Machine Learning- Exercise 2
Theoretical Part

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Multiclass Logistic Regression

(a) $P(Y=y|X=\boldsymbol{x}) = softmax(\boldsymbol{Wx}+\boldsymbol{b})_{[y]} = \frac{e^{w_{[y]}x+b_{[y]}}}{\sum_i e^{w_{[i]}x+b_{[i]}}}$, where $[i]$ indicates the i-th index of the vector / matrix, **W** the weights matrix and **b** the bias vector.

(b) $\Theta^* = \underset{\Theta}{argmin}\frac{1}{m}\sum_{t=1}^{m} -\log\left(P(Y_t=y|X_t=\boldsymbol{x})\right) =$

$\underset{\Theta}{argmin}\sum_t -\log\left(softmax(\boldsymbol{Wx}+\boldsymbol{b})_{[y]}\right) \overset{\boldsymbol{w}:=\boldsymbol{W}_{[y]},b:=\boldsymbol{b}_{[y]}}{\triangleq}$

$\underset{\Theta}{argmin}\sum_t -\log\left(\frac{e^{\boldsymbol{wx}+b}}{\sum_i e^{\boldsymbol{W}_{[i]}\boldsymbol{x}+\boldsymbol{b}_{[i]}}}\right) = \underset{\Theta}{argmin}\sum_t -\frac{\boldsymbol{wx}+b}{\log(\sum_i \boldsymbol{W}_{[i]}\boldsymbol{x}+\boldsymbol{b}_{[i]})}$

(c) I'll calculate the gradients of the average loss for all the training examples w.r.t each weight ($w_1,\dots,w_n - \ n\ is\ the\ number\ of\ classes$) and the bias ($b$):

Weights: $\nabla_{w_i} loss(\boldsymbol{W};\boldsymbol{b}) = \nabla_{w_i}\frac{1}{m}\sum_{t=1}^{m} -\log\left(P(Y_t=y|X_t=\boldsymbol{x})\right) =$
$\frac{1}{m}\sum_{t=1}^{m}\nabla_{w_i} -\log\left(P(Y_t=y|X_t=\boldsymbol{x})\right) = \frac{1}{m}\sum_{t=1}^{m}\nabla_{w_i} -\log\left(\frac{e^{\boldsymbol{w_y x}+b_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\right) =$
$\frac{1}{m}\sum_{t=1}^{m} -\nabla_{w_i}\log\left(e^{\boldsymbol{w_y x}+b_y}\right) + \nabla_{w_i}\log\left(\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right) =$
$\frac{1}{m}\sum_{t=1}^{m} -\nabla_{w_i}(\boldsymbol{w_y x}+b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{w_i}\left(\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right)$

We'll call the derivative of the loss of example t as $\nabla_{w_i} loss_t(\boldsymbol{W};\boldsymbol{b}) = -\nabla_{w_i}(\boldsymbol{w_y x}+b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{w_i}\left(\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right)$, so in total $\nabla_{w_i} loss(\boldsymbol{W};\boldsymbol{b}) = \frac{1}{m}\sum_{t=1}^{m} loss_t(\boldsymbol{W};\boldsymbol{b})$. For $i=y$ (where y is the correct tag), $loss_t(\boldsymbol{W};\boldsymbol{b}) = -\nabla_{w_y}(\boldsymbol{w_y x}+b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{w_y}\left(\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right) = -\boldsymbol{x} + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\left(\sum_{j=1}^{n}\nabla_{w_y} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right) = -\boldsymbol{x} + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\left(\nabla_{w_y} e^{\boldsymbol{W}_0\boldsymbol{x}+\boldsymbol{b}_0} + \cdots\nabla_{w_y} e^{\boldsymbol{W}_y\boldsymbol{x}+\boldsymbol{b}_y} + \cdots\ \nabla_{w_y} e^{\boldsymbol{W}_n\boldsymbol{x}+\boldsymbol{b}_n}\right) = -\boldsymbol{x} + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\left(e^{\boldsymbol{W}_y\boldsymbol{x}+\boldsymbol{b}_y}\boldsymbol{x}\right) = \boldsymbol{x}\left(\frac{e^{\boldsymbol{W}_y\boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}} - 1\right) = \boldsymbol{x}\left(softmax_{[y]}(\boldsymbol{Wx}+\boldsymbol{b}) - 1\right)$. For $i \neq y$: $loss_t(\boldsymbol{W};\boldsymbol{b}) = -\nabla_{w_i}(\boldsymbol{w_y x}+b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{w_i}\left(\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right) = -0 + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\left(\sum_{j=1}^{n}\nabla_{w_y} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}\right) = \frac{e^{\boldsymbol{W}_i\boldsymbol{x}+\boldsymbol{b}_i}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j\boldsymbol{x}+\boldsymbol{b}_j}}\boldsymbol{x} = softmax(\boldsymbol{Wx}+\boldsymbol{b})_{[i]} * \boldsymbol{x}$.

In conclusion, $\nabla_{w_i} loss(\boldsymbol{W};\boldsymbol{b}) = \frac{1}{m}\sum_{t=1}^{m} \nabla_{w_i} loss_t(\boldsymbol{W};\boldsymbol{b})$,

where $\nabla_{w_i} loss_t(\boldsymbol{W};\boldsymbol{b}) = \begin{cases} -\boldsymbol{x} + \frac{e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\boldsymbol{x}, i = y \\ \frac{e^{\boldsymbol{W}_i \boldsymbol{x}+\boldsymbol{b}_i}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\boldsymbol{x} \quad , i \neq y \end{cases}$.

Bias: $\nabla_{b_i} loss(\boldsymbol{W};\boldsymbol{b}) = \nabla_{b_i}\frac{1}{m}\sum_{t=1}^{m} -\log\big(P(Y_t = y|X_t = \boldsymbol{x})\big) = \frac{1}{m}\sum_{t=1}^{m}\nabla_{b_i} - \log\big(P(Y_t = y|X_t = \boldsymbol{x})\big) = \frac{1}{m}\sum_{t=1}^{m}\nabla_{b_i} - \log\left(\frac{e^{\boldsymbol{w}_y \boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\right) =$
$\frac{1}{m}\sum_{t=1}^{m} - \nabla_{b_i}\log\big(e^{\boldsymbol{w}_y \boldsymbol{x}+\boldsymbol{b}_y}\big) + \nabla_{b_i}\log\big(\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}\big) =$
$\frac{1}{m}\sum_{t=1}^{m} -\nabla_{b_i}(\boldsymbol{w_y x} + b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{b_i}\big(\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}\big)$, so like before we'll look at the case $i = y$ for a single example t:
$-\nabla_{b_y}(\boldsymbol{w_y x} + b_y) + \frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\nabla_{b_y}\big(\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}\big) = -1 +$
$\frac{1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\Big(\nabla_{b_y} e^{\boldsymbol{W}_0 \boldsymbol{x}+\boldsymbol{b}_0} + \cdots \nabla_{b_y} e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y} + \cdots \nabla_{b_y} e^{\boldsymbol{W}_n \boldsymbol{x}+\boldsymbol{b}_n}\Big) = -1 +$
$\frac{e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y} * 1}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}} = softmax(\boldsymbol{Wx} + \boldsymbol{b})_{[y]} - 1$, and for $i \neq y$ it's the same without the -1.

For conclusion, $\nabla_{b_i} loss(\boldsymbol{W};\boldsymbol{b}) = \frac{1}{m}\sum_{t=1}^{m} \nabla_{b_i} loss_t(\boldsymbol{W};\boldsymbol{b})$,

where $\nabla_{b_i} loss_t(\boldsymbol{W};\boldsymbol{b}) = \begin{cases} \frac{e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}} - 1, i = y \\ \frac{e^{\boldsymbol{W}_i \boldsymbol{x}+\boldsymbol{b}_i}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}} \quad , i \neq y \end{cases}$.

Now, for the update rules: the general SGD update rule is $\boldsymbol{w_i} = \eta\nabla_{\boldsymbol{w}_i} loss(\boldsymbol{W};\boldsymbol{b})$ for each row $\boldsymbol{w_i}$ in the weight matrix $\boldsymbol{W}$, and $b_i = \eta\nabla_{b_i} loss(\boldsymbol{W};\boldsymbol{b})$ for each element b_i in the bias vector $\boldsymbol{b}$. So the update rules are:

$$\boldsymbol{w_i} = \begin{cases} -\eta\boldsymbol{x} + \eta\frac{e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\boldsymbol{x}, i = y \\ \eta\frac{e^{\boldsymbol{W}_i \boldsymbol{x}+\boldsymbol{b}_i}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}}\boldsymbol{x} \quad , i \neq y \end{cases}$$

$$b_i = \begin{cases} \eta\frac{e^{\boldsymbol{W}_y \boldsymbol{x}+\boldsymbol{b}_y}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}} - \eta, i = y \\ \eta\frac{e^{\boldsymbol{W}_i \boldsymbol{x}+\boldsymbol{b}_i}}{\sum_{j=1}^{n} e^{\boldsymbol{W}_j \boldsymbol{x}+\boldsymbol{b}_j}} \quad , i \neq y \end{cases}$$