Newton's Second Law to Lagrangian and Hamiltonian

Newton's Second Law

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a}$$

$$\mathbf{F} - m\mathbf{a} = 0$$

D'Alembert's Principle

$$(\mathbf{F} - m\mathbf{a}) \cdot \delta\mathbf{r} = 0$$

Principle of Least Action

$$\int_{t_1}^{t_2} (\mathbf{F} - m\mathbf{a}) \cdot \delta\mathbf{r}\, dt = 0$$

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \delta\mathbf{r}\, dt - \int_{t_1}^{t_2} m\mathbf{a} \cdot \delta\mathbf{r}\, dt = 0$$

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \delta\mathbf{r}\, dt \coloneqq \int_{t_1}^{t_2} -\delta V\, dt = \delta \int_{t_1}^{t_2} -V\, dt$$

$$-\int_{t_1}^{t_2} m\mathbf{a} \cdot \delta\mathbf{r}\, dt = -\int_{t_1}^{t_2} m\dot{\mathbf{v}} \cdot \delta\mathbf{r}\, dt$$

$$= -[m\mathbf{v} \cdot \delta\mathbf{r}]_{t_1}^{t_2} + \int_{t_1}^{t_2} m\mathbf{v} \cdot \frac{d}{dt}(\delta\mathbf{r})\, dt$$

$$\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = 0 \Longrightarrow [m\mathbf{v}\cdot\delta\mathbf{r}]_{t_1}^{t_2} = 0$$

$$\Longrightarrow -\int_{t_1}^{t_2} m\mathbf{a}\cdot\delta\mathbf{r}\,dt = \int_{t_1}^{t_2} m\mathbf{v}\cdot\delta\mathbf{v}\,dt$$

$$\delta(v^2) = \delta(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\delta\mathbf{v}$$

$$\Longrightarrow \int_{t_1}^{t_2} m\mathbf{v}\cdot\delta\mathbf{v}\,dt = \int_{t_1}^{t_2} \frac{m}{2}\delta(\mathbf{v}\cdot\mathbf{v})\,dt$$

$$= \delta\int_{t_1}^{t_2} \frac{m}{2}v^2\,dt \coloneqq \delta\int_{t_1}^{t_2} T\,dt$$

$$\Longrightarrow \int_{t_1}^{t_2} (\mathbf{F} - m\mathbf{a})\cdot\delta\mathbf{r}\,dt = \delta\int_{t_1}^{t_2} (T - V)\,dt = 0$$

$$\delta\int_{t_1}^{t_2} (T - V)\,dt \coloneqq \delta\int_{t_1}^{t_2} \mathcal{L}\,dt \coloneqq \delta S = 0$$

$$\delta\int_{t_1}^{t_2} \mathcal{L}\,dt = \int_{t_1}^{t_2} \big(\mathcal{L}(\mathbf{q} + \boldsymbol{\varepsilon}, \dot{\mathbf{q}} + \dot{\boldsymbol{\varepsilon}}, t) - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)\big)\,dt$$

$$= \sum_i \int_{t_1}^{t_2} \left(\varepsilon_i \partial_{q_i}\mathcal{L} + \dot{\varepsilon}_i \partial_{\dot{q}_i}\mathcal{L}\right)dt$$

$$= \sum_i \left(\left[\varepsilon_i \partial_{\dot{q}_i}\mathcal{L}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\varepsilon_i \partial_{q_i}\mathcal{L} - \varepsilon_i \dot{\partial}_{\dot{q}_i}\mathcal{L}\right)dt \right)$$

$$\boldsymbol{\varepsilon}(t_1) = \boldsymbol{\varepsilon}(t_2) = 0 \Longrightarrow \left[\varepsilon_i \partial_{\dot{q}_i} \mathcal{L}\right]_{t_1}^{t_2} = 0$$

$$\Longrightarrow \delta S = \sum_i \int_{t_1}^{t_2} \varepsilon_i \left(\partial_{q_i} \mathcal{L} - \dot{\partial}_{\dot{q}_i} \mathcal{L}\right) dt = 0$$

$$\delta S = 0 \text{ for all } \boldsymbol{\varepsilon} \Longrightarrow \dot{\partial}_{\dot{q}_i} \mathcal{L} = \partial_{q_i} \mathcal{L}$$

Lagrangian to Hamiltonian

$$\partial_{\dot{q}_i} \mathcal{L} \coloneqq p_i, \qquad \partial_{q_i} \mathcal{L} = \frac{d}{dt}\left(\partial_{\dot{q}_i} \mathcal{L}\right) = \dot{p}_i$$

$$\delta \mathcal{L} = \sum_i \left(\left(\partial_{q_i} \mathcal{L}\right) \delta q_i + \left(\partial_{\dot{q}_i} \mathcal{L}\right) \delta \dot{q}_i\right) + (\partial_t \mathcal{L})\, \delta t$$

$$= \sum_i (\dot{p}_i \delta q_i + p_i \delta \dot{q}_i) + (\partial_t \mathcal{L})\, \delta t$$

$$= \sum_i (\dot{p}_i \delta q_i + \delta(p_i \dot{q}_i) - \dot{q}_i \delta p_i) + (\partial_t \mathcal{L})\, \delta t$$

$$\delta \mathcal{L} - \sum_i \delta(p_i \dot{q}_i) = \sum_i (\dot{p}_i \delta q_i - \dot{q}_i \delta p_i) + (\partial_t \mathcal{L})\, \delta t$$

$$\delta\left(\sum_i p_i \dot{q}_i - \mathcal{L}\right) \coloneqq \delta \mathcal{H} = \sum_i (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i) - (\partial_t \mathcal{L})\, \delta t$$

$$\partial_{p_i} \mathcal{H} = \dot{q}_i, \qquad \partial_{q_i} \mathcal{H} = -\dot{p}_i, \qquad \partial_t \mathcal{H} = -\partial_t \mathcal{L}$$

Standard Hamiltonian

$$\mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t) = T(\dot{\mathbf{q}}) - V(\mathbf{q},t) = \sum_i \frac{m_i}{2}{\dot{q}_i}^2 - V(\mathbf{q},t)$$

$$p_i = \partial_{\dot{q}_i}\mathcal{L} = m_i\dot{q}_i \Longrightarrow \dot{q}_i = \frac{p_i}{m_i}$$

$$\mathcal{H}(\mathbf{q},\mathbf{p},t) = \sum_i p_i\dot{q}_i - \mathcal{L} = \sum_i \frac{{p_i}^2}{2m_i} + V(\mathbf{q},t) = T(\mathbf{p}) + V(\mathbf{q},t)$$

Poisson Bracket

$$\{A,B\} \coloneqq \sum_i \Big((\partial_{q_i}A)(\partial_{p_i}B) - (\partial_{p_i}A)(\partial_{q_i}B) \Big)$$

$$\{A,B\} = -\{B,A\}$$

$$\{A+B,C\} = \{A,C\} + \{B,C\}$$

$$\{AB,C\} = B\{A,C\} + A\{B,C\}$$

$$\big\{A,\{B,C\}\big\} + \big\{B,\{C,A\}\big\} + \big\{C,\{A,B\}\big\} = 0$$

$$\dot{A}(\mathbf{q},\mathbf{p},t) = \sum_i \Big((\partial_{q_i}A)\,\dot{q}_i + (\partial_{p_i}A)\,\dot{p}_i \Big) + \partial_t A$$

$$= \sum_i \Big((\partial_{q_i}A)(\partial_{p_i}\mathcal{H}) - (\partial_{p_i}A)(\partial_{q_i}\mathcal{H}) \Big) + \partial_t A = \{A,\mathcal{H}\} + \partial_t A$$