

# Random Walks on Simple Two-Dimensional Manifolds

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# Overview

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- ▶ Random
- ▶ Walk
- ▶ Simple
- ▶ Two-Dimensional
- ▶ Manifolds

*Coordinate Patch*  $\mu : U \rightarrow V$  : continuous functions mapping from  $U \subseteq \mathbb{R}^2$  to a subset of the surface  $V$

*Chart* : covers entire surface

Regular Surfaces:

- ▶ Differentiable — the coordinate functions of  $\mu$  in  $\mathbb{R}^3$  have continuous partial derivatives for all orders
- ▶ Homeomorphic —  $\mu$  and its inverse are continuous
- ▶ Satisfies the Regularity Condition — The differential of  $\mu$  is a one-to-one linear transformation

$$\begin{aligned}\phi : \mathbb{R}^2 &\rightarrow P \\ \phi(u, v) &= (u, v, 0)\end{aligned}$$

$$\sigma : \mathbb{R}^2 \rightarrow S$$

$$\sigma(u, v) = \left( \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)$$

$$\begin{aligned}\tau : [0, 1) \times [0, 1) &\rightarrow T(R, r) \\ \tau(u, v) &= \left( (R + r \cos(2\pi v)) \cos(2\pi u), \right. \\ &\quad (R + r \cos(2\pi v)) \sin(2\pi u), \\ &\quad \left. r \sin(2\pi v) \right)\end{aligned}$$

1. Extend definition of line to other surfaces
2. Assume a path is a geodesic contained in a coordinate patch
3. Derive geodesic equations for coordinate functions of path



# Geodesic Equations

$$u'' + \frac{\mu_{uu} \cdot \mu_u}{\mu_u \cdot \mu_u} (u')^2 + \frac{\mu_{vv} \cdot \mu_u}{\mu_u \cdot \mu_u} (v')^2 + 2 \frac{\mu_{uv} \cdot \mu_u}{\mu_u \cdot \mu_u} u' v' = 0$$
$$v'' + \frac{\mu_{uu} \cdot \mu_v}{\mu_v \cdot \mu_v} (u')^2 + \frac{\mu_{vv} \cdot \mu_v}{\mu_v \cdot \mu_v} (v')^2 + 2 \frac{\mu_{uv} \cdot \mu_v}{\mu_v \cdot \mu_v} u' v' = 0$$

$$\Gamma_{jk}^i = \frac{\mu_{jk} \cdot \mu_i}{\mu_i \cdot \mu_i} \quad \text{where } i, j, k \in \{u, v\}$$

↓ abuse of symbols

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k \in \{1,2\}} \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

# Stepping Method

## Runge-Kutta 4th Order Method (RK4)

$$\frac{dy}{dt} = F(y) \quad y_0 = y(0)$$

Numerically solve up to  $t = h$  with  $N$  iterations.

$$\delta \leftarrow h/N$$

$$y \leftarrow y_0$$

*loop N times:*

$$k_1 \leftarrow F(y)$$

$$k_2 \leftarrow F(y + (\delta/2)k_1)$$

$$k_3 \leftarrow F(y + (\delta/2)k_2)$$

$$k_4 \leftarrow F(y + \delta k_3)$$

$$y \leftarrow y + (\delta/6)(k_1 + 2k_2 + 2k_3 + k_4)$$

# Stepping Method

Define

$$p = \frac{du}{dt} \quad \text{and} \quad q = \frac{dv}{dt}$$

Then the geodesic equations become

$$\frac{du}{dt} = p$$

$$\frac{dv}{dt} = q$$

$$\frac{dp}{dt} = -\Gamma_{uu}^u p^2 - 2\Gamma_{uv}^u pq - \Gamma_{vv}^u q^2$$

$$\frac{dq}{dt} = -\Gamma_{uu}^v p^2 - 2\Gamma_{uv}^v pq - \Gamma_{vv}^v q^2$$

## Random Walks

## Method



- ▶ Collection of every step point
- ▶ Number of steps in RK4
- ▶ Simplifications due to symmetry
  - ▶ Plane with radius representation
  - ▶ Sphere with polar angle representation
- ▶ Method of “compressing” the data

# Plane

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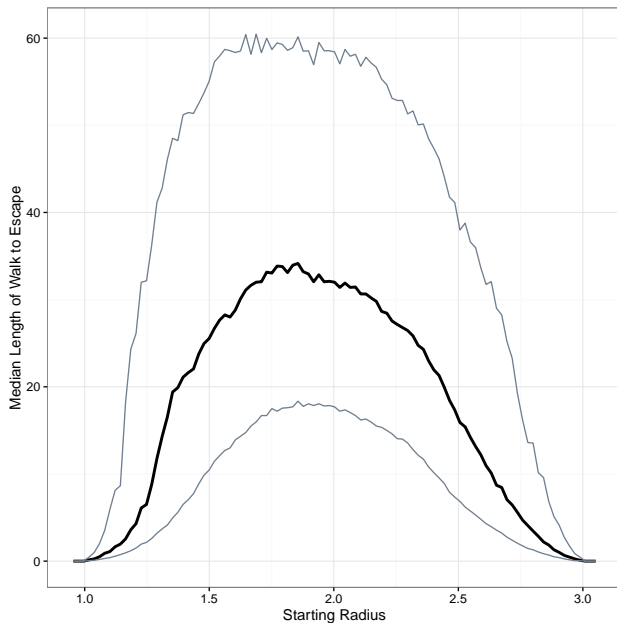
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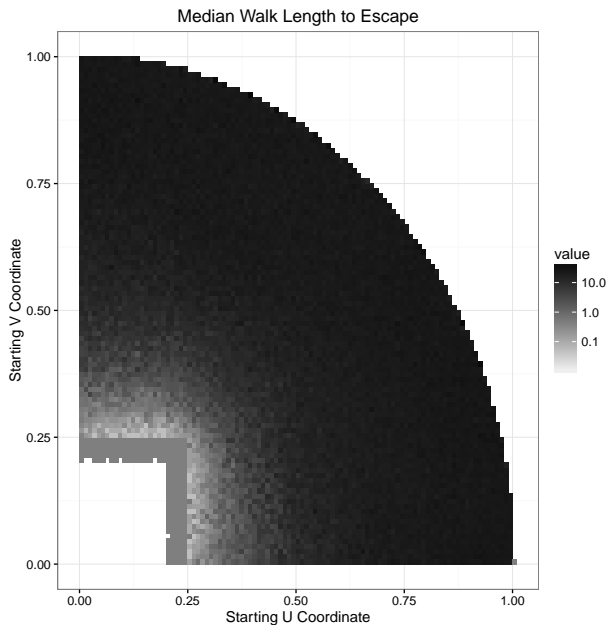
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# Plane





# Sphere

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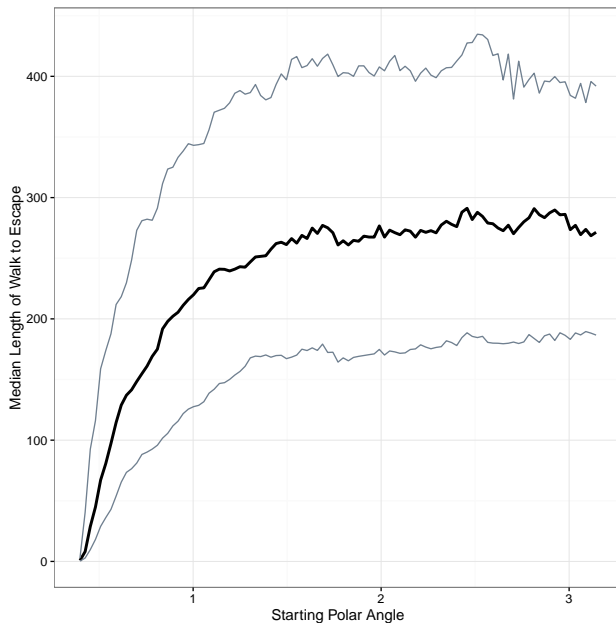
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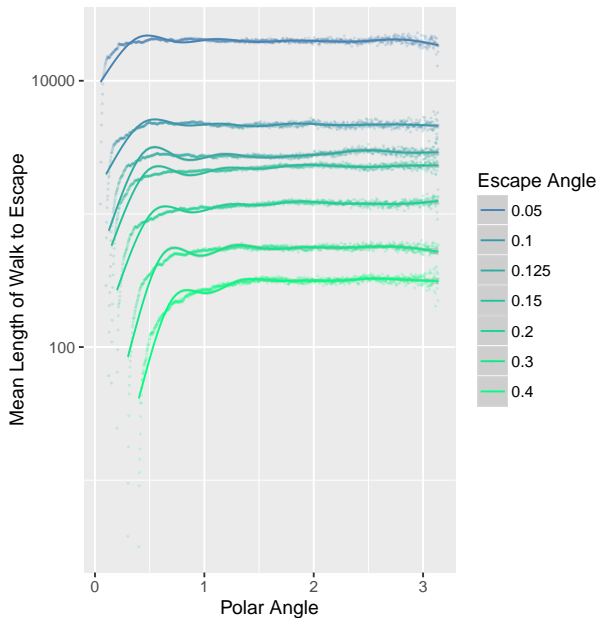
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# Sphere



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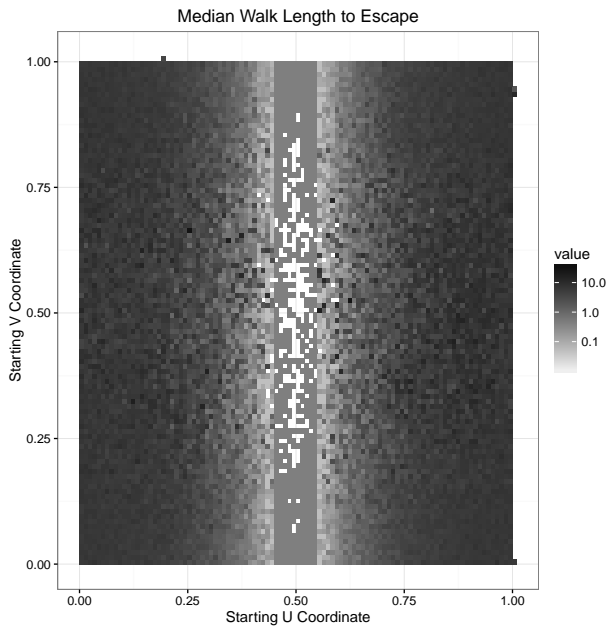
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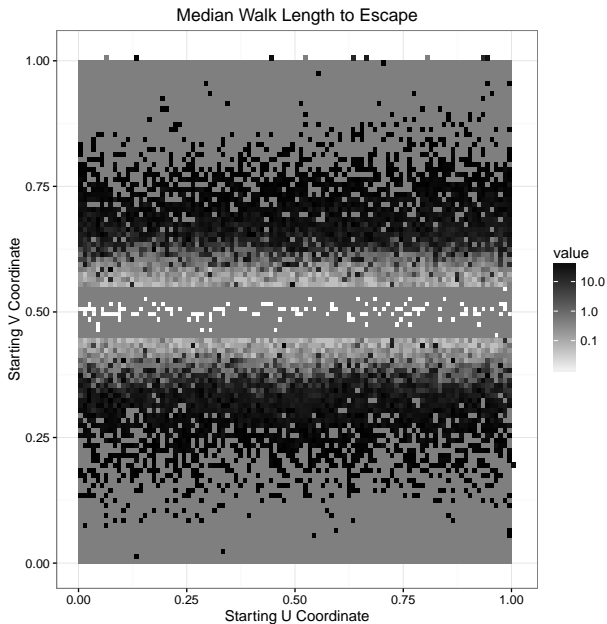
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# Overall Package

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## Package Attributes

- ▶ Versatility
- ▶ Flexibility
- ▶ Speed

## Specific Parts

- ▶ Stepper
- ▶ Coordinate Wrappers
- ▶ Escape Checks

# Questions?