soleadea

LEVEL 1: QUANTITATIVE METHODS

Reading 4 (4th out of 7): PROBABILITY DISTRIBUTIONS

Difficulty: **medium**

Benchmark Study Time: **3.5h**

2022

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INTENTIONALLY

EMPTY

THIS E-BOOK:

- is a selective summary of the corresponding Reading in your CFA® Program Curriculum,
- provides place for your own notes,
- helps you structure your study and revision time!

How to use this e-book to maximize your knowledge retention:

1. **Print** the e-book in duplex and bind it to keep **all** important info for this Reading **in one place.**
2. **Read** this e-book, best twice, to grasp the idea of what this Reading is about.
3. **Study** the Reading from your curriculum. **Here add** your notes, examples, formulas, definitions, etc.
4. **Review** the Reading using this e-book, e.g. write your summary of key concepts or revise the formulas at the end of this e-book (if applicable).
5. **Done?** Go to your study plan and change the Reading's status to **green**:
(it will make your Chance-to-Pass-Score™ grow ☺).
6. **Come back** to this e-book from time to time to **regularly review for knowledge retention!**

NOTE: While studying or reviewing this Reading, you can use the tables at the end of this e-book and mark your study/review sessions to hold yourself accountable.

PROBABILITY DISTRIBUTION – BASICS

Random variable

A random variable is a variable that can take different numerical values depending on the situation.

Probability function

The probability function helps us determine the probability of a random variable taking on a given value.
The notation:

$$P(X = x)$$

stands for the probability that a **random variable X takes on the value x**. For a discrete random variable, the probability function is labeled **p(x)**. This notation is the probability that a random variable X takes on the value x. In the case of a **continuous random variable**, the probability function is labeled **f(x)**, and it is called the **probability density function** or simply density.

2 key characteristics of the probability function:

1. The probability of an event is always equal to or greater than 0 and less than or equal to 1. If an event is impossible, its probability equals 0. And if an event is certain, its probability is 1. If an event may or may not occur, its probability is somewhere between 0 and 1.
2. The sum of the probabilities of all possible values that a random variable X can take on equals 1.

Cumulative distribution function

The cumulative distribution function tells us what the probability that the random variable takes on the value less than or equal to some value x is. Note that the probability that a random variable takes on a value less than or equal to x is equal to the sum of the probabilities of all values of the variable that are less than or equal to x.

$$F(x) = P(X \leq x).$$

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TYPES OF DISTRIBUTIONS

Discrete uniform distribution

A dice roll is an example of a discrete uniform distribution, i.e. a distribution in which the probability that the variable takes on any possible value is the same. Variables in a discrete uniform distribution are called discrete uniform random variables. This means they can take on a specified number of equally probable values.

Binomial distribution

Bernoulli random variable & Bernoulli trial

A random variable that can take on only two outcomes is called a Bernoulli random variable.

A **Bernoulli trial** is a test with two possible outcomes we call success and failure, which are mutually exclusive and complementary. Usually, the probability of success is denoted as **p** and the probability of failure is denoted as **q** ($q = 1 - p$).

An example of a Bernoulli trial: flipping a coin

Binomial random variable

The number of successes in a sequence of "n" Bernoulli trials is called a **binomial random variable**.

A binomial random variable is the sum of **Bernoulli random variables**.

the outcome of a single trial is a success → Bernoulli random variable=1

the outcome of a single trial is a failure → Bernoulli random variable=0

If:

there are 'n' Bernoulli trials,

X = binomial random variable,

Ys = Bernoulli random variables (taking values of 1 or 0),

then:

$$X = Y_1 + Y_2 + \cdots + Y_{n-1} + Y_n$$

An example of a binomial random variable outcome: we get 4 heads when we flip a coin 10 times

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Binomial distribution

The binomial distribution is based on two assumptions:

- the trials are independent.
- the probability of success is constant for all trials.

A binomial distribution is characterized by two parameters n and p.

$$X \sim B(n, p)$$

Where:

- n – number of trials,
- p – probability of success.

The probability of **x successes in n** trials:

$$p(x) = P(X = x) = \binom{n}{x} \times p^x \times (1-p)^{n-x} = \frac{n!}{(n-x)! \times x!} \times p^x \times (1-p)^{n-x}$$

Mean & Variance of binomial random variable

single trial:

$$\text{mean} = p$$

$$\text{variance} = p \times (1-p) = p \times q$$

'n' trials:

$$\text{mean} = n \times p$$

$$\text{variance} = n \times p \times (1-p) = n \times p \times q$$

Continuous random variable

A continuous random variable can take on any values from a specified range. The probability function of a continuous random variable is called the probability density function or simply density. When we're dealing with a continuous random variable, the probability that the random variable takes on a precisely specified value is always 0. However, we can calculate the probability that a random variable takes on a value between two different values A and B. The probability that the random variable takes on a value between A and B equals the area under the graph of the density function between points A and B.

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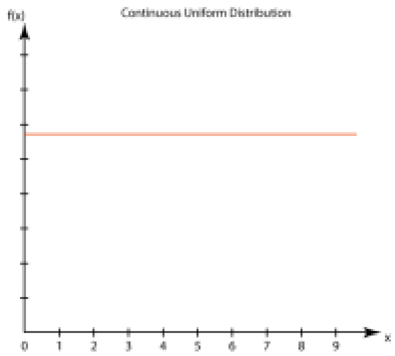
CONTINUOUS UNIFORM DISTRIBUTION

- For any two intervals of the continuous uniform distribution that have the same length, the probability that the random variable takes on the value from any of these two intervals is the same. It is because the density function of the continuous uniform distribution is parallel to the X-axis.

- The density function for a uniform random variable takes the following form:

$$f(X) = \frac{1}{B - A} \text{ if } A < X < B \text{ or } 0 \text{ in other cases}$$



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NORMAL DISTRIBUTION

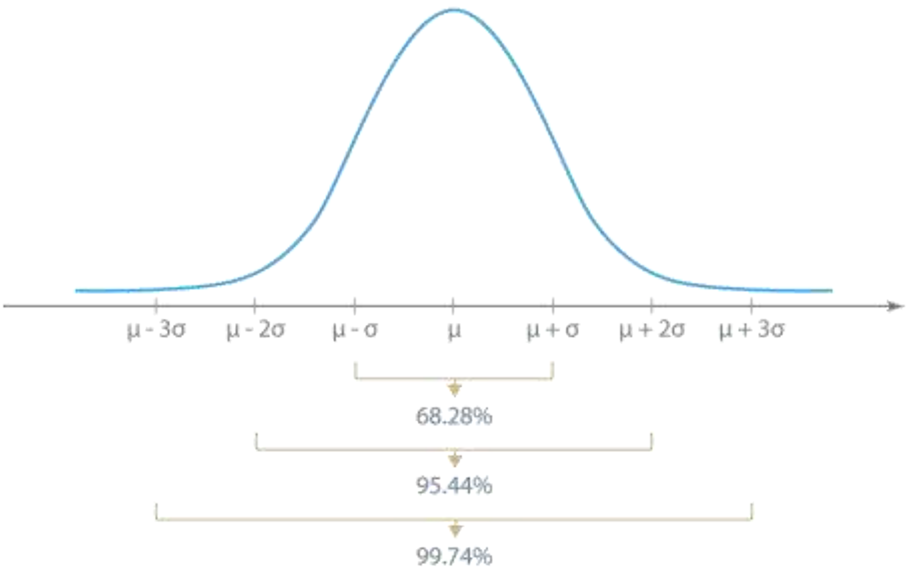
Characteristics of a normal distribution

Key characteristics of a normal distribution:

1. A normal distribution is completely characterized by two parameters: mean and variance (standard deviation):

$$X \sim N(\mu, \sigma)$$

2. A normal density function is symmetrical and bell-shaped, whatever its parameters. Because its graph plots symmetrically, the mean equals the mode and the median. Each normal distribution has a kurtosis of 3.
3. A linear combination of normally distributed random variables also has a normal distribution.



A **univariate distribution** describes a single random variable, for example the return on a company's stock.

A **multivariate normal distribution** describes a group of random variables related to each other. Multivariate distributions are often used in practice, for example in the portfolio theory.

Standard normal distribution

Since the normal distribution is characterized by two parameters (mean and variance), there is an infinite number of normal distributions. The one characterized by the mean equal to 0 and the standard deviation of 1 is called the unit normal distribution or the standard normal distribution. A standard normal random variable is denoted as Z. The probabilities for the standard normal distribution can be found in tables of the cumulative distribution function for the standard normal random variable.

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$$Z \sim N(0,1)$$

For a standard normal random variable:

$$\mu = 0$$

$$\sigma^2 = 1$$

$$\sigma = 1$$

Every normal distribution can be transformed into the standard normal distribution. The adaptation of a normal random variable of any normal distribution to the standard normal random variable is called standardizing. Standardizing is carried out based on this formula:

$$Z = \frac{(X - \mu)}{\sigma}$$

Where:

- X – value of a normal random variable,
- Z –value of the standard normal random variable,
- μ – mean of the normal random variable X,
- σ – standard deviation of the normal random variable X.

Safety-first rules

Safety-first rules are downside risk measures because they concern shortfall risk.

Shortfall risk is the risk that the value of a portfolio falls below a pre-defined threshold level in a specified time period.

According to **Roy's safety-first criterion**, we should choose portfolios characterized by the highest safety-first ratio given by the formula:

$$SFRatio = \frac{E(R_P) - R_t}{\sigma_P}$$

Where:

- SF_{ratio} – safety-first ratio,
- $E(R_p)$ – expected return on the portfolio,
- R_t – shortfall level (minimal return rate on the portfolio accepted by an investor).
- σ_p – portfolio risk.

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Value at Risk (VaR)

Value at Risk is a measure of downside risk because it is the measure that informs us about the minimal amount of money we can lose with a given probability, e.g.

If 5% VaR for a portfolio is USD 500,000 for 1 month →

→ there is a 5% probability that losses will exceed USD 500,000 in one month

Lognormal distribution vs Normal distribution

<u>Lognormal distribution</u>

If X is a random variable that follows a normal distribution, then e^X is a random variable that follows a lognormal distribution.

$$e = \text{Euler number} = 2.718$$

When X is a random variable following a lognormal distribution, then $\ln X$ is a normally distributed random variable.

The lognormal distribution:

1. is skewed to the right.
2. on the left, it is bounded by 0.
3. is described by two parameters of the associated normal distribution (the mean and the variance).

$$\text{mean of a lognormal random variable} = e^{\mu+0.5\sigma^2}$$

$$\text{variance of a lognormal random variable} = \left(e^{2\mu+\sigma^2}\right) \times \left(e^{\sigma^2} - 1\right)$$

Where:

- μ – mean of the associated normal distribution,
- σ^2 –variance of the associated normal distribution.

lognormal distribution ⟷ prices

normal distribution ⟷ returns

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Continuously compounded return (CCR) vs Holding period return (HPR)

$$e^{CCR} = 1 + HPR$$

$$CCR = \ln(1 + HPR)$$

Monte Carlo simulation

Monte Carlo simulation is used to simulate complex processes whose results are hard to predict using analytical methods. This method is based on repeated trials in which you draw pre-defined parameters affecting, e.g. the price of a security. Before we start the simulation, it is necessary to determine the probability distributions for the parameters used in the simulation. Trials are repeated several thousand times and every single time a valuation is conducted. Finally, the expected value of the security is calculated.

Monte Carlo is used among other things to:

- value securities, including complex derivatives and strategies that involve them.
- simulate trading and investment strategies.
- estimate value at risk (also known as VaR).
- value assets with distributions other than normal.

However, there may be some limitations to this method because:

- Monte Carlo simulation is a complex and laborious process (however, with modern computers it can be conducted a lot faster than before).
- The simulation results are only as good as the assumptions that you make.
- Monte Carlo simulation provides only statistical estimates and not exact results.

Historical simulation is based on historical data and therefore it reflects changes in the distribution of risk parameters that affect valuation over time. Each set of parameters that affect the valuation process will take into account these changes. In **Monte Carlo simulation**, each trial is based on the pre-defined probability distributions, so it doesn't respect the kind of changes that take place in these parameters. A big disadvantage of historical simulation are general doubts as to whether the future price of assets can be estimated using historical data. What's more, some rare events may never occur in the historical data set and, thus, they won't be taken into account in the analysis. If, however, such events with a very low probability happen, then the real result may differ significantly from the one that was estimated with the use of historical simulation. Monte Carlo method doesn't have this flaw. In addition, Monte Carlo simulation makes it possible to control the parameters. Thanks to it, we can conduct conditional simulations that give answers to different questions or scenarios we may have.

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Summarizing key concepts:

☐ Probability function & Cumulative distribution function

My summary:

☐ Discrete uniform distribution

My summary:

☐ Binomial distribution

My summary:

☐ Continuous uniform distribution

My summary:

☐ Normal distribution

My summary:

☐ Standard normal distribution

My summary:

☐ Safety-first rules

My summary:

☐ Lognormal distribution vs Normal distribution

My summary:

☐ Monte Carlo simulation

My summary:

Reviewing formulas:

$$p(x) = P(X = x) = \binom{n}{x} \times p^x \times (1-p)^{n-x} = \frac{n!}{(n-x)! \times x!} \times p^x \times (1-p)^{n-x}$$

Write down the formula:

$$\text{SFRatio} = \frac{E(R_P) - R_t}{\sigma_P}$$

Write down the formula:

$$\text{mean of a lognormal random variable} = e^{\mu + 0.5\sigma^2}$$

$$\text{variance of a lognormal random variable} = \left(e^{2\mu + \sigma^2}\right) \times (e^{\sigma^2} - 1)$$

Write down the formula:

$$e^{CCR} = 1 + HPR$$

$$CCR = \ln(1 + HPR)$$

Write down the formula:

Keeping myself accountable:

TABLE 1 | STUDY

When you sit down to study, you may want to **try the Pomodoro Technique** to handle your study sessions: study for 25 minutes, then take a 5-minute break. Repeat this 25+5 study-break sequence all throughout your daily study session.

✔ Tick off as you proceed.

POMODORO TIMETABLE: study-break sequences (25′ + 5′)													
date		date		date		date		date		date		date	
25′		25′		25′		25′		25′		25′		25′	
5′		5′		5′		5′		5′		5′		5′	
25′		25′		25′		25′		25′		25′		25′	
5′		5′		5′		5′		5′		5′		5′	
25′		25′		25′		25′		25′		25′		25′	
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25′		25′		25′		25′		25′		25′		25′	
5′		5′		5′		5′		5′		5′		5′	

TABLE 2 | REVIEW

Never ever neglect revision! Though it's not the most popular thing among CFA candidates, regular revision is what makes the difference. If you want to pass your exam, **schedule & do your review sessions.**

REVIEW TIMETABLE: When did I review this Reading?													
date		date		date		date		date		date		date	
date		date		date		date		date		date		date	