1) The k-th frame is taken while the camera is moving from $o(k)$ to $o(k+1)$. Therefore, the effective point spread function will be given by:

$$h_k(x) = \int_k^{k+1} \tau_{o(t)} box(x)\, dt$$

where τ_p is the (2D) translation function as defined in the lectures, and the box (2D) function represents a single's sensor exposure – as assumed the camera optics (PSF) to be ideal anti-aliasing filer.

2) Using the Fourier transform linearity property:

$$H_k(\xi) = \big(\mathcal{F}(h_k)\big)(\xi) = \mathcal{F}\left(\int_k^{k+1} \tau_{o(t)} box\, dt\right)(\xi) = \int_k^{k+1} \mathcal{F}\big(\tau_{o(t)} box\big)(\xi)\, dt$$

Using the known fact that $\mathcal{F}(box)(\xi) = sinc(\xi)$ and the Fourier transform of a function under translation:

$$= \int_k^{k+1} sinc(\xi) e^{-2\pi i \xi^T o(t)}\, dt$$

3) Using the convolution theorem:

$$G_k[\omega] = \mathcal{F}\big((f * h_k)\big)[\omega] = \big(\mathcal{F}(f) \cdot \mathcal{F}(h_k)\big)[\omega]$$

Plugging the result from the previous question:

$$= F[\omega] \cdot \left(\int_k^{k+1} sinc(\omega) e^{-2\pi i \omega^T o(t)}\, dt\right) = F[\omega] \cdot \left(\int_k^{k+1} e^{-2\pi i \omega^T o(t)}\, dt\right) sinc(\omega)$$

Define the kernel to be: $P_k[\omega] = \int_k^{k+1} e^{-2\pi i \omega^T o(t)}\, dt$, and we get:

$$G_k[\omega] = F[\omega] \cdot P_k[\omega] sinc(\omega)$$

4) Using triangle inequality:

$$|P_k[\omega]| = \left|\int_k^{k+1} e^{-2\pi i \omega^T o(t)}\, dt\right| \le \int_k^{k+1} \left|e^{-2\pi i \omega^T o(t)}\right| dt$$

Since $\left|e^{i\theta}\right| = 1$, we get:

$$= \int_k^{k+1} 1\, dt = (k+1) - k = 1$$

5) We substitute the given $o(t)$:

$$P_k[\omega] = \int_k^{k+1} e^{-2\pi i\omega^T(t-k-0.5)ve_1}\, dt = \int_k^{k+1} e^{-2\pi i(t-k-0.5)v\omega_1}\, dt$$

$$= \int_k^{k+1} e^{-2\pi itv\omega_1} \cdot e^{2\pi iv\omega_1(k+0.5)}\, dt = e^{2\pi iv\omega_1(k+0.5)} \int_k^{k+1} e^{-2\pi itv\omega_1}\, dt$$

$$= e^{2\pi iv\omega_1(k+0.5)} \frac{1}{-2\pi iv\omega_1}\left[e^{-2\pi itv\omega_1}\Big|_k^{k+1}\right]$$

$$= e^{2\pi iv\omega_1(k+0.5)} \frac{e^{-2\pi iv\omega_1(k+1)} - e^{-2\pi iv\omega_1 k}}{-2\pi iv\omega_1}$$

$$= e^{2\pi iv\omega_1(k+0.5)} \frac{e^{-2\pi iv\omega_1 k}\left(e^{-2\pi iv\omega_1} - 1\right)}{-2\pi iv\omega_1} = \frac{e^{\pi iv\omega_1}\left(1 - e^{-2\pi iv\omega_1}\right)}{2\pi iv\omega_1}$$

$$= \frac{e^{\pi iv\omega_1} - e^{-\pi iv\omega_1}}{2\pi iv\omega_1} = \frac{\sin(\pi v\omega_1)}{\pi v\omega_1} = sinc(v\omega_1)$$

Where the last two equalities are based on the formulas:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

Therefore,

$$|P_k[\omega]| = |sinc(v\omega_1)|$$

Unlike our expectations, the relationship is not monotonic. Instead, we have a sinusoidal relation with monotonic decaying trend.

6) We substitute the given $o(t)$:

$$P_k[\omega] = \int_k^{k+1} e^{-2\pi i\omega^T o(t)}\, dt = \int_k^{k+1} e^{-2\pi i\omega^T(q+tv)}\, dt = \int_k^{k+1} e^{-2\pi i\omega^T q} \cdot e^{-2\pi i\omega^T vt}\, dt$$

$$= e^{-2\pi i\omega^T q} \cdot \int_k^{k+1} e^{-2\pi i\omega^T vt}\, dt = e^{-2\pi i\omega^T q} \cdot \frac{1}{-2\pi i\omega^T v}\left[e^{-2\pi i\omega^T vt}\Big|_k^{k+1}\right]$$

$$= e^{-2\pi i\omega^T q} \cdot \frac{1}{-2\pi i\omega^T v}\left[e^{-2\pi i\omega^T v(k+1)} - e^{-2\pi i\omega^T vk}\right]$$

$$= e^{-2\pi i\omega^T q} \cdot \frac{1}{-2\pi i\omega^T v} \cdot e^{-2\pi i\omega^T vk}\left[e^{-2\pi i\omega^T v} - 1\right]$$

$$= \frac{1 - e^{-2\pi i\omega^T v}}{2\pi i\omega^T v} \cdot e^{-2\pi i\omega^T(q+vk)} = \frac{e^{\pi i\omega^T v} - e^{-\pi i\omega^T v}}{2\pi i\omega^T v} \cdot e^{-\pi i\omega^T v} \cdot e^{-2\pi i\omega^T(q+vk)}$$

$$= sinc(\omega^T v) \cdot e^{-2\pi i\omega^T(q+v(k+0.5))}$$

Note that for the previous question we need to define:

$$v = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \qquad q = -(k+0.5)\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$$

substituting those values we indeed get the result of the previous question.

7) As a motivation we use the result of question 3:

$$G_k[\omega] = F[\omega] \cdot P_k[\omega] sinc(\omega) \quad \Longrightarrow \quad F[\omega] = \frac{G_k[\omega]}{P_k[\omega] sinc(\omega)}$$

For a naïve solution we could estimate the original image as an average of the blurred images:

$$F[\omega] \overset{naive}{\approx} \frac{1}{n}\sum_{k=1}^{n} \frac{G_k[\omega]}{P_k[\omega] sinc(\omega)} = \frac{1}{n sinc(\omega)} \sum_{k=1}^{n} \frac{G_k[\omega]}{P_k[\omega]}$$

The original image could then be calculated using the inverse Fourier transform:

$$f(x) \approx \mathcal{F}^{-1}\left(\frac{1}{sinc(\omega)} \sum_{k=1}^{n} \frac{1}{P_k[\omega]} \cdot G_k[\omega]\right) \overset{\text{def}}{=} \mathcal{F}^{-1}\left(\frac{1}{sinc(\omega)} \sum_{k=1}^{n} w_k[\omega] \cdot G_k[\omega]\right)$$

In our case, since the movement is of constant velocity we have:

$$\forall k, t \in [k, k+1): \quad o(t) = q_k + t v_k$$

$$P_k[\omega] = \ sinc(\omega^T v_k) \cdot e^{-2\pi i \omega^T (q_k + v_k (k+0.5))}$$

And since the images are all pre-aligned, we have:

$$\forall k: \quad o(k+0.5) = 0 \quad \Longrightarrow \quad q_k + (k+0.5) v_k = 0 \quad \Longrightarrow \quad q_k = -(k+0.5) v_k$$

By combining these results:

$$P_k[\omega] = sinc(\omega^T v_k) \cdot e^{-2\pi i \omega^T (-(k+0.5) v_k + v_k (k+0.5))} = sinc(\omega^T v_k)$$

Therefore, out weights become:

$$w_k[\omega] = \frac{1}{n \cdot sinc(\omega^T v_k)}$$

This means that for each frequency, frames that have larger magnitude are less attenuated by the movement blurring. Since we cannot compute w_k (v_k are unknowns) we can estimate their effect using the magnitude of the frames:

$$\widetilde{w}_k[\omega] = \frac{|G_k[\omega]|}{\sum_{k'=1}^{n} |G_{k'}[\omega]|}$$

Where the purpose of the normalization is to make sure the weights induce an average over the frames.

In order to achieve even stronger weighting of the less attenuated frames, we can use a different aggregation method, for example a different norm average:

$$\widetilde{w}_k^p[\omega] = \frac{|G_k[\omega]|^p}{\sum_{k'=1}^{n} |G_{k'}[\omega]|^p}$$

By setting $p = 1$ we get our original weighting $\widetilde{w}_k$, and by increasing the value of p we get more skewed weighting towards the larger values (for $p \to \infty$ we reach the max-norm).

All in all, we achieve the final estimation of the image:

$$f(x) \approx \mathcal{F}^{-1}\left(\frac{1}{sinc(\omega)} \sum_{k=1}^{n} \frac{|G_k[\omega]|^p}{\sum_{k'=1}^{n} |G_{k'}[\omega]|^p} G_k[\omega]\right)$$