Regularity Theory for Linear and Nonlinear Poisson-Type Equations

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What is this talk about?

Today's talk is based on my Master's thesis:

Title: Calderón-Zygmund estimates for the Poisson equation

Advisors: Clara Torres-Latorre, Xavier Ros-Oton

We will see:

- What is the *Poisson equation*? What is a PDE?
- What is an *estimate*?
- What are the *Calderón-Zygmund estimates*?

PDEs and the Poisson equation

- **1747**. Origin of PDEs: the problem of the vibrating string (a.k.a. the wave equation)
- A Partial Differential Equation (PDE) is an equation involving a function and its partial derivatives.

PDEs and the Poisson equation

- A second-order divergence form elliptic PDE can be written as

$$-\mathrm{div}(A\nabla u) = f$$

The Laplace and the Poisson equation

The most characteristic second-order divergence form elliptic equations are the Laplace equation

$$\Delta u = 0$$

and **the Poisson** equation

$$\Delta u = f$$

Beginning regularity theory

- **1900**. Second International Congress of Mathematics in Paris. The Hilbert program.

Hilbert's 19th problem (regularity)

Let $L : \mathbb{R}^n \to \mathbb{R}$ smooth and uniformly convex. Let $\Omega \subset \mathbb{R}^n$. Consider energy functionals of the form

$$J(u) := \int_\Omega L(\nabla u)\, dx$$

Are all local minimisers smooth?

Beginning regularity theory

Why study regularity? For instance, when we face a physical problem modelled with a PDE, we ask ourselves:

Will u be a smooth function, or will it develop singularities?

- If u is smooth $\implies$ Nice!
- If u develops singularities... It is not always bad! It can be a:
  - Bug: the model does not work.
  - Feature: there is an actual singularity, like the Big Bang.

A priori estimates

First steps in regularity theory:

- **1906**. Introduction of *a priori* estimates (Bernstein)

A priori estimates

Estimating the size of a solution to a PDE or its derivatives before said solution is known to exist.

A priori estimates

- First work on *a priori* estimates: C^k estimates for the Laplace equation.

C^k estimates for the Laplace equation (Bernstein)

Let $\Delta u = 0$ on B_1 in the weak sense. Then for all $k \in \mathbb{N}$:

$$\|u\|_{C^k(B_{1/2})} \leq C(n,k)\|u\|_{L^1(B_1)}$$

A priori estimates

Warning

The C^k estimates don't work for all second-order elliptic equations!

- In the Poisson equation

$$\Delta u = f$$

if f is C^0, u is not necessarily $C^2 \Longrightarrow$ More tools are needed!

Schauder estimates

- **1934.** To deal with second-order elliptic equations, the space $C^{2,\alpha}$ is better than C^2 (Schauder).

Schauder estimates

For the Dirichlet problem for a linear elliptic equation of second order with $C^{0,\alpha}$ coefficients,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^{0}(\Omega)}\right)$$

L^p estimates

- Classical solutions $\Longrightarrow$ Weak solutions.
- Use of Sobolev spaces and L^p theory $\Longrightarrow$ need for L^p estimates.
- **1952.** Study of singular integral operators and L^p estimates for the Newtonian potential (Calderón-Zygmund)

Calderón-Zygmund estimates

For the Poisson equation, $\Delta u = f$, we know:

- $f \in C^0 \not\Longrightarrow u \in C^2$
- $f \in C^{0,\alpha} \Longrightarrow u \in C^{2,\alpha}$

What happens if $f \in L^p$?

Goal: Given $\Delta u = f$ and $f \in L^p \Longrightarrow u$ is $W^{2,p}$

And do we have estimates on the derivatives?

Yes! $\|D^2 u\|_{L^p} \leq C\left(\|f\|_{L^p} + \|u\|_{L^p}\right)$

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Calderón-Zygmund

via interpolation

The Calderón-Zygmund inequality, v1

Calderón-Zygmund inequality, v1

Let Ω be a bounded domain. Let $f \in L^p(\Omega)$ with $1 < p < \infty$. Let u be the Newtonian potential of f. Then $u \in W^{2,p}(\Omega)$, $\Delta u = f$ a.e. in Ω, and

$$\left\|D^2 u\right\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

This approach is based on

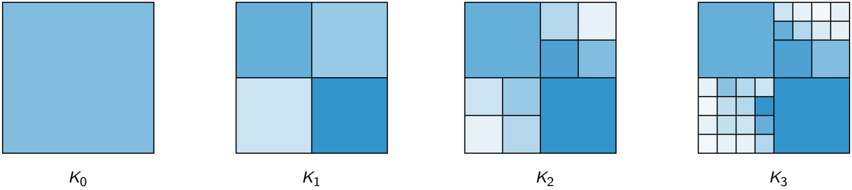
David Gilbarg and Neil S. Trudinger
Elliptic Partial Differential Equations of Second Order
2nd edition. *Berlin, Springer*, 2001.

To prove it we need:

- the Calderón-Zygmund decomposition
- interpolation of L^p spaces

Calderón-Zygmund decomposition

Idea: Given a function f and a cube K_0, subdivide K_0 and the resulting cubes when the density of f is below certain threshold.



We end up with:

- Good cubes, G: the density is below the threshold.
- Bad cubes: F: the density is above the threshold.

Then f has "nice properties" on the subcubes.

Interpolation of L^p spaces

Marcinkiewicz interpolation theorem (kind of)

If T is a bounded linear mapping on both L^q and L^r, it can be extended to a bounded linear mapping on L^p for all $q < p < r$.

Calderón-Zygmund inequality, v1

Calderón-Zygmund inequality, v1

Let Ω be a bounded domain. Let $f \in L^p(\Omega)$ with $1 < p < \infty$. Let u be the Newtonian potential of f. Then $u \in W^{2,p}(\Omega)$, $\Delta u = f$ a.e. in Ω, and

$$\left\|D^2 u\right\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

What does this mean?

$$\left.\begin{aligned} u = \Gamma * f \\ f \in L^p(\Omega) \end{aligned}\right\} \Longrightarrow u \in W^{2,p}(\Omega)$$

Calderón-Zygmund

a geometric approach

The Calderón-Zygmund inequality, v2

Calderón-Zygmund inequality, v2

Let $f \in L^p(B_1)$. Let u be a solution to $\Delta u = f$ in B_1. Then

$$\left\|D^2 u\right\|_{L^p(B_{1/2})} \leq C \left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)} \right)$$

This approach is based on

Lihe Wang
A Geometric Approach to the Calderón-Zygmund Estimates
Acta Mathematica Sinica, English Series, 19 Jan. 2003, 381-396

To prove it we need:

- the Hardy-Littlewood maximal funcion.

Using the Hardy-Littlewood maximal function

Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then its **maximal function** is defined as

$$\mathcal{M}u(x) = \sup_{r>0} \fint_{B_r(x)} |u|$$

Key fact

The measures of the superlevelsets of u and of $\mathcal{M}u(x)$ decay roughly in the same way.

The Calderón-Zygmund inequality, v2

Calderón-Zygmund inequality, v2

Let $f \in L^p(B_1)$. Let u be a solution to $\Delta u = f$ in B_1. Then

$$\left\|D^2 u\right\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right)$$

What does this mean?

$$\left.\begin{aligned} u \text{ solves } \Delta u = f \\ f \in L^p(B_1) \end{aligned}\right\} \Longrightarrow u \in W^{2,p}(B_{1/2})$$

What's the difference?

Comparison

We have two different results. Let u solve

$$\Delta u = f$$

in B_1 for $f \in L^p(B_1)$. Then:

1. If u is the Newtonian potential $\Longrightarrow u \in W^{2,p}(B_1)$ and

$$\|Du\|_{L^p(B_1)} \leq C\|f\|_{L^p(B_1)}$$

2. If u is any function $\Longrightarrow u \in W^{2,p}(B_{1/2})$ and

$$\left\|D^2u\right\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right)$$

The price we pay for a more regular solution is a reduction of the domain...

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Problem to solve

Goal: study the existence and regularity of nontrivial solutions to

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where we consider

- u a positive function,
- Ω a bounded, star-shaped domain.

Where does the equation come from?

For the constrained minimisation problem:

$$\min \left\{ \int_\Omega |\nabla u|^2 \,\middle|\, u \in H_0^1(\Omega) \text{ such that } \|u\|_{p+1} = 1 \right\}$$

The associated Euler-Lagrange equations are (kind of)[1]

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

[1] After some algebraic manipulation, the use of orthogonality in Hilbert spaces and the weak maximum principle for subharmonic functions.

Existence of solutions

Using the direct method of the Calculus of Variations and Rellich-Kondrachov's embedding, we see that a solution exists as long as

$$1 < p < \frac{n+2}{n-2}$$

Next questions:

- When $1 < p < \frac{n+2}{n-2}$, what can we say about the regularity of the solutions?
  - We will see that u is actually C^∞!
- When $p \geq \frac{n+2}{n-2}$, what happens?
  - For $p > \frac{n+2}{n-2}$ there are no solutions.
  - For $p = \frac{n+2}{n-2}$ it depends on the domain.

Regularity of solutions

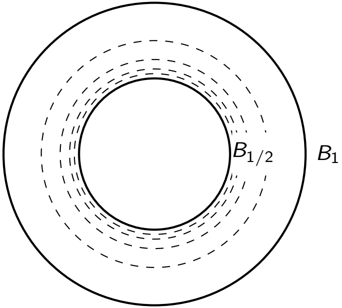
when $1 < p < \frac{n+2}{n-2}$

Case $1 < p < \frac{n+2}{n-2}$: regularity of solutions

We will use a **bootstrapping argument** to see the following chain of embeddings:

$$u \in H_0^1(B_1) \implies u \in C^0 \implies u \in C^1 \implies \cdots \implies u \in C^\infty(B_{1/2})$$

in the sequence of balls $\{B^k\} = \{B_{1/2+1/2^k}\}$



Bootstrapping

Recall Sobolev's embedding

$$W^{l,p} \subset W^{k,p^*} \quad p^* = \frac{np}{n-(l-k)p} \quad l > k \quad p < n$$

The solution to $-\Delta u = u^p$ in B_1 is $H_0^1(B_1)$. Then:

$$u \in H_0^1 \xRightarrow{Sobolev} u \in L^{q_0} \xRightarrow{equation} u^p \in L^{q_0/p} \xRightarrow{C-Z} u \in W^{2,q_0/p}.$$

We repeat

$$u \in W^{2,q_0/p} \xRightarrow{Sobolev} u \in L^{q_1} \xRightarrow{equation} u^p \in L^{q_1/p} \xRightarrow{C-Z} u \in W^{2,q_1/p},$$

with $q_1 > q_0$.

Bootstrapping

We keep repeating

$$u \in W^{2,q_{j-1}/p} \xrightarrow{Sobolev} u \in L^{q_j} \xrightarrow{equation} u \in L^{q_j/p} \xrightarrow{C-Z} u \in W^{2,q_j/p}$$

until $q_j > np/2$.

Recall Morrey's embedding

$$W^{1,p} \subset C^{0,1-n/p} \qquad p > n$$

Let q_N be the first $q_j > np/2$. Then

$$u \in W^{2,q_N/p} \xrightarrow{Morrey} u \in C^{0,1-q_N/p} \Rightarrow u \in C^0 \Rightarrow u \in L^\infty \Rightarrow u \in W^{2,p}$$

for all p.

Bootstrapping

We have seen

- $u \in H_0^1 \implies u \in C^0$ (and $u \in W^{2,p}$ for all p)

Analogously we see

- $\nabla u \in H^1 \implies \nabla u \in C^0 \implies u \in C^1$
- $D^2 u \in H^1 \implies D^2 u \in C^0 \implies \nabla u \in C^1 \implies u \in C^2$

  $\dots$
- $D^k u \in H^1 \implies D^k u \in C^0 \implies \cdots \implies u \in C^k$ for all k.

Therefore, we end up with

$$\boxed{u \in H_0^1(B_1) \implies u \in C^\infty(B_{1/2})}$$

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Enter nonlinearity

What happens to the results when we consider the **nonlinear setting**?

Nonlinear $W^{2,p}$ estimates

Let u be a bounded solution to the fully nonlinear PDE $F(D^2u, x) = f(x)$ in B_1. Let $n < p < \infty$. Assume $f \in L^p$. Then, under some technical assumptions, u is in $W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|f\|_{L^p(B_1)} + \sup_{\partial B_1} |u| \right).$$

Enter nonlinearity

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$$\boldsymbol{\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|f\|_{L^p(B_1)} + \sup_{\partial B_1} |u| \right)}.$$

Final thoughts

- Lawrence C. Evans: "*There is in truth no central core theory of nonlinear PDE, nor can there be*".

Linear estimates

$$\left\|D^2 u\right\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right)$$

Nonlinear estimates

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \sup_{\partial B_1}|u|\right)$$

- Understanding the linear setting gives us a **good intuition** on how things should look like.

Thank you!