since $y = \bigsqcup \beta \circ \alpha$, it suffices to prove that $\beta(\alpha_i) \sqsubseteq s$ for every $i : I$. Let $i : I$ be arbitrary and let $\gamma_j : J \to B$ be some approximating family for $\beta(\alpha_i)$. Then it is enough to establish $\beta(\gamma_j) \sqsubseteq s$ for every $j : J$. But we know that $\gamma_j \ll^B \alpha_i$, so $\beta_{\gamma_j} \sqsubseteq s$ by definition of (†) and the fact that s is the supremum of (†). ◁

Finally, from $y \sqsubseteq s$ and $x \ll y$, it follows that there must exist $b : B$ and $i : I$ such that: $x \sqsubseteq \beta(b) \ll \beta(\alpha_i) \ll y$, which finishes the proof. ◀

▶ **Lemma 55** (Binary interpolation). *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$ and let $x, y, z : D$. If $x, y \ll z$, then there exists some $b : B$ such that $x, y \ll \beta(b) \ll z$.*

Proof. Let $x, y, z : D$ such that $x, y \ll z$. By unary interpolation, there are $b_x, b_y : B$ such that $x \ll \beta(b_x) \ll z$ and $y \ll \beta(b_y) \ll z$. Since β is a basis, there exists a family $\alpha : I \to B$ such that $\beta(\alpha_i) \ll z$ for every $i : I$, and $\beta \circ \alpha$ is directed and has supremum z. Since $\beta(b_x) \ll z$, there must exists $i_x : I$ with $\beta(b_x) \sqsubseteq \beta(\alpha_{i_x})$. Similarly, there exists $i_y : I$ such that $\beta(b_y) \sqsubseteq \beta(\alpha_{i_y})$. By directedness of $\beta \circ \alpha$, there exists $k : I$ with $\beta(\alpha_{i_x}), \beta(\alpha_{i_y}) \sqsubseteq \beta(\alpha_k)$. Hence,

$$x \ll \beta(b_x) \sqsubseteq \beta(\alpha_{i_x}) \sqsubseteq \beta(\alpha_k) \ll z \qquad \text{and} \qquad y \ll \beta(b_y) \sqsubseteq \beta(\alpha_{i_y}) \sqsubseteq \beta(\alpha_k) \ll z,$$

so that $x, y \ll \beta(\alpha_k) \ll z$, as wished. ◀

6.3 Algebraic Dcpos

We now turn to a particular class of continuous dcpos, called algebraic dcpos.

▶ **Definition 56** (Algebraic dcpo). *A dcpo D is* algebraic *if there exists some basis $\beta : B \to D$ for it such that $\beta(b)$ is compact for every $b : B$.*

▶ **Lemma 57.** *Let D be a $\mathcal{V}$-dcpo. Then D is algebraic if and only if there exists $\beta : B \to D$ with $B : \mathcal{V}$ such that*

(i) *every element $\beta(b)$ is compact;*

(ii) *for every $x : D$, there exists $\alpha : I \to B$ with $I : \mathcal{V}$ such that $x = \bigsqcup \beta \circ \alpha$.*

Proof. We just need to show that having $\beta : B \to D$ and $\alpha : I \to B$ such that every element $\beta(b)$ is compact and $x = \bigsqcup \beta \circ \alpha$, already implies that $\beta(\alpha_i) \ll x$ for every $i : I$. But if $i : I$, then $\beta(\alpha_i) \ll \beta(\alpha_i) \sqsubseteq \bigsqcup \beta \circ \alpha = x$ by compactness of $\beta(\alpha_i)$, so Lemma 42(iii) now finishes the proof. ◀

▶ **Example 58** ($\mathcal{L}_{\mathcal{U}}(X)$ is algebraic). Let $X : \mathcal{U}$ be a set and consider $\mathcal{L}_{\mathcal{U}}(X) : \mathcal{U}\text{-DCPO}_{\mathcal{U}^+,\mathcal{U}^+}$. The basis $[\bot, \eta_X] : (\mathbf{1}_{\mathcal{U}} + X) \to \mathcal{L}_{\mathcal{U}}(X)$ exhibits $\mathcal{L}_{\mathcal{U}}(X)$ as an algebraic dpco.

Proof. By Example 46, the elements $\bot$ and $\eta_X(x)$ (with $x : X$) are all compact, so it remains to show that $\mathbf{1}_{\mathcal{U}} + X$ is indeed a basis. Recalling Lemmas 19 and 20, we can write any element $(P, \varphi) : \mathcal{L}_{\mathcal{V}}(X)$ as the directed join $\bigsqcup([\bot, \eta_X] \circ \alpha)$ with $\alpha :\equiv [\mathsf{id}, \varphi] : (\mathbf{1}_{\mathcal{U}} + P) \to (\mathbf{1}_{\mathcal{U}} + X)$. By Lemma 57 the proof is finished. ◁

▶ **Example 59** ($\mathcal{P}_{\mathcal{U}}(X)$ is algebraic)**.** Let $X : \mathcal{U}$ be a set and consider $\mathcal{P}_{\mathcal{U}}(X) : \mathcal{U}\text{-}\mathsf{DCPO}_{\mathcal{U}^+,\mathcal{U}}$. The basis $\iota : \mathsf{List}(X) \to \mathcal{P}_{\mathcal{U}}(X)$ that maps a finite list to a Kuratowski finite subset exhibits $\mathcal{P}_{\mathcal{U}}(X)$ as an algebraic dpco.

Proof. By Example 48, the element $\iota(l)$ is compact for every list $l : \mathsf{List}(X)$, so it remains to show that $\mathsf{List}(X)$ is indeed a basis. In the proof of Example 48, we saw that every $\mathcal{U}$-subset A of X can be expressed as the directed supremum $\bigsqcup \iota \circ \alpha$ where $\alpha : \mathsf{List}(\mathbb{T}(A)) \to \mathsf{List}(X)$ takes a list $[(x_0, p_0), \ldots, (x_{n-1}, p_{n-1})]$ to the list $[x_0, \ldots, x_{n-1}]$. Another application of Lemma 57 now finishes the proof. ◁

▶ **Example 60** (Scott's D_∞ is algebraic)**.** The pointed dcpo $D_\infty : \mathcal{U}_0\text{-}\mathsf{DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ with $D_\infty \cong D_\infty^{D_\infty}$ from Section 5.2 is algebraic. We postpone the proof until Section 6.4, since we will need some additional results on locally small dcpos.

6.4 Ideal Completion

Finally, we consider how to build dcpos from posets, or more generally from abstract bases, using the rounded ideal completion [1, Section 2.2.6]. Given our definition of the notion of dcpo, the reader might expect us to define ideals using families rather than subsets. However, we use subsets for extensionality reasons. Two subsets A and B of some X are equal exactly when $x \in A \iff x \in B$ for every $x : X$. However, given two (directed) families $\alpha : I \to X$ and $\beta : J \to X$, it is of course not the case (it does not even typecheck) that $\alpha = \beta$ when $\Pi_{i:I}\exists_{j:J}\alpha_i = \beta_j$ and $\Pi_{j:J}\exists_{i:I}\beta_j = \alpha_i$ hold. We could try to construct the ideal completion by quotienting the families, but then it seems impossible to define directed suprema in the ideal completion without resorting to choice.

▶ **Definition 61** (Abstract basis)**.** *A pair $(B, \prec)$ with $B : \mathcal{V}$ and $\prec$ taking values in $\mathcal{V}$ is called a $\mathcal{V}$-*abstract basis *if:*

(i) *$\prec$ is proposition-valued;*
(ii) *$\prec$ is transitive;*
(iii) *$\prec$ satisfies nullary interpolation, i.e. for every $x : B$, there exists some $y : B$ with $y \prec x$;*
(iv) *$\prec$ satisfies binary interpolation, i.e. for every $x, y : B$ with $x \prec y$, there exists some $z : B$ with $x \prec z \prec y$.*

▶ **Example 62.** Let D be a $\mathcal{V}$-dcpo with a basis $\beta : B \to D$, By Lemmas 42, 53 and 55, the pair $\left(B, \ll^B\right)$ is an example of a $\mathcal{V}$-abstract basis.

▶ **Example 63.** Any preorder $(P, \sqsubseteq)$ with $P : \mathcal{V}$ and $\sqsubseteq$ taking values in $\mathcal{V}$ is a $\mathcal{V}$-abstract basis, since reflexivity implies both interpolation properties.

For the remainder of this section, fix some arbitrary $\mathcal{V}$-abstract basis $(B, \prec)$.

▶ **Definition 64** (Directed subset)**.** *Let A be a $\mathcal{V}$-subset of B. Then A is* directed *if A is inhabited (i.e. $\exists_{x:B}\, x \in A$ holds) and for every $x, y \in A$, there exists some $z \in A$ such that $x, y \sqsubseteq z$.*

▶ **Definition 65** (Ideal, lower set)**.** *Let A be a $\mathcal{V}$-subset of B. Then A is an* ideal *if A is a directed subset of B and A is a* lower set*, i.e. if $x \prec y$ and $y \in A$, then $x \in A$ as well.*

▶ **Construction 66** (Rounded ideal completion $\mathsf{Idl}(B, \prec)$)**.** We construct a $\mathcal{V}$-dcpo, known as the *(rounded) ideal completion* $\mathsf{Idl}(B, \prec) : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{V}^+,\mathcal{V}}$ of $(B, \prec)$. The carrier is given by the type $\sum_{I:B\to\mathcal{V}} \mathsf{is\text{-}ideal}(I)$ of ideals on $(B, \prec)$. The order is given by subset inclusion $\subseteq$. If we have a directed family $\alpha : A \to \mathsf{Idl}(B, \prec)$ of ideals (with $A : \mathcal{V}$), then the subset given by $\lambda x.\exists_{a:A} x \in \alpha_a$ is again an ideal and the supremum of α in $\mathsf{Idl}(B, \prec)$. ⌟

▶ **Lemma 67** (Rounded ideals)**.** *The ideals of* $\mathsf{Idl}(B, \prec)$ *are* rounded*. That is, if $I : \mathsf{Idl}(B, \prec)$ and $x \in I$, then there exists some $y \in I$ with $x \prec y$.*

Proof. Immediate from the fact that ideals are directed sets. ◀

▶ **Definition 68** (Principal ideal $\downarrow x$)**.** *We write $\downarrow(-) : B \to \mathsf{Idl}(B, \prec)$ for the map that takes $x : B$ to the* principal ideal *$\lambda y.y \prec x$.*

▶ **Lemma 69.** *Let $I : \mathsf{Idl}(B, \prec)$ be an ideal. Then I may be expressed as the supremum of the directed family $(x, p) : \mathbb{T}(I) \mapsto \downarrow x : \mathsf{Idl}(B, \prec)$, which we will denote by $I = \bigsqcup_{x\in I} \downarrow x$.*

Proof. Directedness of the family follows from the fact that I is a directed subset. Since I is a lower set, $\downarrow x \subseteq I$ holds for every $x \in I$, establishing $\bigsqcup_{x\in I} \downarrow x \subseteq I$. The reverse inclusion follows from Lemma 67. ◀

We wish to prove that $\mathsf{Idl}(B, \prec)$ is continuous with basis $\downarrow(-) : B \to \mathsf{Idl}(B, \prec)$. To this end, it is useful to express $\ll_{\mathsf{Idl}(B,\prec)}$ in more elementary terms.

▶ **Lemma 70.** *Let $I, J : \mathsf{Idl}(B, \prec)$ be two ideals. Then $I \ll J$ holds if and only there exists $x \in J$ such that $I \subseteq \downarrow x$.*

Proof. The left-to-right implication follows immediately from Lemma 69.

For the converse, note that $I \ll J$ is a proposition, so we may assume that we have $x \in J$ with $I \subseteq \downarrow x$. Now let $\alpha : A \to \mathsf{Idl}(B, \prec)$ be a directed family such that $J \subseteq \bigsqcup \alpha$. Then there must exist some $a : A$ for which $x \in \alpha_a$. But $I \subseteq \downarrow x$ and α_a is a lower set, so $I \subseteq \alpha_a$. ◀

▶ **Theorem 71.** *The map $\downarrow(-) : B \to \mathsf{Idl}(B, \prec)$ is a basis for $\mathsf{Idl}(B, \prec)$. Thus, $\mathsf{Idl}(B, \prec)$ is a continuous $\mathcal{V}$-dcpo.*

Proof. Let $I : \mathsf{Idl}(B, \prec)$ be arbitrary. By Lemma 69 we can express I as the supremum $\bigsqcup_{x\in I} \downarrow x$, so it is enough to prove that $\downarrow x \ll I$ for every $x \in I$. But this follows from Lemmas 67 and 70. ◀

▶ **Lemma 72.** *If $\prec$ is reflexive, then the compact elements of $\mathsf{Idl}(B, \prec)$ are exactly the principal ideals and $\mathsf{Idl}(B, \prec)$ is algebraic.*

Proof. Immediate from Lemma 70. ◀

▶ **Theorem 73.** *The ideal completion is the free dcpo on a small poset. That is, if we have a poset $(P, \sqsubseteq)$ with $P : \mathcal{V}$ and $\sqsubseteq$ taking values in $\mathcal{V}$, then for every $D : \mathcal{V}\text{-}\mathsf{DCPO}_{\mathcal{U},\mathcal{T}}$ and monotone function $f : P \to D$, there is a unique continuous function $\overline{f} : \mathsf{Idl}(P, \sqsubseteq) \to D$ such that*

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\downarrow(-)"'] & & D \\ & \mathsf{Idl}(P, \sqsubseteq) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

commutes.

Proof. Given $(P, \sqsubseteq)$, D and f as in the theorem, we define $\overline{f}$ by mapping an ideal I to the supremum of the directed (since I is an ideal) family $\mathbb{T}(I) \xrightarrow{\mathsf{pr}_1} P \xrightarrow{f} D$.

Commutativity of the diagram expresses that $f(x) = \bigsqcup_{y \sqsubseteq x} f(y)$ for every $x : P$. By antisymmetry of $\sqsubseteq$, it suffices to prove $f(x) \sqsubseteq \bigsqcup_{y \sqsubseteq x} f(y)$ and $\bigsqcup_{y \sqsubseteq x} f(y) \sqsubseteq f(x)$. The first holds by reflexivity of $\sqsubseteq$ and the second holds because f is monotone.

Uniqueness of $\overline{f}$ follows easily using Lemma 69. Finally, continuity of $\overline{f}$ is not hard to establish either. ◀

▶ **Definition 74** (Continuous retract, section, retraction)**.** *A $\mathcal{V}$-dcpo D is a* continuous retract *of another $\mathcal{V}$-dcpo E if we have continuous functions $s : D \to E$ (the* section*) and $r : E \to D$ (the* retraction*) such that $r(s(x)) = x$ for every $x : D$.*

▶ **Theorem 75.** *If E is a dcpo with basis $\beta : B \to D$ and D is a continuous retract of E with retraction r, then $r \circ \beta$ is a basis for D.*

Proof. Let E be a dcpo with basis $\beta : B \to D$ and suppose that we have continuous retraction $r : E \to D$ with continuous section $s : D \to E$. Given $x : D$, there exists some approximating family $\alpha : I \to B$ for $s(x)$. We claim that α is an approximating family for x as well, i.e.

(i) $r(\beta(\alpha_i)) \ll x$ for every $i : I$ and

(ii) $\bigsqcup r \circ \beta \circ \alpha = x$.

The second follows from continuity of r, since: $\bigsqcup r \circ \beta \circ \alpha = r(\bigsqcup \beta \circ \alpha) = r(s(x)) = x$. For (i), suppose that $i : I$ and that $\gamma : J \to D$ is a directed family satisfying $x \sqsubseteq \bigsqcup \gamma$. We must show that there exists $j : J$ with $r(\beta(\alpha_I)) \sqsubseteq \gamma_j$. By continuity of s, we get $s(x) \sqsubseteq \bigsqcup s \circ \gamma$. Hence, since $\beta(\alpha_i) \ll s(x)$, there must exist $j : J$ with $\beta(\alpha_i) \sqsubseteq s(\gamma_j)$. Thus, by monotonicity of r, we get the desired $r(\beta(\alpha_i)) \sqsubseteq r(s(\gamma_j)) = \gamma_j$. ◀

We now turn to locally small dcpos, as they allow us to find canonical approximating families, which is used in the proof of Theorem 78.

▶ **Lemma 76.** *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$. The following are equivalent:*

(i) *D is locally small;*

(ii) *$\beta(b) \ll x$ has size $\mathcal{V}$ for every $x : D$ and $b : B$.*

Proof. Recalling Lemma 51, the type $x \sqsubseteq y$ is equivalent to $\prod_{b:B}(\beta(b) \ll x \to \beta(b) \ll y)$ for every $x, y : D$. Thus, (ii) implies (i). Conversely, assume that D is locally small and let $x : D$ and $b : B$. We claim that $\beta(b) \ll x$ is equivalent to $\exists_{b':B}\big(b \ll^{B} b' \times \beta(b') \sqsubseteq_{\mathsf{small}} x\big) : \mathcal{V}$. The left-to-right implication is given by Lemma 54, and the converse by Lemma 42(iii). ◀

▶ **Lemma 77.** *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$. If D is locally small, then an element $x : D$ is the supremum of the large directed family $(\sum_{b:B} \beta(b) \ll x) \xrightarrow{\mathsf{pr}_1} B \xrightarrow{\beta} D$. Moreover, if D is locally small, then this directed family is small.*

Proof. The family $\mathsf{pr}_1 \circ \beta$ is directed by the nullary (Lemma 53) and binary (Lemma 55) interpolation properties. Now suppose that D is locally small. By Lemma 76, we have $I : \mathcal{V}$ and $\alpha : I \to D$ directed such that $\bigsqcup \alpha$ is the supremum of $(\sum_{b:B} \beta(b) \ll x) \xrightarrow{\mathsf{pr}_1} B \xrightarrow{\beta} D$. Since $\beta : B \to D$ is a basis of D, we see that $x \sqsubseteq \bigsqcup \alpha$. For the reverse inequality, it suffices to show that $\beta(b) \sqsubseteq x$ for every $b : B$ with $\beta(b) \ll x$. But this follows from Lemma 42(ii). ◀

▶ **Theorem 78.** *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$ and suppose that D is locally small. Then D is a continuous retract of the algebraic $\mathcal{V}$-dcpo $\mathsf{Idl}\big(B, \sqsubseteq^{B}\big)$ (recall Lemma 52).*

Proof. By Lemma 72, $\mathsf{Idl}\big(B, \sqsubseteq^B\big)$ is indeed algebraic. Let D be a $\mathcal{V}$-dcpo satisfying the hypotheses of the lemma. Let $\ll_{\mathsf{small}} : B \to D \to \mathcal{V}$ be such that $(b \ll_{\mathsf{small}} x) \simeq (\beta(b) \ll x)$ for every $x : D$ and $b : B$.

For every $x : D$, we can consider the subset $\twoheaddownarrow x$ given by $\lambda(b : B).b \ll_{\mathsf{small}} x$. We show that it is an ideal. By Lemma 77 it is a directed subset. And if $b \in \twoheaddownarrow x$ and $b' \sqsubseteq^B b$, then $b' \in \twoheaddownarrow x$ as well by virtue of Lemma 42(iii). So $\twoheaddownarrow x$ is a lower set, and indeed an ideal.

We claim that the map $\twoheaddownarrow(-)$ is continuous. By Lemma 43, it is monotone. Thus, we are left to show that if $\alpha : I \to D$ is directed, then $\twoheaddownarrow(\bigsqcup \alpha) \subseteq \bigsqcup_{i:I} \twoheaddownarrow \alpha_i$. Let $b \in \twoheaddownarrow(\bigsqcup \alpha)$, i.e. $b \in B$ such that $b \ll_{\mathsf{small}} \bigsqcup \alpha$. By Lemma 54, there exists $b' : B$ with $b \ll^B b' \ll_{\mathsf{small}} \bigsqcup \alpha$. Hence, there must exist $i : I$ such that $\beta(b) \ll \beta(b') \sqsubseteq \alpha_i$, thus, $b \in \twoheaddownarrow \alpha_i$ and $\twoheaddownarrow(-)$ is indeed continuous.

Next, define $r : \mathsf{Idl}\big(B, \sqsubseteq^B\big) \to D$ using Theorem 73 as the unique continuous function such that

$$\begin{tikzcd} B \arrow[rr, "\beta"] \arrow[dr, "\twoheaddownarrow(-)"'] & & D \\ & \mathsf{Idl}\big(B, \sqsubseteq^B\big) \arrow[ur, "r"'] & \end{tikzcd}$$

commutes, i.e. r maps an ideal I to the directed supremum $\bigsqcup_{b \in I} \beta(b)$ in D.

Finally, we show that $\twoheaddownarrow(-)$ is a section of r. That is, the equality $\bigsqcup_{b \ll_{\mathsf{small}} x} \beta(b) = x$ holds for every $x : D$. But this is exactly Lemma 77. ◀

One may wonder how restrictive the condition that D is locally small is. We note that if X is a set, then $\mathcal{L}_{\mathcal{V}}(X)$ (by Lemma 18) and $\mathcal{P}_{\mathcal{V}}(X)$ are examples of locally small $\mathcal{V}$-dcpos. A natural question is what happens with exponentials. In general, E^D may fail to be locally small even when both D and E are. However, we do have the following result.

▶ **Lemma 79.** *Let D and E be $\mathcal{V}$-dcpos. Suppose that D is continuous and E is locally small. Then E^D is locally small.*

Proof. Since being locally small is a proposition, we may assume that we are given a basis $\beta : B \to D$ of D. We claim that for every two continuous functions $f, g : D \to E$ we have an equivalence

$$\left(\prod_{x:D} f(x) \sqsubseteq_E g(x)\right) \simeq \left(\prod_{b:B} f(\beta(b)) \sqsubseteq_{\mathsf{small}} g(\beta(b))\right).$$

Since $B : \mathcal{V}$ and $\sqsubseteq_{\mathsf{small}}$ takes values in $\mathcal{V}$, the second type is also in $\mathcal{V}$. For the equivalence, note that the left-to-right implication is trivial. For the converse, assume the right-hand side and let $x : D$. By continuity of D, there exists some approximating family $\alpha : I \to B$ for x. We use it as follows:

$$\begin{aligned} f(x) &= f\Big(\bigsqcup \beta \circ \alpha\Big) \\ &= \bigsqcup_{i:I} f(\beta(\alpha_i)) && \text{(by continuity of } f\text{)} \\ &\sqsubseteq \bigsqcup_{i:I} g(\beta(\alpha_i)) && \text{(by assumption)} \\ &= g\Big(\bigsqcup \beta \circ \alpha\Big) && \text{(by continuity of } g\text{)} \\ &= g(x), \end{aligned}$$

which finishes the proof. ◀

Moreover, the (co)limit of locally small dcpos is locally small.

▶ **Lemma 80.** *Given a system $(D_i, \varepsilon_{i,j}, \pi_{i,j})$ as in Section 5.1, if every D_i is locally small, then so is D_∞.*

Finally, the requirement that D is locally small is necessary, in the following sense.

▶ **Lemma 81.** *Let D be a $\mathcal{V}$-dcpo with basis $\beta : B \to D$. Suppose that D is a continuous retract of $\mathsf{Idl}\big(B, \sqsubseteq^B\big)$. Then D is locally small.*

Proof. Let $s : D \to \mathsf{Idl}\big(B, \sqsubseteq^B\big)$ be a section of a map $r : \mathsf{Idl}\big(B, \sqsubseteq^B\big) \to D$, with both maps continuous. Then $x \sqsubseteq_D y$ holds if and only if $s(x) \sqsubseteq_{\mathsf{Idl}(B, \sqsubseteq^B)} s(y)$. Since $\mathsf{Idl}\big(B, \sqsubseteq^B\big)$ is locally small, so must D. ◀

We have now developed the theory sufficiently to give a proof of Example 60.

Proof of Example 60 (Scott's D_∞ is algebraic). Firstly, notice that D_0 is not just a $\mathcal{U}_0$-dcpo, but in fact a $\mathcal{U}_0$-sup lattice, i.e. it has joins for all families indexed by types in $\mathcal{U}_0$. Moreover, since joins in exponentials are given pointwise, every D_n is in fact a $\mathcal{U}_0$-sup lattice. In particular, every D_n has all finite joins. Hence, if we have $\alpha : I \to D_n$ with $I : \mathcal{U}_0$, then we can consider the directed family $\overline{\alpha} : \overline{I} \to D_n$ with $\overline{I} :\equiv \sum_{k:\mathbf{N}}(\mathsf{Fin}\, k \to D_n)$ and $\overline{\alpha}$ mapping a pair (k, f) to the finite join $\bigvee_{0 \le i < k} f(i)$. Moreover, if every α_i is compact, then so is every $\overline{\alpha}_{\overline{i}}$, since finite joins of compact elements are compact again. We show this explicitly for binary joins from which the general case follows by induction. If $a, b : D_n$ are compact and $a, b \sqsubseteq a \vee b \sqsubseteq \bigsqcup \gamma$ for some directed family $\gamma : J \to D_n$, then by compactness of a and b, there exist $j_a, j_b : J$ such that $a \sqsubseteq \gamma_{j_a}$ and $b \sqsubseteq \gamma_{j_b}$. By directedness of γ, there exists $k : J$ with $a, b \sqsubseteq \gamma_k$. Hence, $a \vee b \sqsubseteq \gamma_k$, as desired.

▷ Claim. Every D_n is locally small and has a basis $\beta_n : B_n \to D_n$ of compact elements.

Proof. We prove this by induction. For $n = 0$, this follows from Lemma 18 and Example 58. Now suppose that B_m is locally small and has a basis $\beta_m : B_m \to D_m$. By Lemma 79, the dcpo $B_{m+1} \equiv B_m^{B_m}$ is locally small. If we have $a, b : B_m$, then we define the continuous *step function* $(a \Rightarrow b) : D_m \to D_m$ by $x \mapsto \bigvee_{\beta_m(a) \sqsubseteq x} \beta_m(b)$, which is well-defined since D_m is locally small. We are going to show that $a \Rightarrow b$ is compact for every $a, b : B_m$ and that every $f : D_{m+1}$ is the join of certain step functions. To this end, we first observe that

$$(a \Rightarrow b \sqsubseteq f) \iff (\beta_m(b) \sqsubseteq f(\beta_m(a))), \tag{†}$$

which follows from the fact that continuous functions are monotone.

For compactness, suppose that $a \Rightarrow b \sqsubseteq \bigsqcup_{i:I} f_i$. By (†) we have $\beta_m(b) \sqsubseteq \bigvee_{i:I}(f_i(\beta_m(a)))$. By compactness of $\beta_m(b)$, there exists $i : I$ such that $\beta_m(b) \sqsubseteq f_i(\beta_m(a))$ already. Using (†) once more, we get the desired $a \Rightarrow b \sqsubseteq f_i$.

Now let $f : D_m \to D_m$ be continuous. We claim that f is the join of the step-functions below it, i.e. $f = \bigvee_{a,b:B_m, a \Rightarrow b \sqsubseteq f} a \Rightarrow b$, which is well-defined, since D_{m+1} is locally small. One inequality clearly holds as we are only considering step-functions below f. For the reverse inequality, let $x : D_m$ be arbitrary. By Lemma 77, we have:

$$x = \bigsqcup_{\substack{a':B_m \\ \beta_m(a') \ll x}} \beta_m(a') \quad \text{and} \quad f(x) = \bigsqcup_{\substack{b':B_m \\ \beta_m(b') \ll f(x)}} \beta_m(b'). \tag{‡}$$

Hence, it suffices to show that $\beta_m(b') \sqsubseteq \bigvee_{a,b:B_m, a \Rightarrow b \sqsubseteq f}(a \Rightarrow b)(x)$ whenever $b' : B_m$ is such that $\beta_m(b') \ll f(x)$. By (†) and the definition of a step-function it is enough to find $a' : B_m$

such that $\beta_m(b') \sqsubseteq f(\beta_m(a'))$ and $\beta_m(a') \sqsubseteq x$. Using (‡), our assumption $\beta_m(b') \ll f(x)$ and continuity of f, we get that there exists $a' : B_m$ with $\beta_m(a') \ll x$ (and thus $\beta_m(a') \sqsubseteq x$) and $b_m(b') \sqsubseteq f(\beta_m(a'))$, as desired.

Thus, by the paragraph preceding the claim, D_{m+1} has a basis of compact elements: $\beta_{m+1} : (\sum_{k:\mathbf{N}}(\mathsf{Fin}(k) \to (D_m \times D_m))) \to D_{m+1}$ with $\beta_{m+1}(k, \lambda i.(a_i, b_i)) :\equiv \bigvee_{0 \leq i < k} a_i \Rightarrow b_i$, finishing the proof of the claim. $\lhd$

Finally, we show that a basis of compact elements for D_∞ is $\beta_\infty : (B_\infty :\equiv \sum_{n:\mathbf{N}} B_n) \to D_\infty$ where $\beta_\infty(n, b) :\equiv \varepsilon_{n,\infty}(\beta_n(b))$. We first check compactness by showing that if $x : D_n$ is compact, then so is $\varepsilon_{n,\infty}(x)$. This follows easily from the fact that $(\varepsilon_{n,\infty}, \pi_{n,\infty})$ is an embedding-projection. For if $\alpha : I \to D_\infty$ is directed and $\varepsilon_{n,\infty}(x) \sqsubseteq \bigsqcup \alpha$, then $x = \pi_{n,\infty}(\varepsilon_{n,\infty}(x)) \sqsubseteq \pi_{n,\infty}(\bigsqcup \alpha) = \bigsqcup \pi_{n,\infty} \circ \alpha$, by continuity of $\pi_{n,\infty}$. Thus, by compactness of x, there exist $i : I$ such that $x \sqsubseteq \pi_{n,\infty}(\alpha_i)$ already. Hence, $\varepsilon_{n,\infty}(x) \sqsubseteq \varepsilon_{n,\infty}(\pi_{n,\infty}(\alpha_i)) \sqsubseteq \alpha_i$, so $\varepsilon_{n,\infty}(x)$ is indeed compact. Now let $\sigma : D_\infty$ be arbitrary. As mentioned in the proof of Theorem 37, we have $\sigma = \bigsqcup_{n:\mathbf{N}} \varepsilon_{n,\infty}(\sigma_n)$. By Lemma 77 and the claim, we can express every $\sigma_n : D_n$ as $\bigsqcup_{b:B_n, \beta_n(b) \ll \sigma_n} \beta_n(b)$. Hence,

$$\sigma = \bigsqcup_{n:\mathbf{N}} \varepsilon_{n,\infty}\left(\bigsqcup_{\substack{b:B_n \\ \beta_n(b) \ll \sigma_n}} \beta_n(b)\right) = \bigsqcup_{n:\mathbf{N}} \bigsqcup_{\substack{b:B_n \\ \beta_n(b) \ll \sigma_n}} \varepsilon_{n,\infty}(\beta_n(b))$$

by continuity of $\varepsilon_{n,\infty}$. Thus, σ may be expressed as the supremum of the directed family $\left(\sum_{n:\mathbf{N}} \sum_{b:B_n} \beta_n(b) \ll \sigma_n\right) \to B_\infty \xrightarrow{\beta_\infty} D_\infty$. (And in light of Lemma 76 and the claim, the type $\sum_{n:\mathbf{N}} \sum_{b:B_n} \beta_n(b) \ll \sigma_n$ can be replaced by a type in $\mathcal{U}_0$.) Finally, using Lemma 57, we see that D_∞ is indeed algebraic. $\lhd$

We end this section by describing an example of a continuous dcpo, built using the ideal completion, that is not algebraic. In fact, this dcpo has no compact elements at all.

▶ **Example 82** (A continuous dcpo that is not algebraic)**.** We inductively define a type and an order representing dyadic rationals $m/2^n$ in the interval $(-1, 1)$ for integers m, n. The intuition for the upcoming definitions is the following. Start with the point 0 in the middle of the interval (represented by center below). Then consider the two functions (respectively represented by left and right below)

$$\begin{aligned} &l, r : (-1, 1) \to (-1, 1) \\ &l(x) = (x - 1)/2 \\ &r(x) = (x + 1)/2 \end{aligned}$$

that generate the dyadic rationals. Observe that $l(x) < 0 < r(x)$ for every $x : (-1, 1)$. Accordingly, we inductively define the following types.

▶ **Definition 83** (Dyadics $\mathbb{D}$)**.** *The type of* dyadics $\mathbb{D} : \mathcal{U}_0$ *is the inductive type with three constructors:*

$$\mathsf{center} : \mathbb{D} \qquad \mathsf{left} : \mathbb{D} \to \mathbb{D} \qquad \mathsf{right} : \mathbb{D} \to \mathbb{D}.$$

▶ **Definition 84** (Order $\prec$ on $\mathbb{D}$)**.** *Let* $\prec : \mathbb{D} \to \mathbb{D} \to \mathcal{U}_0$ *be inductively defined as:*

$$\begin{array}{lll} \mathsf{center} \prec \mathsf{center} :\equiv \mathbf{0} & \mathsf{left}\, x \prec \mathsf{center} :\equiv \mathbf{1} & \mathsf{right}\, x \prec \mathsf{center} :\equiv \mathbf{0} \\ \mathsf{center} \prec \mathsf{left}\, y :\equiv \mathbf{0} & \mathsf{left}\, x \prec \mathsf{left}\, y :\equiv x \prec y & \mathsf{right}\, x \prec \mathsf{left}\, y :\equiv \mathbf{0} \\ \mathsf{center} \prec \mathsf{right}\, y :\equiv \mathbf{1} & \mathsf{left}\, x \prec \mathsf{right}\, y :\equiv \mathbf{1} & \mathsf{right}\, x \prec \mathsf{right}\, y :\equiv x \prec y. \end{array}$$

One then shows that $\prec$ is proposition-valued, transitive, irreflexive, trichotomous, dense and that it has no endpoints. *Trichotomy* means that exactly one of $x \prec y$, $x = y$, $y \prec x$ holds. *Density* says that for every $x, y : \mathbb{D}$, there exists some $z : \mathbb{D}$ such that $x \prec z \prec y$. Finally, *having no endpoints* means that for every $x : \mathbb{D}$, there exist some $y, z : \mathbb{D}$ with $y \prec x \prec z$. Using these properties, we can show that $(\mathbb{D}, \prec)$ is a $\mathcal{U}_0$-abstract basis. Thus, taking the rounded ideal completion, we get $\mathsf{Idl}(\mathbb{D}, \prec) : \mathcal{U}_0\text{-}\mathsf{DCPO}_{\mathcal{U}_1,\mathcal{U}_0}$, which is continuous with basis $\downarrow(-) : \mathbb{D} \to \mathsf{Idl}(\mathbb{D}, \prec)$ by Theorem 71. But $\mathsf{Idl}(\mathbb{D}, \prec)$ cannot be algebraic, since none of its elements are compact. Indeed suppose that we had an ideal I with $I \ll I$. By Lemma 70, there would exist $x \in I$ with $I \sqsubseteq \downarrow x$. But this implies $x \prec x$, but $\prec$ is irreflexive, so this is impossible.

7 Conclusion and Future Work

We have developed domain theory constructively and predicatively in univalent foundations, including Scott's D_∞ model of the untyped λ-calculus, as well as notions of continuous and algebraic dcpos. We avoid size issues in our predicative setting by having large dcpos with joins of small directed families. Often we find it convenient to work with locally small dcpos, whose orders have small truth values.

In future work, we wish to give a predicative account of the theory of algebraic and continuous exponentials, which is a rich and challenging topic even classically. We also intend to develop applications to topology and locale theory. It is also important to understand when classical theorems do not have constructive and predicative counterparts. For instance, Zorn's Lemma doesn't imply excluded middle but it implies propositional resizing [9] and we are working on additional examples.

We have formalized the following in Agda [10], in addition to the Scott model of PCF and its computational adequacy [8, 18]:

1. dcpos,
2. limits and colimits of dcpos, Scott's D_∞,
3. lifting and exponential constructions,
4. pointed dcpos have subsingleton joins (in the right universe),
5. way-below relation, continuous, algebraic dcpos, interpolation properties,
6. abstract bases and rounded ideal completions (including its universal property),
7. continuous dcpos are continuous retract of their ideal completion, and hence of algebraic dcpos,
8. ideal completion of dyadics, giving an example of a non-algebraic, continuous dcpo.

In the near future we intend to complete our formalization to also include Theorems 21, 23 and 25, Examples 59 and 60, and Lemma 79.

References

1 S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

2 Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994. doi:10.1017/CBO9780511600579.

3 Roberto M. Amadio and Pierre-Louis Curien. *Domains and Lambda-Calculi*, volume 46 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1998. doi:10.1017/CBO9780511983504.

4 Andrej Bauer and Iztok Kavkler. A constructive theory of continuous domains suitable for implementation. *Annals of Pure and Applied Logic*, 159(3):251–267, 2009. `doi:10.1016/j.apal.2008.09.025`.

5 Nick Benton, Andrew Kennedy, and Carsten Varming. Some domain theory and denotational semantics in Coq. In Stefan Berghofer, Tobias Nipkow, Christian Urban, and Makarius Wenzel, editors, *Theorem Proving in Higher Order Logics (TPHOLs 2009)*, volume 5674 of *Lecture Notes in Computer Science*, pages 115–130. Springer, 2009. `doi:10.1007/978-3-642-03359-9_10`.

6 The Agda community. Agda wiki. `https://wiki.portal.chalmers.se/agda/pmwiki.php`.

7 Thierry Coquand, Giovanni Sambin, Jan Smith, and Silvio Valentini. Inductively generated formal topologies. *Annals of Pure and Applied Logic*, 124(1–3):71–106, 2003. `doi:10.1016/s0168-0072(03)00052-6`.

8 Tom de Jong. The Scott model of PCF in univalent type theory. November 2019. `arXiv:1904.09810`.

9 Tom de Jong. Domain theory in predicative Univalent Foundations. Abstract for a talk at *HoTT/UF 2020* on 7 July, 2020. URL: `https://hott-uf.github.io/2020/HoTTUF_2020_paper_14.pdf`.

10 Tom de Jong. Formalisation of domain theory in constructive and predicative univalent foundations. `https://github.com/tomdjong/TypeTopology`, June 2020. Agda development.

11 Robert Dockins. Formalized, Effective Domain Theory in Coq. In Gerwin Klein and Ruben Gamboa, editors, *Interactive Theorem Proving (ITP 2014)*, volume 8558 of *Lecture Notes in Computer Science*, pages 209–225. Springer, 2014. `doi:10.1007/978-3-319-08970-6_14`.

12 Martín H. Escardó. The proof of the interpolation property of the way-below relation of a continuous poset. Unpublished note. The proof has been included in Escardó's *Function-space compactifications of function spaces* and in the book *Continuous lattices and domains* by Gierz et. al., March 2000. URL: `https://www.cs.bham.ac.uk/~mhe/papers/interpolation.pdf`.

13 Martín H. Escardó and Cory M. Knapp. Partial Elements and Recursion via Dominances in Univalent Type Theory. In Valentin Goranko and Mads Dam, editors, *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, volume 82 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 21:1–21:16. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2017. `doi:10.4230/LIPIcs.CSL.2017.21`.

14 Martín Hötzel Escardó. Introduction to Univalent Foundations of Mathematics with Agda. February 2020. `arXiv:1911.00580`.

15 Martín Hötzel Escardó. Various new theorems in constructive univalent mathematics written in Agda. `https://github.com/martinescardo/TypeTopology`, June 2020. Agda development.

16 G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003. `doi:10.1017/CBO9780511542725`.

17 Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott. *A Compendium of Continuous Lattices*. Springer Berlin Heidelberg, 1980. `doi:10.1007/978-3-642-67678-9`.

18 Brendan Hart. Investigating Properties of PCF in Agda. MSci project, School of Computer Science, University of Birmingham, 2020. Report and Agda code available at `https://github.com/BrendanHart/Investigating-Properties-of-PCF`.

19 Michael Hedberg. A type-theoretic interpretation of constructive domain theory. *Journal of Automated Reasoning*, 16(3):369–425, 1996. `doi:10.1007/BF00252182`.

20 J. M. E. Hyland. First steps in synthetic domain theory. In A. Carboni, M. C. Peddicchio, and G. Rosolini, editors, *Category Theory*, volume 1488 of *Lecture Notes in Mathematics*, pages 131–156. Springer, 1991. `doi:10.1007/bfb0084217`.

21 Peter Johnstone and André Joyal. Continuous categories and exponentiable toposes. *Journal of Pure and Applied Algebra*, 25(3):255–296, 1982. `doi:10.1016/0022-4049(82)90083-4`.

22 Tatsuji Kawai. Predicative theories of continuous lattices. June 2020. `arXiv:2006.05642`.

23 Nicolai Kraus, Martín Escardó, Thierry Coquand, and Thorsten Altenkirch. Notions of Anonymous Existence in Martin-Löf Type Theory. *Logical Methods in Computer Science*, 13(1), 2017. doi:10.23638/LMCS-13(1:15)2017.

24 David Lidell. Formalizing domain models of the typed and the untyped lambda calculus in Agda. Master's thesis, Chalmers University of Technology and University of Gothenburg, 2020. Final version in preparation. Agda code available at https://github.com/DoppeD/CwfsOfDomains.

25 John Longley and Dag Normann. *Higher-Order Computability*. Springer, 2015. doi:10.1007/978-3-662-47992-6.

26 Maria Emilia Maietti and Silvio Valentini. Exponentiation of Scott formal topologies. volume 73, pages 111–131, 2004. Proceedings of the Workshop on Domains VI. doi:10.1016/j.entcs.2004.08.005.

27 Michael Makkai and Robert Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989. URL: https://doi.org/10.1090/conm/104, doi:10.1090/conm/104.

28 Sara Negri. Continuous lattices in formal topology. In Eduardo Giménez and Christine Paulin-Mohring, editors, *Types for Proofs and Programs (TYPES 1996)*, volume 1512 of *Lecture Notes in Computer Science*, pages 333–353. Springer, 1998. doi:10.1007/BFb0097800.

29 Sara Negri. Continuous domains as formal spaces. *Mathematical Structures in Computer Science*, 12(1):19–52, 2002. doi:10.1017/S0960129501003450.

30 Dirk Pattinson and Mina Mohammadian. Constructive Domains with Classical Witnesses. June 2020. arXiv:1910.04948.

31 G. D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5(3):223–255, 1977. doi:10.1016/0304-3975(77)90044-5.

32 Bernhard Reus. Formalizing synthetic domain theory — the basic definitions. *Journal of Automated Reasoning*, 23(3-4):411–444, 1999. doi:10.1023/A:1006258506401.

33 Bernhard Reus and Thomas Streicher. General synthetic domain theory — a logical approach. *Mathematical Structures in Computer Science*, 9(2):177—-223, 1999. doi:10.1017/S096012959900273X.

34 G. Rosolini. Categories and effective computations. In David H. Pitt, Axel Poigné, and David E. Rydeheard, editors, *Category Theory and Computer Science*, volume 283 of *Lecture Notes in Computer Science*, pages 1–11. Springer, 1987.

35 Giuseppe Rosolini. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986.

36 Giovanni Sambin. Intuitionistic formal spaces—a first communication. In *Mathematical logic and its applications*, pages 187–204. Springer, 1987. doi:10.1007/978-1-4613-0897-3_12.

37 Giovanni Sambin, Silvio Valentini, and Paolo Virgili. Constructive domain theory as a branch of intuitionistic pointfree topology. *Theoretical Computer Science*, 159(2):319–341, 1996. doi:10.1016/0304-3975(95)00169-7.

38 Dana S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer, 1972. doi:10.1007/BFB0073967.

39 Dana S. Scott. Lectures on a mathematical theory of computation. In Manfred Broy and Gunther Schmidt, editors, *Theoretical Foundations of Programming Methodology: Lecture Notes of an International Summer School, directed by F. L. Bauer, E. W. Dijkstra and C. A. R. Hoare*, volume 91 of *NATO Advanced Study Institutes Series*, pages 145–292. Springer, 1982. doi:10.1007/978-94-009-7893-5_9.

40 Dana S. Scott. A type-theoretical alternative to ISWIM, CUCH, OWHY. *Theoretical Computer Science*, 121(1):411–440, 1993. doi:10.1016/0304-3975(93)90095-B.

41 Andrew W. Swan. Choice, collection and covering in cubical sets. Talk in the electronic *HoTTEST* seminar, 6 November. Slides at https://www.uwo.ca/math/faculty/kapulkin/seminars/hottestfiles/Swan-2019-11-06-HoTTEST.pdf. Video recording at https://www.youtube.com/watch?v=r9KbEOzyr1g, 2019.

42 Taichi Uemura. Cubical Assemblies, a Univalent and Impredicative Universe and a Failure of Propositional Resizing. In Peter Dybjer, José Espírito Santo, and Luís Pinto, editors, *24th International Conference on Types for Proofs and Programs (TYPES 2018)*, volume 130 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 7:1–7:20. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2019. `doi:10.4230/LIPIcs.TYPES.2018.7`.

43 The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

44 Vladimir Voevodsky. Resizing rules — their use and semantic justification. Slides from a talk at TYPES, Bergen, 11 September, 2011. URL: `https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2011_Bergen.pdf`.

45 Vladimir Voevodsky. An experimental library of formalized mathematics based on the univalent foundations. *Mathematical Structures in Computer Science*, 25(5):1278–1294, 2015.

46 Vladimir Voevodsky, Benedikt Ahrens, Daniel Grayson, et al. UniMath — a computer-checked library of univalent mathematics. Available at `https://github.com/UniMath/UniMath`.