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Valuation of Commodity-Based Swing Options

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In the energy markets, in particular the electricity and natural gas markets, many contracts incorporate flexibility-of-delivery options known as "swing" or "take-or-pay" options. Subject to daily as well as periodic constraints, these contracts permit the option holder to repeatedly exercise the right to receive greater or smaller amounts of energy. We extract market information from forward prices and volatilities and build a pricing framework for swing options based on a one-factor mean-reverting stochastic process for energy prices that explicitly incorporates seasonal effects. We present a numerical scheme for the valuation of swing options calibrated for the case of natural gas.

Key words: energy prices; seasonality; one-factor model; numerical valuations; dynamic programming; binomial forest
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1. Introduction

Due to the complex patterns of consumption and the limited storability of energy, many contracts in the energy markets have been designed to allow flexibility of delivery with respect to both the timing and the amount of energy used. Under a regulated environment, pricing such contracts has not been an issue because prices were set by regulators under the assumption of cost recovery, meaning that if the set price turned out to favor either the producer or the consumer, future prices were adjusted to compensate for the over- or underpayment. With the transition to a deregulated environment such compensation will no longer be possible, and contracts will need to be priced according to their financial risks. Historically, the contracts that have allowed the most flexibility, and consequently are the most complex, have been known as "swing" or "take-or-pay," and have occasionally been called "variable volume" or "variable take." Providing their owner with flexibility-of-delivery options, swings permit the option holder to repeatedly exercise the right to receive greater or smaller amounts of energy, subject to daily as well as periodic (monthly or semiannual) constraints. Due to their nonstandard nature, these options are indeed "exotic," but what renders them particularly interesting is that they have a natural raison d'être in the marketplace: They address the need to hedge in a market subject to frequent, but not pervasive, price- and demand-spiking behavior that is typically followed by reversion to normal levels.[1]

In this paper, we develop a framework for the pricing of swing options in the context of a one-factor, seasonal, mean-reverting stochastic process for the underlying commodity price, from the point of view of a profit-maximizing agent. Such an agent is not legally or physically precluded from selling excess amounts he or she cannot consume. As a result, any exercise amount is chosen solely for economic reasons. We also calibrate the seasonal, mean-reverting

[1] Consider, for example, a risk-averse economic agent who is short of energy in a typical 22-business-days summer month. Such an agent would be concerned with energy prices spiking on multiple days in the month, should hot temperatures prevail. Full protection can be attained by a strip of 22 daily European options, but that constitutes excessive protection, as the likelihood of such numerous hot days is small. Acquiring the option to exercise on 10 of those 22 days might be sufficient protection. The agent can buy 10 identical American options whose exercise period covers the 22-business-days summer month. However, the agent would still overpay for his/her desired protection. These American options have the same optimal exercise time, but the agent, either not able to exercise all the options on the same day (because of supply constraints) or not willing to do so, would pay a premium it cannot recover. A swing with 10 rights is the perfect hedging instrument.

model for the stochastic process describing the underlying commodity price for the case of natural gas using observed market prices for futures and options contracts, we implement the numerical scheme for pricing swing options, and we provide numerical examples.

Descriptions of swing options, as well as other options traded in the energy markets, have attracted a lot of interest from participants in the energy markets. Joskow (1985) examines specific coal contracts and shows that most have take-or-pay provisions. Joskow (1987) looks at more general coal contracts and notes that they usually include delivery schedules with minimum and maximum production and take obligations. Kaminski and Gibner (1995) provide descriptions of several exotic options traded in the energy markets. Barbieri and Garman (1996) and Garman and Barbieri (1997) focus on swing options and describe several variants, but without discussing how to value them in an efficient manner. Thompson (1995) considers special cases of take-or-pay contracts and, for these specific structures, extends a lattice-based valuation approach introduced by Hull and White (1993). Pilipovic and Wengler (1998) also discuss special cases of swing options that can be solved with simple procedures. The main contribution of our paper is to provide an efficient valuation framework for the most general case of a swing option, as well as to propose and calibrate a stochastic process appropriate for energy prices.

Swing options and their variants have a potentially wide array of application. For example, a variant of swing options, called a flexi-option, has been used in interest rate risk management. Other applications of swing options include the valuation of storage facilities and the option to repeatedly shut down services. Common options in supply chain management can also be thought of as swing options. Anupindi and Bassok (1999) discuss multiperiod supply contracts with different degrees of flexibility under uncertain demand that is independent and identically distributed across periods. The valuation framework we present in this paper also applies in this situation, and allows generalizations along the directions of state-dependent demand uncertainty and restrictions in the total quantities supplied over multiple periods.

This paper is organized as follows: Section 2 provides a definition of swing options, discusses several of their properties, and introduces a dynamic programming framework for their valuation. Section 3 describes a one-factor, seasonal, mean-reverting model for the spot price of the underlying commodity, introduces a pricing framework for futures and European options on futures, and provides empirical calibration results for the case of natural gas. Section 4 concentrates on the valuation of swing options under the one-factor model, describes the numerical scheme, and provides numerical examples. Section 5 concludes.

2. The Swing Options

2.1. Definitions

A swing contract is often bundled together with a standard base-load forward contract that specifies, for a given period and a predetermined price, the amount of the commodity to be delivered over that period. The swing portion allows flexibility in the delivery amount around the amount of the base-load contract.

There are many types of swing options, but they all share a few common characteristics. If 0 is the time when the contract is written, the option takes effect during a period $[T_1, T_2]$, $0 \leq T_1 < T_2$. This period usually coincides with the period for the base-load contract. Within this period, the swing entitles the owner to exercise up to N rights. These rights can have different meanings leading to different variants of swings. In all cases, a right can be exercised only at a discrete set of dates $\{\tau_1, \dots, \tau_n\}$ with $T_1 \leq \tau_1 < \tau_2 < \cdots < \tau_n \leq T_2$, with at most one right exercised on any given date. Moreover, if a right is exercised on a given date, there is a "refraction" time Δt_R, which limits the next time a right can be exercised. If $\Delta t_R \leq \min_{1\leq j\leq n-1}(\tau_{j+1} - \tau_j)$, then this restriction is redundant; otherwise, this refraction constraint would need to be included in the contract.

The two main categories of contracts depend on the duration of the effect associated with the exercise of a right.

Local effect: The exercise of a right modifies the delivery volume only on the date of exercise; i.e., the delivery reverts to the level specified in the base-load contract thereafter.

Global effect: The exercise of a right modifies the delivery volume beginning on the exercise date—i.e., the delivery remains at the new level until the next exercise, if any.

In the remainder of this paper we will concentrate on the first category of contracts. The pricing of contracts in the second category is similar, but contains enough different subtleties to warrant separate treatment. From now on, when we refer to swing contracts we refer to flexible contracts of the first category.

As indicated before, there exist many different variants, depending on the exact specifications of the rights. We assume that each right, if exercised on a given date, allows the holder of the swing contract to choose an incremental volume that may be positive or negative. When positive, the holder receives an increased amount of the underlying commodity while, when negative, the holder delivers that amount or, equivalently, decreases the base-load volume. In addition, in case of an exercise at date τ_j, $1 \le j \le n$, physical constraints restrict the chosen incremental volume to take values in the following intervals:

$$[l_j^1, l_j^2) \cup (l_j^3, l_j^4],$$

where the bounds are specified in the contract and are such that $l_j^1 \le l_j^2 \le 0 \le l_j^3 \le l_j^4$.

The total volume delivered over $[T_1, T_2]$ via the swing contract is typically restricted between bounds specified in the contract. Violation of this overall constraint might be allowed, but would lead to penalties settled at expiration (either a one-time penalty or a per-unit violation penalty). The penalty could be predetermined at the initialization of the contract or depend on the value of a random variable observable at expiration T_2 (such as the spot price at expiration, or the maximum spot price over $[T_1, T_2]$, or the average spot price over this period).

All these various possibilities can be captured in the contract by the specification of a general penalty function φ, where $\varphi(V)$ is the total penalty cost to be paid by the holder of the contract at time T_2 for a total demand of V units over $[T_1, T_2]$. For example, for a contract that specifies that the total volume delivered by the swing needs to be in the interval [Min, Max], with a fixed penalty of C_1 dollars if below Min, and a per unit penalty of P_{T_2} (the unit spot price of the underlying commodity at time T_2) if above Max, the function φ is defined by

$$\varphi(V) = \begin{cases} C_1 & \text{if } V < \text{Min}, \\ 0 & \text{if Min} \le V \le \text{Max}, \\ P_{T_2}(V - \text{Max}) & \text{if } V > \text{Max}. \end{cases}$$

For another example, assume that the contract specifies that the total volume delivered by the swing has to be in the interval [Min, Max], and that this is an absolute constraint. Then, the function φ would be defined as

$$\varphi(V) = \begin{cases} \infty & \text{if } V < \text{Min}, \\ 0 & \text{if Min} \le V \le \text{Max}, \\ \infty & \text{if } V > \text{Max}. \end{cases}$$

To complete the description of the swing, one needs to specify a "strike" price at which one unit of commodity will be exchanged at the time of the exercise of a right. There are many possibilities: One could use a predetermined strike price K, fixed at the initialization of the contract; or one could use a strike price observable at a future date (e.g., the commodity spot or T_2-futures price at time T_1); or variable strike prices either known at the initialization of the contract or observable at future dates.

2.2. Mathematical Description of the Standard Swing Option

The main input parameters associated with a standard swing contract are:

- Time at which the contract is written and priced: 0.
- Consumption interval: $[T_1, T_2]$.
- Possible exercise dates: $\{\tau_1, \dots, \tau_n\} \in [T_1, T_2]$.
- Number of rights: $N \le n$.
- Refraction period: Δt_R.
- Volume constraints at τ_j: $[l_j^1, l_j^2) \cup (l_j^3, l_j^4]$, with $l_j^1 \le l_j^2 \le 0 \le l_j^3 \le l_j^4$.
- Penalty function, depending on the total demand over $[T_1, T_2]$: φ.
- Strike price K, or term structure of strike prices K_t, $t \in \{\tau_1, \dots, \tau_n\}$.

For $1 \le j \le n$, define the exercise decision variables as follows:

$$\chi_j^+ = \begin{cases} 1 & \text{if the holder of the swing contract} \\ & \quad \text{exercises for more volume on date } \tau_j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_j^- = \begin{cases} 1 & \text{if the holder of the swing contract} \\ & \quad \text{exercises for less volume on date } \tau_j, \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding volume decisions

$$V_j^+ = \begin{cases} \text{incremental volume bought} & \text{if } \chi_j^+ = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_j^- = \begin{cases} \text{incremental volume sold} & \text{if } \chi_j^- = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following set of equations provides a precise mathematical description of the constraints associated with a standard swing option:

$$0 \le \chi_j^+ + \chi_j^- \le 1 \quad \text{for all } 1 \le j \le n,$$

$$(\chi_i^+ + \chi_i^-) + (\chi_j^+ + \chi_j^-) \le 1 + \frac{\tau_j}{\tau_i + \Delta t_R} \quad \text{for all } 1 \le i < j \le n,$$

$$0 \le \sum_{i=j}^{n} (\chi_j^+ + \chi_j^-) \le N,$$

$$l_j^3 \chi_j^+ \le V_j^+ \le l_j^4 \chi_j^+ \quad \text{for all } 1 \le j \le n,$$

$$l_j^1 \chi_j^- \le V_j^- \le l_j^2 \chi_j^- \quad \text{for all } 1 \le j \le n.$$

2.3. Properties of Swing Options

There are several properties of swing options independent of the stochastic model for the price of the underlying commodity. Let us first focus on a simple standard swing contract.

2.3.1. A Simple Swing Contract. Following the specifications of §2.2, consider a simple case with N rights, each giving the option of buying one extra unit of commodity at strike price K, with no overall constraints on the total number of extra units bought over $[T_1, T_2]$.

1. For $N = 1$ (one exercise right), the value of the swing option equals that of a conventional American-style call option (more precisely a Bermudan option because of the restriction of the exercise space to a set of discrete dates).

2. An upper bound to the value of the swing option with N exercise rights is given by N identical Bermudan options. While the Bermudan options could (and optimally would) be exercised simultaneously, the swing option permits the exercise of only one right on each exercise date and imposes a refraction period as well.

3. A lower bound to the value of the swing option is given by the maximum value of a strip of N European options covering the same length of time and amount, where the maximum is taken over all possible sets of N distinct exercise dates. This lower bound corresponds to the best set of *predetermined* exercise dates, whereas the swing's exercise dates cover the entire time range.

4. For the case where $N = n$, i.e., when the number of rights is equal to the number of exercise dates, the value of the swing option is equal to the value of a strip of European options.

5. Without any penalty for overall consumption, the swing will be exercised in "bang-bang" fashion, i.e., either at the highest or lowest level allowed by the local constraint.

2.3.2. General Properties. The properties discussed above for the case of the simple swing do not hold in general. When the penalty function is nonzero, there is no obvious correspondence between the value of European or American/Bermudan options and the value of the swing. Moreover, it is no longer necessarily true that swings are exercised in bang-bang fashion. However, the following properties hold:

1. Under the assumption that the stochastic process for the price of the underlying commodity exhibits constant returns to scale, and that the penalties are of the unit type,[2] the value of the swing option is homogeneous of degree one in prices and penalties:

$$f(cP_t, cK, c\varphi) = cf(P_t, K, \varphi), \quad c > 0,$$

where f is the value of the swing. To show this, note that the value is obviously homogeneous when one uses the same exercise policy under both scales. The result then follows from the fact that one can use the same *optimal* exercise policies under both scales.

2. The value of the option is homogeneous of degree one in quantities:

$$f(c \cdot \min, c \cdot \max, c \cdot \text{Max}, c \cdot \text{Min}) = cf(\min, \max, \text{Max}, \text{Min}), \quad c > 0.$$

The argument for the validity of this property is quite similar to the one given in Property 1. One simply has to argue that an optimal exercise policy under one scale can be rescaled to become an optimal exercise policy under the other scale.

These two general properties significantly reduce the computations for swing prices, as one can work in one scale of prices and quantities and imply swing prices for all other scales.

2.4. Valuing Swing Options via Dynamic Programming

The complexities of swing options—specifically, the constraints explained in §2.2—require a modification of the dynamic programming techniques used to price American-style options. Whereas an American option can be exercised only once, a swing option has multiple exercise rights, and it also has constraints on total volume delivered. Apart from the underlying spot price, the following two state variables are necessary to price a swing: number of exercise rights left and usage level so far. Assuming appropriate discretization of the usage-level variable, swing options can be priced through a binomial/trinomial *forest*—a multiple-layer tree extension of the traditional binomial/trinomial tree dynamic programming approach.[3]

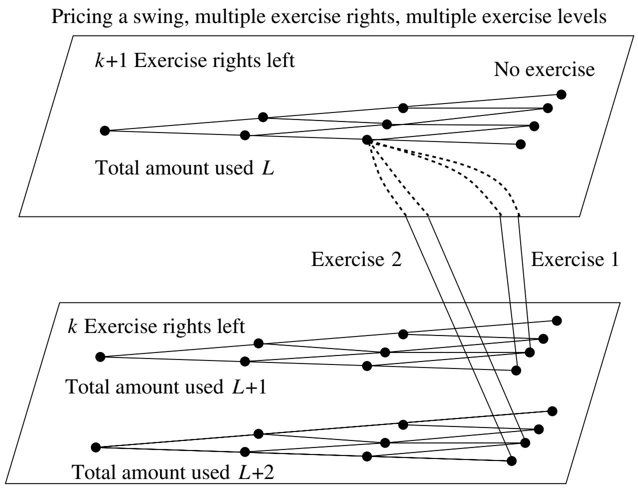
The intuition behind the valuation of swings is as follows. The procedure starts from the option's expiration date and works backward in time to value the instrument using "backward induction" in three dimensions: Price, Number of Exercise Rights Left, and Usage Level.[4] At each date the possibility of an exercise is considered by taking the maximum value over staying in the current tree, i.e., not exercising a swing right, or jumping down to the tree with the

[2] Under unit penalties, either a fixed amount per unit, or an amount that is linear with respect to the final underlying commodity price, is paid for each unit in excess of, or deficient to, the overall limits.

[3] In option pricing, dynamic programming with additional state variables has been used in the case of pricing American lookback options (Hull and White 1993), "shout" options (Cheuk and Vorst 1997), as well as swings with the number of possible exercise dates equal to the number of exercise rights (Thompson 1995).

[4] The Price "dimension" may be represented by more than one state variable. Such a situation arises, for example, when the price process depends on multiple random factors. While the case with multiple random factors is conceptually similar to the one we discuss, the computational burden increases with the additional random factors.

Figure 1 **Connection Between the Level with $k+1$ Exercise Rights Left and the Level with k Exercise Rights Left in the Swing Forest**



next lower number of exercises left and appropriate usage level. If k rights of the swing have been exercised, then the exercise of an additional right for an amount A would leave the swing holder with the value of the immediate exercise plus a forward starting swing, after the refraction period Δt_R, with $k+1$ rights exercised (or $N-k-1$ rights left) and an amount already used augmented by A. The concept is described graphically in Figure 1.[5]

To numerically price a swing option, we discretize the usage amount delivered in each possible exercise date.[6] Assuming that the owner of the swing can only choose among, at most, L different usage amounts each time a swing right is exercised, where the amount used is one of L consecutive integer multiples of a minimum usage amount, the number of possible usage levels after exercising k swing rights is $k\times(L-1)+1$. For each possible combination of rights left and usage level, we construct a tree based on the stochastic process for the price of the underlying asset. The total number of trees necessary to price a swing with N exercise rights is then given by:

$$\begin{aligned}\text{total number of trees} &= \sum_{k=0}^{N}[k(L-1)+1]\\ &= \frac{(L-1)(N+1)(N+2)}{2}\approx\frac{N^2L}{2},\end{aligned}$$

where $\approx$ signifies the asymptotic limit for large values of N, L. If for each possible exercise date the number of nodes associated with the underlying spot price is less than or equal to J, each tree has no more than $n\times J$ decision nodes, where n is the number of possible exercise dates. At each decision node we need to compare the value of the swing under all possible decisions, i.e., the L possible exercise amounts plus the possibility of not exercising, by computing expected values in respective trees. The total number of computations of expected values is then equal to the number of trees times the number of decision nodes per tree times the number of comparisons per decision node, and is $\approx nJN^2L^2/2$.

The computer memory necessary if one wants to keep in memory all the trees is, at most, a multiple of the number of trees times the number of decision nodes, which is $\approx nJN^2L/2$. However, there are considerable savings possible, because only two levels corresponding to different numbers of exercise rights left need to be in memory at any time. This reduces the memory requirements to $\approx 2nJNL$. Further savings in computer memory are possible with the caveat that more computations may be necessary.

In the appendix, we discuss the convergence of the swing price as the time interval between nodes tends to zero.

3. A One-Factor Model for Energy Prices

Before presenting numerical examples of pricing swing options, we offer a model for natural gas prices that we calibrate to data. The model describes the behavior of an underlying spot price P_t through a one-factor mean-reverting stochastic process, and is an extension of models discussed in Schwartz (1997) and Schwartz and Smith (2000).[7] We formulate the stochastic process directly under a given market-defined martingale probability measure Q. This measure is such that all tradable instruments, such as futures, forwards and options, have prices that are described by stochastic processes that, when discounted, are martingales under Q. We do not assume that the spot price P_t corresponds to a tradable instrument (nor do we assume that it is observable), so

[5] The figure implicitly assumes that the refraction period equals the period between possible exercise dates. To deal with situations where the refraction period is greater, one would need to introduce an additional state variable that keeps track of the time left until the option holder is allowed to exercise another swing right.

[6] The discretization does not imply that that the optimal exercise amount is limited to the discrete amounts prescribed by the step-size in the discretization scheme. One could interpolate among the swing values for different usage levels to calculate the optimal exercise amount.

[7] See also the works of Pilipovic (1997), Barz (1998), and Deng (1999, 2000). Manoliu and Tompaidis (2002) provide an extension to a multifactor model for energy prices. While for spot electricity prices a one-factor mean-reverting model may be inadequate due to the existence of large in magnitude and short in duration price spikes, a one-factor mean-reverting diffusion model is plausible for the price of monthly futures contracts for natural gas. We also point out that for our dataset, discussed in §3.3, there are very small differences in the performance of calibrated one-factor and two-factor models, due to the lack of long-term options data.

its discounted price process will not necessarily be a martingale under Q. Nevertheless, in an abuse of notation, we will refer to Q as the risk-neutral measure in the remainder of this paper.

The intuition behind having a nontradable spot instrument lies in the limited storability of energy. An amount of natural gas or electricity delivered at one time is not equivalent to the same amount delivered at another time. Absence of asset substitution across time appears in other commodities as well, and has been modeled in the literature by a derived quantity, the "convenience yield," as discussed in Gibson and Schwartz (1990). We do not introduce a convenience yield, but it is easy to see that the process for the underlying spot price could be transformed into a martingale under Q with the addition of a convenience yield term. This possibility suggests that convenience yields can be understood in terms of limited asset substitution across time.

3.1. Formulation

Let P_t denote the spot price at time t. An example for P_t is the value of a unit of energy delivered a fixed time after time t, e.g., the following hour or day.

We describe P_t by the product of a deterministic seasonality factor f_t and a random factor describing the deseasonalized spot price D_t:

$$P_t = f_t D_t. \tag{1}$$

The period of the seasonal pattern in the spot price can be set to unity without loss of generality; i.e., $f_{t+1} = f_t$. To avoid redundancy we impose a normalization condition

$$\int_0^1 \ln f_t \, dt = 0. \tag{2}$$

We assume that the logarithm of the deseasonalized spot price $X_t = \ln D_t$ reverts to a long-term average level ξ, according to an Ornstein-Uhlenbeck process:

$$dX_t = \kappa(\xi - X_t)\, dt + \sigma_X \, dZ_t, \tag{3}$$

where $(Z_t)_t$ is a standard Brownian motion under the risk-neutral measure Q. The mean-reversion rate κ and instantaneous volatility σ_X are assumed constant.

Given information at time 0, the random variable X_t is normally distributed under the risk-neutral measure with mean

$$E_Q(X_t \mid X_0) = e^{-\kappa t} X_0 + \xi(1 - e^{-\kappa t})$$

and variance

$$\mathrm{Var}_Q(X_t \mid X_0) = (1 - e^{-2\kappa t}) \frac{\sigma_X^2}{2\kappa}.$$

Accordingly, the deseasonalized spot price and the spot price are lognormally distributed with mean

$$\begin{aligned} E_Q(D_t \mid X_0) &= \exp\{E_Q(X_t \mid X_0) + \tfrac{1}{2}\mathrm{Var}_Q(X_t \mid X_0)\} \\ &= \exp\left\{ e^{-\kappa t} X_0 + \xi(1 - e^{-\kappa t}) + \tfrac{1}{2}(1 - e^{-2\kappa t}) \frac{\sigma_X^2}{2\kappa} \right\}, \\ E_Q(P_t \mid X_0) &= f_t E_Q(D_t \mid X_0). \end{aligned}$$

3.2. Valuation of Futures and Options on Futures

Under the assumption that interest rates depend deterministically on time, futures prices are equal to forward prices, and denoting by $F(t, T)$ the price at time t for a forward contract that matures at time T, we have

$$F(t, T) = E_Q(P_T \mid \mathcal{F}_t) \quad \text{for } t \le T, \tag{4}$$

where $\mathcal{F}_t$ represents all the information available up to time t. Under the one-factor model we have

$$\begin{aligned} \ln[F(t, T)] &= \ln[E_Q(P_T \mid \mathcal{F}_t)] \\ &= \ln f_T + E_Q(X_T \mid \mathcal{F}_t) + \tfrac{1}{2}\mathrm{Var}_Q(X_T \mid \mathcal{F}_t) \\ &= \ln f_T + e^{-\kappa(T-t)} X_t + \xi(1 - e^{-\kappa(T-t)}) \\ &\quad + \frac{\sigma_X^2}{4\kappa}[1 - e^{-2\kappa(T-t)}]. \end{aligned} \tag{5}$$

Using Equations (3) and (5) and applying Itô's lemma, the futures price follows the stochastic process[8]

$$dF(t, T) = F(t, T)\sigma_X e^{-\kappa(T-t)} \, dZ_t. \tag{6}$$

From Equations (4) and (6), it is clear that the futures price $F(t, T)$ is a martingale under the risk-neutral measure Q.

To value European options on futures, we can exploit the fact that the futures price $F(t, T)$ is lognormally distributed. The price C_0, at time 0, of a European call with expiration at time t and strike K on a futures contract that matures at time T is given by Black's formula:

$$\begin{aligned} C_0 &= e^{-rt} E_Q([F(t, T) - K]_+ \mid \mathcal{F}_0) \\ &= e^{-rt}(F(0, T)N(d) - KN[d - \sigma_1(t, T)]), \end{aligned}$$

[8] The simplest way to derive Equation (6) is the following: From Equations (3) and (5) we have

$$\begin{aligned} d\ln F(t, T) &= \kappa e^{-\kappa(T-t)} X_t \, dt + e^{-\kappa(T-t)} \, dX_t - \xi\kappa e^{-\kappa(T-t)} \, dt \\ &\quad - \frac{\sigma_X^2}{2} e^{-2\kappa(T-t)} \, dt \\ &= e^{-\kappa(T-t)} \sigma_X \, dZ_t - \frac{\sigma_X^2}{2} e^{-2\kappa(T-t)} \, dt. \end{aligned}$$

On the other hand, postulating Equation (6) for dF, we have

$$d\ln F = \frac{dF}{F} - \frac{(dF)^2}{2F^2} = e^{-\kappa(T-t)} \sigma_X \, dZ_t - \frac{\sigma_X^2}{2} e^{-2\kappa(T-t)} \, dt.$$

where N is the cumulative standard normal distribution, and d is given by

$$d = \frac{\ln[F(0,T)/K)}{\sigma_1(t,T)} + \tfrac{1}{2}\sigma_1(t,T), \tag{7}$$

where

$$\sigma_1^2(t,T) = \mathrm{Var}_Q[\ln F(t,T) \mid \mathcal{F}_0] = e^{-2\kappa(T-t)}(1-e^{-2\kappa t})\frac{\sigma_X^2}{2\kappa}. \tag{8}$$

The annualized implied volatility is given by $\sigma_1(t,T)/\sqrt{t}$. The implied volatility tends to zero as $t^{-1/2}$ as the time to the expiration of the option increases. The intuition behind the decline of the implied volatility is that, in the long term, the mean reversion dominates and the volatility tends to the volatility of the mean level ξ, which in our one-factor model is zero.[9]

3.3. Empirical Calibration

We have obtained futures prices and implied volatilities for options on futures on natural gas. Our dataset covers the period from 9/2/97 to 9/4/98, and was obtained from the Bloomberg service. For each trading date, we have the futures prices for delivery of natural gas for the following 36 months. Delivery of natural gas takes place at Henry Hub throughout the delivery month at the price at which the futures contract settles on its last trading day, i.e., the third-to-last business day before the beginning of the delivery month. Prices are quoted in \$/MMBTU (dollars per million British Thermal Units). The dataset also contains the implied volatility for the options on the following month's futures contract (the option of the shortest expiration).

To calibrate the one-factor model to the natural gas price, we assumed that P represents the futures price for delivery of gas over the next month, starting in three business days.[10] For the functional form of the seasonality factor f, we use a function that is piecewise constant with 12 different values, one for each month of the year. The normalization condition, Equation (2), was imposed on the values of the seasonality factor

$$\sum_{i=1}^{12} \ln(f_{i/12}) = 0. \tag{9}$$

[9] This decline of the long-term implied volatility is a major drawback of a one-factor model, and indicates that the model would be inappropriate for pricing long-term options.

[10] The price P can be thought of as the one-month commodity swap price exchanged for the (random) daily spot price of natural gas. Ignoring intramonth discounting, P is the risk-neutral expectation of the average natural gas spot price for the month. The definition implies that P is observable only one day per month.

The parameters that were calibrated include the 12 values for the seasonality factor, the volatility σ_X, the mean-reversion rate κ, the long-term level ξ, and the initial value of the deseasonalized spot price X_0. An additional constraint was imposed on the calibrated short-term volatility by setting it equal to the implied volatility

$$\sigma_{\text{implied}}^2 = (1-e^{-2\kappa t})\frac{\sigma_X^2}{2\kappa t}, \tag{10}$$

where t is the time to the expiration of the option. The objective function that was minimized under constraints (9) and (10) was the sum of the absolute difference between the calibrated and the actual futures prices over all the available maturity dates.

The calibration was performed for every Monday and Friday in the dataset and 103 sets of calibrated parameters were obtained. Empirical results are summarized in Table 1 and illustrated in Figure 2.

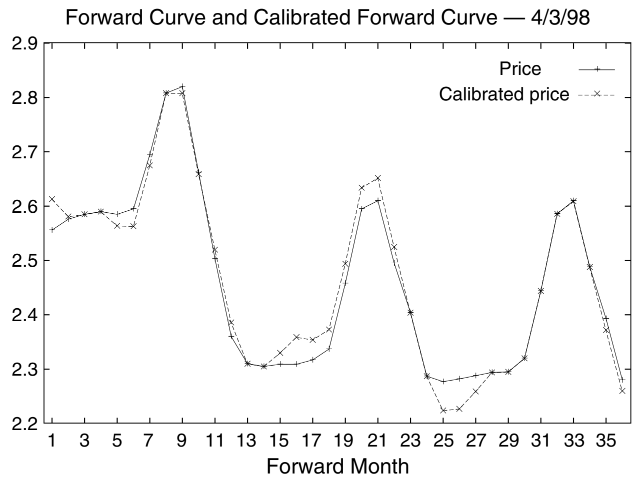
The in-sample error of the calibration is quantified by the average error per futures contract, which, over the whole sample, was 1.95 cents (a little less than 1%), while the biggest average error on any date was 3.8 cents per futures contract, and the smallest average error was 0.5 cents per contract.

The fluctuation of the calibrated parameters across these 103 dates provides an estimate for the out-of-sample performance of the calibrated model. Overall, the long-term average for the deseasonalized futures' natural gas price was approximately \$2.31 per MMBTU, and 95% of the observations were between \$2.21 and \$2.40 per MMBTU. The volatility σ_X fluctuated throughout the year, indicating an additional seasonal pattern with rapid mean reversion, which we did not account for. The mean-reversion rate was the hardest to estimate, due to the absence of reliable long-term implied volatilities in our dataset. The seasonality factor was remarkably stable, varying less than 1.5% throughout the period. The main mode of

Table 1 **Parameter Values for Natural Gas Prices**

Parameter	Average value	Standard deviation
Long-term log level (ξ)	0.802	0.028
Mean-reversion rate (κ)	3.4	2.1
Volatility (σ_X)	59%	14%
Long-term average ($\xi + \sigma_X^2/4\kappa$)	0.836	0.021
January factor	1.107	0.011
February factor	1.061	0.013
March factor	1.010	0.0049
April factor	0.9628	0.0058
May factor	0.9526	0.0067
June factor	0.9528	0.0065
July factor	0.9564	0.0064
August factor	0.9593	0.0044
September factor	0.9623	0.0046
October factor	0.9731	0.0062
November factor	1.029	0.0095
December factor	1.092	0.0078

Figure 2 Comparison Between the Forward and the Calibrated Forward Curve on 4/3/98



Notes. The average error per futures contract was 1.8 cents. The apparent downward drift in the figure is due to the initially high spot price, which, according to Equation (3) reverts to the long-term spot price level.

change of the seasonality factor appears to be a steepening (flattening) mode that makes the December, January, and February contracts more expensive relative to the summer contracts. We note that the values of the calibrated parameters are consistent with the values estimated in Manoliu and Tompaidis (2002) using Kalman filtering.

4. Numerical Method for Pricing Energy Derivatives

4.1. Tree-Building Procedure

Hull and White (1994) develop a procedure for building trinomial trees that can be adjusted to approximate the stochastic process for the deseasonalized spot price, D, starting from the stochastic process for its logarithm, $X = \ln D$:

$$dX_t = -\kappa(X_t - \xi)\,dt + \sigma_X\,dZ_t.$$

There are two stages in the construction.

The first stage is to build a trinomial tree for the process $(X_t^*)_t$ satisfying

$$dX_t^* = -\kappa X_t^*\,dt + \sigma_{X^*}\,dZ_t$$

with $X_0^* = 0$ and $\sigma_{X^*} = \sigma_X$. In an abuse of notation we will use X^* to describe the generic tree variable associated with the stochastic process $(X_t^*)_t$. The tree is symmetric around the value $X^* = 0$, and the nodes are evenly spaced in t and X^* at intervals of lengths δt and δX^*, where δt is the length of each time step and δX^* is taken to be $\sigma_{X^*}\sqrt{3\delta t}$. Denote by (i, j) the node for which $t = i\,\delta t$ and $X^* = j\,\delta X^*$. If the three branches emanating from (i, j) are referred to as "upper/middle/lower," then one of the following forms of branching is allowed to emanate from (i, j), depending on the value of j:

(a) "up one/straight along/down one" (standard form);

(b) "up two/up one/straight along";

(c) "straight along/down one/down two."

The latter two (nonstandard) forms are used to incorporate mean reversion when the spot price is very low or very high.

Let p_u, p_m, and p_d denote the probabilities along the upper, middle, and lower branches. For each of the branching forms (a), (b), and (c), these can be calculated by noting that the variable $X_{t+\delta t}^* - X_t^*$ is normally distributed, with expected value equal to $-\kappa X_t^*\,\delta t$ and variance $\sigma_{X^*}^2\,\delta t$ (neglecting terms of order higher than δt). Let $x = \kappa j\,\delta t$.

If the branching at node (i, j) is of the form (a), the probabilities are

$$p_u = \tfrac{1}{6} + \frac{x(x-1)}{2}, \quad p_m = \tfrac{2}{3} - x^2, \quad p_d = \tfrac{1}{6} + \frac{x(x+1)}{2}.$$

If the branching is of the form (b), the probabilities are

$$p_u = \tfrac{1}{6} + \frac{x(x+1)}{2}, \quad p_m = -\tfrac{1}{3} - x(x+2),$$
$$p_d = \tfrac{7}{6} + \frac{x(x+3)}{2}.$$

Finally, for form (c), the probabilities are

$$p_u = \tfrac{7}{6} + \frac{x(x-3)}{2}, \quad p_m = -\tfrac{1}{3} - x(x-2),$$
$$p_d = \tfrac{1}{6} + \frac{x(x-1)}{2}.$$

To keep these probabilities positive, it is required that the maximum value J of the absolute value of the integers $|j|$ used in the tree be between $0.184/(\kappa\,\delta t)$ and $0.577/(\kappa\,\delta t)$. For simplicity, we set J to be the smallest integer greater than $0.184/(\kappa\,\delta t)$. Thus, for each i the tree will have nodes (i, j) with $-n_i \le j \le n_i$, where $n_i = \min(i, J)$. For most of the nodes, namely for (i, j) with $|j| < J$, the branching used is of the standard form (a). It switches to nonstandard ones when $j = \pm J$, namely, to form (b) when $j = -J$ and to form (c) when $j = J$.[11]

The second stage in the construction of a tree for the logarithm of the deseasonalized spot price X_t is to displace the nodes (i, j) at time $t = i\,\delta t$ by a certain amount a_i, to incorporate the drift. Essentially,

[11] There are alternative ways to determine the maximum value of J. For example, one can check whether standard branching of form (a) would lead to "probabilities" that are greater than 1 or less than 0, in which case one can switch to the nonstandard types of branching.

the shifts a_i are determined so that the deseasonalized forward prices calculated by the numerical algorithm match the initial deseasonalized forward curve.[12] The tree for X has the same transition probabilities as the tree for X^*, but the branches are "shifted" in the new tree.

To define the shifts a_i, we first define auxiliary variables $B_{i,j}$ for each node (i,j). Let $B_{0,0}=1$. For each j, $-n_{i+1} \le j \le n_{i+1}$, define

$$B_{i+1,j} = \sum_k B_{i,k} b_{i+1,j}(k),$$

where $b_{i+1,j}(k)$ is the probability of moving from node (i,k) to node $(i+1,j)$. Its value is set to zero if node $(i+1,j)$ is not connected to node (i,k). The auxiliary variable $B_{i,j}$ corresponds to the probability that node (i,j) will be reached.

Once $B_{i,j}$ have been defined, a_i is given by

$$F_i/f_i = E_Q(P_i/f_i) = E_Q(D_i) = \sum_{j=-n_i}^{n_i} B_{i,j} e^{X^*_{i,j}+a_i},$$

where F_i is the forward price with maturity date $i\,\delta t$, f_i is the seasonal index for the maturity date $i\,\delta t$, and F_i/f_i is the deseasonalized forward price. Hence, we can express a_i as

$$a_i = \ln\left(\frac{F_i}{f_i}\right) - \ln\left(\sum_{j=-n_i}^{n_i} B_{i,j} e^{X^*_{i,j}}\right).$$

In Figures 3–5 we illustrate, through a numerical example, the construction of the trinomial trees. The time δt between the nodes is one month, and the current time is the last day that the October forward contract is traded. The spot price corresponds to the forward price of the contract maturing today. The term structures of forward prices and seasonality factors are given in Table 2. The long-term mean-reversion level ξ is 0.8, the mean-reversion rate $\kappa = 3$, and the volatility $\sigma^*_X = 60\%$.

The increments in the X^* direction are $\delta X^* = 0.3$, and in the time direction $\delta t = 1/12 = 0.0833$. The biggest integer J is $J = 1$. The tree for the probabilities $B_{i,j}$ of reaching node (i,j) are shown in Figure 3. The tree for X^* is shown in Figure 4. To match the forward prices, we adjust the values on the X^* tree by the quantities $a_0 = 0.8995$, $a_1 = 0.8608$, $a_2 = 0.8377$, and $a_3 = 0.8186$. We note that the option payoff can be calculated from the deseasonalized spot price, rather than from the seasonal spot price, using the relationship $P_t = f_t D_t$. Note that this transformation reduces the problem to one where the underlying stochastic process is continuous, but where the option payoff depends on the seasonality factor that corresponds to the exercise time.

[12] The initial deseasonalized forward curve is defined as F_t/f_t, where F_t is the initial forward price for time t and f_t the seasonality factor.

Figure 3 The Probabilities $B_{i,j}$ for Reaching Node (i,j) in a Four-Month Trinomial Tree for Natural Gas Prices

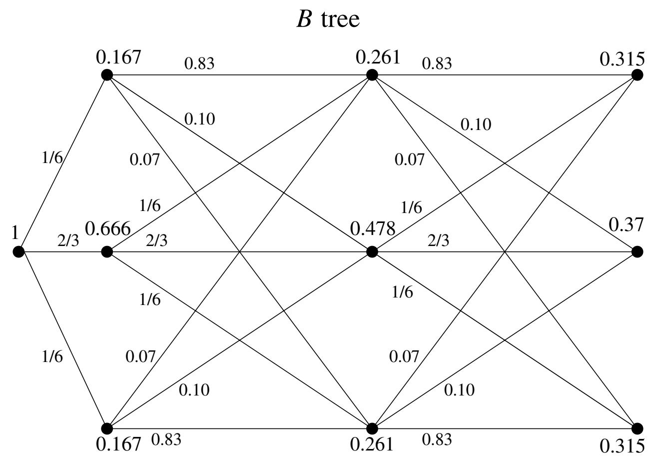
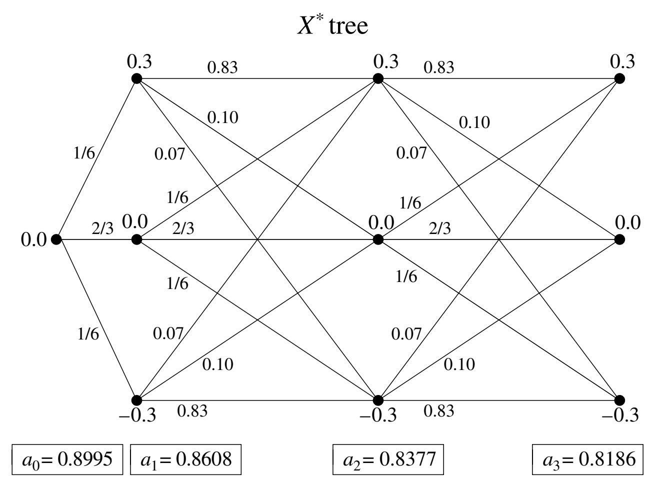
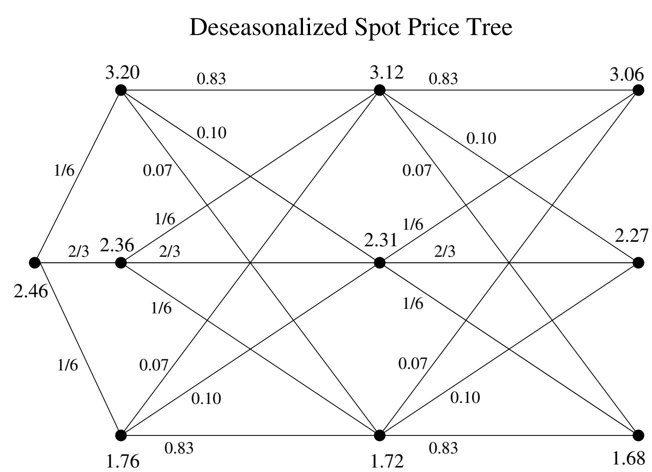


Figure 4 The First Stage in Building a Four-Month Trinomial Tree for the Price of Natural Gas with Mean Reversion



Note. Trinomial tree for the adjusted logarithm of the deseasonalized spot price X^*.

Figure 5 Trinomial Tree for the Deseasonalized Spot Price $D = \exp(X)$



Note. The seasonal spot price at node (i,j), can be obtained from the relationship $S_{i,j} = f_i \times D_{i,j}$, where f_i is the seasonality factor for date i.

Table 2 Term Structures of Forward Prices and Seasonality Factors

Month	Forward price (per MMBTU)	Seasonality factor
October	\$2.36	0.96
November	\$2.45	1.02
December	\$2.58	1.09
January	\$2.59	1.11

4.2. A Numerical Example for Pricing a Swing Option

In this example, we consider a simplified swing option where we have four exercise dates but can exercise at most two swing rights; each exercise permits the purchase of either one or two MMBTUs. Exercise can occur at the last day of the month that the following month's forward contract is traded. To value such an option, envisage three trinomial trees—one each for: no exercise rights left; one exercise right left; and two exercise rights left—layered one above the other. The interest rate is 5% per year, and the other parameters are the same as in the example presented in the previous section. The logarithm of the deseasonalized spot price tree is shown in Figure 5.

We consider two swing price structures:

(a) The strike is fixed at \$2.40 per MMBTU.

(b) The strike is set at-the-money-forward; i.e., for delivery in October it is set at \$2.36, for delivery in November at \$2.45, for delivery in December at \$2.58, and for delivery in January at \$2.59.

Figure 6 values the swing with the fixed strike and includes:

1. The bottom level shows the swing payoffs with zero rights left. The zeros in the figure are attributable to the absence of possible actions to be taken at the zero level, and the absence of penalties.

2. The midlevel shows the tree when one right has been exercised and two MMBTUs bought. In this tree, the value for the top node in the second month is calculated by

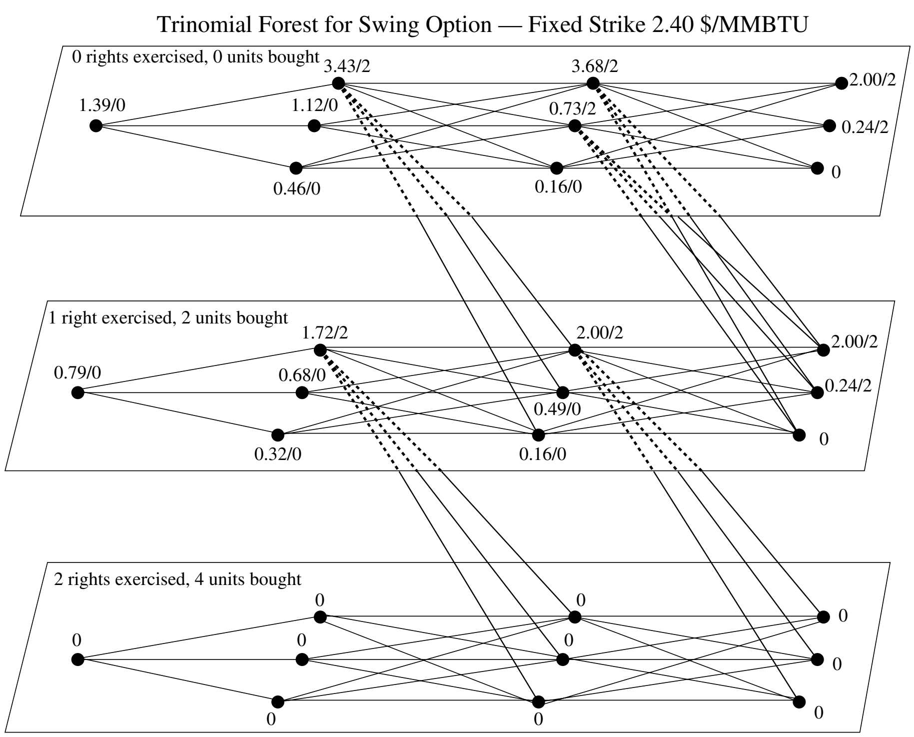
$$
\begin{aligned}
1.72 = \max\{&2\cdot(1.02\cdot 3.20 - 2.40) + 0,\\
&(0.83\cdot 2.00 + 0.10\cdot 0.49 + 0.07\cdot 0.16)e^{-0.05/12}\}\\
= \max\{&2\cdot(3.26-2.40), 1.71\} = \max\{1.72, 1.71\},
\end{aligned}
$$

which shows that it is optimal to exercise the remaining swing at that node for the maximum possible amount of two MMBTUs rather than wait. The value $3.26 = 1.02\cdot 3.20$ is obtained by multiplying the deseasonalized spot price by the seasonality factor for November.

3. Finally, the top level values the swing with two exercise rights left, in each node taking the greater of:

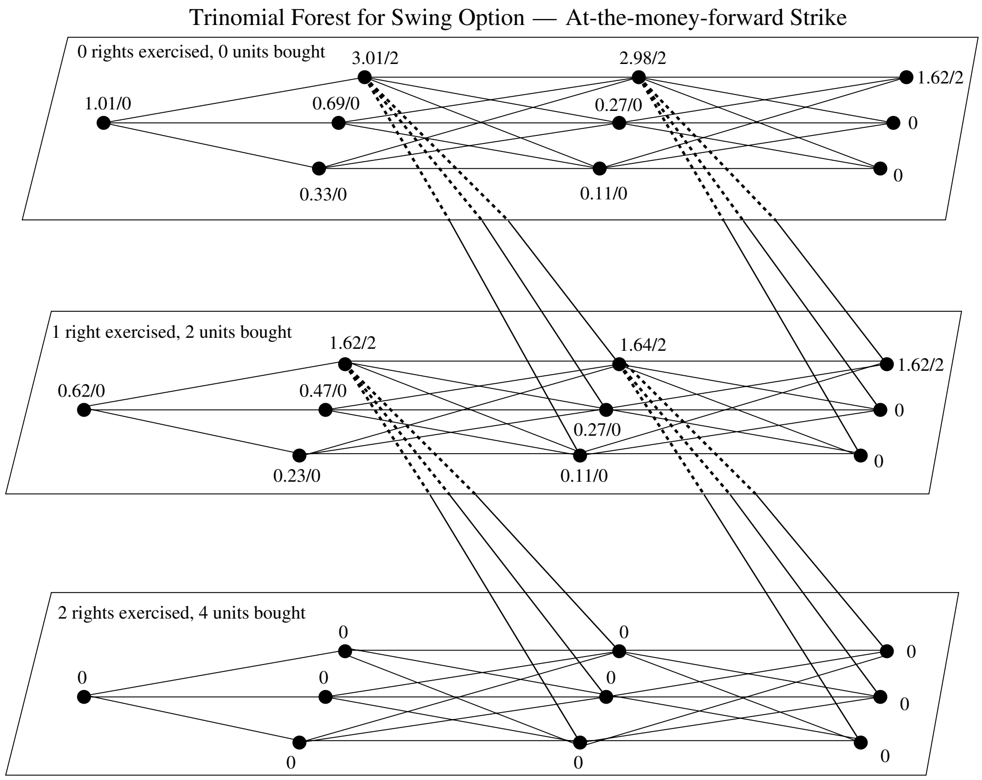
(a) Exercise now + risk-neutral expected present value of value in the tree below;

Figure 6 A Four-Month Trinomial Forest for Pricing a Swing with Two Rights Left



Notes. Fixed strike price of \$2.40 per MMBTU. No penalties, maximum amount bought at each exercise: Two MMBTU.

Figure 7 A Four-Month Trinomial Forest for Pricing a Swing with Two Rights Left



Notes. Strike set at-the-money forward. No penalties, maximum amount bought at each exercise: Two MMBTU.

(b) Defer exercise and take expected present value of next three nodes in the same tree.

In this tree, then, the value 3.43, on the top node for the second month, is obtained by

$$
\begin{aligned}
3.43 = \max\{&2\cdot(1.02\cdot 3.20 - 2.40) \\
&+ (0.83\cdot 2.00 + 0.10\cdot 0.49 + 0.07\cdot 0.16)e^{-0.05/12}, \\
&(0.83\cdot 3.68 + 0.10\cdot 0.73 + 0.07\cdot 0.16)e^{-0.05/12}\} \\
= \max&\{3.43, 3.12\}.
\end{aligned}
$$

Working backward, the value of the swing option is \$1.39. Expressed as a percentage of the spot natural gas price (the one corresponding to the October forward contract), the value of the swing is 59%.

Note that due to the mean reversion in the spot price, swing rights are exercised early for large deviations from the forward price. To compare with the upper and lower bounds, the value of a three month Bermudan option that can be exercised monthly for up to two units is worth \$0.79 (33% of the unit spot price), which is the same as the price at the root node of the tree with one right left to exercise. The value of the European options expiring in one, two, and three months are \$0.30 (13%), \$0.63 (27%), and \$0.71 (30%), respectively. Therefore, the lower bound for the swing price is \$1.34 (57%) and the upper bound is \$1.58 (66%).

The calculations for the swing with the strike set at-the-money-forward are shown in Figure 7. The value of the swing is \$1.01 (43%), while the value of the Bermudan option is \$0.62 (26%), and the values of the European options expiring in one, two, and three months are \$0.27 (11%), \$0.42 (18%), and \$0.50 (21%), respectively. The lower bound in this case is \$0.92 (39%) and the upper bound is \$1.24 (52%).

We also examined the interesting issue of whether there exists a unique optimal threshold value for the early exercise of call options in the environment considered in our paper: Geometric Brownian Motion versus mean reversion, and alternating fixed and variable seasonality factors and exercise prices; i.e., whether, if it is optimal to exercise a swing right for S^*, it is also optimal to exercise such a right for all $S \geq S^*$. While in general we do not have such results, and indeed multiple optimal thresholds can be demonstrated for certain parameter values, seasonality factors, and exercise prices, we *are* able to demonstrate a single threshold in the case of:

1. Geometric Brownian Motion.
2. Mean reversion when the mean-reversion rate is "sufficiently" large.

Finally, we were unable to demonstrate a violation of single threshold when the strike prices are the same.[13]

5. Concluding Remarks

In this paper, we have presented and tested a general valuation framework of a common and important form of options found in the energy sector—swing options, which permit their holders to buy or sell energy subject to both daily and periodic limits. The valuation methodology is based on the use of multilayered trinomial trees, which both discretizes the stochastic process and permits the valuation of an option requiring multiple decision variables. To ground the results firmly in both theory and empirical applicability, this paper has also proposed and tested a one-factor mean-reverting process for energy prices that explicitly incorporates seasonal effects.

This paper has concentrated on the case of a profit-maximizing agent whose specific consumption needs are irrelevant. However, many end users of swing options could be legally or technically precluded from selling excess amounts they cannot consume. In that situation, the exercise amount is constrained by the option holder's ability and need to consume energy. Often the daily needed quantity is itself unpredictable and, very frequently, weather-related.[14] Under such conditions, the pricing and hedging framework would need to be extended once an adequate market measure is chosen. This choice is intimately linked to the possibility of hedging the "private" quantity uncertainty of the buyer, for example by using *weather derivatives*. While the techniques developed in this paper can still be useful, the overall pricing and hedging framework faces the same conceptual difficulties encountered in real options valuation and hedging, for which both private and public risks are present.

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Appendix. Convergence of the Numerical Algorithm

We note that the stochastic process for the logarithm of the deseasonalized spot price follows a mean-reverting Ornstein-Uhlenbeck stochastic process. The numerical algorithm presented in §4.1 corresponds exactly to the trinomial tree construction proposed by Hull and White (1994), where the logarithm of the deseasonalized spot price plays the role of the short-term interest rate, and where the deseasonalized spot price plays the role of a discount bond. The weak convergence of this numerical algorithm has been established for the case of European options by Lesne et al. (2000). We note that the results of Lesne et al. do not directly apply to approximations to the seasonal spot price, due to the discontinuity of the seasonality factor. However, they do apply to approximations to the deseasonalized spot price, a useful fact which we take advantage of as explained below.

We use induction to establish weak convergence for swing options with multiple exercise rights. Starting with the option with no exercise rights left, and for any usage level, we have that the value is equal to the discounted expected value of the terminal date penalty corresponding to the usage level. By the Lesne et al. (2000) result, the value computed by the numerical approximation converges to the continuous time value as $\delta t \to 0$. Next, we consider the value for the swing with one exercise right left. On the first-to-last exercise date before expiration, the option value is the greater of the value obtained by immediate exercise for an allowed usage amount and that obtained by the discounted expected value of the terminal payoff if the option is not exercised (note that the time between exercise dates is finite). By the Lesne et al. (2000) result, we have that the discounted expected value of the terminal payoff computed by the numerical approximation converges to the continuous-time discounted expected value. The value of immediate exercise, on the other hand, is equal to the amount received from exercise plus the discounted expected value of a swing option with no exercise rights left, and, thus, converges to the continuous time value. Because the maximum function is continuous with respect to its arguments, the value computed by the numerical approximation converges to the continuous time value. By induction, the numerical approximation converges for earlier exercise dates, as well as for multiple swing rights.

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[13] Details are available from the authors.

[14] See also Jaillet et al. (1998a, b) for some related practical discussion on this topic.

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