Random Walks on Simple Two-Dimensional Manifolds

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Overview

Introduction

- Random
- Walk
- Simple
- Two-Dimensional
- Manifolds

Regular Surfaces

Coordinate Patch $\mu : U \to V$: continuous functions mapping from $U \subseteq \mathbb{R}^2$ to a subset of the surface V

Chart : covers entire surface

Regular Surfaces:

- Differentiable — the coordinate functions of μ in $\mathbb{R}^3$ have continuous partial derivatives for all orders
- Homeomorphic — μ and its inverse are continuous
- Satisfies the Regularity Condition — The differential of μ is a one-to-one linear transformation

Charts

$$\phi : \mathbb{R}^2 \to P$$
$$\phi(u, v) = (u, v, 0)$$

Charts

$$\sigma : \mathbb{R}^2 \to S$$

$$\sigma(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)$$

Charts

$$\tau : [0,1) \times [0,1) \to T(R,r)$$

$$\tau(u,v) = \Big((R + r\cos(2\pi v))\cos(2\pi u), \\ (R + r\cos(2\pi v))\sin(2\pi u), \\ r\sin(2\pi v) \Big)$$

Geodesic Equations

1. Extend definition of line to other surfaces
2. Assume a path is a geodesic contained in a coordinate patch
3. Derive geodesic equations for coordinate functions of path

Geodesic Equations

$$u'' + \frac{\mu_{uu} \cdot \mu_u}{\mu_u \cdot \mu_u}(u')^2 + \frac{\mu_{vv} \cdot \mu_u}{\mu_u \cdot \mu_u}(v')^2 + 2\frac{\mu_{uv} \cdot \mu_u}{\mu_u \cdot \mu_u}u'v' = 0$$
$$v'' + \frac{\mu_{uu} \cdot \mu_v}{\mu_v \cdot \mu_v}(u')^2 + \frac{\mu_{vv} \cdot \mu_v}{\mu_v \cdot \mu_v}(v')^2 + 2\frac{\mu_{uv} \cdot \mu_v}{\mu_v \cdot \mu_v}u'v' = 0$$

Christoffel Symbols

$$\Gamma^i_{jk} = \frac{\mu_{jk} \cdot \mu_i}{\mu_i \cdot \mu_i} \quad \text{where } i, j, k \in \{u, v\}$$

$$\Downarrow \text{abuse of symbols}$$

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k \in \{1,2\}} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

Stepping Method

Runge-Kutta 4th Order Method (RK4)

$$\frac{dy}{dt} = F(y) \quad y_0 = y(0)$$

Numerically solve up to $t = h$ with N iterations.

$\delta \leftarrow h/N$

$y \leftarrow y_0$

loop N *times:*

$k_1 \leftarrow F(y)$

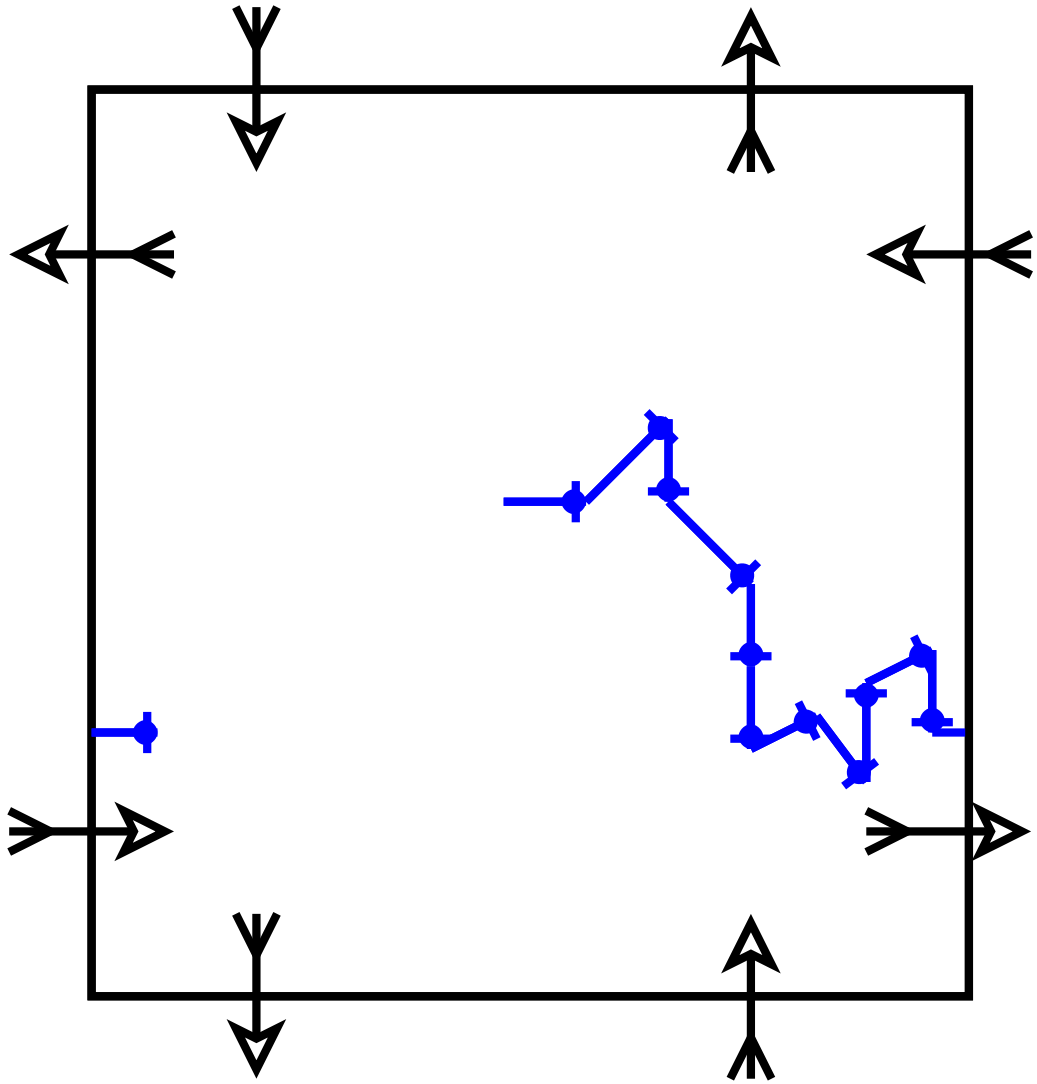
$k_2 \leftarrow F\left(y + (\delta/2)k_1\right)$

$k_3 \leftarrow F\left(y + (\delta/2)k_2\right)$

$k_4 \leftarrow F\left(y + \delta k_3\right)$

$y \leftarrow y + (\delta/6)(k_1 + 2k_2 + 2k_3 + k_4)$

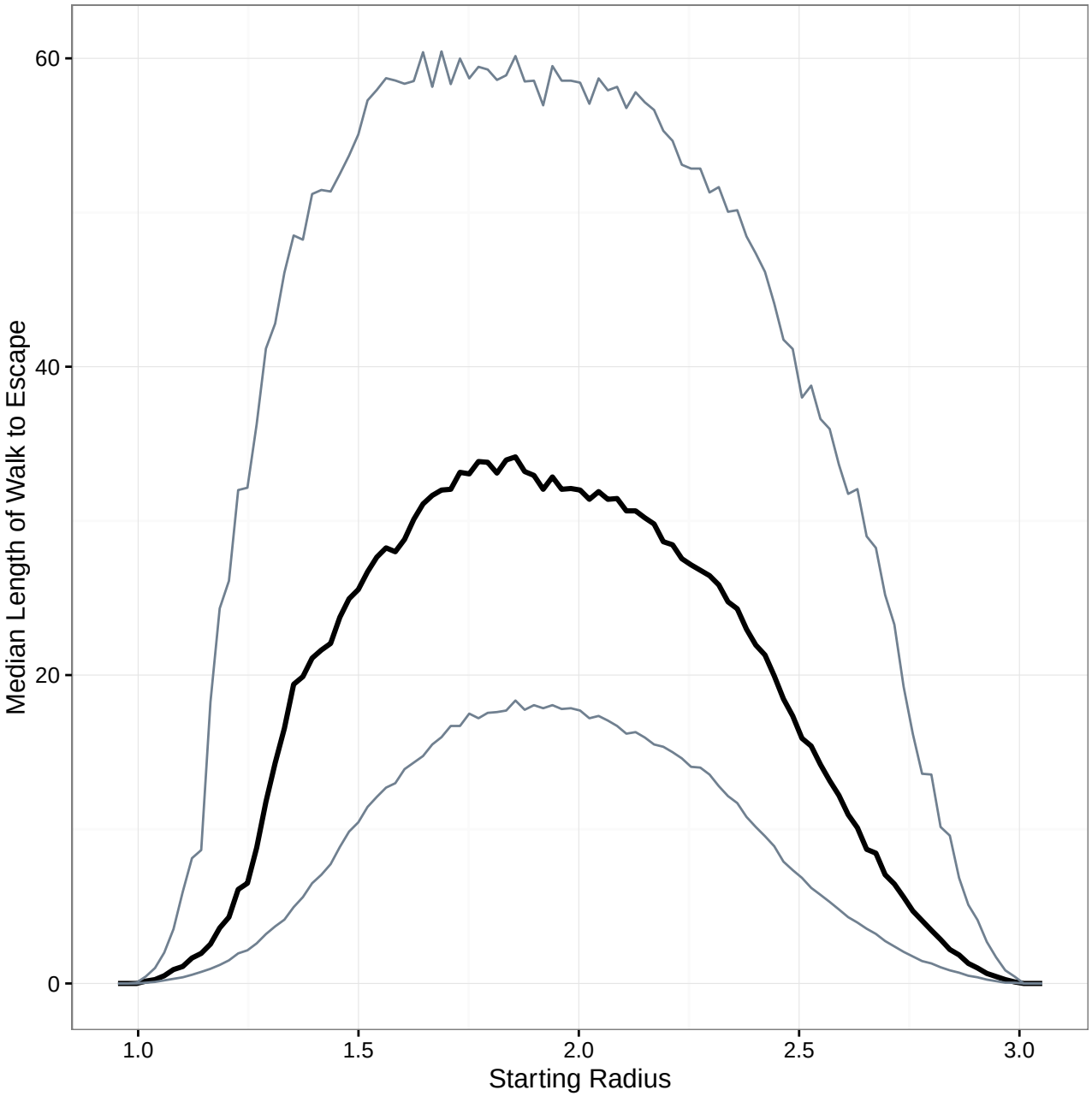
Coordinate Wrapping



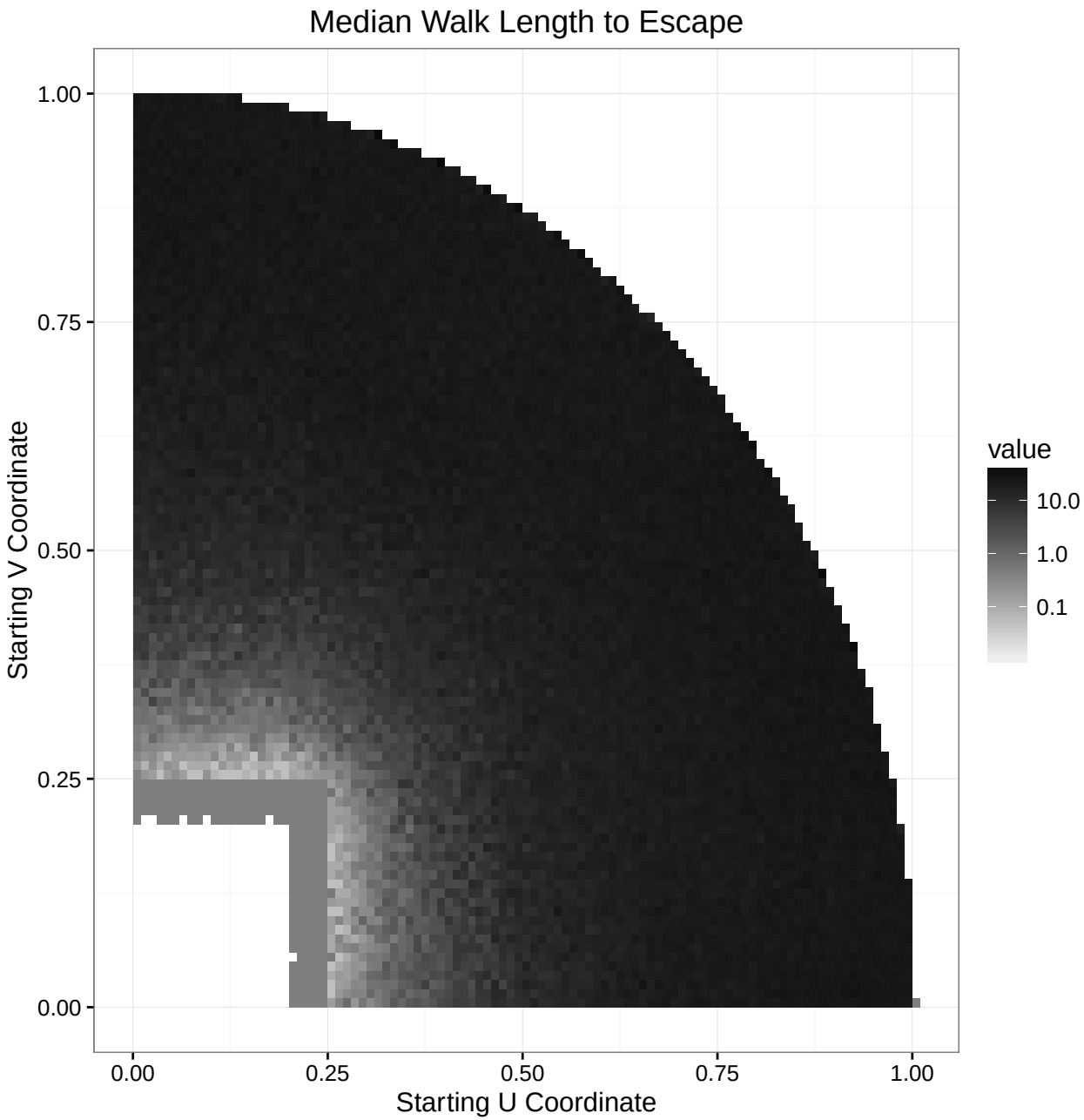
Optimizations

- Collection of every step point
- Number of steps in RK4
- Simplifications due to symmetry
  - Plane with radius representation
  - Sphere with polar angle representation
- Method of “compressing” the data

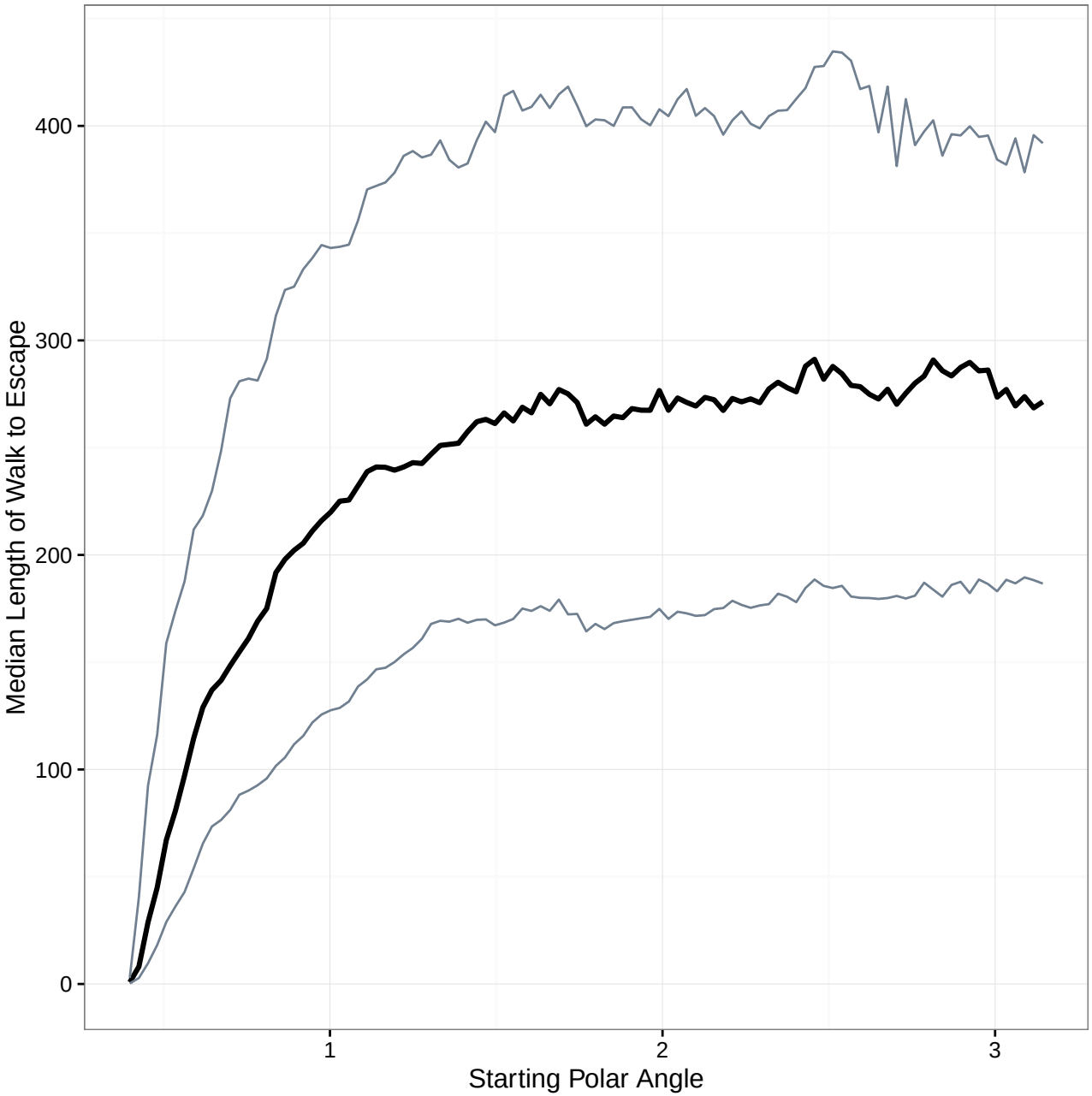
Plane


60
40
20
0
Median Length of Walk to Escape
1.0
1.5
2.0
2.5
3.0
Starting Radius

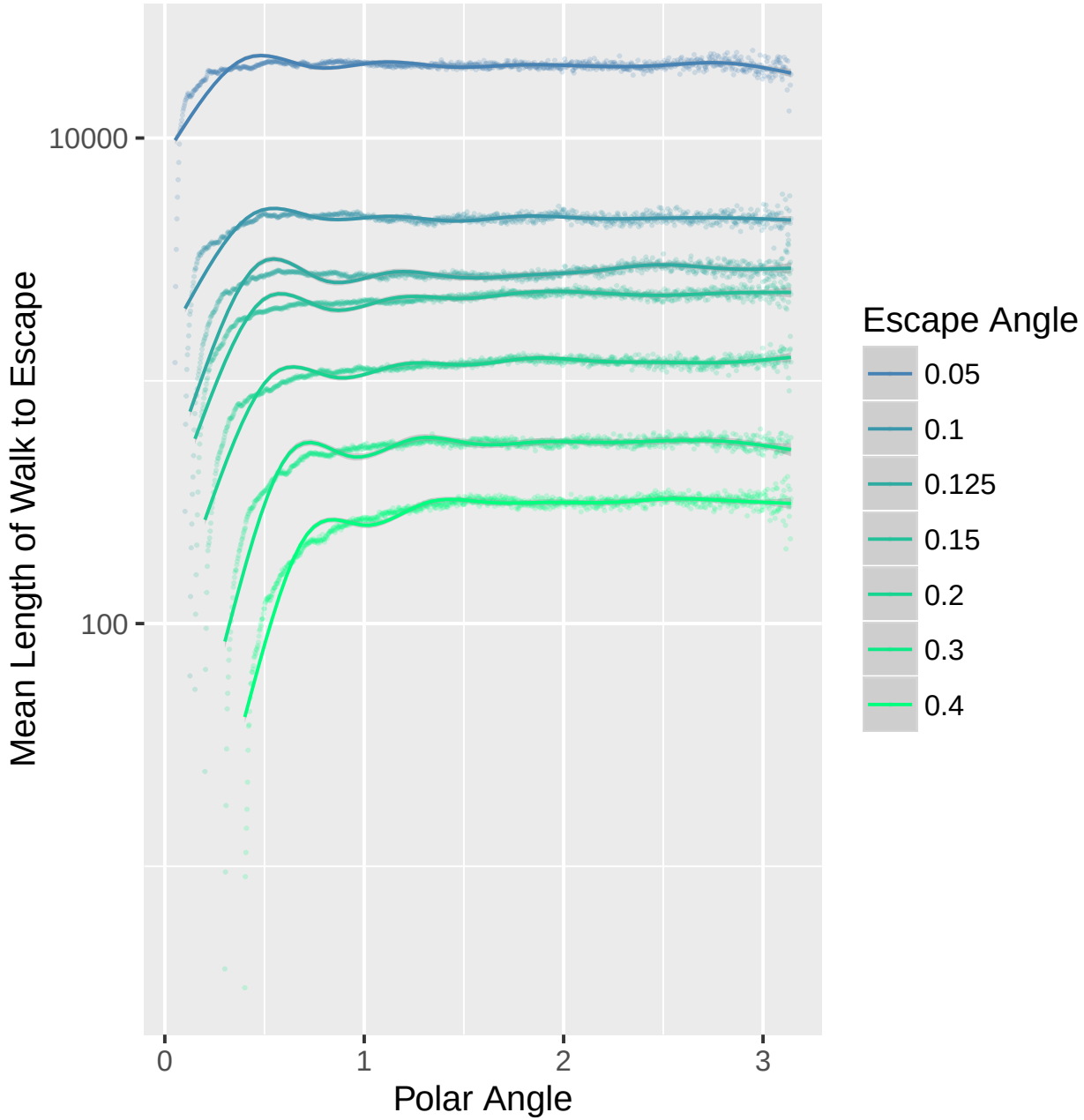
Plane


Median Walk Length to Escape
Starting V Coordinate
Starting U Coordinate
0.00
0.25
0.50
0.75
1.00
value
10.0
1.0
0.1

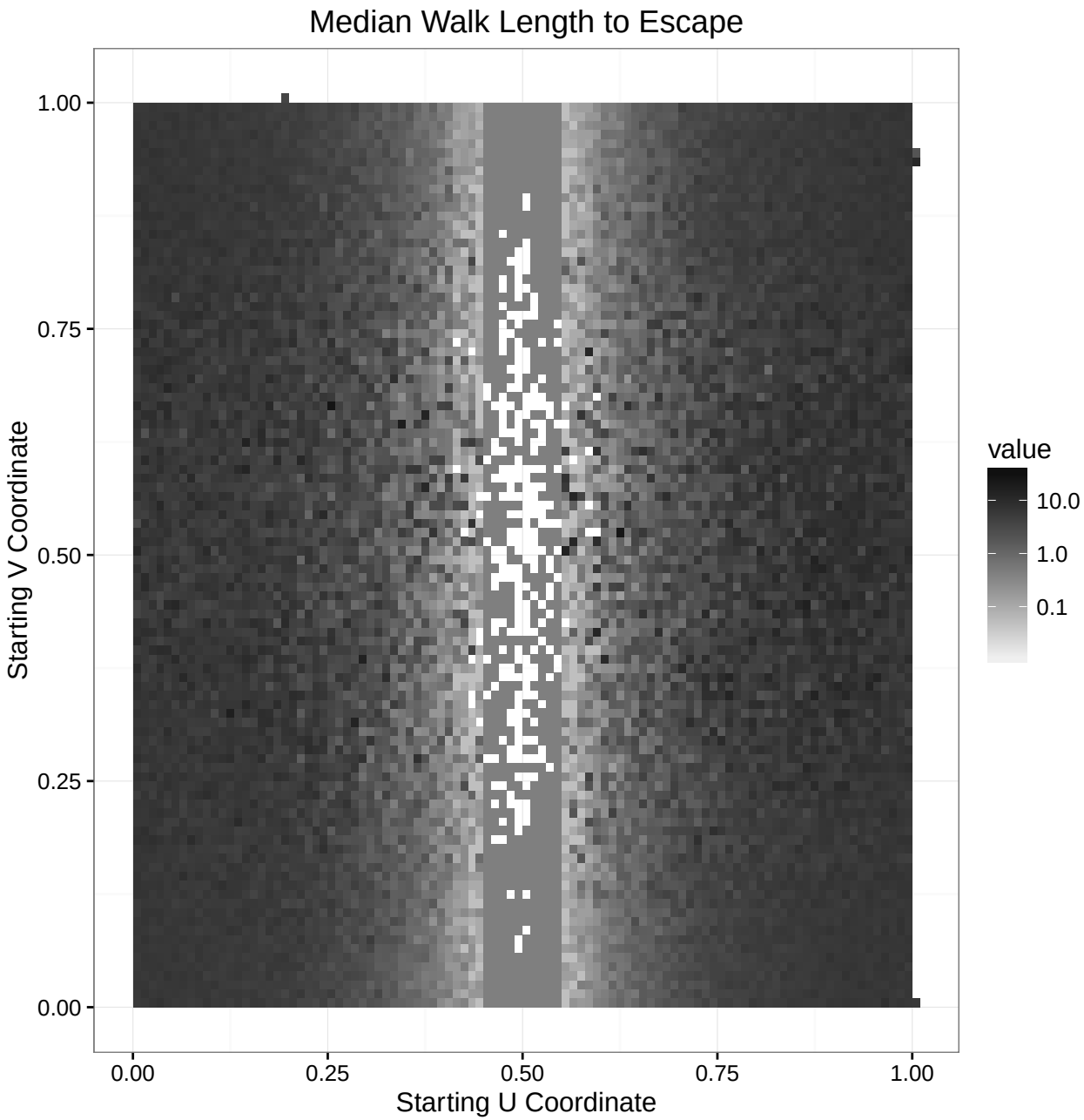
Sphere


Median Length of Walk to Escape
0
100
200
300
400
1
2
3
Starting Polar Angle

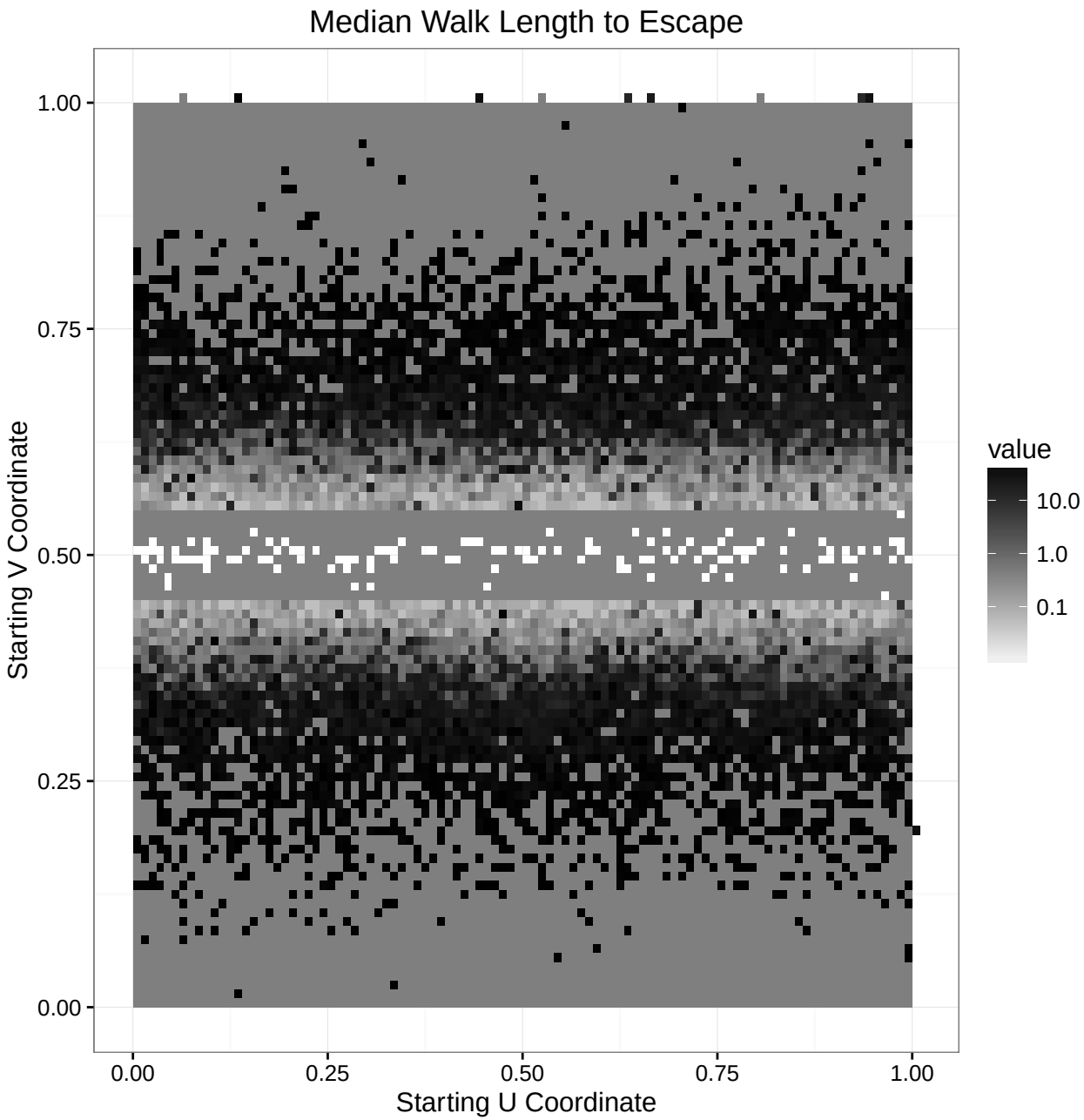
Sphere


10000
100
Mean Length of Walk to Escape
0
1
2
3
Polar Angle
Escape Angle
0.05
0.1
0.125
0.15
0.2
0.3
0.4

Torus


Median Walk Length to Escape
Starting V Coordinate
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
Starting U Coordinate
value
10.0
1.0
0.1

Torus


Median Walk Length to Escape
Starting V Coordinate
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
Starting U Coordinate
value
10.0
1.0
0.1

Overall Package

Package Attributes

- Versatility
- Flexibility
- Speed

Specific Parts

- Stepper
- Coordinate Wrappers
- Escape Checks

Questions?