The Parker Advantage Dice Conjecture

In order to calculate the mean of the highest roll of m dice, each n-sided, we first write the down the probability of the highest roll being equal to k:

$$P_k = \frac{1}{n^m}\left[k^m - (k-1)^m\right] \tag{1}$$

$$P_k = \frac{1}{n^m}\sum_{s=0}^{m-1}(-1)^{m+s+1}\binom{m}{s}k^s, \tag{2}$$

where we've used the binomial theorem in (2).

To calculate the mean, we have a sum over k weighted by the probabilities,

$$\overline{k} = \sum_{k=1}^{n} P_k\, k \tag{3}$$

$$\overline{k} = \frac{1}{n^m}\sum_{k=1}^{n}\sum_{s=0}^{m-1}(-1)^{m+s+1}\binom{m}{s}k^{s+1} \tag{4}$$

$$\overline{k} = \frac{1}{n^m}\sum_{s=0}^{m-1}(-1)^{m+s+1}\binom{m}{s}H_n^{(-s-1)}, \tag{5}$$

where $H_n^{(m)}$ is the generalised Harmonic number of order m of n.

Expanding $H_n^{(m)}$ in n^{-1} at $n \to \infty$ yields

$$H_n^{(m)} = \zeta(m) + \frac{1}{n^m}\left[\frac{n}{1-m} + \frac{1}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right], \tag{6}$$

where $\zeta(m)$ is the Riemann zeta function evaluated at m, which can be used to approximate $H_n^{(m)}$ for large n.

Plugging this result into (5), all terms within the sum involving a power of n smaller than m will vanish for large n. First discarding the pieces that won't contribute for any value of s in the sum, we get

$$\overline{k} \sim \frac{1}{n^m} \sum_{s=0}^{m-1} (-1)^{m+s+1} \binom{m}{s} \left[n^{s+1} \left(\frac{n}{s+2} + \frac{1}{2} \right) \right]. \tag{7}$$

Collecting all of the pieces that survive, we are left with

$$\overline{k} \sim \frac{1}{n^m} \left[-\binom{m}{m-2} n^{m-1} \frac{n}{m} + \binom{m}{m-1} n^m \left(\frac{n}{m+1} + \frac{1}{2} \right) \right] \tag{8}$$

$$\overline{k} \sim -\frac{1}{2} (m-1) + m \left(\frac{n}{m+1} + \frac{1}{2} \right) \tag{9}$$

$$\overline{k} \sim \frac{mn}{m+1} + \frac{1}{2}, \tag{10}$$

thus proving the Parker advantage dice conjecture.