Financial Engineering & Risk Management

The Multi-Period Binomial Model

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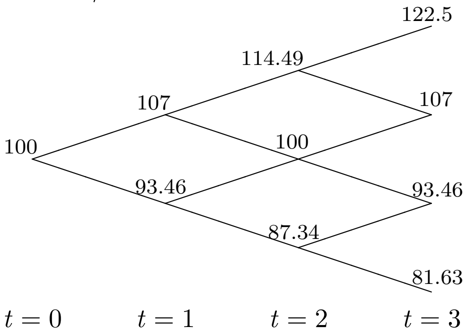
A 3-period Binomial Model

Recall $R = 1.01$ and $u = 1/d = 1.07$.

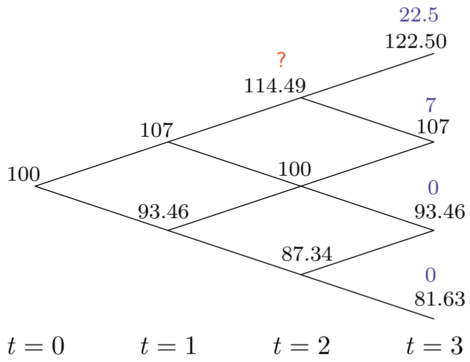


Just a series of 1-period models spliced together!

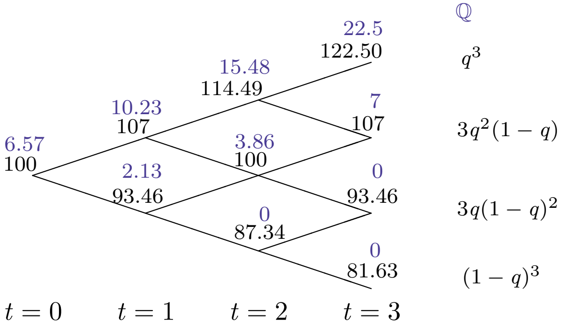
- all the results from the 1-period model apply
- just need to multiply 1-period probabilities along branches to get probabilities in multi-period model.

Pricing a European Call Option

Assumptions: expiration at $t = 3$, strike $= \$100$ and $R = 1.01$.



Pricing a European Call Option



- We can also calculate the price as

$$C_0 = \frac{1}{R^3} \mathsf{E}_0^{\mathbb{Q}} \left[\max(S_T - 100,\ 0) \right] \tag{1}$$

  - this is risk-neutral pricing in the binomial model
  - avoids having to calculate the price at every node.
- How would you find a replicating strategy?
  - to be defined and discussed in another module.

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What's Going On?

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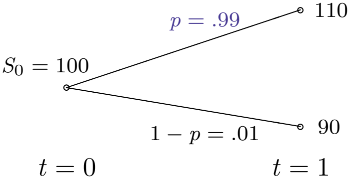
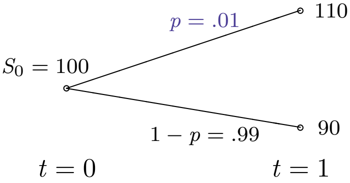
What's Going On?

- Stock ABC



- Stock XYZ



Question: What is the price of a call option on ABC with strike $K = \$100$?

Question: What is the price of a call option on XYZ with strike $K = \$100$?

What's Going On?

- Saw earlier

$$\begin{aligned} C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\ &= \frac{1}{R}\left[q\,C_u + (1-q)\,C_d\right] \\ &= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1] \end{aligned}$$

- So it appears that p doesn't matter!
- This is true ...
- ... but it only appears surprising because we are asking the wrong question!

Another Surprising Result?

$R = 1.02$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price: K = 95

			24.10
		19.22	11.00
	14.76	7.08	0.00
11.04	4.56	0.00	0.00
t=0	t=1	t=2	t=3

$R = 1.04$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price: K = 95

			24.10
		21.01	11.00
	18.19	8.76	0.00
15.64	6.98	0.00	0.00
t=0	t=1	t=2	t=3

Question: So the option price increases when we increase R. Is this surprising?

(See *"Investment Science"* (OUP) by D. G. Luenberger for additional examples on the binomial model.)

Existence of Risk-Neutral Probabilities $\Leftrightarrow$ No-Arbitrage

Recall our analysis of the binomial model:

- no arbitrage $\Leftrightarrow d < R < u$
- any derivative security with time T payoff, C_T, can be priced using

$$C_0 = \frac{1}{R^n} \mathsf{E}_0^{\mathbb{Q}}[C_T] \tag{2}$$

where $q > 0$, $1 - q > 0$ and $n = \#$ of periods.

(If Δt is the length of a period, then $T = n \times \Delta t$.)

In fact for any model if there exists a risk-neutral distribution, $\mathbb{Q}$, such that (2) holds, then arbitrage cannot exist. Why?

Reverse is also true: if there is no arbitrage then a risk-neutral distribution exists.

Together, these last two statements are often called the first fundamental theorem of asset pricing.

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Pricing American Options

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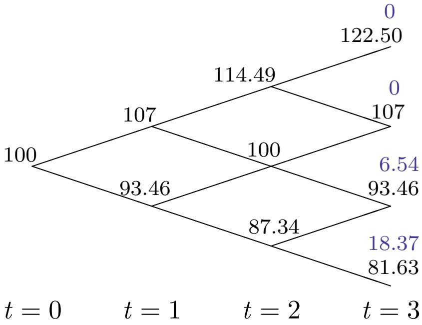
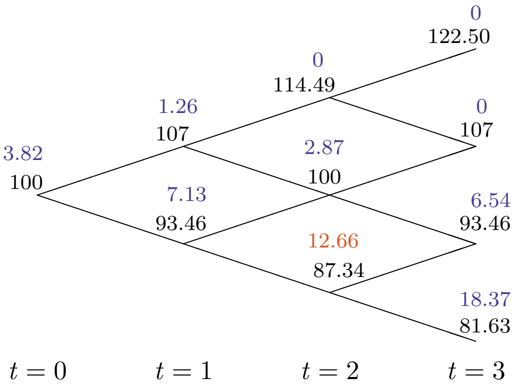
Pricing American Options

- Can also price American options in same way as European options
  - but now must also check if it's optimal to early exercise at each node.
- But recall never optimal to early exercise an American call option on non-dividend paying stock.

e.g. Price American put option: expiration at $t = 3$, $K = \$100$ and $R = 1.01$.



Pricing American Options



- Price option by working backwards in binomial the lattice.

$$\text{e.g.} \qquad 12.66 = \max\left[12.66,\ \frac{1}{R}\left(q \times 6.54 + (1-q) \times 18.37\right)\right]$$

A Simple Die-Throwing Game

Consider the following game:

1. You can throw a fair 6-sided die up to a maximum of three times.
2. After any throw, you can choose to 'stop' and obtain an amount of money equal to the value you threw.

   e.g. if 4 thrown on second throw and choose to 'stop', then obtain $\$4$.

Question: If you are risk-neutral, how much would you pay to play this game?

Solution: Work backwards, starting with last possible throw:

1. You have just 1 throw left so fair value is 3.5.
2. You have 2 throws left so must figure out a strategy determining what to do after 1^{st} throw. We find

$$\text{fair value} = \frac{1}{6} \times (4 + 5 + 6) + \frac{1}{2} \times 3.5 = 4.25.$$

3. Suppose you are allowed 3 throws. Then ...

Question: What if you could throw the die 1000 times?

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Replicating Strategies in the Binomial Model

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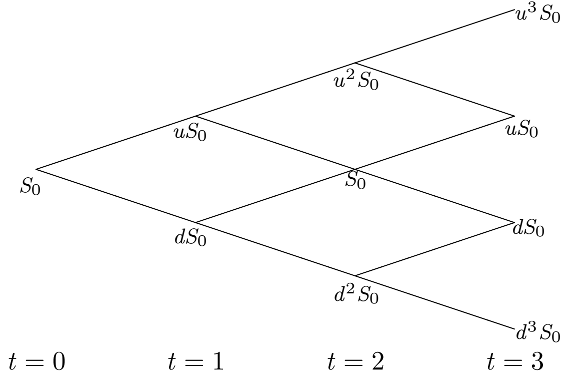
Trading Strategies in the Binomial Model

- Let S_t denote the stock price at time t.
- Let B_t denote the value of the cash-account at time t
  - assume without any loss of generality that $B_0 = 1$ so that $B_t = R^t$
  - so now explicitly viewing the cash account as a security.
- Let x_t denote # of shares held between times $t-1$ and t for $t = 1, \ldots, n$.
- Let y_t denote # of units of cash account held between times $t-1$ and t for $t = 1, \ldots, n$.
- Then $\boldsymbol{\theta}_t := (x_t, y_t)$ is the portfolio held:
  (i) immediately **after** trading at time $t-1$ so it is known at time $t-1$
  (ii) and immediately **before** trading at time t.
- $\boldsymbol{\theta}_t$ is also a random process and in particular, a **trading strategy**.

Trading Strategies in the Binomial Model



Self-Financing Trading Strategies

Definition. The value process, $V_t(\boldsymbol{\theta})$, associated with a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, is defined by

$$V_t = \begin{cases} x_1 S_0 + y_1 B_0 & \text{for } t = 0 \\ x_t S_t + y_t B_t & \text{for } t \geq 1. \end{cases} \tag{3}$$

Definition. A self-financing trading strategy is a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. In particular, a self-financing strategy satisfies

$$V_t \;=\; x_{t+1} S_t + y_{t+1} B_t \,, \qquad t = 1, \ldots, n-1. \tag{4}$$

The definition states that the value of a self-financing portfolio just before trading is equal to the value of the portfolio just after trading

– so no funds have been deposited or withdrawn.

Self-Financing Trading Strategies

Proposition. If a trading strategy, $\boldsymbol{\theta}_t$, is self-financing then the corresponding value process, V_t, satisfies

$$V_{t+1} - V_t \;=\; x_{t+1}\,(S_{t+1} - S_t) \;+\; y_{t+1}\,(B_{t+1} - B_t)$$

so that changes in portfolio value can only be due to capital gains or losses and not the injection or withdrawal of funds.

Proof. For $t \geq 1$ we have

$$\begin{aligned} V_{t+1} - V_t &= (x_{t+1}S_{t+1} + y_{t+1}B_{t+1}) \;-\; (x_{t+1}S_t + y_{t+1}B_t) \\ &= x_{t+1}(S_{t+1} - S_t) \;+\; y_{t+1}(B_{t+1} - B_t) \end{aligned}$$

and for $t = 0$ we have

$$\begin{aligned} V_1 - V_0 &= (x_1S_1 + y_1B_1) - (x_1S_0 + y_1B_0) \\ &= x_1(S_1 - S_0) + y_1(B_1 - B_0). \end{aligned}$$

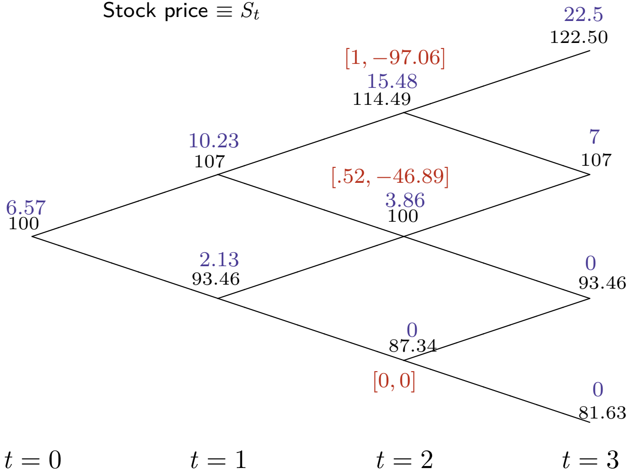
$\square$

Risk-Neutral Price $\equiv$ Price of Replicating Strategy

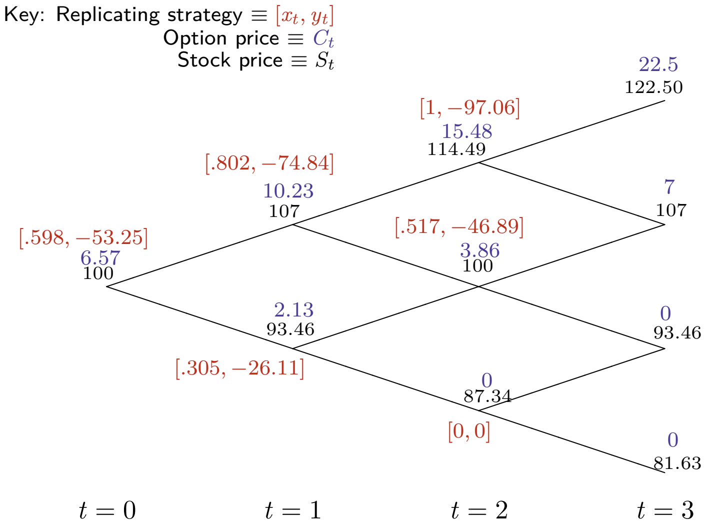
- We have seen how to price derivative securities in the binomial model.
- The key to this was the use of the 1-period risk neutral probabilities.
- But we first priced options in 1-period models using a replicating portfolio
  - and we did this without needing to define risk-neutral probabilities.
- In the multi-period model we can do the same, i.e., can construct a self-financing trading strategy that replicates the payoff of the option
  - this is called dynamic replication.
- The initial cost of this replicating strategy must equal the value of the option
  - otherwise there's an arbitrage opportunity.
- The dynamic replication price is of course equal to the price obtained from using the risk-neutral probabilities and working backwards in the lattice.
- And at any node, the value of the option is equal to the value of the replicating portfolio at that node.

The Replicating Strategy For Our European Option

Key: Replicating strategy $\equiv [x_t, y_t]$
Option price $\equiv C_t$
Stock price $\equiv S_t$



The Replicating Strategy For Our European Option



e.g. $.802 \times 107 + (-74.84) \times 1.01 = 10.23$ at upper node at time $t = 1$

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Including Dividends

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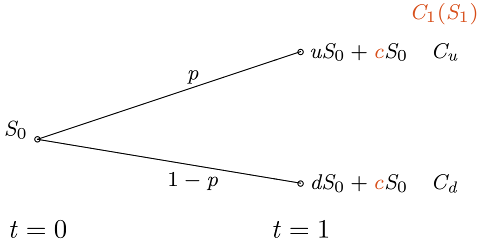
Including Dividends



- Consider again 1-period model and assume stock pays a proportional dividend of cS_0 at $t = 1$.
- No-arbitrage conditions are now $d + c < R < u + c$.
- Can use same replicating portfolio argument to find price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$.
- Set up replicating portfolio as before:

$$
\begin{aligned}
uS_0x + cS_0x + Ry &= C_u \\
dS_0x + cS_0x + Ry &= C_d
\end{aligned}
$$

Derivative Security Pricing with Dividends

- Solve for x and y as before and then must have $C_0 = xS_0 + y$.
- Obtain

$$\begin{aligned} C_0 &= \frac{1}{R}\left[\frac{R-d-c}{u-d}C_u + \frac{u+c-R}{u-d}C_d\right] \qquad (5) \\ &= \frac{1}{R}\left[q C_u + (1-q) C_d\right] \\ &= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1]. \end{aligned}$$

- Again, can price any derivative security in this 1-period model.
- Multi-period binomial model assumes a proportional dividend in each period
  - so dividend of cS_i is paid at $t = i+1$ for each i.
- Then each embedded 1-period model has identical risk-neutral probabilities
  - and derivative securities priced as before.
- In practice dividends are not paid in every period
  - and are therefore just a little more awkward to handle.

The Binomial Model with Dividends

- Suppose the underlying security does not pay dividends. Then

$$S_0 = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right] \tag{6}$$

  - this is just risk-neutral pricing of European call option with $K = 0$.

- Suppose now underlying security pays dividends in each time period.
- Then can check (6) no longer holds.
- Instead have

$$S_0 = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n} + \sum_{i=1}^{n} \frac{D_i}{R^i}\right] \tag{7}$$

  - D_i is the dividend at time i
  - and S_n is the ex-dividend security price at time n.

- Don't need any new theory to prove (7)
  - it follows from risk-neutral pricing and observing that dividends and S_n may be viewed as a portfolio of securities.

Viewing a Dividend-Paying Security as a Portfolio

- To see this, we can view the i^{th} dividend as a separate security with value

$$P_i \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{D_i}{R^i}\right].$$

- Then owner of underlying security owns a "portfolio" of securities at time 0
  - value of this "portfolio" is $\sum_{i=1}^n P_i \;+\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right]$.
- But value of underlying security is S_0.
- Therefore must have

$$S_0 \;=\; \sum_{i=1}^n P_i \;+\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n}{R^n}\right]$$

which is (7).

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Pricing Forwards and Futures

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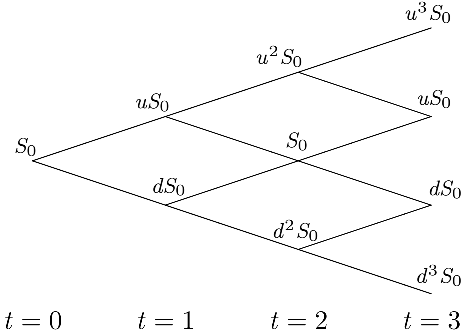
Pricing Forwards in the Binomial Model

- Have an n-period binomial model with $u = 1/d$.



- Consider now a forward contract on the stock that expires after n periods.
- Let G_0 denote date $t = 0$ "price" of the contract.
- Recall G_0 is chosen so that contract is initially worth zero.

Pricing Forwards in the Binomial Model

- Therefore obtain

$$0 = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n - G_0}{R^n}\right]$$

so that

$$G_0 = \mathsf{E}_0^{\mathbb{Q}}\left[S_n\right]. \tag{8}$$

- Again, (8) holds whether the underlying security pays dividends or not.

What is a Futures "Price"?

- Consider now a futures contract on the stock that expires after n periods.
- Let F_t be the date t "price" of the futures contract for $0 \leq t \leq n$.
- Then $F_n = S_n$. Why?
- A common misconception is that:

  (i) F_t is how much you must pay at time t to buy one contract

  (ii) or how much you receive if you sell one contract

  This is **false!**
- A futures contract always costs nothing.
- The "price", F_t is only used to determine the cash-flow associated with holding the contract
  - so that $\pm(F_t - F_{t-1})$ is the payoff received at time t from a long or short position of one contract held between $t-1$ and t.
- In fact a futures contract can be characterized as a security that:

  (i) is always worth zero

  (ii) and that pays a dividend of $(F_t - F_{t-1})$ at each time t.

Pricing Futures in the Binomial Model

- Can compute time $t = n - 1$ futures price, F_{n-1}, by solving

$$0 \quad = \quad \mathsf{E}_{n-1}^{\mathbb{Q}}\left[\frac{F_n - F_{n-1}}{R}\right]$$

  to obtain $F_{n-1} = \mathsf{E}_{n-1}^{\mathbb{Q}}[F_n]$.

- In general we have $F_t = \mathsf{E}_t^{\mathbb{Q}}[F_{t+1}]$ for $0 \leq t < n$ so that

$$\begin{aligned} F_t &= \mathsf{E}_k^{\mathbb{Q}}[F_{t+1}] \\ &= \mathsf{E}_t^{\mathbb{Q}}[\mathsf{E}_{t+1}^{\mathbb{Q}}[F_{t+2}]] \\ &\vdots \qquad \vdots \\ &= \mathsf{E}_t^{\mathbb{Q}}[\mathsf{E}_{t+1}^{\mathbb{Q}}[\ \cdots\ \mathsf{E}_{n-1}^{\mathbb{Q}}[F_n]]]. \end{aligned}$$

Pricing Futures in the Binomial Model

- Law of iterated expectations then implies $F_t \;=\; \mathsf{E}_t^{\mathbb{Q}}\left[F_n\right]$
  - so the futures price process is a $\mathbb{Q}$-martingale.
- Taking $t = 0$ and using $F_n = S_n$ we also have

$$F_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[S_n\right]. \tag{9}$$

- Note that (9) holds whether the security pays dividends or not
  - dividends only enter through $\mathbb{Q}$.
- Comparing (8) and (9) and we see that $F_0 = G_0$ in the binomial model
  - not true in general.

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The Black-Scholes Model

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The Black-Scholes Model

Black and Scholes assumed:

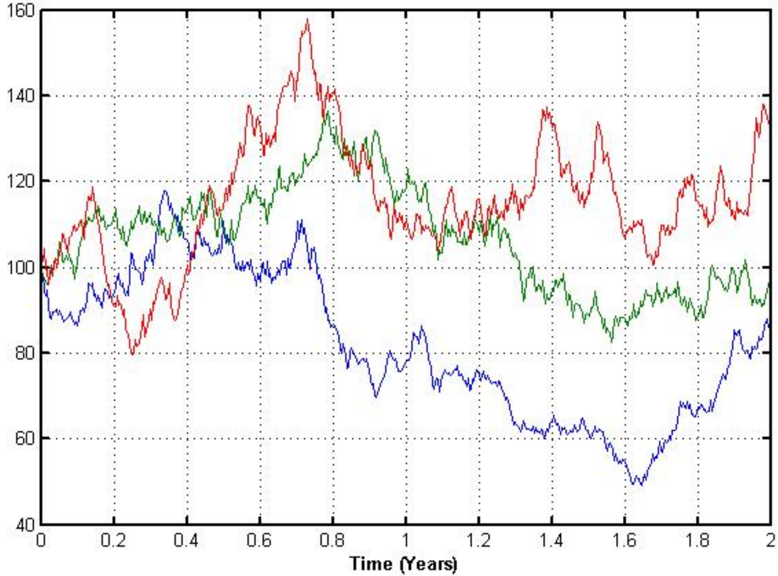
1. A continuously-compounded interest rate of r.
2. Geometric Brownian motion dynamics for the stock price, S_t, so that

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

   where W_t is a standard Brownian motion.

3. The stock pays a dividend yield of c.
4. Continuous trading with no transactions costs and short-selling allowed.

Sample Paths of Geometric Brownian Motion



The Black-Scholes Formula

- The Black-Scholes formula for the price of a European call option with strike K and maturity T is given by

$$C_0 \;=\; S_0 e^{-cT} N(d_1) \;-\; K e^{-rT} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = \mathsf{P}(N(0,1) \le d)$.

- Note that μ does not appear in the Black-Scholes formula
  - just as p is not used in option pricing calculations for the binomial model.
- European put option price, P_0, can be calculated from put-call parity

$$P_0 + S_0 e^{-cT} \;=\; C_0 + K e^{-rT}.$$

The Black-Scholes Formula

- Black-Scholes obtained their formula using a similar replicating strategy argument to the one we used for the binomial model.
- In fact, can show that under the Black-Scholes GBM model

$$C_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K,\,0)\right]$$

where under $\mathbb{Q}$

$$S_t \;=\; S_0\, e^{(r-c-\sigma^2/2)t+\sigma W_t}.$$

Calibrating a Binomial Model

- Often specify a binomial model in terms of Black-Scholes parameters:
  1. r, the continuously compounded interest rate.
  2. σ, the annualized volatility.
- Can convert them into equivalent binomial model parameters:
  1. $R_n = \exp\left(r\frac{T}{n}\right)$, where $n =$ number of periods in binomial model
  2. $R_n - c_n = \exp\left((r-c)\frac{T}{n}\right) \approx 1 + r\frac{T}{n} - c\frac{T}{n}$
  3. $u_n = \exp\left(\sigma\sqrt{\frac{T}{n}}\right)$
  4. $d_n = 1/u_n$

  and now price European and American options, futures etc. as before.
- Then risk-neutral probabilities calculated as

$$q_n = \frac{e^{(r-c)\frac{T}{n}} - d_n}{u_n - d_n}.$$

- Spreadsheet calculates binomial parameters this way
  - binomial model prices converge to Black-Scholes prices as $n \to \infty$.

The Binomial Model as $\Delta t \to 0$

- Consider a binomial model with n periods
  - each period corresponds to time interval of $\Delta t := T/n$.
- Recall that we can calculate European option price with strike K as

$$C_0 = \frac{1}{R^n} \mathsf{E}_0^{\mathbb{Q}} \left[\max(S_T - K,\ 0) \right] \tag{10}$$

- In the binomial model can write (10) as

$$\begin{aligned} C_0 &= \frac{1}{R_n^n} \sum_{j=0}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} \max(S_0 u_n^j d_n^{n-j} - K,\ 0) \\ &= \frac{S_0}{R_n^n} \sum_{j=\eta}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} u_n^j d_n^{n-j} - \frac{K}{R_n^n} \sum_{j=\eta}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} \end{aligned}$$

  where $\eta := \min\{ j : S_0 u_n^j d_n^{n-j} \geq K \}$.

- Can show that if $n \to \infty$ then C_0 converges to the Black-Scholes formula.

Some History

- Bachelier (1900) perhaps first to model Brownian motion
  - modeled stock prices on the Paris Bourse
  - predated Einstein by 5 years.
- Samuelson (1965) rediscovered the work of Bachelier
  - proposed geometric Brownian motion as a model for security prices
  - succeeded in pricing some kinds of warrants
  - was Merton's doctoral adviser
- Itô (1950's) developed the Itô or stochastic calculus
  - the main mathematical tool in finance
  - Itô's Lemma used later by Black-Scholes-Merton
  - Doeblin (1940) recently credited with independently developing stochastic calculus
- Black-Scholes-Merton (early 1970's) published their papers
- Many other influential figures
  - Thorpe (card-counting and perhaps first to discover Black-Scholes formula?)
  - Cox and Ross
  - Harrison and Kreps
  - . . .

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An Example: Pricing a European Put on a Futures Contract

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Pricing a European Put on a Futures Contract

- We can also price an option on a futures contract.
- In fact many of the most liquid options are options on futures contracts

  e.g. S&P 500, Eurostoxx 50, FTSE 100 and Nikei 225.

  - in these cases the underlying security is not actually traded.
- Consider the following parameters:

  $S_0 = 100$, $n = 10$ periods, $r = 2\%$, $c = 1\%$ and $\sigma = 20\%$

  futures expiration $=$ option expiration $= T = .5$ years.
- Futures price lattice obtained using $S_n = F_n$ and then

$$F_t = \mathsf{E}_t[F_{t+1}] \quad \text{for } 0 \le t < n.$$

- Obtain a put option value of 5.21.

Pricing a European Put on a Futures Contract

- In practice we don't need a model to price liquid options
  - market forces, i.e. supply and demand, determines the price
  - which in this case amounts to determining σ or the implied volatility.
- Models are required to hedge these options however
  - and price exotic or illiquid derivative securities.
- Will return to this near end of course.