# Schwarzschild Geodesics

## Metric

$$x^{\mu} = (ct, r, \varphi)$$

$$c^2 d\tau^2 = c^2 \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\varphi^2$$

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{r_s}{r}\right) & 0 & 0 \\ 0 & \left(1 - \frac{r_s}{r}\right)^{-1} & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

$$r_s = \frac{2GM}{c^2}$$

## Lagrangian

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$$

$$\mathcal{L} = -mc\sqrt{c^2\left(1 - \frac{r_s}{r}\right) - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\varphi}^2}$$

## Newtonian Approximation

$$\mathcal{L} \approx -mc^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GMm}{r}$$

## Christoffel Symbols

$$\Gamma^{t}{}_{\mu\nu} = \begin{bmatrix} 0 & \dfrac{r_s}{2r(r-r_s)} & 0 \\ \dfrac{r_s}{2r(r-r_s)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}{}_{\mu\nu} = \begin{bmatrix} \dfrac{r_s(r-r_s)}{2r^3} & 0 & 0 \\ 0 & \dfrac{r_s}{2r(r_s-r)} & 0 \\ 0 & 0 & r_s - r \end{bmatrix}$$

$$\Gamma^{\varphi}{}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r^{-1} \\ 0 & r^{-1} & 0 \end{bmatrix}$$

## Geodesic Equations

$$c\frac{d^2t}{d\tau^2} = -c\Gamma^{t}{}_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} - c\Gamma^{t}{}_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

$$\frac{d^2t}{d\tau^2} = \frac{r_s}{r(r_s-r)}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

$$\frac{d^2r}{d\tau^2} = -c^2\Gamma^{r}{}_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau} - \Gamma^{r}{}_{rr}\frac{dr}{d\tau}\frac{dr}{d\tau} - \Gamma^{r}{}_{\varphi\varphi}\frac{d\varphi}{d\tau}\frac{d\varphi}{d\tau}$$

$$\frac{d^2r}{d\tau^2} = \frac{c^2r_s(r_s-r)}{2r^3}\left(\frac{dt}{d\tau}\right)^2 + \frac{r_s}{2r(r-r_s)}\left(\frac{dr}{d\tau}\right)^2 + (r-r_s)\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\frac{d^2\varphi}{d\tau^2} = -\Gamma^{\varphi}{}_{\varphi r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} - \Gamma^{\varphi}{}_{r\varphi}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}$$

$$\frac{d^2\varphi}{d\tau^2} = -\frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}$$

## Hamiltonian

$$\mathcal{H} = \dot{r}\partial_{\dot{r}}\mathcal{L} + \dot{\varphi}\partial_{\dot{\varphi}}\mathcal{L} - \mathcal{L}$$

$$\mathcal{H} = \left(1 - \frac{r_s}{r}\right)\frac{mc^3}{\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}}$$

## Conserved Quantities

$$E = \left(1 - \frac{r_s}{r}\right)\frac{mc^3}{\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}}$$

$$\frac{E}{mc^2} = \left(1 - \frac{r_s}{r}\right)\frac{c}{\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}$$

$$L = \frac{mcr^2\dot{\varphi}}{\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}}$$

$$\frac{L}{m} = r^2\dot{\varphi}\frac{c}{\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} = r^2\frac{d\varphi}{dt}\frac{dt}{d\tau} = r^2\frac{d\varphi}{d\tau}$$

## Timelike Equation of Motion

$$c^2 = c^2\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(1-\frac{r_s}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_s}{r}\right)\left(r^2\left(\frac{d\varphi}{d\tau}\right)^2 + c^2\right)$$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2c^2} - c^2\left(1-\frac{r_s}{r}\right) - \frac{L^2}{m^2r^2} + \frac{r_sL^2}{m^2r^3}$$

## Radial Trajectory

$$L = 0 \Longrightarrow \left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2c^2} - c^2\left(1-\frac{r_s}{r}\right)$$

## Radial Freefall from Infinity

$$\left(\frac{dr}{d\tau}\right)_0 = 0 \Longrightarrow E = mc^2$$

$$\frac{dr}{d\tau} = -c\sqrt{\frac{r_s}{r}}$$

$$\frac{dt}{d\tau} = \left(1-\frac{r_s}{r}\right)^{-1}$$

$$\dot{r} = \frac{d\tau}{dt}\frac{dr}{d\tau} = -c\left(1-\frac{r_s}{r}\right)\sqrt{\frac{r_s}{r}}$$

## Circular Orbit

$$\frac{dr}{d\tau} = 0 \Longrightarrow r = R$$

$$\frac{E^2}{mc^2} = mc^2\left(1 - \frac{r_s}{R}\right) + \frac{L^2}{mR^2} - \frac{r_s L^2}{mR^3}$$

## Effective Potential

$$V_{eff} = \frac{E^2 - m^2c^4}{2mc^2} = \frac{L^2}{2mR^2} - \frac{GMm}{R} - \frac{GML^2}{mc^2R^3}$$

$$\partial_R V_{eff} = \frac{GMm}{R^2} + \frac{3GML^2}{mR^4c^2} - \frac{L^2}{mR^3}$$

$$\partial_R V_{eff} = 0 \Longrightarrow L^2 = \frac{GMm^2c^2R^2}{3GM - c^2R} \Longrightarrow R > \frac{3r_s}{2}$$

$$R < 3r_s \Longrightarrow {\partial_R}^2 V_{eff} < 0$$

$$R \geq 3r_s \Longrightarrow {\partial_R}^2 V_{eff} \geq 0$$

## Circular Orbit Time Dilation

$$0 = c^2\frac{r_s(r_s - r)}{2r^3}\left(\frac{dt}{d\tau}\right)^2 + (r - r_s)\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{c^2 r_s}{2r^3}\left(\frac{dt}{d\tau}\right)^2$$

$$\Rightarrow c^2 = c^2\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - r^2\frac{c^2 r_s}{2r^3}\left(\frac{dt}{d\tau}\right)^2$$

$$\gamma = \frac{dt}{d\tau} = \left(1-\frac{3r_s}{2r}\right)^{-\frac{1}{2}}$$

## Stationary Time Dilation

$$dr = d\varphi = 0 \Longrightarrow r = R$$

$$c^2 d\tau^2 = c^2\left(1-\frac{r_s}{R}\right)dt^2$$

$$\gamma = \frac{dt}{d\tau} = \left(1-\frac{r_s}{R}\right)^{-\frac{1}{2}}$$

## Stationary Proper Acceleration

$$\frac{dr}{d\tau} = \frac{d\varphi}{d\tau} = 0 \Longrightarrow r = R$$

$$A^r = c^2 \Gamma^r{}_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau}$$

$$A^r = \frac{c^2 r_s}{2R^2}$$

$$A = \sqrt{g_{rr}A^r A^r}$$

$$A = \frac{c^2 r_s}{2R^2\sqrt{1-\frac{r_s}{R}}}$$

# Null Geodesics

$$c^2 d\tau^2 = c^2 \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\varphi^2 = 0$$

## Null Metric

$$x^\mu = (r, \varphi)$$

$$c^2 dt^2 = \left(1 - \frac{r_s}{r}\right)^{-2} dr^2 + \left(1 - \frac{r_s}{r}\right)^{-1} r^2 d\varphi^2$$

$$g_{\mu\nu} = \begin{bmatrix} \left(1 - \frac{r_s}{r}\right)^{-2} & 0 \\ 0 & \left(1 - \frac{r_s}{r}\right)^{-1} r^2 \end{bmatrix}$$

## Lightlike Equation of Motion

$$c^2 = \left(1 - \frac{r_s}{r}\right)^{-2} \dot{r}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} r^2 \dot{\varphi}^2$$

## Radial Photon Trajectory

$$\dot{\varphi} = 0$$

$$c^2 = \left(1 - \frac{r_s}{r}\right)^{-2} \dot{r}^2$$

$$\dot{r} = \pm c \left(1 - \frac{r_s}{r}\right)$$

## Christoffel Symbols

$$\Gamma^{r}{}_{\mu\nu} = \begin{bmatrix} \dfrac{r_s}{r(r_s - r)} & 0 \\ 0 & \dfrac{3r_s}{2} - r \end{bmatrix}$$

$$\Gamma^{\varphi}{}_{\mu\nu} = \begin{bmatrix} 0 & \dfrac{2r - 3r_s}{2r(r - r_s)} \\ \dfrac{2r - 3r_s}{2r(r - r_s)} & 0 \end{bmatrix}$$

## Geodesic Equations

$$\ddot{x}^{\mu} = -\Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$$

$$\ddot{r} = -\Gamma^{r}{}_{rr}\dot{r}\dot{r} - \Gamma^{r}{}_{\varphi\varphi}\dot{\varphi}\dot{\varphi}$$

$$\ddot{r} = \frac{r_s\dot{r}^2}{r(r - r_s)} + \left(r - \frac{3r_s}{2}\right)\dot{\varphi}^2$$

$$\ddot{\varphi} = -\Gamma^{\varphi}{}_{\varphi r}\dot{\varphi}\dot{r} - \Gamma^{\varphi}{}_{r\varphi}\dot{r}\dot{\varphi}$$

$$\ddot{\varphi} = \frac{(3r_s - 2r)\dot{r}\dot{\varphi}}{r(r - r_s)}$$

## Photon Sphere

$$\dot{r} = \ddot{r} = 0 \Longrightarrow r = R$$

$$0 = \dot{\varphi}^2\left(R - \frac{3r_s}{2}\right) \Longrightarrow R = \frac{3r_s}{2}$$