Calculus 2 - Exercise 7

All exercises should be submitted by June 4th at 23:00.
Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. Write your solution clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

<u>Questions:</u>

1. Determine whether each of the following series converges or diverges. Prove your statements (you don't have to compute the value of the series).
   (a) (5 points) $\sum_{n=1}^{\infty} \frac{9+n^2}{n^4+3n^2+4}$
   (b) (5 points) $\sum_{n=2}^{\infty} \left(\frac{n-1}{n+1}\right)^{n^2-n}$
   (c) (5 points) $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)^{n+2}}$
   (d) (5 points) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$
   (e) (10 points) $\sum_{n=1}^{\infty} \frac{a^n}{n\cdot \sqrt[n]{n}}$, $\forall a > 0$
   (f) (5 points) $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)$
   Hint: consider the harmonic series.

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$a_{n+1} = 1 + \sqrt{3a_n^2 - 1} \ , a_1 = 1$$

(a) (5 points) Prove that the sequence $(a_n)_{n=1}^{\infty}$ is increasing.

(b) (5 points) Prove that $\lim\limits_{n\to\infty} a_n = \infty$.

(c) (5 points) Prove that the series $\sum\limits_{n=1}^{\infty} \frac{1}{a_n}$ is convergent.

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers and let $S_N = \sum\limits_{n=1}^{N} a_n$ for every $N \in \mathbb{N}$. Suppose that $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(a) (5 points) Prove that the series $\sum\limits_{n=2}^{\infty} \left[\frac{1}{S_{n-1}} - \frac{1}{S_n}\right]$ is convergent.

(b) (5 points) Prove that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{S_n^2}$ is convergent.

(c) (5 points) Conclude that the series $\sum\limits_{n=1}^{\infty} \frac{1}{\left(\cos 1 + \cos\left(\frac{1}{2}\right) + \ldots + \cos\left(\frac{1}{n}\right)\right)^2}$ is convergent.

4. (10 points) Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence of positive real numbers. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{1}{a_n^n}$ is convergent if and only if there exists a $k \in \mathbb{N}$ for which $a_k > 1$.

5. Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ be two sequences of non-negative real numbers. Prove or disprove each of the following statements:

(a) (5 points) Suppose that $\exists L \in \mathbb{R}$ for which $\lim\limits_{n\to\infty} b_n = L$ and that $\sum\limits_{n=1}^{\infty} a_n < \infty$. Then $\sum\limits_{n=1}^{\infty} (a_n \cdot b_n) < \infty$.

(b) (5 points) Suppose that $a_n = \frac{1}{n} e^{-a_{n-1}}$ for every $2 \le n \in \mathbb{N}$, then $\sum\limits_{n=1}^{\infty} a_n = \infty$.

(c) (5 points) If $\sum\limits_{n=1}^{\infty} a_n < \infty$ and $a_n > 0$ for every $n \in \mathbb{N}$, then there exists an $k \in \mathbb{N}$ for which $(a_n)_{n=k}^{\infty}$ is decreasing.

(d) (10 points) Let $f : [-1, 1] \to \mathbb{R}$. Suppose that $f(0) = 0$, f is differntiable at 0 and $f'(0) > 0$. Then $\sum\limits_{n=1}^{\infty} f\left(\frac{1}{n^2}\right)$ is convergent.