LEVEL 1: QUANTITATIVE METHODS

Reading 3 (3rd out of 7): BASICS OF PROBABILITY

Difficulty: **hard**

Benchmark Study Time: **3.5h**

2022

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INTENTIONALLY EMPTY

THIS E-BOOK:

- is a selective summary of the corresponding Reading in your CFA® Program Curriculum,
- provides place for your own notes,
- helps you structure your study and revision time!

How to use this e-book to maximize your knowledge retention:

1. **Print** the e-book in duplex and bind it to keep **all** important info for this Reading **in one place.**
2. **Read** this e-book, best twice, to grasp the idea of what this Reading is about.
3. **Study** the Reading from your curriculum. **Here add** your notes, examples, formulas, definitions, etc.
4. **Review** the Reading using this e-book, e.g. write your summary of key concepts or revise the formulas at the end of this e-book (if applicable).
5. **Done?** Go to your study plan and change the Reading's status to **green**: (it will make your Chance-to-Pass-Score™ grow ☺).
6. **Come back** to this e-book from time to time to **regularly review for knowledge retention!**

NOTE: While studying or reviewing this Reading, you can use the tables at the end of this e-book and mark your study/review sessions to hold yourself accountable.

INTRODUCTION TO PROBABILITY

Definitions

A **random variable** is a variable that can take different numerical values depending on the case (example: stock return).

The set of all possible values or outcomes that a random variable can take is called a **sample space**.

An **event** is a subset of a sample space.

If a set of events contains all the possible outcomes of a random variable, that is the entire sample space, we're dealing with a **set of exhaustive events**.

If two events can't occur at the same time, they're called **mutually exclusive events**.

Properties of a probability

1. The probability of any event must be a value between 0 and 1:

$$0 \leq \mathrm{P(A)} \leq 1$$

2. The sum of the probabilities of a set of **mutually exclusive and exhaustive events** always **equals** 1.

Types of probabilities

Objective probability

2 types of objective probability:

- a **priori probability**, which is based on logical analysis,
- an **empirical probability**, which is calculated as a relative frequency of occurrence of some events based on historical data.

Subjective probability

Subjective probability depends on individual beliefs, judgments, intuitions, and experience.

Odds for an event vs Odds against an event

- If the probability of winning a bet is 1/5th → odds for the event are 1 to 4 → odds against the event are 4 to 1,
- If the probability of winning a bet is 2/7th → odds for the event are 2 to 5 → odds against the event are 5 to 2,
- If odds for winning a bet are 3 to 4 → the probability of winning is 3/7th → odds against winning are 4 to 3,
- If odds against winning a bet are 8 to 1 → the probability of winning is 1/9th → odds for winning are 1 to 8.

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

PROBABILITY – FORMULAS

Conditional probability

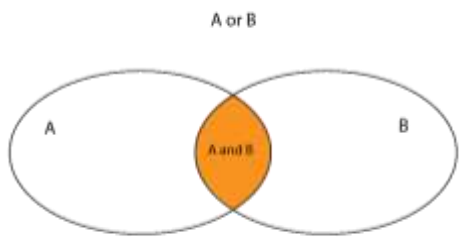
Conditional probability states that the probability of event A, given that event B has occurred ($\mathrm{P(A\backslash B)}$), equals the ratio of the joint probability of events A and B ($\mathrm{P(AB)}$) and the probability of event B ($\mathrm{P(B)}$).

$$\mathrm{P(A\backslash B)} = \frac{\mathrm{P(AB)}}{\mathrm{P(B)}}$$

Rules for computing probabilities

Addition rule for probabilities

The addition rule for probabilities allows us to determine the probability that event A or B occurs, or that both A and B occur ($\mathrm{P(A\ or\ B)}$).



$$\mathrm{P(A\ or\ B)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(AB)}$$

Multiplication rule for probabilities

The joint probability of events A and B ($\mathrm{P(AB)}$) equals the probability of event A given event B occurs ($\mathrm{P(A\backslash B)}$) multiplied by the probability of event B ($\mathrm{P(B)}$).

$$\mathrm{P(AB)} = \mathrm{P(A\backslash B)} \times \mathrm{P(B)}$$

Dependent events vs Independent events

A dependent event is an event whose probability changes depending on the occurrence of other events. An independent event is an event whose probability is not affected by whether or not another event occurs. Events A and B are independent only if:

$$\mathrm{P(A\backslash B)} = \mathrm{P(A)} \quad \text{or} \quad \mathrm{P(B\backslash A)} = \mathrm{P(B)} \quad \text{or} \quad \mathrm{P(AB)} = \mathrm{P(A)} \times \mathrm{P(B)}$$

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Total probability rule

$$P(A) = P(AS_1) + P(AS_2) + \cdots + P(AS_n) == P(A\backslash S_1) \times P(S_1) + P(A\backslash S_2) \times P(S_2) + \cdots + P(A\backslash S_n) \times P(S_n)$$

Where:

- $S_1, S_2, \ldots, S_n$ – mutually exclusive and exhaustive scenarios.

BAYES' FORMULA

According to Bayes' formula:

$$P(B\backslash A) = \frac{P(A\backslash B) \times P(B)}{P(A)}$$

OR

$$\text{updated probability of an event given the new information} =$$
$$= \left(\frac{\text{probability of the new information given event}}{\text{unconditional probability of the new information}}\right) \times\times (\text{prior probability of event})$$

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CHARACTERISTICS OF A RANDOM VARIABLE

Expected value of a random variable

The expected value of a random variable is the probability-weighted average of the possible outcomes of the random variable. The expected value is denoted as E(X).

$$E(X) = P(X_1) \times X_1 + P(X_2) \times X_2 + \cdots + P(X_n) \times X_n = \sum_{i=1}^{n} P(X_i) \times X_i$$

Where:

- X_i – outcome 'i' of the random variable X,
- $P(X_i)$ – probability of outcome 'i'.

Conditional expected value

Let's consider a random variable X and a scenario S.

$$E(X|S) = P(X_1|S) \times X_1 + P(X_2|S) \times X_2 + \cdots + P(X_n|S) \times X_n$$

If we define scenarios that are **mutually exclusive and exhaustive**, we'll be able to compute the expected value of the variable using the **total probability rule for expected value**.

Total probability rule for expected value

Two scenarios:

- S_1 – increase in GDP in the analyzed period will exceed 4%,
- S_2 – increase in GDP in the analyzed period will be equal to or less than 4%.

The expected value of a random variable:

$$E(X) = E(X|S_1) \times P(S_1) + E(X|S_2) \times P(S_2)$$

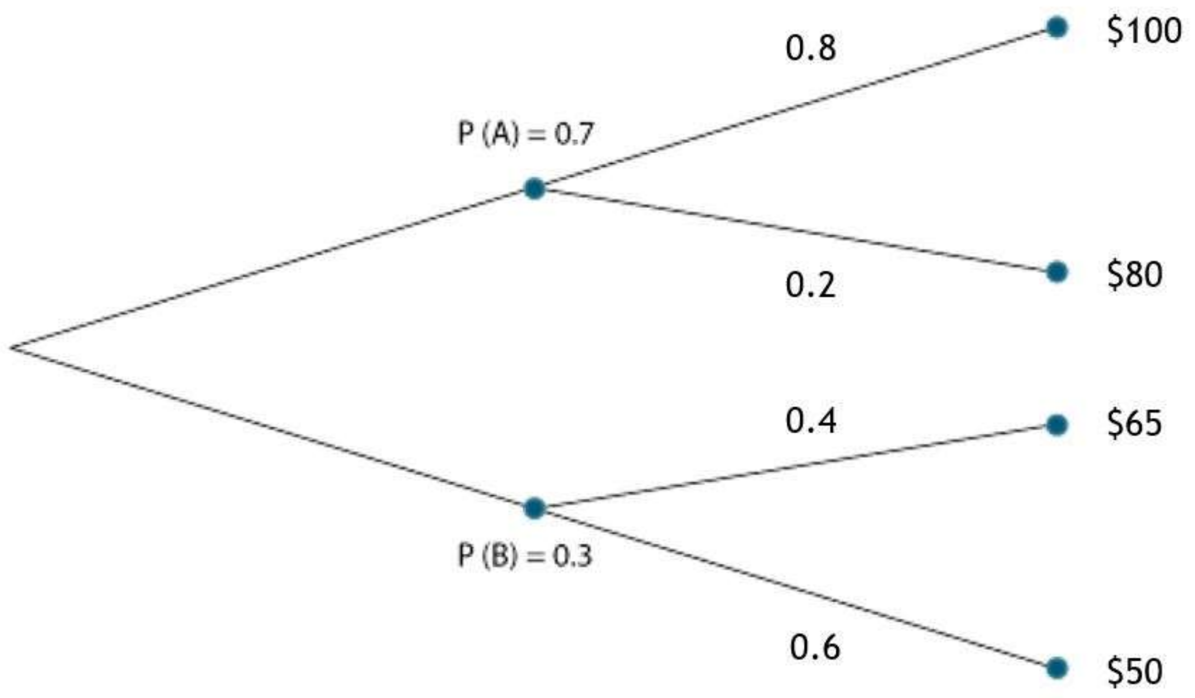
More than two scenarios:

$$E(X) = E(X|S_1) \times P(S_1) + E(X|S_2) \times P(S_2) + \cdots + E(X|S_n) \times P(S_n)$$

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

Using tree diagram

Example:



The tree diagram shows various prices of a share in relation to two different events (A, B) and their probabilities.

What is the expected value of the share?

Solution:

$$\mathrm{E(share\ price)} = \mathrm{E(share\ price} \mid \mathrm{A)} \times \mathrm{P(A)} + \mathrm{E(share\ price} \mid \mathrm{B)} \times \mathrm{P(B)}$$

Where:

$$\mathrm{E(share\ price} \mid \mathrm{A)} = 0.8 \times \$100 + 0.2 \times \$80 = \$96$$

$$\mathrm{E(share\ price} \mid \mathrm{B)} = 0.4 \times \$65 + 0.6 \times \$50 = \$56$$

So:

$$\mathrm{E(share\ price)} = \mathrm{E(share\ price} \mid \mathrm{A)} \times \mathrm{P(A)} + \mathrm{E(share\ price} \mid \mathrm{B)} \times \mathrm{P(B)} = \$96 \times 0.7 + \$56 \times 0.3 = \$84$$

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

Variance of a random variable

The variance of a random variable is the expected value of squared deviations from the random variable's expected value.

$$\sigma^2(X) = E\left\{[X - E(X)]^2\right\}$$

$$\sigma^2(X) = P(X_1) \times [X_1 - E(X)]^2 + P(X_2) \times [X_2 - E(X)]^2 + \cdots + P(X_n) \times [X_n - E(X)]^2$$

Where:

- X_i – outcome 'i' of the random variable X.
- $P(X_i)$ – probability of outcome 'i'.

Portfolio expected return and variance

Portfolio expected return

$$E(R_p) = \sum_{i=1}^{n} w_i \times E(R_i)$$

Where:

- w_i – weight of asset 'i' in the portfolio,
- $E(R_i)$ – expected return on asset 'i'.

Covariance

Covariance shows us if deviations from expected values are linearly associated. If they are associated and if both variables deviate above the expected value, or if they simultaneously deviate below the expected value, covariance is positive. If the variables deviate in opposite directions, covariance is negative. The greater the deviations in the same direction (both positive and negative), the greater the covariance. A covariance of zero shows that there is no linear association between the variables. Covariance can take any value from minus infinity to plus infinity and it's an intermediate step in computing the correlation coefficient, which is easier to interpret.

$$Cov(R_i, R_j) = E\left[\left(R_i - E(R_i)\right) \times \left(R_j - E\left(R_j\right)\right)\right]$$

Where:

- R_i, R_j – random variables,
- $E(R_i)$ – expected return on asset 'i',
- $Cov(R_i, R_j)$ – covariance between two variables.

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

Correlation coefficient

Correlation coefficient can take on values ranging from (- 1) to (+ 1).

$$\rho\left(R_i, R_j\right) = \frac{\mathrm{Cov}(R_i, R_j)}{\sigma(R_i) \times \sigma(R_j)}$$

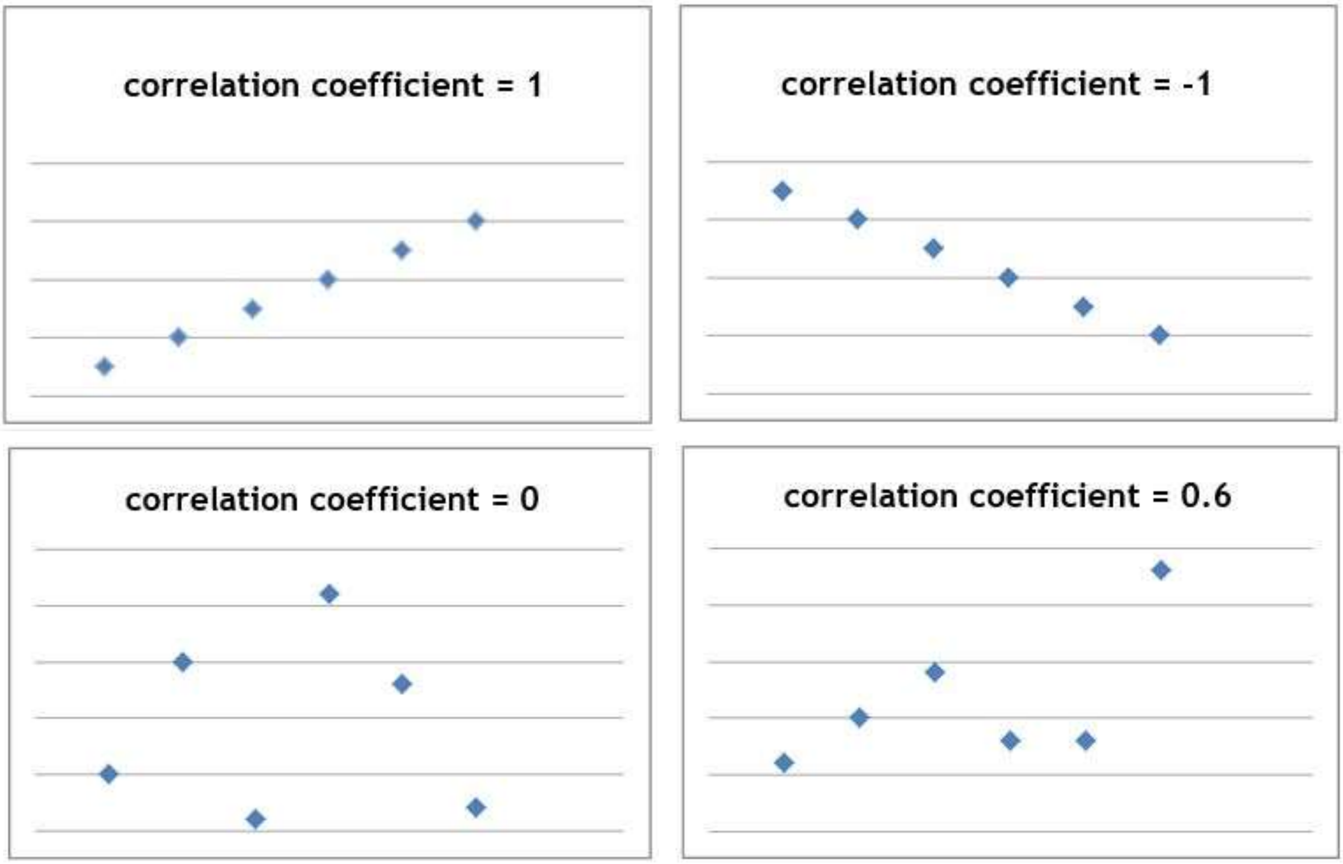
Where:

- R_i, R_j – random variables,
- $\rho\left(R_i, R_j\right)$ – correlation coefficient between two variables,
- $\mathrm{Cov}(R_i, R_j)$ – covariance between two variables.

Interpretation:

- A correlation coefficient of **+1** (a perfect positive correlation) means that there is a linear association between the variables.
- A correlation coefficient of **-1** (a perfect negative correlation) means that there is an inverse linear association between the variables.
- The closer the correlation coefficient to zero, the smaller the association between the variables. If the correlation coefficient equals zero, the variables are not correlated, so there is no linear association between them.

Scatter plots:



HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

Portfolio variance

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \times w_j \times \mathrm{Cov}(R_i, R_j)$$

Where:

- σ_p^2 – variance of the portfolio,
- w_i – weight of asset 'i' in the portfolio,
- $\mathrm{Cov}(R_i, R_j)$ – covariance between two assets,
- $\mathrm{Cov}(R_i, R_i) = \sigma_i^2$.

The lower the correlation between pairs of assets in a portfolio,

the lower the variance of the portfolio.

Calculating covariance using joint probability function

$$\mathrm{Cov}(R_A, R_B) = \sum_{i=1}^{n} \sum_{j=1}^{n} P(R_{A,i}, R_{B,i}) \times \left(R_{A,i} - E(R_A)\right) \times \left(R_{B,i} - E(R_B)\right)$$

Where:

- R_A, R_B – random variables,
- $P(R_{A,i}, R_{B,i})$ – probability that the variable R_A will equal $R_{A,i}$ and at the same time the variable R_B will equal $R_{B,i}$.

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

PRINCIPLES OF COUNTING

N factorial

N factorial allows us to compute the total number of possible n-element sequences comprising all the elements of a set of n elements.

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$$

Combination formula

We use the combination formula when we want to choose **r objects from an n-element set**. In the case of the combination formula, **the order** in which the elements in the subsets are listed **doesn't matter**, so the result doesn't depend on the arrangement of elements in the subsets.

$$C_n^r = \binom{n}{r} = \frac{n!}{(n-r)! \times r!}$$

Permutation formula

If we want to know the number of possible **r-element subsets of an n-element set** in which the **order** of elements **matters**, we need to use the permutation formula.

$$P_n^r = \frac{n!}{(n-r)!}$$

HERE KNOWLEDGE RETENTION HAPPENS | WRITE: notes, examples, formulas, definitions, relations, etc.

Summarizing key concepts:

☐ Probability – definitions

My summary:

☐ Properties of probability

My summary:

☐ Odds for an event vs Odds against an event

My summary:

☐ Conditional probability

My summary:

☐ Addition rule for probabilities

My summary:

☐ Multiplication rule for probabilities

My summary:

☐ Dependent events vs Independent events

My summary:

☐ Total probability rule

My summary:

☐ Bayes' formula

My summary:

☐ Expected value of a random variable

My summary:

☐ Using tree diagram

My summary:

☐ Variance of a random variable

My summary:

☐ Portfolio expected return & variance

My summary:

☐ Covariance & Correlation coefficient

My summary:

☐ Scatter plot – interpretation

My summary:

☐ N factorial

My summary:

☐ Combination formula

My summary:

☐ Permutation formula

My summary:

Reviewing formulas:

$$P(A\backslash B) = \frac{P(AB)}{P(B)}$$

Write down the formula:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

Write down the formula:

$$P(AB) = P(A\backslash B) \times P(B)$$

Write down the formula:

$$P(A) = P(AS_1) + P(AS_2) + \cdots + P(AS_n) == P(A\backslash S_1) \times P(S_1) + P(A\backslash S_2) \times P(S_2) + \cdots + P(A\backslash S_n) \times P(S_n)$$

Write down the formula:

$$P(B\backslash A) = \frac{P(A\backslash B) \times P(B)}{P(A)}$$

Write down the formula:

$$E(X) = P(X_1) \times X_1 + P(X_2) \times X_2 + \cdots + P(X_n) \times X_n = \sum_{i=1}^{n} P(X_i) \times X_i$$

Write down the formula:

$$E(X|S) = P(X_1|S) \times X_1 + P(X_2|S) \times X_2 + \cdots + P(X_n|S) \times X_n$$

Write down the formula:

$$E(X) = E(X|S_1) \times P(S_1) + E(X|S_2) \times P(S_2) + \cdots + E(X|S_n) \times P(S_n)$$

Write down the formula:

$$E(R_p) = \sum_{i=1}^{n} w_i \times E(R_i)$$

Write down the formula:

$$Cov(R_i, R_j) = E\left[\left(R_i - E(R_i)\right) \times \left(R_j - E\left(R_j\right)\right)\right]$$

Write down the formula:

$$\rho\left(R_i, R_j\right) = \frac{Cov(R_i, R_j)}{\sigma(R_i) \times \sigma(R_j)}$$

Write down the formula:

$$\sigma_p^2 = \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} w_i \times w_j \times Cov(R_i, R_j)$$

Write down the formula:

$$Cov(R_A, R_B) = \sum_{i=1}^{n} \sum_{j=1}^{n} P(R_{A,i}, R_{B,i}) \times \left(R_{A,i} - E(R_A)\right) \times \left(R_{B,i} - E(R_B)\right)$$

Write down the formula:

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$$

Write down the formula:

$$C_n^r = \binom{n}{r} = \frac{n!}{(n-r)! \times r!}$$

Write down the formula:

$$P_n^r = \frac{n!}{(n-r)!}$$

Write down the formula:

Keeping myself accountable:

TABLE 1 | STUDY

When you sit down to study, you may want to **try the Pomodoro Technique** to handle your study sessions: study for 25 minutes, then take a 5-minute break. Repeat this 25+5 study-break sequence all throughout your daily study session.

✔ Tick off as you proceed.

POMODORO TIMETABLE: study-break sequences (25′ + 5′)													
date		date		date		date		date		date		date	
25′		25′		25′		25′		25′		25′		25′	
5′		5′		5′		5′		5′		5′		5′	
25′		25′		25′		25′		25′		25′		25′	
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25′		25′		25′		25′		25′		25′		25′	
5′		5′		5′		5′		5′		5′		5′	

TABLE 2 | REVIEW

Never ever neglect revision! Though it's not the most popular thing among CFA candidates, regular revision is what makes the difference. If you want to pass your exam, **schedule & do your review sessions.**

REVIEW TIMETABLE: When did I review this Reading?													
date		date		date		date		date		date		date	
date		date		date		date		date		date		date	