

# Financial Engineering and Risk Management

## Mean Variance Optimization

**Martin Haugh**

**Garud Iyengar**

Columbia University  
Industrial Engineering and Operations Research

# Overview

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- Assets and portfolios
- Quantifying random asset and portfolio returns: mean and variance
- Mean-variance optimal portfolios
- Efficient frontier
- Sharpe ratio and Sharpe optimal portfolios
- Market portfolio
- Capital Asset Pricing Model

# Assets and portfolios

Asset  $\equiv$  anything we can purchase

- Random price  $P(t)$
- Random gross return  $R(t) = \frac{P(t+1)}{P(t)}$
- Random net return:  $r(t) = R(t) - 1 = \frac{P(t+1) - P(t)}{P(t)}$

Total wealth  $W > 0$  distributed over  $d$  assets

- $w_i$  = dollar amount in asset  $i$ :  $w_i > 0 \equiv$  long,  $w_i < 0 \equiv$  short
- Net return on a position  $w$

$$r_w(t) = \frac{\sum_{i=1}^d R_i(t) w_i - \sum_{i=1}^d w_i}{W} = \frac{\sum_{i=1}^d r_i(t) w_i}{\sum_{i=1}^d w_i} = \sum_{i=1}^d r_i(t) \cdot \underbrace{\frac{w_i}{W}}_{x_i}$$

- **portfolio** vector  $\mathbf{x} = (x_1, \dots, x_d)$ : each component can be +ve/-ve
  - $x_i$  = fraction invested in asset  $i \Rightarrow \sum_{i=1}^d x_i = 1$

# How does one deal with randomness?

Random net return on the portfolio  $r_x = \sum_{i=1}^d r_i x_i$

How does one “quantify” random returns ?

- Maximize expected return  $\mathbb{E}[r_x]$ ?
- Should one worry about **spread** around the mean?
- How does one quantify the **spread**?

# Random returns on assets and portfolios

## Parameters defining asset returns

- Mean of asset returns:  $\mu_i = \mathbb{E}[r_i(t)]$
- Variance of asset returns:  $\sigma_i^2 = \mathbf{var}(r_i(t))$
- Covariance of asset returns:  $\sigma_{ij} = \mathbf{cov}(r_i(t), r_j(t)) = \rho_{ij}\sigma_i\sigma_j$
- Correlation of asset returns  $\rho_{ij} = \mathbf{cor}(r_i(t), r_j(t))$

All parameters assumed to be constant over time.

## Parameters defining portfolio returns

- Expected return on a portfolio  $\mathbf{x} = (x_1, \dots, x_d)^\top$

$$\mu_x = \mathbb{E}[r_x(t)] = \sum_{i=1}^d \mathbb{E}[r_i(t)]x_i = \sum_{i=1}^d \mu_i x_i$$

- Variance of the return on portfolio  $\mathbf{x}$ :

$$\sigma_x^2 = \mathbf{var}(r_x(t)) = \mathbf{var}\left(\sum_{i=1}^d r_i x_i\right) = \sum_{i=1}^d \sum_{j=1}^d \mathbf{cov}(r_i(t), r_j(t)) x_i x_j$$

## Example

$d = 2$  assets with Normally distributed returns  $\mathcal{N}(\mu, \sigma^2)$

$$r_1 \sim \mathcal{N}(1, 0.1) \quad r_2 \sim \mathcal{N}(2, 0.5) \quad \mathbf{cor}(r_1, r_2) = -0.25$$

Parameters

$$\begin{aligned} \mu_1 &= 1 & \mu_2 &= 2 \\ \sigma_1^2 &= \mathbf{var}(r_1) = 0.1 & \sigma_2^2 &= \mathbf{var}(r_2) = 0.5 \\ \sigma_{12} &= \mathbf{cov}(r_1, r_2) = \mathbf{cor}(r_1, r_2)\sigma_1\sigma_2 = -0.25\sqrt{0.05} = 0.0559 \end{aligned}$$

Portfolio:  $(x, 1 - x)$

$$\begin{aligned} \mu_x &= \sum_{i=1}^d \mu_i x_i = x + 2(1 - x) \\ \sigma_x^2 &= \sum_{i,j=1}^d \sigma_{ij} x_i x_j = \sum_{i=1}^d \sigma_i^2 x_i^2 + 2 \sum_{j>i} \sigma_{ij} x_i x_j \\ &= 0.1x^2 + 0.5(1 - x)^2 + 2(0.0559)x(1 - x) \end{aligned}$$

# Diversification reduces uncertainty

$d$  assets each with  $\mu_i \equiv \mu$ ,  $\sigma_i \equiv \sigma$ ,  $\rho_{ij} = 0$  for all  $i \neq j$

Two different portfolios

- $\mathbf{x} = (1, 0, \dots, 0)^\top$ : everything invested in asset 1
- $\mathbf{y} = \frac{1}{d}(1, 1, \dots, 1)^\top$ : equal investment in all assets.

Expected returns of the two portfolios

- $\mu_x = \mathbb{E}[\sum_{i=1}^d \mu_i x_i] = \mu_1 = \mu$
- $\mu_y = \mathbb{E}[\sum_{i=1}^d \mu_i y_i] = \frac{1}{d} \sum_{i=1}^d \mu_i = \mu$

Both have the same expected return!

Variance of returns of the two portfolios

- $\sigma_x^2 = \mathbf{var}(\sum_{i=1}^d r_i x_i) = \sigma^2$
- $\sigma_y^2 = \mathbf{var}(\sum_{i=1}^d r_i y_i) = \sum_{i=1}^d \sigma^2 (\frac{1}{d})^2 = \frac{\sigma^2}{d}$

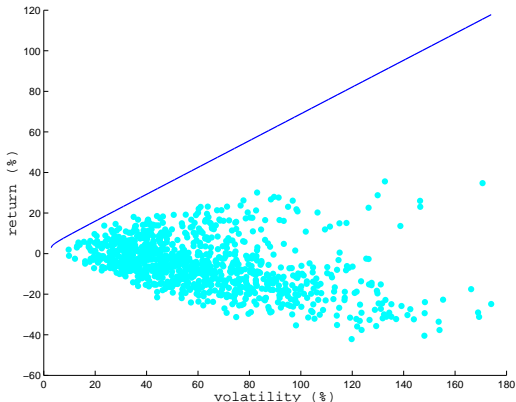
Diversified portfolio has a much lower variance!

# Markowitz mean-variance portfolio selection

Markowitz (1954) proposed that

- “Return” of a portfolio  $\equiv$  Expected return  $\mu_x$
- “Risk” of a portfolio  $\equiv$  volatility  $\sigma_x$

Efficient frontier



Efficient frontier  $\equiv$  max return  
for a given risk

How does one characterize the  
efficient frontier?

How does one compute effi-  
cient/optimal portfolios?



# Mean variance formulations

- Minimize risk ensuring return  $\geq$  target return

$$\begin{aligned} \min_x \quad & \sigma_x^2 \\ \text{s.t.} \quad & \mu_x \geq r \end{aligned} \quad \equiv \quad \begin{aligned} \min_x \quad & \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \sum_{i=1}^d \mu_i x_i \geq r \\ & \sum_{i=1}^d x_i = 1. \end{aligned}$$

- Maximize return ensuring risk  $\leq$  risk budget

$$\begin{aligned} \max_x \quad & \mu_x \\ \text{s.t.} \quad & \sigma_x^2 \leq \bar{\sigma}^2 \end{aligned} \quad \equiv \quad \begin{aligned} \max_x \quad & \sum_{i=1}^d \mu_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \leq \bar{\sigma}^2, \\ & \sum_{i=1}^d x_i = 1. \end{aligned}$$

- Maximize a risk-adjusted return

$$\begin{aligned} \max_x \quad & \mu_x - \tau \sigma_x^2 \\ \text{s.t.} \quad & \sum_{i=1}^d x_i = 1. \end{aligned} \quad \equiv \quad \begin{aligned} \max_x \quad & \sum_{i=1}^d \mu_i x_i - \tau \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \right) \\ \text{s.t.} \quad & \sum_{i=1}^d x_i = 1. \end{aligned}$$

$\tau \equiv$  risk-aversion parameter

# Financial Engineering and Risk Management

Efficient frontier

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# Mean-variance for 2-asset market

$d = 2$  assets

- Asset 1: mean return  $\mu_1$  and variance  $\sigma_1^2$
- Asset 2: mean return  $\mu_2$  and variance  $\sigma_2^2$
- Correlation between asset returns  $\rho$

Portfolio:  $(x, 1 - x)$

$$\mu_x = \sum_{i=1}^d \mu_i x_i = \mu_1 x + \mu_2 (1 - x)$$

$$\begin{aligned}\sigma_x^2 &= \sum_{i,j=1}^d \sigma_{ij} x_i x_j = \sum_{i=1}^d \sigma_i^2 x_i^2 + 2 \sum_{j>i} \sigma_{ij} x_i x_j \\ &= \sigma_1^2 x^2 + \sigma_2^2 (1 - x)^2 + 2\rho\sigma_1\sigma_2 x(1 - x)\end{aligned}$$

# Mean-variance for 2-asset market

Minimize risk formulation for the mean-variance portfolio selection problem

$$\begin{aligned} \min_x \quad & \sigma_1^2 x^2 + \sigma_2^2 (1-x)^2 + 2\rho\sigma_1\sigma_2 x(1-x) \\ \text{s.t.} \quad & \mu_1 x + \mu_2 (1-x) = r \end{aligned}$$

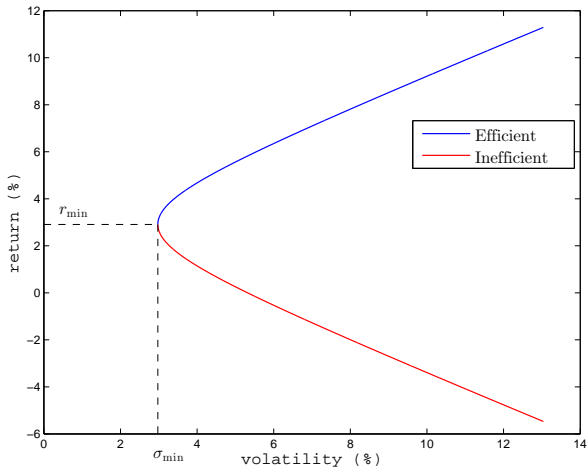
Expected return constraint:  $x = \frac{r - \mu_2}{\mu_1 - \mu_2}$

Variance:

$$\begin{aligned} \sigma_r^2 &= \sigma_1^2 \left( \frac{r - \mu_2}{\mu_1 - \mu_2} \right)^2 + \sigma_2^2 \left( \frac{\mu_1 - r}{\mu_1 - \mu_2} \right)^2 + 2\rho\sigma_1\sigma_2 \left( \frac{r - \mu_2}{\mu_1 - \mu_2} \right) \left( \frac{\mu_1 - r}{\mu_1 - \mu_2} \right) \\ &= ar^2 + br + c \end{aligned}$$

Explicit expression for the variance as a function of target return  $r$ .

# Efficient frontier



Only the top half is efficient! why did we get the bottom?

How does one solve the  $d$  asset problem?

# Computing the optimal portfolio

Mean-variance portfolio selection problem

$$\begin{aligned}\sigma^2(r) = \min_x \quad & \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \sum_{i=1}^d \mu_i x_i = r \\ & \sum_{i=1}^d x_i = 1.\end{aligned}$$

Form the Lagrangian with **Lagrange multipliers**  $u$  and  $v$

$$L = \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j - v \left( \sum_{i=1}^d \mu_i x_i - r \right) - u \left( \sum_{i=1}^d x_i - 1 \right)$$

Setting  $\frac{\partial L}{\partial x_i} = 0$  for  $i = 1, \dots, d$  gives  $d$  equations

$$2 \sum_{j=1}^d \sigma_{ij} x_j - v \mu_i - u = 0, \quad \text{for all } i = 1, \dots, d \quad (*)$$

Can solve the  $d + 2$  equations in  $d + 2$  variables:  $x_1, \dots, x_d$ ,  $u$  and  $v$ .

**Theorem.** A portfolio  $x$  is mean-variance optimal if, and only if, it is **feasible** and there exists  $u$  and  $v$  satisfying  $(*)$ .

# Computing the optimal portfolio

Matrix formulation

$$\underbrace{\begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & \dots & 2\sigma_{1d} & -\mu_1 & -1 \\ 2\sigma_{21} & 2\sigma_{22} & \dots & 2\sigma_{2d} & -\mu_2 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2\sigma_{d1} & 2\sigma_{d2} & \dots & 2\sigma_{dd} & -\mu_d & -1 \\ \mu_1 & \mu_2 & \dots & \mu_d & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \\ v \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r \\ 1 \end{bmatrix}}_{\mathbf{b}}$$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \\ v \\ u \end{bmatrix} = \mathbf{A}^{-1} \mathbf{b}$$

# Two fund theorem

Fix two different target returns:  $r_1 \neq r_2$

Suppose

- $\mathbf{x}^{(1)} = (x_1^{(1)}, \dots, x_d^{(1)})^\top$  optimal for  $r_1$ : Lagrange multipliers  $(v_1, u_1)$
- $\mathbf{x}^{(2)} = (x_1^{(2)}, \dots, x_d^{(2)})^\top$  optimal for  $r_2$ : Lagrange multipliers  $(v_2, u_2)$

Consider any other return  $r$

- Choose  $\beta = \frac{r-r_1}{r_2-r_1}$
- Consider the position:  $\mathbf{y} = (1 - \beta)\mathbf{x}^{(1)} + \beta\mathbf{x}^{(2)}$

$\mathbf{y}$  is a portfolio

$$\sum_{i=1}^d y_i = (1 - \beta) \sum_{i=1}^d x_i^{(1)} + \beta \sum_{i=1}^d x_i^{(2)} = (1 - \beta) + \beta = 1$$

$\mathbf{y}$  is feasible for target return  $r$

$$\sum_{i=1}^d \mu_i y_i = (1 - \beta) \sum_{i=1}^d \mu_i x_i^{(1)} + \beta \sum_{i=1}^d \mu_i x_i^{(2)} = (1 - \beta)r_1 + \beta r_2 = r$$



## Two fund theorem (contd)

Set  $v = (1 - \beta)v_1 + \beta v_2$  and  $u = (1 - \beta)u_1 + \beta u_2$ .

$$\begin{aligned} 2 \sum_{j=1}^d \sigma_{ij} y_j - v \mu_i - u &= \sum_{j=1}^d 2 \sigma_{ij} ((1 - \beta)x_j^{(1)} + \beta x_j^{(2)}) \\ &\quad - \mu_i ((1 - \beta)v_1 + \beta v_2) - ((1 - \beta)u_1 + \beta u_2) \\ &= (1 - \beta) \left( 2 \sum_{j=1}^d \sigma_{ij} x_j^{(1)} - v_1 \mu_i - u_1 \right) \\ &\quad + \beta \left( 2 \sum_{j=1}^d \sigma_{ij} x_j^{(2)} - v_2 \mu_i - u_2 \right) = 0 \end{aligned}$$

$y$  is **optimal** for target return  $r$ !

**Theorem** All efficient portfolios can be constructed by diversifying between any **two efficient portfolios** with different expected returns.

Why are there so many funds in the market?

# Efficient frontier

The optimal portfolio for target return  $r$

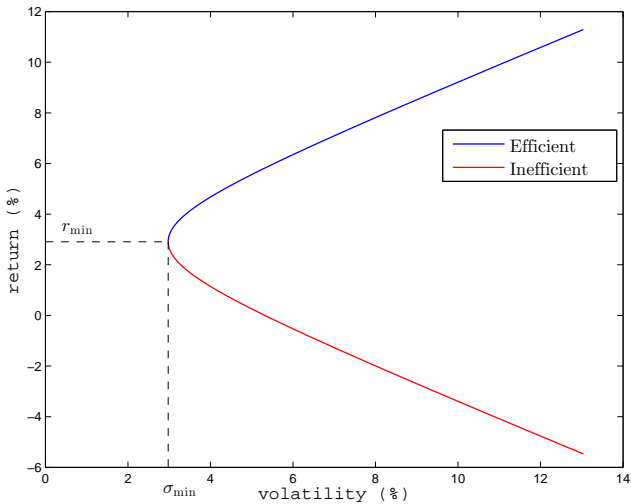
$$\begin{aligned} \mathbf{y}^* &= \left( \frac{r_2 - r}{r_2 - r_1} \right) \mathbf{x}^{(1)} + \left( \frac{r - r_1}{r_2 - r_1} \right) \mathbf{x}^{(2)} \\ &= r \underbrace{\left( \frac{\mathbf{x}^{(2)} - \mathbf{x}^{(1)}}{r_2 - r_1} \right)}_g + \underbrace{\left( \frac{r_2 \mathbf{x}^{(1)} - r_1 \mathbf{x}^{(2)}}{r_2 - r_1} \right)}_h \\ y_i^* &= rg_i + h_i, \quad i = 1, \dots, d. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma^2(r) &= \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} (rg_i + h_i)(rg_j + h_j) \\ &= r^2 \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} g_i g_j \right) + 2r \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} g_i h_j \right) + \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} h_i h_j \right) \end{aligned}$$

The  $d$ -asset frontier has the same structure as the 2-asset frontier.

# Efficient frontier



# Financial Engineering and Risk Management

Mean-variance with a risk-free asset

**Martin Haugh**

**Garud Iyengar**

Columbia University

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# Mean Variance with a risk-free asset

New asset: pays net return  $r_f$  with no risk (deterministic return)

$x_0$  = fraction invested in the risk-free asset

Mean-variance problem:  $x_0$  does **not** contribute to risk.

$$\begin{aligned} \max \quad & (r_f x_0 + \sum_{i=1}^d \mu_i x_i) - \tau \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \right) \\ \text{s. t.} \quad & x_0 + \sum_{i=1}^d x_i = 1. \end{aligned}$$

Only meaningful for  $r \geq r_f$

Substituting  $x_0 = 1 - \sum_{i=1}^d x_i$  we get

$$\max \quad r_f + \sum_{i=1}^d (\mu_i - r_f) x_i - \tau \left( \sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j \right)$$

$\hat{\mu}_i = \mu_i - r_f = \text{excess}$  return of asset  $i$

# Mean-variance optimal portfolio

Taking derivatives we get

$$\hat{\mu}_i - 2\tau \sum_{j=1}^d \sigma_{ij} x_j = 0, \quad i = 1, \dots, d.$$

Matrix formulation

$$2\tau \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \dots & \sigma_{dd} \end{bmatrix}}_V \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}}_{\hat{\mu}} = \underbrace{\begin{bmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_d \end{bmatrix}}_{\hat{\mu}} \Rightarrow \mathbf{x}(\tau) = \frac{1}{2\tau} V^{-1} \hat{\mu}$$

The family of frontier portfolios as a function of  $\tau$ :

$$\left\{ \left( 1 - \sum_{i=1}^d x_i(\tau), \mathbf{x}(\tau) \right) : \tau \geq 0 \right\}$$

# One-fund theorem

The positions in the risky assets in the frontier portfolio

$$\mathbf{x} = \frac{1}{2\tau} \mathbf{V}^{-1} \hat{\boldsymbol{\mu}}$$

do not add up to 1.

Define a **portfolio** of risky assets by dividing  $\mathbf{x}$  by the sum of its components.

$$\mathbf{s}^* = \left( \frac{1}{\sum_{i=1}^d x_i} \right) \mathbf{x} = \left( \frac{1}{\frac{1}{2\tau} \sum_{i=1}^d (\mathbf{V}^{-1} \hat{\boldsymbol{\mu}})_i} \right) \left( \frac{1}{2\tau} \mathbf{V}^{-1} \hat{\boldsymbol{\mu}} \right)$$

The portfolio  $\mathbf{s}^*$  is **independent** of  $\tau$ ! Since  $\sum_{i=1}^d x_i = 1 - x_0$ ,  $\mathbf{x} = (1 - x_0) \mathbf{s}^*$ .

Family of frontier portfolios =  $\{(x_0, (1 - x_0) \mathbf{s}^*) : x_0 \in \mathbb{R}\}$

**Theorem** All efficient portfolios in a market with a risk-free asset can be constructed by diversifying between the risk-less asset and the single portfolio  $\mathbf{s}^*$ .

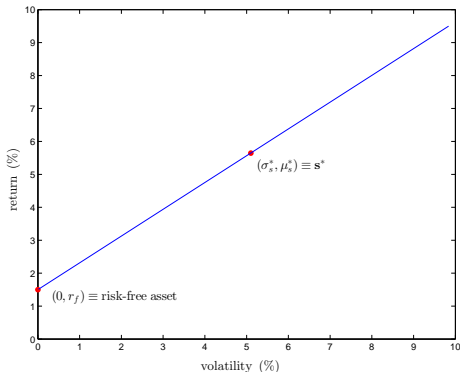
# Efficient frontier with risk-free asset

Return and risk of portfolio  $s^*$ :  $\mu_s^* = \sum_{i=1}^d \mu_i s_i^*$ ,  $\sigma_s^* = \sqrt{\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij}^2 s_i^* s_j^*}$

Return on a generic frontier portfolio:  $x_0$  in risk-free and  $(1 - x_0)$  in  $s^*$

$$\mu_x = x_0 r_f + (1 - x_0) \mu_s^* \quad \sigma_x = (1 - x_0) \sigma_s^*$$

## Efficient Frontier



Straight line with an intercept  $r_f$  at  $\sigma = 0$  and slope

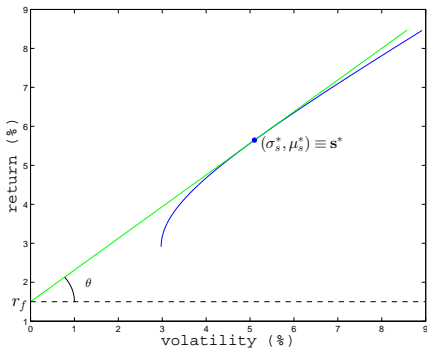
$$m = \frac{\mu_s - r_f}{\sigma_s}$$

How does this relate to the frontier with only risky assets?

Does the portfolio  $s^*$  have an economic interpretation?



# Efficient frontier with risk-free asset



$s^*$  must be an efficient risky portfolio

The efficient frontier with a risk-free asset must be **tangent** to the efficient frontier with only risky assets.

The portfolio  $s^*$  maximizes the angle  $\theta$  or equivalently

$$\tan(\theta) = \frac{\sum_{i=1}^d \mu_i x_i - r_f}{\sqrt{\sum_{i=1}^d \sum_{j=1}^d \sigma_{ij} x_i x_j}} = \frac{\text{expected excess return}}{\text{volatility}}$$

# Sharpe Ratio

**Definition.** The **Sharpe ratio** of a portfolio or an asset is the ratio of the expected excess return to the volatility. The **Sharpe optimal portfolio** is a portfolio that maximizes the Sharpe ratio.

The portfolio  $s^*$  is a Sharpe optimal portfolio

$$s^* = \operatorname{argmax}_{\left\{x: \sum_{i=1}^d x_i = 1\right\}} \left\{ \frac{\mu_x - r_f}{\sigma_x} \right\}$$

Investors diversify between the risk-free asset and the Sharpe optimal portfolio.

The investment in the various risky assets are in fixed proportions ...  
prices/returns should be correlated! This insight leads to the **Capital Asset Pricing Model**.

# **Financial Engineering and Risk Management**

## **Capital Asset Pricing Model**

**Martin Haugh**

**Garud Iyengar**

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Industrial Engineering and Operations Research

# Market Portfolio

**Definition.** Let  $C_i$ ,  $i = 1, \dots, d$ , denote the market capitalization of the  $d$  assets. Then the **market portfolio**  $\mathbf{x}^{(m)}$  is defined as follows.

$$x_i^{(m)} = \frac{C_i}{\sum_{j=1}^d C_j}, \quad i = 1, \dots, d.$$

Let  $\mu_m$  denote the expected net rate of return on the market portfolio, and let  $\sigma_m$  denote the volatility of the market portfolio.

Suppose all investors in the market are mean-variance optimizers. Then all of them invest in the Sharpe optimal portfolio  $\mathbf{s}^*$ . Let

$w^{(k)}$  = wealth of the  $k$ -th investor

$x_0^{(k)}$  = fraction of wealth of the  $k$ -investor in the risk-free asset

Then

$$C_i = \sum_k w^{(k)} (1 - x_0^{(k)}) s_i^*$$

The market portfolio  $\mathbf{x}^{(m)}$  = Sharpe optimal portfolio  $\mathbf{s}^*$ !

# Capital Market Line

Capital market line is another name for the efficient frontier with risk-free asset

Recall: Efficient frontier = line through the points  $(0, r_f)$  and  $(\sigma_m, \mu_m)$

Slope of the capital market line

$$m_{\text{CML}} = \frac{\mu_m - r_f}{\sigma_m} = \text{maximum achievable Sharpe ratio}$$

$m_{\text{CML}}$  is frequently called the price of risk. It is used to compare projects.

**Example.** Suppose the price of a share of an oil pipeline venture is \$875. It is expected to yield \$1000 in one year, but the volatility  $\sigma = 40\%$ . The current interest rate  $r_f = 5\%$ , the expected rate of return on the market portfolio  $\mu_m = 17\%$  and the volatility of the market  $\sigma_m = 12\%$ . Is the oil pipeline worth considering?

$$r_{\text{oil}} = \frac{1000}{875} - 1 = 14\% \ll \bar{r} = r_f + \left( \frac{\mu_m - r_f}{\sigma_m} \right) \sigma = 45\%$$

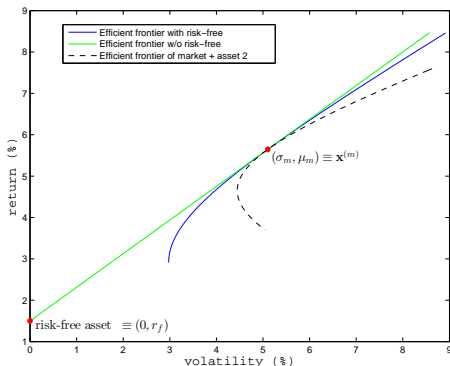
Not worth considering!

# Inferring asset returns from market returns

An asset is a portfolio: asset  $j \equiv \mathbf{x}^{(j)} = (0, \dots, 1, \dots, 0)^\top$ , 1 in the  $j$ -th position.

Diversify between  $\mathbf{x}^{(j)}$  and market portfolio  $\mathbf{x}^{(m)}$ :  $\gamma \mathbf{x}^{(j)} + (1 - \gamma) \mathbf{x}^{(m)}$

- return  $\mu_\gamma = \gamma \mu_j + (1 - \gamma) \mu_m$
- volatility  $\sigma_\gamma = \sqrt{\gamma^2 \sigma_j^2 + (1 - \gamma)^2 \sigma_m^2 + 2 \sigma_{jm} \gamma (1 - \gamma)}$



# All three curves are tangent at $(\sigma_m, r_m)$

Slope of the capital market line

$$m_{\text{CML}} = \frac{\mu_m - r_f}{\sigma_m}$$

Slope of the frontier generated by asset  $j$  and market portfolio  $x^{(m)}$

$$\begin{aligned} \frac{d\mu_\gamma}{d\sigma_\gamma} &= \frac{\frac{d\mu_\gamma}{d\gamma}}{\frac{d\sigma_\gamma}{d\gamma}} = \frac{\mu_j - \mu_m}{\frac{\gamma\sigma_j^2 - (1-\gamma)\sigma_m^2 + (1-\gamma)\sigma_{jm} - \gamma\sigma_{jm}}{\sqrt{\gamma^2\sigma_j^2 + (1-\gamma)^2\sigma_m^2 + 2\sigma_{jm}\gamma(1-\gamma)}}} \\ \left. \frac{d\mu_\gamma}{d\sigma_\gamma} \right|_{\gamma=0} &= \frac{\mu_j - \mu_m}{\frac{\sigma_{jm} - \sigma_m^2}{\sigma_m}} \end{aligned}$$

Equating slopes at  $\gamma = 0$  we get the following result:

$$\mu_j - r_f = \underbrace{\left( \frac{\sigma_{jm}}{\sigma_m^2} \right)}_{\text{beta of asset } j} (\mu_m - r_f)$$

This pricing formula is called the **Capital Asset Pricing Model (CAPM)**.

# Connecting CAPM to regression

Regress the excess return  $r_j - r_f$  of asset  $j$  on the excess market return  $r_m - r_f$

$$(r_j - r_f) = \alpha + \beta(r_m - r_f) + \epsilon_j$$

Parameter estimates

- coefficient  $\beta_j = \frac{\text{cov}(r_j - r_f, r_m - r_f)}{\text{var}(r_m - r_f)} = \frac{\sigma_{jm}}{\sigma_m^2}$
- intercept  $\alpha_j = (\mathbb{E}[r_j] - r_f) - \beta(\mathbb{E}[r_m] - r_f) = (\mu_j - r_f) - \beta(\mu_m - r_f)$ .
- residuals  $\epsilon_j$  and  $(r_m - r_f)$  are uncorrelated, i.e.  $\text{cor}(\epsilon_j, r_m - r_f) = 0$ .
- CAPM implies that  $\alpha_j = 0$  for all assets.
- Effective relation:  $r_j - r_f = \beta_j(r_m - r_f) + \epsilon_j$

Decomposition of risk

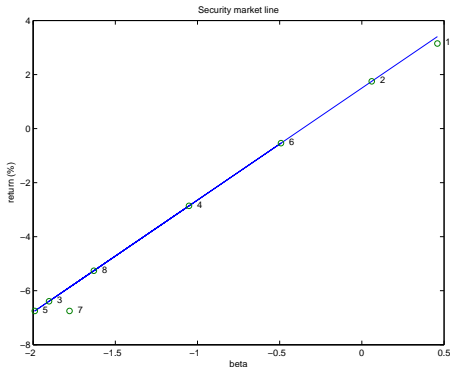
$$\begin{aligned}\text{var}(r_j - r_f) &= \beta_j^2 \text{var}(r_m - r_f) + \text{var}(\epsilon) \\ \sigma_j^2 &= \underbrace{\beta_j^2 \sigma_m^2}_{\text{market risk}} + \underbrace{\text{var}(\epsilon)}_{\text{residual risk}}\end{aligned}$$

Only compensated for taking on market risk and not residual risk



# Security Market Line

Plot of the historical returns on an asset vs  $r_f + \beta(\mu_m - r_f)$



The assets are labeled in the order they appears in the spreadsheet.

All assets should lie on the security line if CAPM holds. So why the discrepancy?

# Assumptions underlying CAPM

- All investors have identical information about the uncertain returns.
- All investors are mean-variance optimizers (or the returns are Normal)
- The markets are in **equilibrium**.

Leveraging deviations from the security market line

- Jensen's index or alpha

$$\alpha = (\hat{\mu}_j - r_f) - \beta_j(\hat{\mu}_m - r_f)$$

hold long if positive, short otherwise

- Sharpe ratio of a stock

$$s_j = \frac{\hat{\mu}_j - r_f}{\hat{\sigma}_j}$$

hold long if  $> m_{\text{MCM}}$ , short otherwise.

# CAPM as a pricing formula

Suppose the payoff from an investment in 1yr is  $X$ . What is the fair price for this investment.

Let  $r_X = \frac{X}{P} - 1$  denote the net rate of return on  $X$ . The beta of  $X$  is given by

$$\beta_X = \frac{\text{cov}(r_X, r_m)}{\sigma_m^2} = \frac{1}{P} \frac{\text{cov}(X, r_m)}{\sigma_m^2}$$

Suppose CAPM holds. Then  $\mu_X = \mathbb{E}[r_X]$  must lie on the security market line, i.e.

$$\begin{aligned}\mu_X &= r_f + \beta_X(r_m - r_f) \\ \frac{\mathbb{E}[X]}{P} - 1 &= r_f + \frac{1}{P} \frac{\text{cov}(X, r_m)}{\text{var}(r_m)} (\mu_m - r_f)\end{aligned}$$

Rearranging terms:

$$P = \frac{\mathbb{E}[X]}{1 + r_f} - \frac{\text{cov}(X, r_m)}{(1 + r_f)\text{var}(r_m)} (\mu_m - r_f)$$