Riemannian Geometry

Coordinate Transformation

$$dy^m = \frac{\partial y^m}{\partial x^n} dx^n$$

$$T^{m_1 \dots m_a}_{n_1 \dots n_b}(\mathbf{y}) = \frac{\partial y^{m_1}}{\partial x^{r_1}} \dots \frac{\partial y^{m_a}}{\partial x^{r_a}} \frac{\partial x^{s_1}}{\partial y^{n_1}} \dots \frac{\partial x^{s_b}}{\partial y^{n_b}} T^{r_1 \dots r_a}_{s_1 \dots s_a}(\mathbf{x})$$

Metric Tensor

$$ds^2 = g_{mn}(\mathbf{x})\, dx^m dx^n$$

$$g^{mr} g_{rn} \coloneqq \delta^m_n \Longrightarrow g^{mn} = (g^{-1})_{mn}$$

$$g_{m_1 r_1} \dots g_{m_a r_a} g^{n_1 s_1} \dots g^{n_b s_b} T^{r_1 \dots r_c}_{s_1 \dots s_d} \coloneqq T^{n_1 \dots n_b r_{b+1} \dots r_c}_{m_1 \dots m_a s_{a+1} \dots s_d}$$

Length of a Curve

$$L = \int_C ds = \int_C \sqrt{g_{mn} dx^m dx^n}$$

Local Normal Coordinates

$$\mathbf{x} = \mathbf{X} \rightarrow \boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}$$

$$g_{mn}(\boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}) = \delta_{mn}, \qquad \frac{\partial g_{mn}}{\partial \xi_{\mathbf{X}}{}^r}(\boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}) = 0$$

Covariant Derivative

$$\frac{\partial T^{m_1\dots m_a}_{n_1\dots n_b}}{\partial {\xi_{\mathbf{X}}}^r} \text{ is a tensor at } \boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}$$

$$\Rightarrow \mathcal{D}_r T^{m_1\dots m_a}_{n_1\dots n_b}(\mathbf{x}=\mathbf{X}) \coloneqq \frac{\partial x^{m_1\dots m_a}}{\partial {\xi_{\mathbf{X}}}^{k_1\dots k_a}} \frac{\partial {\xi_{\mathbf{X}}}^{l_1\dots l_b}}{\partial x^{n_1\dots n_b}} \frac{\partial {\xi_{\mathbf{X}}}^s}{\partial x^r} \frac{\partial T^{k_1\dots k_a}_{l_1\dots l_b}}{\partial {\xi_{\mathbf{X}}}^s}(\boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0})$$

$$\frac{\partial T^{m_1\dots m_a}_{n_1\dots n_b}}{\partial {\xi_{\mathbf{X}}}^r}(\boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}) = \frac{\partial}{\partial {\xi_{\mathbf{X}}}^r}\left[\frac{\partial {\xi_{\mathbf{X}}}^{m_1\dots m_a}}{\partial x^{k_1\dots k_a}} \frac{\partial x^{l_1\dots l_b}}{\partial {\xi_{\mathbf{X}}}^{n_1\dots n_b}} T^{k_1\dots k_a}_{l_1\dots l_b}\right](\mathbf{x}=\mathbf{X})$$

$$= \frac{\partial {\xi_{\mathbf{X}}}^{m_1\dots m_a}}{\partial x^{k_1\dots k_a}} \frac{\partial x^{l_1\dots l_b}}{\partial {\xi_{\mathbf{X}}}^{n_1\dots n_b}} \frac{\partial x^s}{\partial {\xi_{\mathbf{X}}}^r} \frac{\partial T^{k_1\dots k_a}_{l_1\dots l_b}}{\partial x^s}(\mathbf{x}=\mathbf{X})$$

$$+ \left[\frac{\partial {\xi_{\mathbf{X}}}^{\dots m_a}}{\partial x^{\dots k_a}} \frac{\partial x^s}{\partial {\xi_{\mathbf{X}}}^r} \frac{\partial^2 {\xi_{\mathbf{X}}}^{m_1}}{\partial x^s \partial x^{k_1}} + \dots + \frac{\partial {\xi_{\mathbf{X}}}^{m_1\dots}}{\partial x^{k_1\dots}} \frac{\partial x^s}{\partial {\xi_{\mathbf{X}}}^r} \frac{\partial^2 {\xi_{\mathbf{X}}}^{m_a}}{\partial x^s \partial x^{k_a}}\right] \frac{\partial x^{l_1\dots l_b}}{\partial {\xi_{\mathbf{X}}}^{n_1\dots n_b}} T^{k_1\dots k_a}_{l_1\dots l_b}(\mathbf{x}=\mathbf{X})$$

$$+ \left[\frac{\partial x^{\dots l_b}}{\partial {\xi_{\mathbf{X}}}^{\dots n_b}} \frac{\partial^2 x^{l_1}}{\partial {\xi_{\mathbf{X}}}^r \partial {\xi_{\mathbf{X}}}^{n_1}} + \dots + \frac{\partial x^{l_1\dots}}{\partial {\xi_{\mathbf{X}}}^{n_1\dots}} \frac{\partial^2 x^{l_b}}{\partial {\xi_{\mathbf{X}}}^r \partial {\xi_{\mathbf{X}}}^{n_b}}\right] \frac{\partial {\xi_{\mathbf{X}}}^{m_1\dots m_a}}{\partial x^{k_1\dots k_a}} T^{k_1\dots k_a}_{l_1\dots l_b}(\mathbf{x}=\mathbf{X})$$

$$\Rightarrow \mathcal{D}_r T^{m_1\dots m_a}_{n_1\dots n_b}(\mathbf{x}=\mathbf{X}) = \frac{\partial T^{m_1\dots m_a}_{n_1\dots n_b}}{\partial x^r}(\mathbf{x}=\mathbf{X})$$

$$+ \left[\frac{\partial x^{m_1}}{\partial {\xi_{\mathbf{X}}}^s} \frac{\partial^2 {\xi_{\mathbf{X}}}^s}{\partial x^r \partial x^t} T^{t\dots m_a}_{n_1\dots n_b} + \dots + \frac{\partial x^{m_a}}{\partial {\xi_{\mathbf{X}}}^s} \frac{\partial^2 {\xi_{\mathbf{X}}}^s}{\partial x^r \partial x^t} T^{m_1\dots t}_{n_1\dots n_b}\right](\mathbf{x}=\mathbf{X})$$

$$+ \left[\frac{\partial {\xi_{\mathbf{X}}}^s}{\partial x^{n_1}} \frac{\partial {\xi_{\mathbf{X}}}^t}{\partial x^r} \frac{\partial^2 x^u}{\partial {\xi_{\mathbf{X}}}^s \partial {\xi_{\mathbf{X}}}^t} T^{m_1\dots m_a}_{u\dots n_b} + \dots + \frac{\partial {\xi_{\mathbf{X}}}^s}{\partial x^{n_b}} \frac{\partial {\xi_{\mathbf{X}}}^t}{\partial x^r} \frac{\partial^2 x^u}{\partial {\xi_{\mathbf{X}}}^s \partial {\xi_{\mathbf{X}}}^t} T^{m_1\dots m_a}_{n_1\dots u}\right](\mathbf{x}=\mathbf{X})$$

$$\Gamma^r{}_{mn}(\mathbf{x}=\mathbf{X}) \coloneqq \left[\frac{\partial x^r}{\partial {\xi_{\mathbf{X}}}^s} \frac{\partial^2 {\xi_{\mathbf{X}}}^s}{\partial x^m \partial x^n}\right](\mathbf{x}=\mathbf{X})$$

$$\delta^m{}_n = \frac{\partial x^m}{\partial x^n} = \frac{\partial y^r}{\partial x^n} \frac{\partial x^m}{\partial y^r} \Longrightarrow \frac{\partial \delta^m{}_n}{\partial x^r} = \frac{\partial x^m}{\partial y^s} \frac{\partial^2 y^s}{\partial x^n \partial x^r} + \frac{\partial y^s}{\partial x^n} \frac{\partial y^t}{\partial x^r} \frac{\partial^2 x^m}{\partial y^s \partial y^t} = 0$$

$$\Rightarrow \left[\frac{\partial {\xi_{\mathbf{X}}}^s}{\partial x^m} \frac{\partial {\xi_{\mathbf{X}}}^t}{\partial x^n} \frac{\partial^2 x^r}{\partial {\xi_{\mathbf{X}}}^s \partial {\xi_{\mathbf{X}}}^t}\right](\mathbf{x}=\mathbf{X}) = -\Gamma^r{}_{mn}(\mathbf{x}=\mathbf{X})$$

$$\Rightarrow \mathcal{D}_r T^{m_1\dots m_a}_{n_1\dots n_b} = \partial_r T^{m_1\dots m_a}_{n_1\dots n_b}$$

$$+\Gamma^{m_1}{}_{rs} T^{s\dots m_a}_{n_1\dots n_b} + \cdots + \Gamma^{m_a}{}_{rs} T^{m_1\dots s}_{n_1\dots n_b}$$

$$-\Gamma^{s}{}_{rn_1} T^{m_1\dots m_a}_{s\dots n_b} - \cdots - \Gamma^{s}{}_{rn_b} T^{m_1\dots m_a}_{n_1\dots s}$$

Christoffel Symbols

$$\mathcal{D}_r g_{mn}(\mathbf{x} = \mathbf{X}) = \frac{\partial {\xi_{\mathbf{X}}}^s}{\partial x^r} \frac{\partial {\xi_{\mathbf{X}}}^k}{\partial x^m} \frac{\partial {\xi_{\mathbf{X}}}^l}{\partial x^n} \frac{\partial g_{kl}}{\partial {\xi_{\mathbf{X}}}^s} (\boldsymbol{\xi}_{\mathbf{X}} = \mathbf{0}) = 0$$

$$\Rightarrow \mathcal{D}_r g_{mn} = \frac{\partial g_{mn}}{\partial x^r} - \Gamma^s{}_{rm} g_{sn} - \Gamma^s{}_{rn} g_{ms} = 0$$

$$\Rightarrow \frac{\partial g_{nr}}{\partial x^m} + \frac{\partial g_{mr}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^r} - 2\Gamma^s{}_{mn} g_{sr} = 0$$

$$\Rightarrow \Gamma^r{}_{mn} = \frac{g^{rs}}{2} \left(\frac{\partial g_{ns}}{\partial x^m} + \frac{\partial g_{ms}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^s} \right)$$

Parallel Transport

$$\mathcal{D} V^m = \mathcal{D}_n V^m dx^n = 0 \Longrightarrow dV^m + \Gamma^m{}_{nr} V^n dx^r = 0$$

Geodesic Equations

$$V^m = dx^m \Longrightarrow d^2 x^m + \Gamma^m{}_{nr} dx^n dx^r = 0$$

$$\Rightarrow \frac{d^2 x^m}{ds^2} + \Gamma^m{}_{nr} \frac{dx^n}{ds} \frac{dx^r}{ds} = 0$$

Ricci Identity

$$\mathcal{D}_r\mathcal{D}_s T^{m_1\dots m_a}_{n_1\dots n_b} - \mathcal{D}_s\mathcal{D}_r T^{m_1\dots m_a}_{n_1\dots n_b}$$

$$= \mathcal{R}^{m_1}{}_{trs} T^{t\dots m_a}_{n_1\dots n_b} + \dots + \mathcal{R}^{m_a}{}_{trs} T^{m_1\dots t}_{n_1\dots n_b}$$

$$-\mathcal{R}^{t}{}_{n_1 rs} T^{m_1\dots m_a}_{t\dots n_b} - \dots - \mathcal{R}^{t}{}_{n_b rs} T^{m_1\dots m_a}_{n_1\dots t}$$

Riemann Tensor

$$\mathcal{R}^r{}_{msn} \coloneqq \frac{\partial \Gamma^r{}_{mn}}{\partial x^s} - \frac{\partial \Gamma^r{}_{ms}}{\partial x^n} + \Gamma^r{}_{st}\Gamma^t{}_{mn} - \Gamma^r{}_{nt}\Gamma^t{}_{ms}$$

Flat Space

$$\mathcal{R}^r{}_{msn} = 0 \Longrightarrow \exists \mathbf{y} \text{ where } g_{mn}(\mathbf{y}) = \delta_{mn}$$

Ricci Tensor

$$R_{mn} \coloneqq \mathcal{R}^r{}_{mrn}$$

Scalar Curvature

$$R \coloneqq g^{mn} R_{mn}$$

Bianchi Identities

$$\mathcal{D}_t \mathcal{R}_{rmsn} + \mathcal{D}_n \mathcal{R}_{rmts} + \mathcal{D}_s \mathcal{R}_{rmnt} = 0$$

$$\mathcal{D}_r \left(R^{mn} - \frac{1}{2} g^{mn} R \right) = \mathcal{D}_r R^{mn} - \frac{1}{2} g^{mn} \mathcal{D}_r R = 0$$