Random Walks on Simple Two-Dimensional Manifolds

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Vocabulary

- Random
- Walk
- Simple
- Two-Dimensional
- Manifolds
- Smooth Surfaces
- Geodesic Equations
- Christoffel Symbols

“Smooth” Surfaces

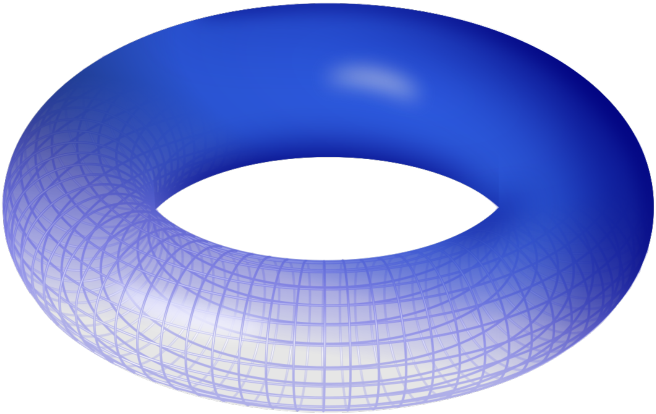

Figure: By Leonid_2 - Own work, CC BY-SA 3.0,
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Geodesic Equations

1. Extend definition of line to other surfaces
2. Assume a path is a geodesic contained in a coordinate patch
3. Derive geodesic equations for coordinate functions of path

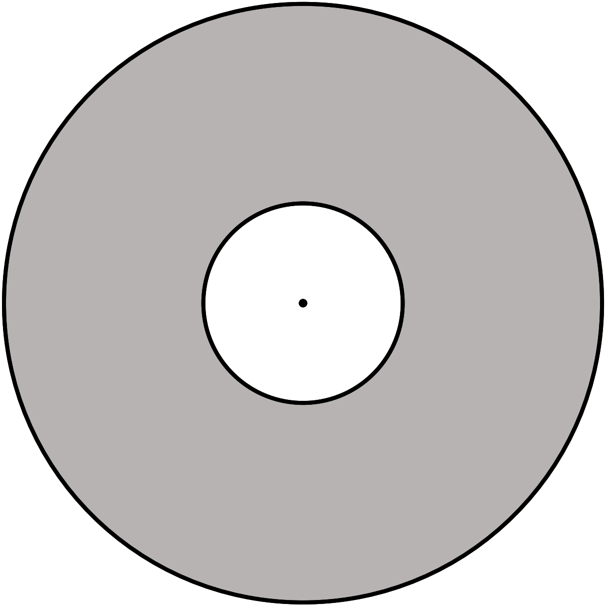
Christoffel Symbols

- Represent surface in geodesic equations
- Characterize properties of surface

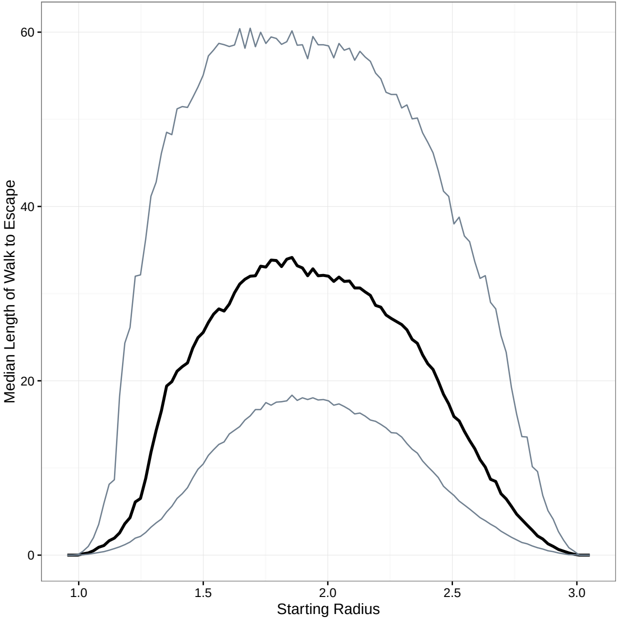
C++ Implementation

- Runge-Kutta 4th Order Method
- Stack Linked List
- Function Pointers

Plane

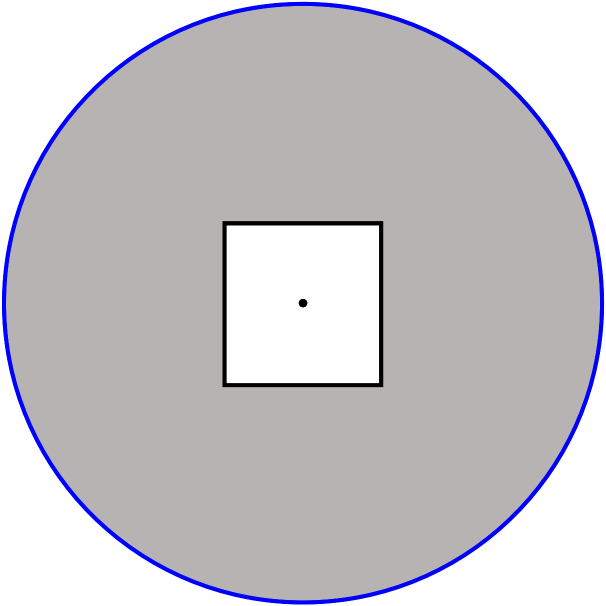


Plane

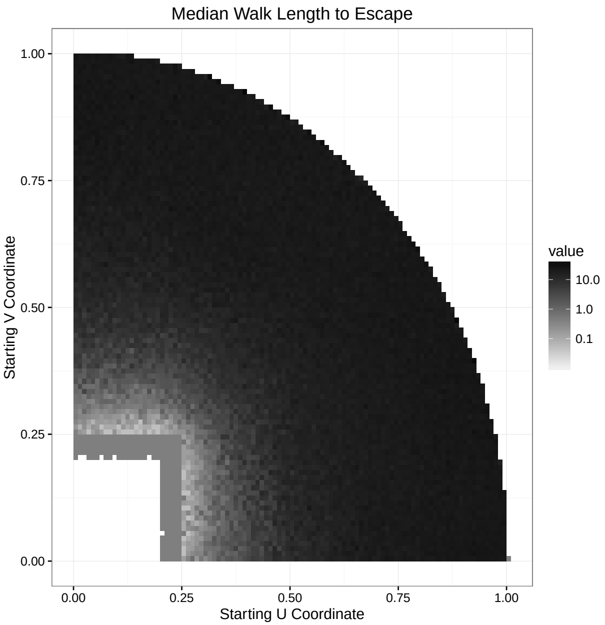


Median Length of Walk to Escape
60
40
20
0
1.0
1.5
2.0
2.5
3.0
Starting Radius

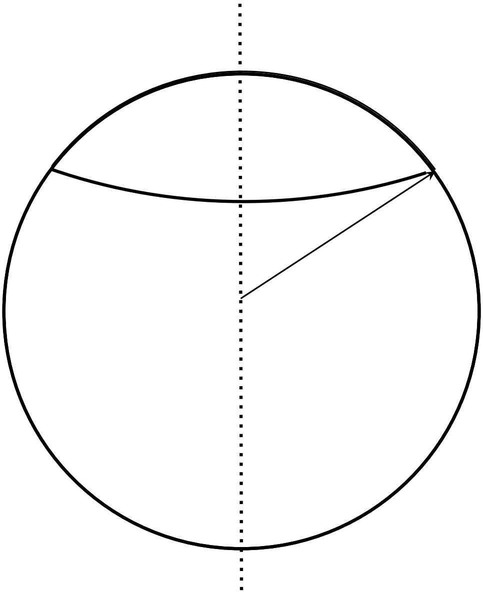
Plane



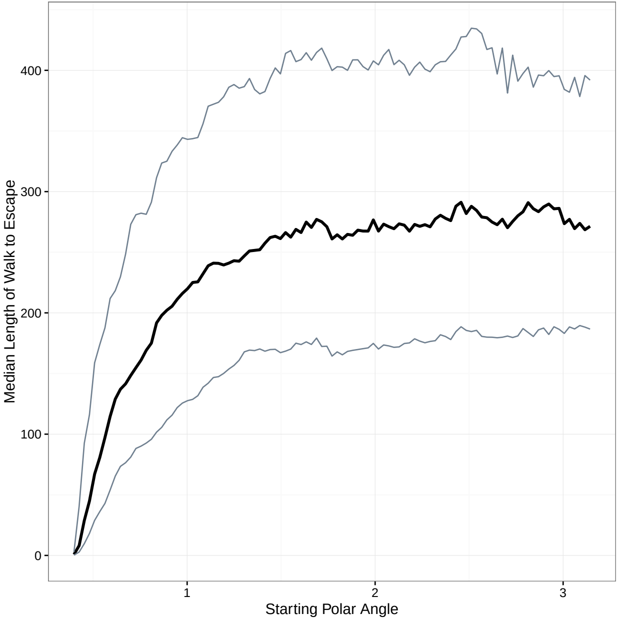
Plane


Median Walk Length to Escape
Starting V Coordinate
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
Starting U Coordinate
value
10.0
1.0
0.1

Sphere

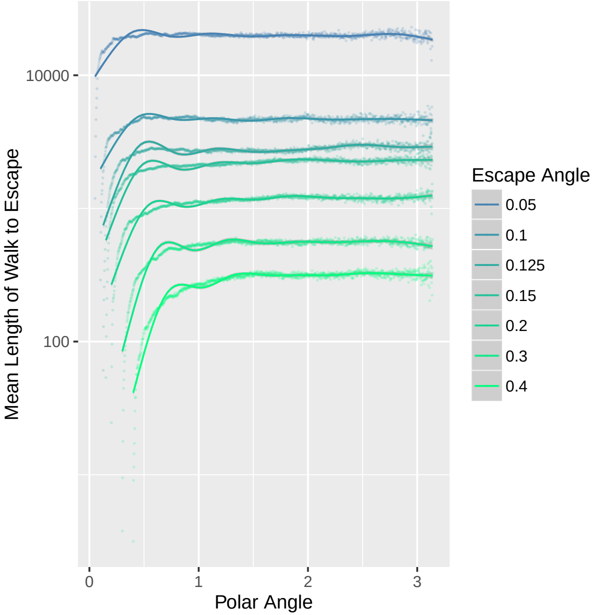


Sphere



Median Length of Walk to Escape
400
300
200
100
0
1
2
3
Starting Polar Angle

Sphere


Mean Length of Walk to Escape
10000
100
0
1
2
3
Polar Angle
Escape Angle
0.05
0.1
0.125
0.15
0.2
0.3
0.4

Questions?

Regular Surfaces

Coordinate Patch $\mu : U \to V$: continuous functions mapping from $U \subseteq \mathbb{R}^2$ to a subset of the surface V

Chart : covers entire surface

Regular Surfaces:

- Differentiable — the coordinate functions of μ in $\mathbb{R}^3$ have continuous partial derivatives for all orders
- Homeomorphic — μ and its inverse are continuous
- Satisfies the Regularity Condition — The differential of μ is a one-to-one linear transformation

Overall Package

Package Attributes

- Versatility
- Flexibility
- Speed

Specific Parts

- Stepper
- Coordinate Wrappers
- Escape Checks

Geodesic Equations

$$u'' + \frac{\mu_{uu} \cdot \mu_u}{\mu_u \cdot \mu_u}(u')^2 + \frac{\mu_{vv} \cdot \mu_u}{\mu_u \cdot \mu_u}(v')^2 + 2\frac{\mu_{uv} \cdot \mu_u}{\mu_u \cdot \mu_u}u'v' = 0$$

$$v'' + \frac{\mu_{uu} \cdot \mu_v}{\mu_v \cdot \mu_v}(u')^2 + \frac{\mu_{vv} \cdot \mu_v}{\mu_v \cdot \mu_v}(v')^2 + 2\frac{\mu_{uv} \cdot \mu_v}{\mu_v \cdot \mu_v}u'v' = 0$$

Christoffel Symbols

$$\Gamma^i_{jk} = \frac{\mu_{jk} \cdot \mu_i}{\mu_i \cdot \mu_i} \quad \text{where } i, j, k \in \{u, v\}$$

$$\Big\Downarrow \text{abuse of symbols}$$

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k \in \{1,2\}} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

Stepping Method

Runge-Kutta 4th Order Method (RK4)

$$\frac{dy}{dt} = F(y) \quad y_0 = y(0)$$

Numerically solve up to $t = h$ with N iterations.

$$
\begin{aligned}
&\delta \leftarrow h/N \\
&y \leftarrow y_0 \\
&\textit{loop } N \textit{ times:} \\
&\qquad k_1 \leftarrow F(y) \\
&\qquad k_2 \leftarrow F\left(y + (\delta/2)k_1\right) \\
&\qquad k_3 \leftarrow F\left(y + (\delta/2)k_2\right) \\
&\qquad k_4 \leftarrow F\left(y + \delta k_3\right) \\
&\qquad y \leftarrow y + (\delta/6)(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Stepping Method

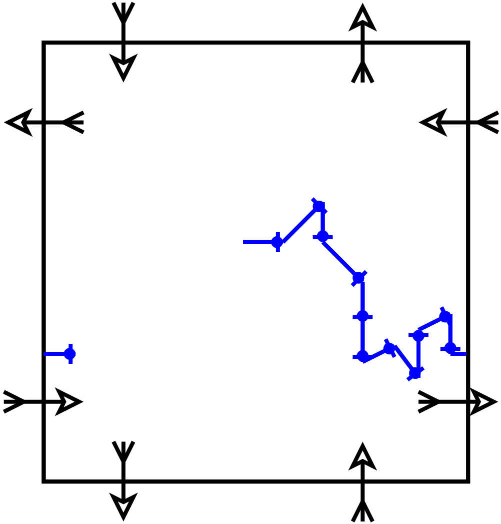
Define

$$p = \frac{du}{dt} \quad \text{and} \quad q = \frac{dv}{dt}$$

Then the geodesic equations become

$$\begin{aligned}
\frac{du}{dt} &= p \\
\frac{dv}{dt} &= q \\
\frac{dp}{dt} &= -\Gamma^u_{uu} p^2 - 2\Gamma^u_{uv} pq - \Gamma^u_{vv} q^2 \\
\frac{dq}{dt} &= -\Gamma^v_{uu} p^2 - 2\Gamma^v_{uv} pq - \Gamma^v_{vv} q^2
\end{aligned}$$

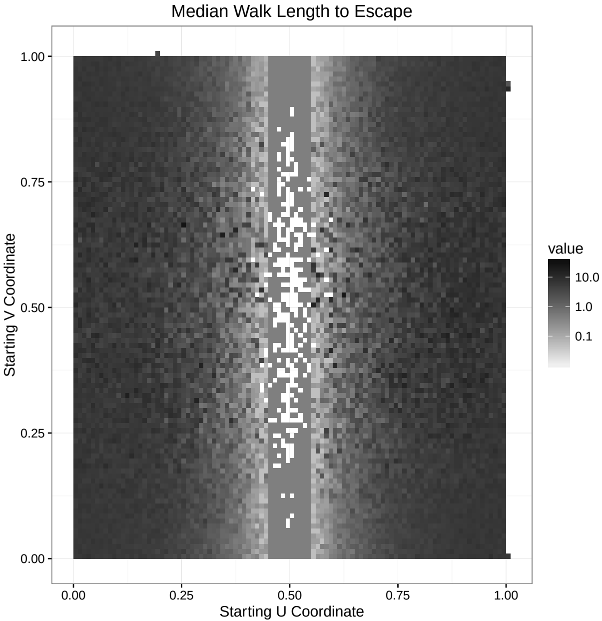
Coordinate Wrapping



Optimizations

- Collection of every step point
- Number of steps in RK4
- Simplifications due to symmetry
  - Plane with radius representation
  - Sphere with polar angle representation
- Method of "compressing" the data

Torus



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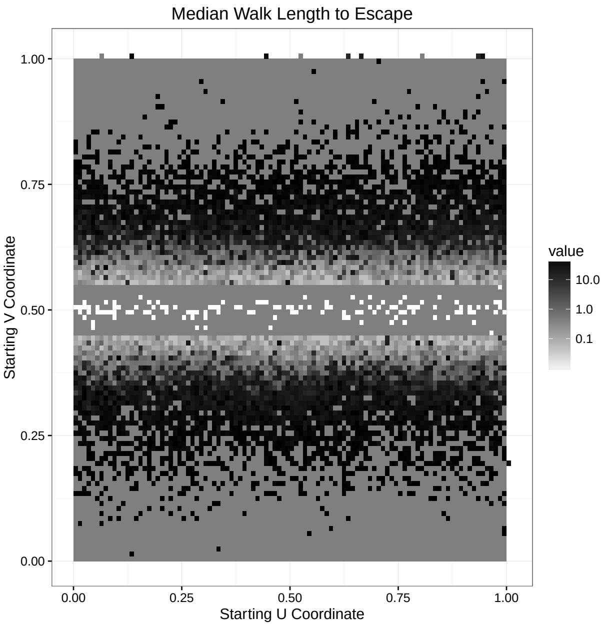
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
Starting U Coordinate
value
10.0
1.0
0.1

Torus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Starting U Coordinate
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value
10.0
1.0
0.1