Special Relativity: Uniform Acceleration and the 'SτVAT' Equations

Tom H. Dodd
Shrewsbury School, The Schools, Shrewsbury, SY3 7BA
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This paper attempts to construct a set of relativistic 'SUVAT' equations which obey the theory of special relativity, rather than the old set of equations which were only valid in the realm of Newtonian mechanics. The final equations that are derived do not take into account initial position or velocity, but do present the opportunity to calculate the time dilation experienced by an accelerating body relative to an observer in an inertial reference frame, as well as the difference between the constant proper acceleration experienced in the accelerating reference frame and the observed acceleration.

1. Introduction

In Newtonian mechanics, a particle moving with uniform acceleration in some frame of reference $\mathcal{R}$ means that the acceleration, ***a***, of the particle measured in that frame is constant. Integrating

$$\boldsymbol{a}(t) = \boldsymbol{a}$$

gives the equation for the velocity, ***v***, of the particle at the time *t*:

$$\boldsymbol{v}(t) = \boldsymbol{u} + \boldsymbol{a}t$$

where ***u*** is the initial velocity of the particle. Integrating once more gives the equation for the position, ***x***, of the particle at the time *t*:

$$\boldsymbol{x}(t) = \boldsymbol{x_0} + \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}t^2$$

where $\boldsymbol{x_0}$ is the initial position of the particle.

If $\boldsymbol{x_0}$ and **u** are both taken to be 0, five simplified versions of the 'SUVAT' equations can be derived:

$$\boldsymbol{v} = \boldsymbol{a}t$$

$$\boldsymbol{s} = \frac{1}{2}\boldsymbol{a}t^2$$

$$\boldsymbol{s} = \frac{1}{2}\boldsymbol{v}t$$

$$\boldsymbol{v}^2 = 2\boldsymbol{as}$$

$$\boldsymbol{s} = \boldsymbol{v}t - \frac{1}{2}\boldsymbol{a}t^2$$

where ***x*** has been replaced by ***s***.

The question is: can one find a set of equations similar to the ones above, but which agree with special relativity?

2. Deriving the 'SτVAT' Equations

Starting with the Lorentz transformations of position and time:

$$\boldsymbol{x}' = \gamma(\boldsymbol{x} - \boldsymbol{v}t)$$

$$t' = \gamma\left(t - \frac{\boldsymbol{vx}}{c^2}\right)$$

where *c* is the speed of light and *γ* is the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}}$$

One can form the equations

$$d\boldsymbol{x}' = \gamma(d\boldsymbol{x} - \boldsymbol{v}dt)$$

$$dt' = \gamma\left(dt - \frac{\boldsymbol{v}d\boldsymbol{x}}{c^2}\right)$$

and use them to find the equation for the relativistic transformation of velocity:

$$\frac{d\boldsymbol{x}'}{dt'} = \boldsymbol{u}' = \frac{\boldsymbol{u} - \boldsymbol{v}}{1 - \frac{\boldsymbol{uv}}{c^2}}$$

One can find the derivative of ***u'*** with respect to *t'*, finding the transformation of acceleration to be:

$$\frac{d\boldsymbol{u}'}{dt'} = \boldsymbol{a}' = \frac{\boldsymbol{a}}{\gamma^3\left(1 - \frac{\boldsymbol{uv}}{c^2}\right)^3}$$

As these equations consider only one spatial dimension, one can allow $|\boldsymbol{v}|^2$ to be equal to $\boldsymbol{v}^2$. If we allow $\boldsymbol{u}$ to be equal to $\boldsymbol{v}$, we can find the proper acceleration, $\boldsymbol{\alpha}$, or the experienced acceleration of a particle observed to have acceleration $\boldsymbol{a}$ and velocity $\boldsymbol{v}$:

$$\boldsymbol{\alpha} = \gamma^3 \boldsymbol{a} = \gamma^3 \frac{d\boldsymbol{v}}{dt}$$

Rearranging and integrating, we find

$$\boldsymbol{\alpha} t = \frac{\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} = \gamma \boldsymbol{v}$$

Rearranging once more, we find the velocity, $\boldsymbol{v}$, at the time t:

$$\boldsymbol{v} = \frac{\boldsymbol{\alpha} t}{\sqrt{1 + \frac{\boldsymbol{\alpha}^2 t^2}{c^2}}}$$

This is the first of the relativistic 'SUVAT' equations. Taking $\boldsymbol{u}$, or the initial velocity, to be 0, the constant of integration is removed. Integrating again gives us the position, $\boldsymbol{s}$, at the time t:

$$\boldsymbol{s} = \frac{c^2}{\boldsymbol{\alpha}} \sqrt{1 + \frac{\boldsymbol{\alpha}^2 t^2}{c^2}} - \frac{c^2}{\boldsymbol{\alpha}}$$

Since $\boldsymbol{x_0}$, or the initial position, is assumed to be equal to 0, the constant of integration is equal to $c^2/\boldsymbol{\alpha}$ so that the position of the particle at time t=0 is equal to 0. A more useful form of the equation can be found by rearranging the equation to make t the subject:

$$t = \frac{\sqrt{\boldsymbol{s}}\sqrt{2c^2 + \boldsymbol{\alpha s}}}{c\sqrt{\boldsymbol{\alpha}}}$$

One can now also find the proper time, τ, or the time elapsed in the accelerating frame of reference. Making use of another equation involving the Lorentz factor

$$\gamma = \frac{dt}{d\tau}$$

one can rewrite the equation relating $\boldsymbol{\alpha}$ and $\boldsymbol{a}$:

$$\boldsymbol{\alpha} = \gamma^3 \frac{d\boldsymbol{v}}{dt} = \gamma^2 \frac{d\boldsymbol{v}}{d\tau}$$

$$\boldsymbol{\alpha} d\tau = \frac{d\boldsymbol{v}}{1 - \frac{\boldsymbol{v}^2}{c^2}}$$

Integrating this equation and then rearranging the result yields

$$\tau = \frac{c}{\boldsymbol{\alpha}} tanh^{-1}\left(\frac{\boldsymbol{v}}{c}\right)$$

where the constant of integration is again ignored so that τ=0 when $\boldsymbol{v}$=0.

At this stage, all of the fundamental 'SτVAT' (pronounced *stow-vat*) equations have been found. Thus it has been found that it is possible, to some extent, that the relativistic variety of the Newtonian 'SUVAT' equations can be derived. This set of equations is most useful when attempting to find the time dilation that occurs, or the difference between the coordinate time, t, and the proper time, τ, when a journey of some distance, $\boldsymbol{s}$, is made with constant acceleration in the accelerating frame, $\boldsymbol{\alpha}$. The observed acceleration, $\boldsymbol{a}$, can be found after finding the final velocity, $\boldsymbol{v}$, and using the equation relating $\boldsymbol{a}$ and $\boldsymbol{\alpha}$.

For example, a journey to the Andromeda galaxy, $2.4 * 10^{22}$ meters away, starting from rest relative to an observer on Earth, where the constant acceleration in the accelerating frame, $\boldsymbol{\alpha}$, is equal to g, the acceleration due to Earth's gravity, would appear to take $8.0 * 10^{13}$ seconds, or just over 2.5 million years, to complete to the observer, while the time experienced by the traveller would only be $4.7 * 10^8$ seconds, or fifteen years. The final acceleration observed back on Earth would be only $5.4 * 10^{-19}$ ms^{-2}, as the final velocity of the traveller would be very close to the speed of light.

Rearranging and mixing these equations may lead to forming a larger set of 'SτVAT' equations, but the four derived in this paper are the ones which could be considered the most useful.