

Random Walks on Simple Two-Dimensional Manifolds

Tom Eichlersmith

Hamline University

`teichlersmith01@hamline.edu`

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- ▶ Random
- ▶ Walk
- ▶ Simple
- ▶ Two-Dimensional
- ▶ Manifolds
- ▶ Smooth Surfaces
- ▶ Geodesic Equations
- ▶ Christoffel Symbols
- ▶ Escape Regions

“Smooth” Surfaces

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Background

Method

Results

Questions

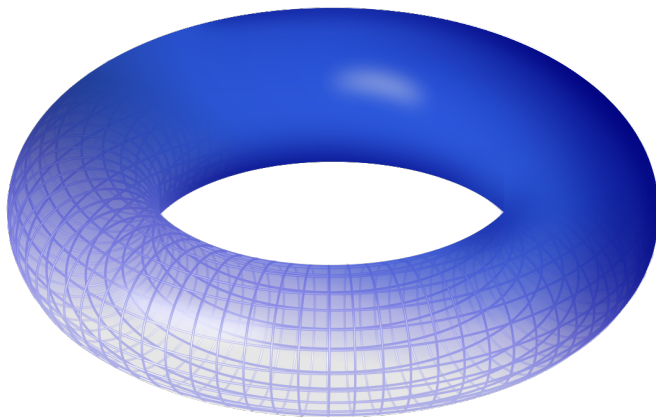


Figure: By Leonid_2 - Own work, CC BY-SA 3.0,
<https://commons.wikimedia.org/w/index.php?curid=8643414>

1. Extend definition of line to other surfaces
2. Assume a path is a geodesic contained in a coordinate patch
3. Derive geodesic equations for coordinate functions of path

Christoffel Symbols

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- ▶ Represent surface in geodesic equations
- ▶ Characterize properties of surface

C++ Implementation

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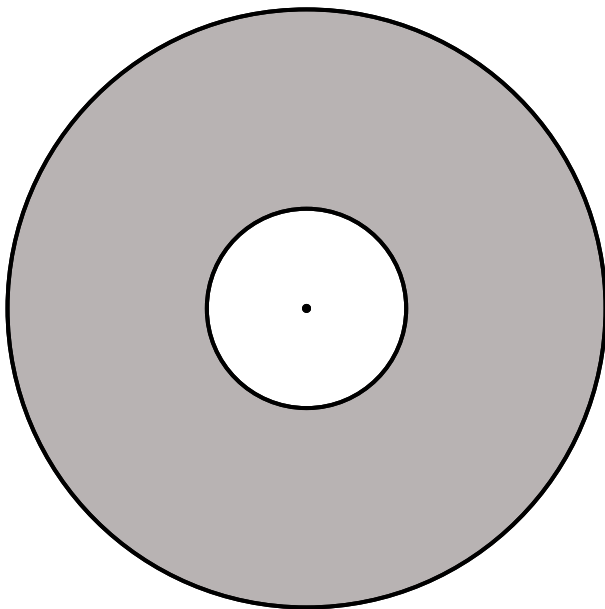
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- ▶ Runge-Kutta 4th Order Method
- ▶ Stack Linked List
- ▶ Function Pointers



Plane

Random Walks

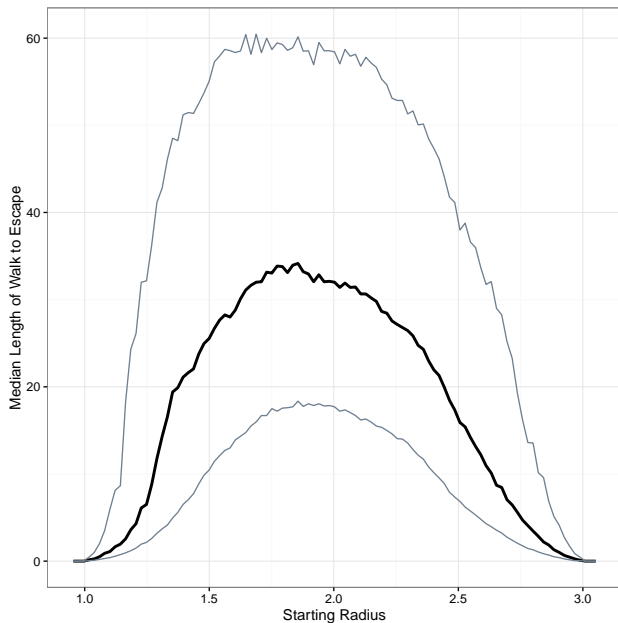
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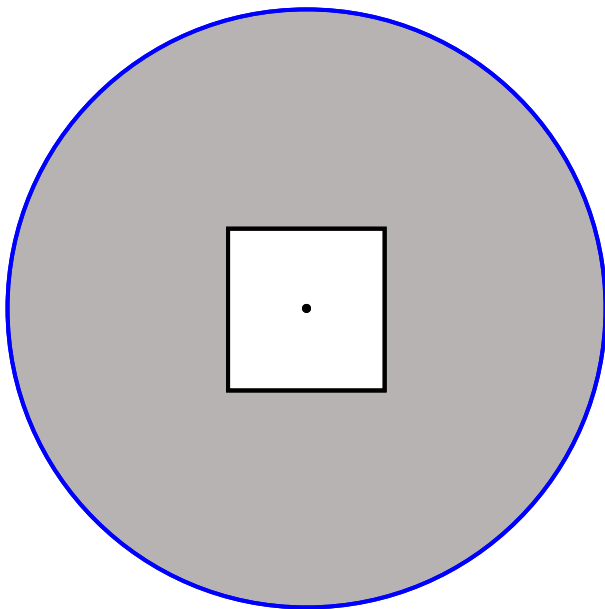
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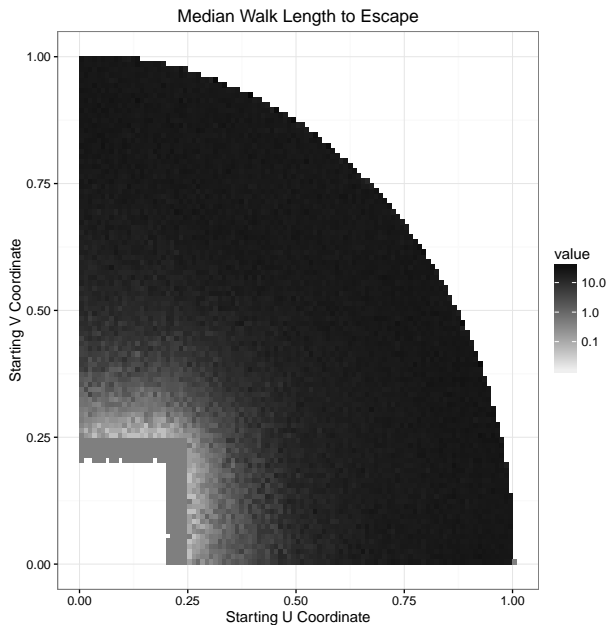
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Background

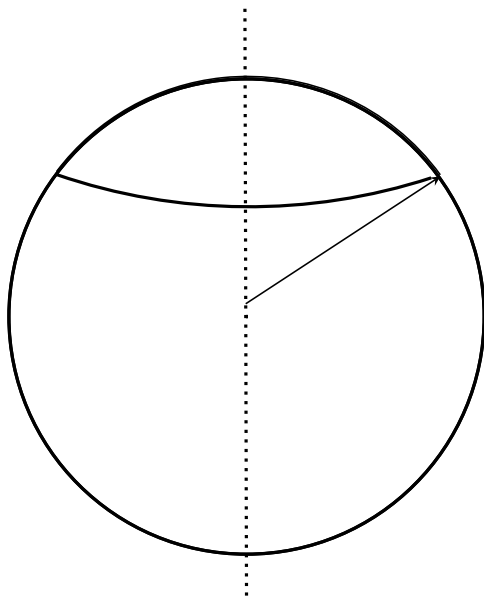
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Sphere



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Random Walks

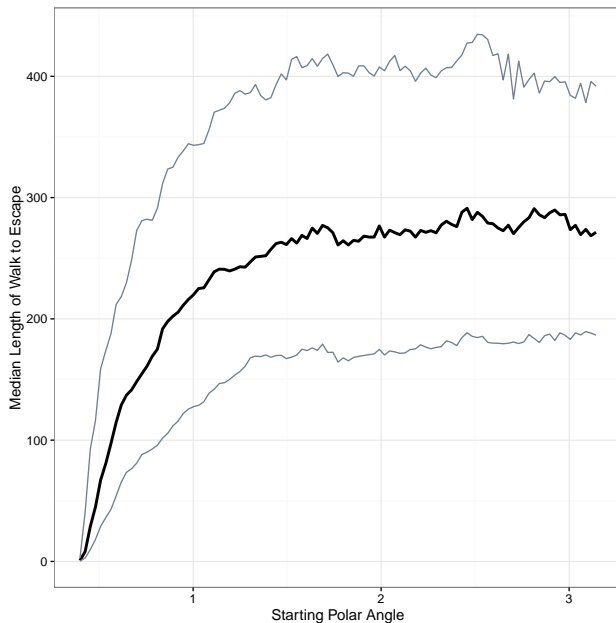
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Background

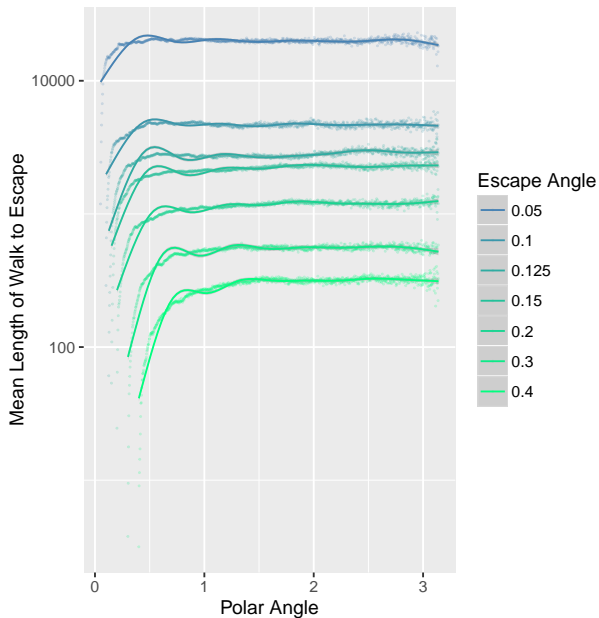
Method

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Sphere



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Questions?

Coordinate Patch $\mu : U \rightarrow V$: continuous functions mapping from $U \subseteq \mathbb{R}^2$ to a subset of the surface V

Chart : covers entire surface

Regular Surfaces:

- ▶ Differentiable — the coordinate functions of μ in \mathbb{R}^3 have continuous partial derivatives for all orders
- ▶ Homeomorphic — μ and its inverse are continuous
- ▶ Satisfies the Regularity Condition — The differential of μ is a one-to-one linear transformation

Package Attributes

- ▶ Versatility
- ▶ Flexibility
- ▶ Speed

Specific Parts

- ▶ Stepper
- ▶ Coordinate Wrappers
- ▶ Escape Checks

$$u'' + \frac{\mu_{uu} \cdot \mu_u}{\mu_u \cdot \mu_u} (u')^2 + \frac{\mu_{vv} \cdot \mu_u}{\mu_u \cdot \mu_u} (v')^2 + 2 \frac{\mu_{uv} \cdot \mu_u}{\mu_u \cdot \mu_u} u' v' = 0$$
$$v'' + \frac{\mu_{uu} \cdot \mu_v}{\mu_v \cdot \mu_v} (u')^2 + \frac{\mu_{vv} \cdot \mu_v}{\mu_v \cdot \mu_v} (v')^2 + 2 \frac{\mu_{uv} \cdot \mu_v}{\mu_v \cdot \mu_v} u' v' = 0$$

$$\Gamma_{jk}^i = \frac{\mu_{jk} \cdot \mu_i}{\mu_i \cdot \mu_i} \quad \text{where } i, j, k \in \{u, v\}$$

\Downarrow abuse of symbols

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k \in \{1,2\}} \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

Stepping Method

Runge-Kutta 4th Order Method (RK4)

$$\frac{dy}{dt} = F(y) \quad y_0 = y(0)$$

Numerically solve up to $t = h$ with N iterations.

$$\delta \leftarrow h/N$$

$$y \leftarrow y_0$$

loop N times:

$$k_1 \leftarrow F(y)$$

$$k_2 \leftarrow F(y + (\delta/2)k_1)$$

$$k_3 \leftarrow F(y + (\delta/2)k_2)$$

$$k_4 \leftarrow F(y + \delta k_3)$$

$$y \leftarrow y + (\delta/6)(k_1 + 2k_2 + 2k_3 + k_4)$$

Define

$$p = \frac{du}{dt} \quad \text{and} \quad q = \frac{dv}{dt}$$

Then the geodesic equations become

$$\frac{du}{dt} = p$$

$$\frac{dv}{dt} = q$$

$$\frac{dp}{dt} = -\Gamma_{uu}^u p^2 - 2\Gamma_{uv}^u pq - \Gamma_{vv}^u q^2$$

$$\frac{dq}{dt} = -\Gamma_{uu}^v p^2 - 2\Gamma_{uv}^v pq - \Gamma_{vv}^v q^2$$

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- ▶ Collection of every step point
- ▶ Number of steps in RK4
- ▶ Simplifications due to symmetry
 - ▶ Plane with radius representation
 - ▶ Sphere with polar angle representation
- ▶ Method of “compressing” the data

