General Relativity

Spacetime Metric

$$c^2 d\tau^2 = -ds^2 \Longrightarrow c^2 d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu$$

$$d\tau^2 \neq 0 \Longrightarrow c^2 = -g_{\mu\nu} u^\mu u^\nu$$

Time-like Geodesics

$$\frac{d^2 x^\mu}{d\tau^2} = A^\mu - \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

Null Geodesics

$$d\tau^2 = 0 \Longrightarrow g_{\mu\nu} dx^\mu dx^\nu = 0$$

$$\Rightarrow \frac{d^2 x^\mu}{d\lambda^2} = \tilde{A}^\mu - \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

Newtonian Limit

$$\frac{d^2 x^i}{dt^2} \approx A^i - c^2 \Gamma^i{}_{tt} \approx A^i + \frac{c^2}{2} \partial_i g_{tt}$$

Time Dilation

$$\gamma = \frac{dt}{d\tau} = \frac{c}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}$$

Lagrangian

$$\mathcal{L} = -mc^2 \frac{d\tau}{dt} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Hamiltonian

$$\mathcal{H} = mc^2 g_{tt} \frac{dt}{d\tau} = \frac{mc^3 g_{tt}}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}$$

Field Equations

$$\mathcal{D}_\mu R^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \mathcal{D}_\nu R, \qquad \mathcal{D}_\mu g^{\mu\nu} = 0, \qquad \mathcal{D}_\mu T^{\mu\nu} = 0$$

$$\Rightarrow R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\Rightarrow 4\Lambda - \mathrm{R} = \kappa T \Longrightarrow R_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right)$$

$$\nabla^2 \varphi = 4\pi G\rho, \qquad g_{tt} \approx -\left(1 + \frac{2\varphi}{c^2}\right), \qquad T_{tt} \approx \rho c^2$$

$$R_{tt} \approx \partial_i {\Gamma^i}_{tt} \approx -\frac{1}{2} \nabla^2 g_{tt} \approx \frac{1}{c^2} \nabla^2 \varphi = \frac{4\pi G\rho}{c^2}$$

$$T \approx g^{tt} T_{tt} \Longrightarrow T_{tt} - \frac{T}{2} g_{tt} \approx \frac{1}{2} T_{tt} \approx \frac{1}{2} \rho c^2 \Longrightarrow \kappa = \frac{8\pi G}{c^4}$$

$$\Rightarrow R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Vacuum Equations

$$T_{\mu\nu} = 0 \Longrightarrow T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} = 0$$

$$\Rightarrow R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Einstein-Hilbert Action

$$S = \int \left[\frac{c^4}{16\pi G} (R - 2\Lambda) + \mathcal{L}_m \right] \sqrt{|\det g|}\, d^4x$$

Weak Field Approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \left| h_{\mu\nu} \right| \ll 1 \Longrightarrow g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$$

$$\Gamma^{\mu}{}_{\alpha\beta} \approx \frac{\eta^{\mu\nu}}{2} \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta} \right)$$

$$R_{\mu\nu} \approx \frac{1}{2} \left[\partial^\sigma \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} \right) - \eta^{\sigma\rho} \partial_\mu \partial_\nu h_{\sigma\rho} \right]$$

$$R \approx \partial^\mu \left(\partial^\nu h_{\mu\nu} - \eta^{\nu\sigma} \partial_\mu h_{\nu\sigma} \right)$$

$$\Rightarrow \frac{1}{2} \left[\partial^\sigma \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} \right) - \eta^{\sigma\rho} \partial_\mu \partial_\nu h_{\sigma\rho} \right]$$

$$- \frac{1}{2} \eta_{\mu\nu} \partial^\sigma \left(\partial^\rho h_{\sigma\rho} - \eta^{\rho\lambda} \partial_\sigma h_{\rho\lambda} \right) + \Lambda \left(\eta_{\mu\nu} + h_{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Gravitational Waves

$$T_{\mu\nu} = \Lambda = 0$$

$$\Rightarrow \partial^{\sigma}\left(\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\mu\sigma} - \partial_{\sigma} h_{\mu\nu}\right) - \eta^{\sigma\rho}\partial_{\mu}\partial_{\nu} h_{\sigma\rho} = 0$$

$$\Rightarrow h_{ij} = h_{ij}^{0} \sin[k(x_k \pm ct) + \phi]$$

$$h_{mt} = h_{mk} = 0, \qquad h_{ii} + h_{jj} = 0$$

Electromagnetic Stress-Energy Tensor

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

$$T_{\mu\nu} = -\frac{1}{\mu_0}\left(F_{\mu\sigma} g^{\sigma\rho} F_{\rho\nu} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} g^{\rho\lambda} F_{\lambda\theta} g^{\theta\sigma}\right)$$