UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

Calderón-Zygmund estimates for the Poisson equation

Master's Thesis

Master's Degree in Advanced Mathematics
and Mathematical Engineering

Student Name: Bartomeu Garau Verger
Supervisors: Clara Torres-Latorre
Xavier Ros-Oton

Department: Mathematics
Academic Year: 2023/2024

A la meva família,
per poder i saber deixar-me marxar;
i en especial al meu padrí Tomeu,
que haurà de veure començar
la meva carrera com a matemàtic
des d'allà dalt.

Contents

Abstract

In this project we study two different proofs of the Calderón-Zygmund estimates for the Poisson equation. Our first approach is that of Gilbarg and Trudinger [GT01], which is based on a real interpolation argument. We also explore the proof of Wang [Wan03], with a geometric approach where the Hardy-Littlewood maximal function plays a crucial role. Moreover, we consider nonlinear elliptic equations. As an application of the estimates, we deduce regularity for solutions to equations of the form

$$-\Delta u = u^p.$$

We also briefly review the known extensions of L^p regularity theory to the nonlinear setting.

CHAPTER 1

Introduction

> *As long as a branch of science offers an abundance of problems, so long it is alive. It is by the solution of problems that the strength of the investigator is hardened.*
>
> — **David Hilbert**
> Second International Congress of Mathematics,
> Paris, 1900

Since the eighteenth century, people have enjoyed the works of Bach, Mozart or Beethoven, have been inspired by them, have fallen in love with them... When most people went to a concert, they were concerned with how good did the music sound, except for a small group of people, that was more concerned on *how was the sound produced*. It is known that violins produce their sound because of the vibrations of their strings, and how tightening or shortening them can vary its sound. However, they did not know that in the 1700's. Not until D'Alembert and Euler got to work.

The problem of the vibrating string (or the wave equation, as it is known nowadays) and the subsequent work by D'Alembert, Euler and many other mathematicians, laid the foundation for the theory of Partial Differential Equations (PDEs). A PDE is an equation that relates a function with its partial derivatives and variables. These kind of equations are an essential tool to model real-life problems in physics and engineering, such as the already mentioned vibrating string, heat distribution, diffusion, electromagnetic fields, fluid dynamics...

During the 18th and 19th centuries, mathematicians developed methods to solve such equations, but everything changed at Second International Congress of Mathematics in 1900 when Hilbert introduced the questions of *existence* and *regularity* of the solutions to variational problems. The Euler-Lagrange equation associated to such problems gives rise to what are known as *elliptic equations*, which will be the focus of this project. The most characteristic examples are the Laplace and the Poisson equations,

$$\Delta u = 0 \quad \text{and} \quad \Delta u = f,$$

respectively. The question of regularity is crucial to understand the physical meaning and implications of the solution. For example, Stephen Hawking and Roger Penrose [HP70] made use of regularity theory (among other things) to prove the singularity theorems that led to the conclusion that a Big Bang-type singularity was necessary to explain the origin of the universe.

The first steps in regularity theory were done by S. Bernstein in [Ber06], where he established the notion of *a priori* estimates for the Laplace equation. This consisted on estimating the size (i.e., the norm) of the solution or its derivatives before said solution is known to exist. Those were C^k estimates, applicable to the Laplace equation because its solutions are C^∞. If we consider the Poisson equation instead, this no longer works. That is, if we have $-\Delta u = f$ with $f \in C^0$, then u is not necessarily C^2. Thus, the invertibility of linear elliptic operators was the next challenge.

Schauder was the first to address that problem in [Sch34]. He noticed that the appropriate space to deal with that problem for second-order elliptic equations was not C^2, but the Hölder space $C^{2,\alpha}$. Here we can find what are known as the Schauder estimates, which state that the Dirichlet problem for a linear elliptic equation of second order with $C^{0,\alpha}$ coefficients can be bounded in the Hölder norm as

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\left(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}\right).$$

Then, he used these estimates to study the problem of existence of solutions for such equations, combining them with the theory of compact operators, making the first major connection between PDEs and Functional Analysis.

Initially, these estimates were done for classical solutions: C^k solutions to equations with k-th order differential operators. Later on, the *weak solutions* appeared, which are functions that, even though they are not as differentiable as classical solutions, they still satisfy some of their properties and have a physical meaning. Even more, in linear problems these solutions usually turn out to be classical ones. Here is where the core of this project lies. Weak solutions rely on the theory of Sobolev spaces, which are the spaces of L^p functions with L^p weak derivatives. Thus, we need a regularity theory in the setting of L^p functions. This is what A. Calderón and A. Zygmund started in [CZ52], studying singular integral operators on $\mathbb{R}^n$.

This project concerns the L^p regularity theory for the Poisson equation. First, in Chapter 2 we review the main concepts of Functional Analysis and elliptic equations that have been mentioned throughout this introduction. Then, in Chapter 3 we develop the L^p regularity theory and prove the Calderón-Zygmund inequality using two different approaches. This inequality allows us to bound the second-order derivatives of the solutions to Poisson equation, showing that they are twice weakly differentiable. We first prove it following the work of Gilbarg and Trudinger in [GT01], using interpolation of L^p spaces. Then, we will prove it *à la* Wang [Wan03], using the distribution function of the Hessian and studying the behaviour of the superlevel sets. Finally, in Chapter 4, we will apply this results to the study of a nonlinear equation.

Nonlinear equations are of great importance, as a great number of problems in physics and engineering are written in terms of nonlinear equations. For example, the famous Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u = \nabla p$$

describe the motion of a viscous fluid substance. In the nonlinear setting we can also find nonlinear elliptic equations of second order. The work started by Bernstein in [Ber06] was extended to this setting by many authors, such as Schauder, Morrey, Nirenberg or Serrin, providing estimates up to the boundary. In this project, following the L^p theory presented in Chapter 3, we will conclude in Chapter 5 with a brief introduction to regularity theory for nonlinear second order elliptic equations and the importance of such problems.

CHAPTER 2

Preliminaries

> *I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove".*
>
> — **G.H. Hardy**
> *Prolegomena to a Chapter on Inequalities*, 1928

And indeed, Harald Bohr was (more or less) right. In this chapter we recall the basic definitions of the Lebesgue and Sobolev spaces, along with a great number of inequalities that have been hunted through the literature and will be the main tools we will use to prove yet another inequality in Chapter 3: the Calderón-Zygmund inequality. This chapter is mainly a collection of basic (yet sophisticated) results from Functional Analysis and Partial Differential Equations courses. For more information on L^p and Sobolev spaces we refer to [Ada75; Bre11; Eva10; GT01], for the Lebesgue differentiation theorem and the Hardy-Littlewood maximal function one may look into [FR22; SS05], and for elliptic equations [GT01; FR22; Eva10].

2.1 L^p spaces

Throughout this chapter we will denote by Ω a bounded domain in $\mathbb{R}^n$ and, unless otherwise stated, we will be considering the Lebesgue measure.

Definition 2.1. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$. We define the L^p spaces as*

$$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_\Omega |u|^p < +\infty \right\},$$

endowed with the norm

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u|^p \right)^{1/p}.$$

For the case $p = \infty$ we define

$$L^\infty(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \exists C \geq 0 \text{ such that } |u(x)| \leq C \text{ a.e. in } \Omega \right\},$$

endowed with the norm

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = \operatorname*{ess\,sup}_C |u(x)| = \inf_C \{|u(x)| \leq C \text{ a.e. in } \Omega\}.$$

L^p spaces are Banach spaces for $1 \leq p \leq \infty$ and, in particular, L^2 is a Hilbert space with the scalar product

$$(u, v) = \int_\Omega uv. \tag{2.1}$$

As a consequence of Riesz representation theorem for L^p spaces, when $1 < p < \infty$, the dual space of L^p can be identified with $L^{p'}$, where p' is chosen so that $1/p + 1/p' = 1$. This allows us to write

$$\|u\|_p = \sup_{\|v\|_{p'} \leq 1} \int_\Omega uv. \tag{2.2}$$

At the same time, this means that the dual of $L^{p'}$ can be identified with L^p, and thus, the L^p spaces are reflexive. That is, $(L^p)^{**} \equiv L^p$.

We will denote by $L^p_{loc}(\Omega)$ the set of functions that are L^p when restricted to all compact subsets K of Ω.

Some useful inequalities to deal with integral estimates for PDEs are the following:

Proposition 2.2 (Hölder's inequality)**.** *Let $1 \leq p \leq \infty$. Let $u \in L^p$ and $v \in L^{p'}$. Then $uv \in L^1$ and*

$$\left| \int_\Omega uv \right| \leq \|u\|_p \|v\|_{p'}. \tag{2.3}$$

Proposition 2.3 (Minkowski's inequality)**.** *Let $1 \leq p \leq \infty$. Let $u, v \in L^p$. Then*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p \tag{2.4}$$

Another well-known result in this setting is the Lebesgue differentiation theorem:

Theorem 2.4 (Lebesgue differentiation theorem)**.** *If $u \in L^1(\Omega)$, then for almost every $x \in \Omega$ we have*

$$\lim_{r \to 0} ⨍_{B_r(x)} |u(x) - u(y)| dy = 0.$$

When this holds at a point $x \in \Omega$, we say that x is a Lebesgue point of u.

In the previous theorem, $⨍_A f$ denotes the mean of f over A, $\frac{1}{|A|}\int_A f$, where $A \subset \mathbb{R}^n$ is any set of finite and positive measure.

2.1.1 Convolution and regularisation

Now we recall the notion of convolution of two functions and its use in the process of mollification.

Definition 2.5. *Let $u \in L^1(\mathbb{R}^n)$ and let $v \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq \infty$. We define their convolution as*

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\,dy.$$

Moreover, the convolution is well-defined, as can be seen in Young's theorem.

Theorem 2.6 (Young)**.** *Let $u \in L^1(\mathbb{R}^n)$ and let $v \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq \infty$. Then for a.e. $x \in \mathbb{R}^n$ the function $y \to u(x-y)v(y)$ is integrable on $\mathbb{R}^n$, $u * v \in L^p(\mathbb{R}^n)$ and*

$$\|u * v\|_p \leq \|u\|_1 \|v\|_p \tag{2.5}$$

We can use the convolution to approximate L^p functions by smooth functions. This can be done by means of mollifiers.

Definition 2.7. *We define $\eta \in C^\infty(\mathbb{R}^n)$ by*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \eta = 1$. Then for each $\epsilon > 0$ we set

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

The function η_ϵ is called the standard mollifier, and it satisfies that $\eta_\epsilon \in C_c^\infty(B(0,\epsilon))$ and $\int_{\mathbb{R}^n} \eta_\epsilon = 1$.

In the following theorem we gather the main properties about the approximation of L^p functions by mollifiers. We define the mollification of u as $u_\epsilon = \eta_\epsilon * u$ and we will denote $\Omega_\epsilon := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \epsilon\}$.

Theorem 2.8. *Let $u \in L^1_{loc}(\Omega)$. Let u_ϵ be the mollification of u in Ω_ϵ. Then*

(i) $u_\epsilon \in C^\infty(\Omega_\epsilon)$;

(ii) $u_\epsilon \to u$ *a.e. as* $\epsilon \to 0$;

(iii) if $u \in C(\Omega)$*, then* $u_\epsilon \to u$ *uniformly on compact subsets of* Ω*; and*

(iv) if $1 \leq p < \infty$ *and* $u \in L^p_{loc}(\Omega)$*, then* $u_\epsilon \to u$ *in* $L^p_{loc}(\Omega)$.

Corollary 2.8.1. *Let $u \in L^p(\mathbb{R}^n)$. Then $u_\epsilon \to u$ in $L^p(\mathbb{R}^n)$.*

2.1.2 Interpolation of L^p functions

When working with measurable functions and studying how they behave, we can use the distribution function, which will be denoted by μ. This function measures the relative size of the function by measuring its superlevel sets.

Definition 2.9. *Let u be a measurable function on Ω. The distribution function μ_u is defined by*

$$\mu_u(\lambda) = |\{x \in \Omega \,|\, |u(x)| > \lambda\}| \tag{2.6}$$

for $\lambda > 0$. If there's no confusion about the function we are measuring, we will denote it simply by μ.

The distribution function is a decreasing function of λ on $(0, \infty)$, and has some interesting properties.

Proposition 2.10. *For any $p > 0$ and $u \in L^p(\Omega)$ we have*

(a) $\mu(\lambda) \leq \lambda^{-p} \int_\Omega |u|^p$

(b) $\|u\|_p^p = p \int_0^\infty \lambda^{p-1} \mu(\lambda)\, d\lambda$

Proof. See section 9 in [GT01]. □

We said that μ is a decreasing function, and Proposition 2.10 makes it even clearer. If we take $u \in L^p(\Omega)$ such that $\|u\|_p = 1$, we would have that $\mu(\lambda) \leq 1/\lambda^p$, which decreases as $\lambda \to \infty$. This motivates the following question: if the distribution function of a $u \in L^p$ is decreasing, can we say that u is L^p? For example, suppose that μ decreases as $\mu(\lambda) \leq 1/\lambda^p$. Then

$$\|u\|_p^p = p \int_0^\infty \lambda^{p-1} \mu(\lambda) \leq p \int_0^\infty \lambda^{-1},$$

which is a diverging integral, so it is not enough to say whether u is in L^p or not. However, it is close. In general, if there exists a constant C such that $\mu(\lambda) \leq C^p/\lambda^p$, we say that u is *weak* L^p.

If we want to go one step further and see if u is L^p instead of weak L^p, we need to set more restrictive conditions on the decay of $\mu(\lambda)$. For example, we can fix $\mu(1)$ as a reference and see that the level sets decay as $\mu(\lambda) \leq \epsilon\mu(1)$, for some $\epsilon := \epsilon(p)$. However, it may not be suitable to have a fixed reference point, so we can scale the estimate to $\mu(\lambda\lambda_0) \leq \epsilon\mu(\lambda_0)$ in order to compare the relative size of u for any given value λ_0. Assuming this level of decay, we have using once again Proposition 2.10 that

$$\begin{aligned}\|u\|_p^p = p\int_0^\infty \lambda^{p-1}\mu(\lambda)d\lambda &\leq p\mu(\Omega)\int_0^1 \lambda^{p-1} + p\sum_{k\geq 1}\int_{\lambda_0^k}^{\lambda_0^{k+1}} \lambda^{p-1}\mu(\lambda)\\ &\leq \mu(\Omega) + p\sum_{k\geq 1}\mu(\lambda_0^k)\int_{\lambda_0^k}^{\lambda_0^{k+1}} \lambda^{p-1}\\ &\leq \mu(\Omega) + (\lambda_0^p - 1)\mu(\Omega)\sum_{k\geq 1}(\epsilon\lambda_0^p)^k.\end{aligned}$$

Therefore, if $\epsilon\lambda_0^p < 1$, the geometric series converges and we can confirm that u is in $L^p(\Omega)$. Therefore, we can see if a function is in L^p or not by an appropriate choice of $\epsilon(p)$.

Another way of using the distribution function to show that a function is L^p is by means of interpolation. In this project we will use Marcinkiewicz's theorem, with a modified version of the statement in [GT01].

Theorem 2.11 (Marcinkiewicz Interpolation Theorem)**.** *Let T be a linear mapping from $L^q(\Omega)\cap L^r(\Omega)$ into itself, $1 \leq q < r \leq \infty$. When $r < \infty$, suppose that there are two constants M_1 and M_2 such that*

$$\mu_{Tu}(\lambda) \leq \left(\frac{M_1\|u\|_q}{\lambda}\right)^q \quad \text{and} \quad \mu_{Tu}(\lambda) \leq \left(\frac{M_2\|u\|_r}{\lambda}\right)^r \tag{2.7}$$

for all $u \in L^q(\Omega)\cap L^r(\Omega)$ and $\lambda > 0$. When $r = \infty$, we simply assume that T is a bounded mapping on $L^\infty(\Omega)$. Then T extends as a bounded linear mapping from L^p into itself for any p such that $q < p < r$ and

$$\|Tu\|_p \leq CM_1^\alpha M_2^{1-\alpha}\|u\|_p \tag{2.8}$$

for all $u \in L^q(\Omega)\cap L^r(\Omega)$, where $\frac{1}{p} = \frac{\alpha}{q} + \frac{1-\alpha}{r}$ and C is a constant depending only on p, q and r (if $r < \infty$).

Proof. Let $\lambda > 0$. For $u \in L^q(\Omega)\cap L^r(\Omega)$ and $\gamma := \gamma(\lambda) > 0$, we set $u = u_1 + u_2$, with

$$u_1(x) = \begin{cases} u(x) & \text{if } |u(x)| > \gamma \\ 0 & \text{if } |u(x)| \leq \gamma \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} 0 & \text{if } |u(x)| > \gamma \\ u(x) & \text{if } |u(x)| \leq \gamma \end{cases}.$$

Then $|Tu| \leq |Tu_1| + |Tu_2|$. First we will assume the case when $r < \infty$. We have that

$$\begin{aligned}\mu(\lambda) = \mu_{Tu}(\lambda) &= |\{x\in\Omega \mid \lambda < |Tu|\}|\\ &\leq \left|\left\{x\in\Omega \;\middle|\; \frac{\lambda}{2} < |Tu_1|\right\}\right| + \left|\left\{x\in\Omega \;\middle|\; \frac{\lambda}{2} < |Tu_2|\right\}\right|\\ &= \mu_{Tu_1}\left(\frac{\lambda}{2}\right) + \mu_{Tu_2}\left(\frac{\lambda}{2}\right)\\ &\leq \left(\frac{M_1\|u_1\|_q}{\lambda/2}\right)^q + \left(\frac{M_2\|u_2\|_r}{\lambda/2}\right)^r = \left(\frac{2M_1}{\lambda}\right)^q\int_\Omega |u_1|^q + \left(\frac{2M_2}{\lambda}\right)^r\int_\Omega |u_2|^r\end{aligned}$$

Now using Proposition (2.10) and the previous inequality, we have

$$\int_\Omega |Tu|^p = p\int_0^\infty \lambda^{p-1}\mu(\lambda)d\lambda$$
$$\leq p(2M_1)^q \int_0^\infty \left(\lambda^{p-1-q}\int_{|u|>\gamma}|u|^q\right)d\lambda + p(2M_2)^r\int_0^\infty\left(\lambda^{p-1-r}\int_{|u|\leq\gamma}|u|^r\right)d\lambda$$

We take $\gamma := \gamma(\lambda)$ such that $\lambda = k\gamma$ for some positive constant k. Plugging this in the previous inequality:

$$\int_\Omega |Tu|^p \leq p(2M_1)^q\int_0^\infty \left((k\gamma)^{p-1-q}\int_{|u|>\gamma}|u|^q\right)kd\gamma$$
$$+ p(2M_2)^r\int_0^\infty\left((k\gamma)^{p-1-r}\int_{|u|\leq\gamma}|u|^r\right)kd\gamma$$
$$= p(2M_1)^q k^{p-q}\int_0^\infty \gamma^{p-1-q}\int_{|u|>\gamma}|u|^q\,d\gamma + p(2M_2)^r k^{p-r}\int_0^\infty\gamma^{p-1-r}\int_{|u|\leq\gamma}|u|^r\,d\gamma$$

Next we rewrite the integrals on the right hand side. The first one would be

$$\int_0^\infty \gamma^{p-1-q}\int_{|u|>\gamma}|u|^q\,d\gamma = \int_\Omega |u|^q\int_0^{|u|}\gamma^{p-1-q}\,d\gamma = \frac{1}{p-q}\int_\Omega |u|^p,$$

and the second one is done analogously. Consequently,

$$\int_\Omega |Tu|^p \leq \left\{\frac{p}{p-q}(2M_1)^q k^{p-q} + \frac{p}{r-p}(2M_2)^r k^{p-r}\right\}\int_\Omega |u|^p = f(k)\int_\Omega |u|^p.$$

This function f takes its minimum at $k_0 = 2M_1^{-p\alpha/(r-p)}M_2^{p(1-\alpha)/(p-q)}$ for $\alpha = \frac{q(r-p)}{p(r-q)}$ and $1-\alpha = \frac{r(p-q)}{p(r-q)}$. Then its minimum value is

$$f(k_0) = 2^p\left[\frac{p}{p-q}M_1^{p\alpha}M_2^{p(1-\alpha)} + \frac{p}{r-p}M_1^{p\alpha}M_2^{p(1-\alpha)}\right] = 2^p\left[\frac{p}{p-q}+\frac{p}{r-q}\right]M^{p\alpha}M^{p(1-\alpha)}.$$

Therefore,

$$\|Tu\|_p \leq CM_1^\alpha M_2^{1-\alpha}\|u\|_p$$

with $C := C(p,q,r) = 2\left[\frac{p}{p-q}+\frac{p}{r-q}\right]^{1/p}$.

Now let us address the case for $r=\infty$. Now as T is a linear bounded mapping on L^∞ we have that

$$\|Tu_2\|_\infty \leq M_2\|u_2\|_\infty \leq M_2\gamma = \frac{\lambda}{2} \tag{2.9}$$

choosing $\gamma = \lambda/(2M_2)$. This way,

$$\mu_{Tu}(\lambda) \leq \mu_{Tu_1}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2M_1}{\lambda}\right)^q\int_\Omega |u_1|^q.$$

Then, using Proposition 2.10, doing the change of variables $\lambda = 2M_2\gamma$ and using Fubini's theorem we end up with

$$\int_\Omega |Tu|^p \leq C^p \int_\Omega |u|^p, \tag{2.10}$$

for $C := C(p,q) = 2\left[\frac{p}{p-1} M_1^q M_2^{p-q}\right]^{1/p}$. □

Thanks to the Marcinkiewicz Interpolation theorem, as long as we are able to show that a linear map is defined on two different L^p spaces and it is bounded, we can assure that it is defined on all the L^p spaces in between.

2.1.3 Hardy-Littlewood maximal function

The Hardy-Littlewood maximal function, as the authors said in [HL30], is most easily grasped when stated in terms of cricket. They do an analogy for the discrete case with the average number of innings a cricket player does in a season and said player's satisfaction (which surprisingly increases if his performance decreases during the season!).
The discrete case has a continuous counterpart for L^p functions, which will be of great use in this project.

Definition 2.12. *Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then, its maximal function is defined as*

$$\mathcal{M}u(x) = \sup_{r>0} ⨍_{B_r(x)} |u| \tag{2.11}$$

When we restrict u to some domain Ω we will denote it by

$$\mathcal{M}_\Omega u(x) := \mathcal{M}(u\chi_\Omega)(x).$$

One of the basic theorems for the Hardy-Littlewood maximal function is the following:

Theorem 2.13 (Weak $1-1$ estimate)**.** *Let $u \in L^1_{loc}(\Omega)$. Then, there exists a constant $C_n > 0$ depending on n, where $n \geq 1$ is the dimension of the space, such that for all $\lambda > 0$:*

$$|\{x \in \Omega \,|\, \mathcal{M}u(x) \geq \lambda\}| \leq \frac{C_n}{\lambda}\|u\|_{L^1(\Omega)}. \tag{2.12}$$

A similar can be stated for different p-norms. For example, consider $u \in L^\infty(\mathbb{R}^n)$. Then

$$\mathcal{M}u(x) = \sup_{r>0} ⨍_{B_r(x)} |u| \leq \|u\|_{L^\infty(\mathbb{R}^n)},$$

therefore

$$\|\mathcal{M}u(x)\|_{L^\infty(\mathbb{R}^n)} \leq \|u(x)\|_{L^\infty(\mathbb{R}^n)}, \tag{2.13}$$

which is the $\infty - \infty$ estimate. Then, as a consequence of it, of the 1-1 estimate, and the Marcinkiewicz interpolation theorem, we have the following estimate:

Theorem 2.14 (Strong $p-p$ estimate)**.** *Let $u \in L^p(\Omega)$ with $1 < p \leq \infty$. Then, there exists a constant $C_{n,p} > 0$ depending on n, p such that*

$$\|\mathcal{M}u(x)\|_{L^p(\Omega)} \leq C_{n,p}\|u\|_{L^p(\Omega)}. \tag{2.14}$$

These theorems, which can be found in [SS05; Wan03], say that the measures of $\{|u(x)| > \lambda\}$ and $\{\mathcal{M}u(x) > \lambda\}$ decay roughly in the same way.

2.2 Sobolev spaces

When we study PDEs, we typically expect our solutions to have some degree of differentiability. Consider, for example the one-dimensional transport equation:

$$\partial_t u(x,t) + \partial_x u(x,t) = 0$$

with initial condition $u(x,0) = f(x)$. This equation describes the transport of the quantity $f(x)$ along the characteristic lines, and its solution is given by $u(t,x) = f(x-t)$.

Since it is a first-order PDE, one might expect to obtain a solution u in C^1. However, note that the initial data f need not be continuous for the equation to make sense. If f is not continuous, the solution will not be in C^1. Therefore, we need more tools, and that is where weak solutions come into play.

Take $\varphi \in C_c^\infty(\Omega)$ and a function $u \in C^\infty(\Omega)$. The integration by parts formula reads

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi dx \quad \text{for } i = 1, ..., n. \tag{2.15}$$

There are no boundary terms because φ is compactly supported in Ω. This formula has no problems as long as $u \in C^k$ for some $k \geq 1$. Even if u is not differentiable but $L^1(\Omega)$, the left hand side still makes sense. In that case, if we find another function $v \in L^1(\Omega)$ such that

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}dx = -\int_\Omega v\varphi dx \quad \text{for } i = 1, ..., n, \tag{2.16}$$

the whole formula makes sense. This function v, if it exists, is called the weak (partial) derivative of u. This definition can be generalised.

Definition 2.15. *Let $k > 0$, $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$. Let $u, v \in L^1_{loc}(\Omega)$. We say that v is the weak (partial) derivative of u if for any test function $\varphi \in C_c^\infty(\Omega)$, the following holds*

$$\int_\Omega uD^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega v\varphi.$$

Moreover, if such v exists, it is unique.

We will denote the weak derivative of u as $v := D^\alpha u$. If the classical derivative of u exists, it coincides with the weak derivative. We can even ask more of these weak derivatives. For example, we can ask them to be integrable, or even L^p for $p > 1$. This leads us to Sobolev spaces.

Definition 2.16 (Sobolev spaces)**.** *Let $k \geq 0$ and $1 \leq p \leq \infty$. We define Sobolev spaces as*

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \leq k \right\},$$

endowed with the norm

$$\|u\|_{W^{k,p}} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}.$$

As for L^p spaces, Sobolev spaces are complete for $1 \leq p \leq \infty$ and reflexive for $1 < p < \infty$. Also, when $p = 2$, the space $W^{k,2}$ is a Hilbert space with the scalar product

$$(u, v) = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v. \tag{2.17}$$

This space is usually denoted by $W^{k,2} \equiv H^k$. Another Banach space is $W_0^{k,p}(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ with respect to the $W^{k,p}$ norm. For the case $p = 2$, it is usually denoted $W_0^{k,2} \equiv H_0^k$. It is important to note that for $u \in W_0^{k,p}$,

$$\|u\|_{W^{k,p}} \sim \|D^k u\|_{L^p}. \tag{2.18}$$

$W_0^{k,p}$ spaces can be roughly defined as the $W^{k,p}$ functions vanishing at the boundary of Ω. They are of great interest since they are the natural space for energy solutions of PDEs. Again, we can also define the local Sobolev spaces, denoted as $W_{loc}^{k,p}(\Omega)$, as the space of functions that are $W^{k,p}$ when restricted to all compact subsets K of Ω.

2.2.1 Approximation by smooth functions

Working with Sobolev functions can be quite difficult due to the definition of weak derivative. In order to ease this task, we will see that Sobolev functions can be approximated by smooth functions. Recall $\Omega_\epsilon := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \epsilon\}$.

Theorem 2.17 (Local approximation by smooth functions)**.** *Assume $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$ and set*

$$u_\epsilon = \eta_\epsilon * u \quad \text{in } \Omega_\epsilon,$$

where $\Omega_\epsilon = \{x \in \Omega \,|\, dist(x, \partial\Omega) > \epsilon\}$. Then $u_\epsilon \in C^\infty(\Omega)$ for each $\epsilon > 0$, and $u_\epsilon \to u$ in $W_{loc}^{k,p}(\Omega)$ as $\epsilon \to 0$.

If we work with Ω a bounded domain instead of an arbitrary one, the approximation holds for general $W^{k,p}$ functions, not only $W_{loc}^{k,p}$.

Theorem 2.18 (Global approximation by smooth functions)**.** *Assume Ω is bounded, and suppose as well that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence of functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.*

We can ask even more of Ω and impose that the boundary is C^1. Then, the Sobolev functions can be approximated both inside the domain and on its boundary.

Theorem 2.19 (Global approximation by smooth functions up to the boundary)**.** *Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence of functions $u_m \in C^\infty(\overline{\Omega})$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.*

2.2.2 Sobolev embeddings and interpolation

In general we say that for any $1 \leq p \leq \infty$, $W^{0,p}(\Omega) = L^p(\Omega)$. When $1 \leq p < \infty$, we also have that $W_0^{0,p}(\Omega) = L^p(\Omega)$, because C_c^∞ is dense in L^p. So for any k we have the basic embeddings

$$W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega) \subset L^p(\Omega), \tag{2.19}$$

and also regarding Sobolev spaces with different order of weak differentiability,

$$W^{k,p}(\Omega) \subset W^{k-1,p}(\Omega) \subset \ldots \subset W^{1,p}(\Omega), \tag{2.20}$$

but there are more interesting embeddings. Embeddings of Sobolev spaces are specially useful in the study of differential and integral equations. We start with Morrey's.

Theorem 2.20 (Morrey's embedding)**.** *Let $p > n$. Then*

$$W^{1,p}(\mathbb{R}^n) \subset C^{0,1-\frac{n}{p}}(\mathbb{R}^n). \tag{2.21}$$

The same holds for any bounded domain Ω with Lipschitz boundary.

What Morrey's embedding theorem states is that once the degree of integrability becomes higher than the dimension of the space, Sobolev functions become Hölder continuous[1]. Whenever the integrability remains below n, we have Sobolev's embedding theorem.

Theorem 2.21 (Sobolev embedding)**.** *Let $k > l \geq 0$ and $p < n$. Let $1 \leq p < q < \infty$ be two real numbers such that*

$$\frac{l-k}{n} = \frac{1}{q} - \frac{1}{p},$$

or said otherwise, $q = \frac{pn}{n+(l-k)p}$. Then

$$W^{k,p}(\mathbb{R}^n) \subset W^{l,q}(\mathbb{R}^n). \tag{2.22}$$

[1] A function is Hölder continuous $C^{0,\alpha}(\Omega)$, with $\alpha \in \mathbb{R}^+$ if there exists a constant C such that

$$|f(x) - f(y)| \leq C\|x-y\|^\alpha \quad \forall x, y \in \Omega.$$

The last two results can be found with much more detail in [Ada75] Chapter 5. An important type of embedding is the compact embedding, as it maps bounded sets to precompact ones. If the embedding goes into a Banach space, the bounded set is mapped to another bounded set. Regarding compact embeddings and Sobolev spaces, we have the Rellich-Kondrachov theorem, which can be found in Chapter 6 of [Ada75].

Theorem 2.22 (Rellich-Kondrachov)**.** *Let Ω be a bounded C^1 domain. Let $k > l \geq 0$ and $p < n$. Let $1 \leq p < q < \infty$ be two real numbers such that*

$$\frac{l-k}{n} < \frac{1}{q} - \frac{1}{p},$$

or said otherwise, $1 \leq q < \frac{pn}{n+(l-k)p}$. Then the embedding

$$W^{k,p}(\mathbb{R}^n) \subset W^{l,q}(\mathbb{R}^n). \tag{2.23}$$

is compact. This is usually denoted as $W^{k,p}(\mathbb{R}^n) \subset\subset W^{l,q}(\mathbb{R}^n)$.

The last theorem we will need regarding embeddings is Poincaré's inequality.

Theorem 2.23 (Poincaré inequality)**.** *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$ with a C^1 boundary. Let $1 \leq p \leq \infty$. Then there exists a constant $C := C(n, p, \Omega)$ such that*

$$\|u - \overline{u}_\Omega\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \tag{2.24}$$

for all $u \in W^{1,p}(\Omega)$, where $\overline{u}_\Omega$ denotes the mean of u over Ω.

However, when we consider $W_0^{1,p}$ functions this theorem can be stated in another way.

Theorem 2.24 (Poincaré's inequality for $W_0^{1,p}$ functions)**.** *Let Ω be a bounded open set. Let $1 \leq p < \infty$. Then there exists a constant $C := C(n, p, \Omega)$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \tag{2.25}$$

for all $u \in W_0^{1,p}(\Omega)$.

2.3 Basics on divergence form elliptic equations

Let $0 < \lambda \leq \Lambda < \infty$. We say that a measurable map $A : \Omega \to \mathbb{R}^{n\times m}$ is $(\lambda, \Lambda) -$ *elliptic* if for a.e. $x \in \Omega$ and al $\xi \in \mathbb{R}^n$

$$\lambda|\xi|^2 \leq \xi \cdot A(x)\xi \quad \text{and} \quad |A(x)\xi| \leq \Lambda|\xi|.$$

Furthermore, let f be, for instance, a $C^\infty(\Omega)$ function and let A be a (λ, Λ)-elliptic coefficient, symmetric matrix. The minimisers to functionals of the kind

$$F(u) = \int_\Omega A\nabla u \cdot \nabla u - fu,$$

are the weak solutions to the *divergence form elliptic PDE*

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega.$$

In particular we will consider the Laplace equation ($A = I$, $f = 0$) and the Poisson equation ($A = I$). For further information on these equations, we refer to Chapter 2 in [Eva10] or Chapters 2 and 4 in [GT01].

2.3.1 The Laplace equation

One of the most famous problems of the Calculus of Variations is that of minimising Dirichlet's energy,

$$E(u) = \int_\Omega |\nabla u(x)|^2 dx, \tag{2.26}$$

whose Euler-Lagrange equations yield the Laplace equation:

$$\Delta u = 0. \tag{2.27}$$

Any C^2 function satisfying this equation is a *harmonic* function, and the solutions to (2.27) are *real analytic*. In particular, any harmonic function is C^∞.

For a fixed point $y \in \Omega$, we can introduce the normalised *fundamental solution* of Laplace's equation as

$$\Gamma(x-y) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x-y|^{2-n} & \text{if } n > 2 \\ \dfrac{1}{2\pi}\log|x-y| & \text{if } n = 2. \end{cases} \tag{2.28}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. We have that Γ is harmonic as long as $x \neq y$, and we have the following estimates for its derivatives:

$$|D_i\Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{1-n} \tag{2.29}$$

$$|D_{ij}\Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{-n} \tag{2.30}$$

If we focus on studying the Dirichlet problem for the Laplace equation in a ball,

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = g & \text{in } \partial B_1 \end{cases}, \tag{2.31}$$

thanks to the Poisson kernel we have that the unique weak solution is

$$u(x) = c_n \int_{\partial B_1} \frac{1-|x|^2}{|x-y|^n} g(y)\, dy, \tag{2.32}$$

where c_n is a positive dimensional constant. This can be scaled to any harmonic function in $\Omega \supset B_r$

$$u(x) = \frac{c_n}{r} \int_{\partial B_r} \frac{r^2 - |x|^2}{|x-y|^n} u(y)\,dy. \tag{2.33}$$

Taking $x = 0$, the previous formula yields the *mean value property* $u(0) = \fint_{\partial B_r} u$. This is another way of characterising harmonic functions. We say that $u \in C^2(\Omega)$ is harmonic in if, and only if it satisfies

$$u(x) = \fint_{\partial B_r(x)} u(y)\,d\sigma(y) = \fint_{B_r(x)} u(y)\,dy \tag{2.34}$$

for any ball $B_r(x) \subset \Omega$. Furthermore, from the mean value property, the maximum principle for harmonic functions follows. That is, for any harmonic function

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{2.35}$$

Moreover, if Ω is connected and there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$, then u is constant within Ω.

After all these, we can prove an important result concerning the behaviour of weak solutions to the Laplace equation.

Theorem 2.25. *Let $\Omega \subset \mathbb{R}^n$ be any open set, and $u \in H^1(\Omega)$ be any function satisfying $\Delta u = 0$ in the weak sense. Then u is C^∞ inside Ω.*
Moreover, if $\Delta u = 0$ in B_1 in the weak sense, then we have the estimates

$$\|u\|_{C^k(B_{1/2})} \le C_k \|u\|_{L^1(B_1)} \tag{2.36}$$

for all $k \in \mathbb{N}$ and for some constant C_k depending only on k and n.

Proof. Assume without loss of generality that $B_1 \subset \Omega$. As u is a weak solution to the Laplace equation on Ω, it can be written as (2.33). Taking $x \in B_{1/2}$, from this formula we see that u is $C^\infty(B_{1/2})$. Therefore, we have

$$\|u\|_{C^k(B_{1/2})} \le C_k \|u\|_{L^\infty(B_1)}.$$

Then, as u is harmonic, if we take $x \in B_{3/4}$, we can write u as

$$u(x) = \fint_{B_{1/4}(x)} u(y)\,dy,$$

hence

$$|u(x)| \le C \int_{B_1} |u(y)|\,dy$$

and thus, we have the estimates (2.36). Even more, as this can be done for any ball in Ω, it follows that u is C^∞ in Ω. $\square$

2.3.2 The Poisson equation

Once we have studied the Laplace equation, we can take the fundamental solution, $\Gamma(x-y)$ and convolute it with an integrable function f. The convolution $\Gamma * f$ is called the *Newtonian potential of* f. If we define

$$u(x) := \Gamma * f = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy, \tag{2.37}$$

and take $f \in C_c^\infty(\Omega)$ instead of integrable, we have that $u \in C^\infty(\overline{\Omega})$. Indeed, by a simple change of variables to avoid differentiating Γ,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy = \int_{\mathbb{R}^n} \Gamma(z)f(x-z)dz,$$

we have that

$$D^k u(x) = \int_{\mathbb{R}^n} \Gamma(z)D^k f(x-z)dz$$

is well defined for all k. Then, in this case[2], if we consider a bounded domain Ω with $\partial\Omega$ regular, the Newtonian potential is a solution to the *Poisson equation*

$$\Delta u = f \quad \text{in } \Omega \tag{2.38}$$

Moreover, if u solves (2.38), it minimises the energy functional

$$I(v) = \int_\Omega \frac{1}{2}|\nabla v(x)|^2 - vf\,dx, \tag{2.39}$$

and, conversely, the minimiser of said functional solves (2.38). It is important to note that not all solutions to the Poisson equation are given by the Newtonian potential, as we will see later in Chapter 3. However, we remark that solutions to the Poisson equation are given by

$$u = \Gamma * f + h,$$

where h is a function such that $\Delta h = 0$.

[2]For the following to hold, it suffices to consider f a bounded, locally Hölder continous function in Ω.

CHAPTER 3

L^p estimates for the Poisson equation

> *One of the most remarkable facts in the elements of the theory of analytic functions appears to me to be this: that there exist partial differential equations whose integrals are all of necessity analytic functions of the independent variables, that is, in short, equations susceptible of none but analytic solutions.*
>
> — **David Hilbert**
> International Congress of Mathematics, 1900

At the Second International Congress of Mathematics in Paris (1900), Hilbert presented the 23 problems that would determine the course of Mathematics during the twentieth century. One of particular interest related to this project is the nineteenth problem, asking whether the solutions of *regular* problems in the calculus of variations are necessarily analytic. More precisely: if we consider local minimisers of a functional of the form

$$J(u) = \int_\Omega L(\nabla u)dx,$$

where $L : \mathbb{R}^n \to \mathbb{R}$ is smooth and uniformly convex, with $\Omega \subset \mathbb{R}^n$, can we say that such minimisers are smooth? That is, are the minimisers of such energy functional real analytic?

The most characteristic examples of functionals encompassed in Hilbert's nineteenth problem are Dirichlet's energy and the energy functional yielding the Poisson equation, which have been introduced in Chapter 2. We have seen that the solutions to the minimisation problem concerning Dirichlet's energy are in fact smooth. Now we ask ourselves: if we consider

$$I(v) = \int_\Omega |\nabla v|^2 - vf\,dx,$$

what can we say about its minismisers? That is, what can we say about the regularity of the solutions to the Poisson equation?

3.1 Solutions and regularity

Our aim is to study *how good* the solutions to the Poisson equation are. Specifically, we are interested in determining whether, as Hilbert stated, they are *regular* enough. Before delving into the question of regularity, we must first define what is a solution of a PDE. When solving a PDE, we hope that the problem under study is *well-posed.* This happens if

(a) the problem has a solution;

(b) the solution is unique; and

(c) the solution is stable.

If any of this conditions is not satisfied, we say that the problem is *ill-posed.*

We aim to solve the PDE in such a way that (a)-(c) hold. However, finding a solution can be challenging. For instance, looking for smooth functions (which would be ideal) may not be realistic, since it imposes restrictive conditions. Instead, we could settle for C^k functions when solving PDEs of order k, to ensure that they can be differentiated and their derivatives are continuous. From now on, this C^k solutions to $k-$th order PDEs will be referred to as *classical solutions.*

Unfortunately, it is not always possible to find such solutions. To address this, we can lower our expectations and broaden the class of potential solutions so we can still find one that satisfies (a)-(c). This broader class is known as weak solutions. Instead of demanding a high degree of differentiability to satisfy the strong form of the equation pointwise and in a classical sense, we look for weakly differentiable solutions that satisfy the equation in its weak form, or in the sense of distributions. Specifically, a weak solution to a PDE is a function that satisfies the integral form of the equation when tested against a set of smooth test functions. Therefore, this approach makes use of the Sobolev spaces theory and the concept of weak derivatives to find a function that meets conditions (a)-(c), even though its not differentiable in a classical sense. These weakly differentiable functions that satisfy the weak form of the equation will be called *weak solutions.*

In the context of second-order elliptic equations, we can either seek classical C^2 solutions or weak $W^{1,p}$ solutions. However, if we prefer not to relax the notion of differentiability too much, we can look for a middle ground: *strong solutions.* Strong solutions take up an intermediate position between classical and weak solutions. They are defined as functions that possess enough regularity to ensure that the strong form of the equation holds pointwise a.e. in the domain. For second-order elliptic equations, this means that we seek $W^{2,p}$ functions, as opposed to weak solutions, which only require $W^{1,p}$ solutions. By seeking strong solutions, we find a balance between more regular solutions and the flexibility provided by weak derivatives. One of the most important results in this setting is the Calderón-Zygmund inequality.

Once we have found a solution, we may still be interested in its degree of differentiability. We can further study if said solution is smoother than what we initially assumed and verify if a weak or strong solution is actually a classical one. This process is known as the study of the *regularity* of solutions to PDEs.

3.2 The Calderón-Zygmund inequality

In [CZ52], Alberto Calderón and Antoni Zygmund studied the existence of singular integrals of L^p functions. In order to do so, they introduced the Calderón-Zygmund cube decomposition, which will be used in this section. What is more important is that they saw that the study of existence of said integrals can be used to study the differentiability of the Newtonian potential, which yields the solution to Poisson's equation. Using part of the work they did, this section is devoted to studying the regularity of solutions to the Poisson equation. More precisely, given the Poisson equation

$$\Delta u = f,$$

we aim to see that if f is an L^p function, u is a $W^{2,p}$ function. That is, if the Laplacian of u is L^p, we want to see that u is actually twice weakly differentiable.

3.2.1 The energy estimates

We start by studying the case $p = 2$. Recall that now we are dealing with L^2, which is a Hilbert space. In this case, the estimates are usually called the *energy estimates*.

Theorem 3.1 (Energy estimates)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $f \in L^2(\Omega)$ and let u be the Newtonian potential of f. Then $u \in W^{2,2}(\Omega)$, $\Delta u = f$ a.e. in Ω and*

$$\|D^2 u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}. \tag{3.1}$$

Furthermore,

$$\int_{\mathbb{R}^n} |D^2 u|^2 = \int_{\Omega} f^2. \tag{3.2}$$

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then $u \in C^\infty(\mathbb{R}^n)$ and $\Delta u = f$ (see Subsection 2.3.2) In particular, for any ball containing the support of f we have that

$$\int_{B_r} (\Delta u)^2 = \int_{B_r} f^2.$$

Integrating by parts,

$$\int_{B_r} |D^2u|^2 = \sum_{i,j} \int_{B_r} (D_{ij}u)^2 = \sum_{i,j} \left[-\int_{B_r} D_{ijj}uD_iu + \int_{\partial B_r} D_{ij}uD_iu \right]$$
$$= \sum_{i,j} \left[\int_{B_r} D_{jj}uD_{ii}u - \int_{\partial B_r} D_iuD_{jj}u + \int_{\partial B_r} D_{ij}uD_iu \right]$$
$$= \int_{B_r} (\Delta u)^2 + \sum_{i,j} \int_{\partial B_r} D_iu[D_{ij}u - D_{jj}u]$$
$$= \int_{B_r} f^2 + \sum_{i,j} \int_{\partial B_r} D_iu[D_{ij}u - D_{jj}u]$$

Using (2.29) and (2.30), we have that $Du = O(r^{1-n})$ and $D^2u = O(r^{-n})$ uniformly on ∂B_r as $r \to \infty$. Therefore

$$\int_{\mathbb{R}^n} |D^2u|^2 = \int_\Omega f^2.$$

Now, this holds for any $f \in C_c^\infty(\Omega)$. To prove the theorem in its full form we proceed with an approximation argument. If we consider the sequence $f_j = f * \eta_{\epsilon_j}$, where $f \in L^2(\Omega)$ and η_{ϵ_j} is the standard mollifier with $\epsilon_j \to 0$ as $j \to \infty$, we have that $f_j \to f$ in $L^2(\Omega)$. Now we take $u_j = \Gamma * f_j \in C^\infty(\Omega)$. By Young's theorem (Theorem 2.6) it follows that $\{u_j\}$ is a Cauchy sequence in the L^2-norm:

$$\|u_j - u_i\|_2 \le \|\Gamma * (f_j - f_i)\|_2 = \|\Gamma\|_1 \|f_j - f_i\|_2 \to 0.$$

Since L^2 is complete, this sequence converges to a limit, $\hat{u}$. Moreover, taking $u = \Gamma * f$ we have that:

$$\|u_j - u\|_2 = \|\Gamma * (f_j - f)\|_2 \le \|\Gamma\|_1 \|f_j - f\|_2 \to 0,$$

since $f_j \to f$. Hence u_j converges to $\hat{u} = u = \Gamma * f$. Now considering the sequence $\{D^2u_j\}$, we have that it is a Cauchy sequence:

$$\|D^2u_j - D^2u_i\|_2 = \|D^2(u_j - u_i)\|_2 \le \|f_j - f_i\|_2 \to 0.$$

Again, on one hand, as L^2 is a Banach space, the sequence converges. On the other hand, by definition of weak derivative, we have that

$$\int_\Omega D^2u_i\varphi = (-1)^2 \int_\Omega u_iD^2\varphi \longrightarrow (-1)^2 \int_\Omega uD^2\varphi = \int_\Omega D^2u\varphi$$

for any test function $\varphi \in C_c^\infty(\Omega)$, since

$$\left| \int_\Omega u_iD^2\varphi - \int_\Omega uD^2\varphi \right| \le \int_\Omega |u_i - u||D^2\varphi| \le \|u_i - u\|_2 \|D^2\varphi\|_2 \longrightarrow 0.$$

So $D^2u_i \to D^2u$. Therefore,

$$\|D^2u\|_2 = \lim_{j\to\infty} \|D^2u_j\|_2 \le \lim_{j\to\infty} \|f_j\|_2 = \|f\|_2.$$

$\square$

3.2.2 An interpolation approach to the general case

Cube decomposition

Let K_0 be a cube in $\mathbb{R}^n$, f a nonnegative integrable function in K_0, and t a positive number such that

$$\int_{K_0} f \leq t|K_0|. \tag{3.3}$$

By bisecting the edges, we can subdivide K_0 into 2^n cubes, K_i. Let us denote by $\mathcal{P}$ the set of subcubes, K_i, where

$$\int_{K_i} f > t|K_i|. \tag{3.4}$$

If there are any subcubes not in $\mathcal{P}$, that is, if there are any subcubes where

$$\int_{K_i} f \leq t|K_i|, \tag{3.5}$$

we bisect them once again into 2^n cubes. We repeat this process indefinitely. Then, for each $K_i \in \mathcal{P}$ we denote by $\tilde{K}_i$ the subcube whose subdivision yields K_i. Hence the ratio between the measure of both cubes ($|\tilde{K}_i|/|K_i|$) will be 2^n, and from (3.4) and (3.5) we have that

$$t < \frac{1}{|K_i|}\int_{K_i} f \leq 2^n t. \tag{3.6}$$

Setting $F = \cup_{K_i \in \mathcal{P}} K_i$ and $G = K_0 - F$, we have

$$f \leq t \quad \text{a.e. in } G \tag{3.7}$$

as a consequence of Lebesgue's differentiation theorem (Theorem 2.4).

The Calderón-Zygmund inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L^p(\Omega)$ for $p \geq 1$. Recall that the Newtonian potential of f is the function u defined by the convolution

$$u(x) = \Gamma * f = \int_\Omega \Gamma(x-y)f(y)dy, \tag{3.8}$$

where Γ is the fundamental solution to Laplace's equation. We now state and prove the Calderón-Zygmund inequality in the style of [GT01].

Theorem 3.2 (Calderón-Zygmund estimate, v1)**.** *Let $f \in L^p(\Omega)$, with $1 < p < \infty$. Let u be the Newtonian potential of f. Then $u \in W^{2,p}(\Omega)$, $\Delta u = f$ a.e. in Ω, and*

$$\|D^2 u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.9}$$

The proof of this theorem relies on the following lemma.

Lemma 3.3. *Let Ω be a bounded domain. Let $f \in L^1(\Omega)$ and u be the Newtonian potential of f. Then*

$$\left|\{x \in \Omega \,|\, |D^2 u| > \lambda\}\right| \leq C\frac{\|f\|_1}{\lambda}. \tag{3.10}$$

Proof. For fixed i, j we define the linear operator

$$\begin{aligned} T : L^2(\Omega) &\longrightarrow L^2(\Omega) \\ f \quad &\longmapsto D_{ij}u \end{aligned} \tag{3.11}$$

We want to see that for all $\lambda > 0$ and $f \in L^2(\Omega)$,

$$\mu(\lambda) := \mu_{Tf}(\lambda) \leq \frac{C\|f\|_1}{\lambda} \tag{3.12}$$

First, we extend f to vanish outside Ω. We fix $\lambda > 0$. Then, we can find a cube K_0 such that

$$\int_{K_0} f \leq \lambda |K_0|.$$

We apply the cube decomposition we saw at the beginning of this subsection to decompose K_0 into a sequence of parallel subcubes $\{K_l\}_{l\in\mathbb{N}}$ such that

$$\lambda < \frac{1}{|K_l|}\int_{K_l} f < 2^n\lambda \tag{3.13}$$

and $|f| < \lambda$ a.e. on $G = K_0 - \cup_l K_l$.
We now split f into a "good part", g, defined as

$$g(x) = \begin{cases} f(x) & \text{for } x \in G \\ \frac{1}{|K_l|}\int_{K_l} f & \text{for } x \in K_l,\ l = 1, 2, \dots \end{cases},$$

and a "bad part", $b = f - g$. This good and bad parts satisfy:

- $|g| \leq 2^n\lambda$, a.e.
- $b(x) = 0$ for $x \in G$.
- $\int_{K_l} b = 0$ for $l = 1, 2, \dots$

Since T is linear, $Tf = Tg + Tb$, and hence,

$$\mu_{Tf}(\lambda) \leq \mu_{Tg}\left(\frac{\lambda}{2}\right) + \mu_{Tb}\left(\frac{\lambda}{2}\right),$$

so in order to prove (3.12) we have to estimate both μ_{Tg} and μ_{Tb}.
First let us estimate Tg. From Proposition 2.10 and the energy estimates (3.2), we have

$$\mu_{Tg}\left(\frac{\lambda}{2}\right) \leq \left(\frac{\|g\|_2}{\lambda/2}\right)^2 = \frac{4}{\lambda^2}\int_\Omega g^2 \leq \frac{2^{n+2}}{\lambda}\int_\Omega |g| \leq \frac{2^{n+2}}{\lambda}\int_\Omega |f|.$$

Setting $C = C(n) := 2^{n+2}$,

$$\mu_{Tg}\left(\frac{\lambda}{2}\right) \leq C\frac{\|f\|_1}{\lambda}. \tag{3.14}$$

So we have the desired estimate for Tg. Now we tackle Tb. Let δ_l be the diameter of K_l and $\overline{y}_l$ the center of K_l. We denote $B_l = B_{\delta_l}(\overline{y}_l)$. Note that $K_l \subset B_l$. Let $F^* = \cup_l B_l$ and $G^* = K_0 - F^*$. As $K_0 \subset F^* \sqcup G^*$, to estimate Tb we have

$$\left|\left\{x \in K_0 \,\middle|\, Tb > \frac{\lambda}{2}\right\}\right| \leq \left|\left\{x \in F^* \,\middle|\, Tb > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in G^* \,\middle|\, Tb > \frac{\lambda}{2}\right\}\right|,$$

thus, we will estimate Tb both on F^* and G^*. We start with F^*. We have that

$$|F^*| = \sum_l |B_l| \leq \alpha(n) \sum_l |K_l| = \alpha(n)|F|, \tag{3.15}$$

where $\alpha(n)$ is a geometric constant concerning the ratio of $|K_l|$ and $|B_l|$. Therefore by (3.15) and (3.13)

$$\left|\left\{x \in F^* \,\middle|\, Tb > \frac{\lambda}{2}\right\}\right| \leq |F^*| \leq \alpha(n)|F| < \frac{\alpha(n)}{\lambda}\int_F f = \alpha(n)\frac{\|f\|_1}{\lambda} \tag{3.16}$$

Now let us estimate Tb on G^*. We write

$$b_l = \begin{cases} b & \text{on } K_l \\ 0 & \text{elsewhere} \end{cases}. \tag{3.17}$$

By (3.17) we write $Tb = \sum_{l\geq 1} Tb_l$. Then we fix an l and cosider a sequence $\{b_{l_m}\}_m$ on $C_c^\infty(K_l)$ converging to b_l in $L^2(\Omega)$ satisfying

$$\int_{K_l} b_{l_m} = \int_{K_l} b_l = 0.$$

Note that if $x \in G^*$, $x \notin K_l$ for any l, so we can express

$$\begin{aligned} Tb_{l_m}(x) &= \int_{K_l} D_{ij}\Gamma(x-y)b_{l_m}(y)dy \\ &= \int_{K_l} D_{ij}\Gamma(x-y)b_{l_m}(y)dy - \int_{K_l} D_{ij}\Gamma(x-\overline{y}_l)b_{l_m}(y)dy \\ &= \int_{K_l} \left[D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\overline{y}_l)\right] b_{l_m}(y)dy, \end{aligned}$$

Then by the mean value theorem for integrals,

$$\begin{aligned} |Tb_{l_m}(x)| &\leq \int_{K_l} |y-\overline{y}_l| \cdot |DD_{ij}\Gamma(x-\hat{y})| \cdot |b_{l_m}(y)|dy \\ &\leq \delta_l \cdot C(n)\int_{K_l} |x-\hat{y}|^{-n-1}|b_{l_m}(y)|dy \\ &\leq \delta_l \cdot C(n)\left[\mathrm{dist}(x,K_l)\right]^{-n-1}\int_{K_l} |b_{l_m}(y)|dy, \end{aligned}$$

where $C(n) = (n-1)/\omega_n$. Then integrating both sides

$$\begin{aligned}\int_{K_0 \setminus B_l} |Tb_{l_m}|\, dx &\leq \delta_l \cdot C(n) \int_{K_0 \setminus B_l} [\mathrm{dist}(x, K_l)]^{-n-1} \left(\int_{K_l} |b_{l_m}|\, dy \right) dx \\ &\leq \delta_l \cdot C(n) \int_{|x| \geq \frac{\delta_l}{2}} |x|^{-n-1} \left(\int_{K_l} |b_{l_m}|\, dy \right) dx \\ &= \delta_l \cdot C(n) \int_{\frac{\delta_l}{2}}^{\infty} r^{-2} \left(\int_{K_l} |b_{l_m}|\, dy \right) dr \\ &= C(n) \int_{K_l} |b_{l_m}|\, dy,\end{aligned}$$

where $C(n) = 2(n-1)/\omega_n$. Taking limits as $m \to \infty$ and summing over l, we end up with

$$\int_{G^*} |Tb| \leq C(n) \int_{F^*} |b| \leq C(n) \int_{F^*} |f|.$$

So, by Proposition 2.10

$$\left| \left\{ x \in G^* \,\middle|\, |Tb| > \frac{\lambda}{2} \right\} \right| \leq \frac{2}{\lambda} \int_{G^*} |Tb| \leq \frac{2}{\lambda} C(n) \int_{F^*} |f| = \frac{c\|f\|_1}{\lambda}. \tag{3.18}$$

In conclusion: by (3.16) and (3.18) we can estimate Tb; and by (3.14) we can estimate Tg. Therefore (3.12) holds, concluding the proof. $\square$

Now, starting from the energy estimates (Theorem 3.1) and using the previous Lemma, we are able to prove the Calderón-Zygmund inequality by means of interpolation.

Proof of Theorem 3.2. For fixed i, j, consider the operator T defined in (3.11). By the energy estimates (Theorem 3.1) we know that

$$\mu_{Tf}(\lambda) \leq \left(\frac{\|f\|_2}{\lambda} \right)^2. \tag{3.19}$$

Then, by Lemma 3.3, it also holds

$$\mu_{Tf}(\lambda) \leq C \frac{\|f\|_1}{\lambda}. \tag{3.20}$$

Now, note that (3.19) and (3.20) fulfill the hypothesis of the Marcinkiewicz interpolation theorem (Theorem 2.11) with $q = 1$ and $r = 2$. Consequently,

$$\|Tf\|_p \leq C(n,p) \|f\|_p \tag{3.21}$$

for all $1 < p \leq 2$ and $f \in L^2(\Omega)$. We now extend this inequality for $p > 2$ by duality. Take any $f, g \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\int_\Omega (Tf)g &= \int_\Omega D_{ij}ug = -\int_\Omega D_iuD_jg = \int_\Omega uD_{ij}g \\ &= \int_\Omega \int_\Omega \Gamma(x-y)f(y)D_{ij}g(x)\,dxdy \\ &= \int_\Omega f(y)\left[D_{ij}\int_\Omega \Gamma(x-y)g(x)\,dx\right]dy = \int_\Omega f(Tg) \\ &\leq \|f\|_p\|Tg\|_{p'}\end{aligned}$$

Thus if $p > 2$, we have

$$\|Tf\|_p = \sup_{\|g\|_{p'}=1}\int_\Omega (Tf)g = \sup_{\|g\|_{p'}=1}\int_\Omega f(Tg) \leq \sup_{\|g\|_{p'}=1}\|Tg\|_{p'}\|f\|_p \leq C(n,p')\|f\|_p. \quad (3.22)$$

Therefore, (3.21) holds for all $f \in C^\infty$, and by (3.22) and approximation, it holds for all $f \in L^p(\Omega)$, with $1 < p < \infty$. □

3.2.3 The general case through a study of the level sets

As usually happens in Mathematics, a result can be proven in more than one way. Recall that our objective is seeing that, given u a solution to the Poisson equation, u is a $W^{2,p}$ function and that the L^p-norm of D^2u is bounded by the L^p-norm of the right-hand-side. One way to do so is with the Calderón-Zygmund cube decomposition, but we can approach the proof from a more geometrical point of view, studying how the level sets decay.

In Chapter 2 we see that a function u is weak L^p if $\mu_u(\lambda)$ decays as C^p/λ^p for some constant C. Moreover, we see that with a more restrictive condition on the decay, $\mu_u(\lambda_0\lambda) \leq \epsilon\mu_u(\lambda)$, and an appropriate choice of ϵ, we can guarantee that u is L^p. Then, we can do the same for D^2u, i.e., check if

$$\left|\left\{|D^2u| > \lambda_0\lambda\right\}\right| \leq \epsilon\left|\left\{|D^2u| > \lambda\right\}\right|. \quad (3.23)$$

However, as u depends on f, f should also appear in (3.23). For instance, we can see if the following holds

$$\left|\left\{|D^2u| > \lambda_0\lambda\right\}\right| \leq \epsilon\left(\left|\left\{|D^2u| > \lambda\right\}\right| + |\{|f| \geq \delta_0\lambda\}|\right), \quad (3.24)$$

for an appropriate $\delta_0 > 0$ to be determined. Unfortunately, this is not true. As we expect D^2u to be in L^p, we cannot hope for pointwise bounds. Nevertheless, we can avoid this problem with the Hardy-Littlewood maximal function, because $\mathcal{M}|D^2u|(x) \leq \lambda$ means that $|D^2u(x_0)| \leq \lambda$ in all scales. Therefore, instead of proving (3.24), we will prove

$$\left|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > \lambda_0\right\}\right| \leq \epsilon\left(\left|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > 1\right\}\right| + \left|B_1 \cap \left\{\mathcal{M}(f^2) > \delta_0^2\right\}\right|\right), \quad (3.25)$$

which can bee seen as an averaged version of (3.24).

Vitali's lemma and ball coverings

In order to prove (3.25), we will cover the set by balls arranged in a way so that they are disjoint and cover a major portion of the set. To do so we will use a modified version of the Vitali covering lemma. First we state the basic covering lemma.

Lemma 3.4 (Vitali). *Let $\mathcal{B}$ be a set of balls in $\mathbb{R}^n$ with bounded radius. Then there is a finite or countable sequence $B_i \in \mathcal{B}$ of disjoint balls such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i\in\mathbb{Z}} 5B_i$$

This way we can arrange a family of disjoint balls such that when enlarged, they cover all the set under consideration. There are many versions of this lemma, but the one that suits us the most is the following:

Theorem 3.5 (Modified Vitali). *Let $0 < \epsilon < 1$. Let $C \subset D \subset B_1$ be two measurable sets with $|C| < \epsilon|B_1|$. If for all $x \in B_1$ and r such that $B_r(x) \subset B_1$ and $|C \cap B_r(x)| \geq \epsilon|B_r|$, $B_r(x) \cap B_1 \subset D$ holds, then $|C| \leq 10^n\epsilon|D|$.*

Proof. We know that $|C| < \epsilon|B_1|$. For a fixed $x \in C$, let $g_C(r) = |C \cap B_r(x)|/|B_r(x)|$. We know that g is a continuous function of the radius. Then for a.e. $x \in C$, since

$$g(2) = \frac{|C \cap B_2|}{|B_2|} < \frac{\epsilon|B_1|}{|B_2|} < \epsilon$$

and

$$\lim_{r\to 0^+} g(r) = \lim_{r\to 0^+} \frac{|C \cap B_r|}{|B_r|} > \lim_{r\to 0^+} \frac{|B_r|}{|B_r|} = 1 > \epsilon,$$

there must exist a $r_x < 2$ such that $|C \cap B_{r_x}(x)| = \epsilon|B_{r_x}|$ and $|C \cap B_r(x)| < \epsilon|B_r|$ for $r_x < r < 2$.

By Vitali's covering lemma (3.4) there exist $x_1, x_2, ... \in C$ such that $B_{r_{x_1}}, B_{r_{x_2}}, ...$ are disjoint and

$$C \subset \bigcup_{i\in I} B_{5r_{x_i}}(x_i).$$

This way $C \subset \cup_{i\in I} B_{5r_{x_i}}(x_i) \cap B_1$, and from the choice of r_x we have

$$|C \cap B_{5r_{x_k}}(x_k)| < \epsilon|B_{5r_{x_k}}(x_k)| = 5^n\epsilon|B_{r_{x_k}}(x_k)| = 5^n|C \cap B_{r_{x_k}}(x_k)|.$$

Also since $x_k \in B_1$ and $r_{x_k} \leq 2$, seeing that the function $g_{B_1}(r) = |B_1 \cap B_r(x_k)|/|B_r|$ is decreasing and $g_{B_1}(2) = |B_1|/|B_2| = 2^{-n}$, we have that $g_{B_1}(r_{x_k}) \geq 2^{-n}$ and so

$$|B_{r_{x_k}}(x_k)| \leq 2^n|B_{r_{x_k}}(x_k) \cap B_1|.$$

To conclude, putting it all together,

$$\begin{aligned}|C| &= |\cup_k B_{5r_{x_k}}(x_k) \cap C| \\ &\leq \sum_k |B_{5r_{x_k}}(x_k) \cap C| \leq 5^n \sum_k \epsilon |B_{r_{x_k}}(x_k)| \\ &\leq 10^n \sum_k \epsilon |B_{r_{x_k}}(x_k) \cap B_1| = 10^n \epsilon |\cup_k B_{r_{x_k}}(x_k) \cap B_1| \\ &\leq 10^n \epsilon |D|\end{aligned}$$

which concludes the proof. □

The Calderón-Zygmund inequality, revisited

Now we aim to prove the Calderón-Zygmund inequality for any solution to the Poisson equation in a ball, as done in [Wan03]. As in the first version of the theorem, the starting point are the energy estimates (Theorem 3.1). Next, we must see how the level sets of D^2u behave.

Lemma 3.6. *There is a constant N_1 so that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if u is a solution of $\Delta u = f$ in $\Omega \supset B_6$ with*

$$\left\{\mathcal{M}|f|^2 \leq \delta^2\right\} \cap \left\{\mathcal{M}|D^2u|^2 \leq 1\right\} \cap B_1 \neq \emptyset \tag{3.26}$$

then

$$\left|\left\{\mathcal{M}|D^2u|^2 > N_1^2\right\} \cap B_1\right| < \epsilon |B_1|. \tag{3.27}$$

Proof. From (3.26) we see that there is an $x_0 \in B_1$ such that

$$\fint_{B_r(x_0)} |D^2u|^2 \leq 1 \quad \text{and} \quad \fint_{B_r(x_0)} |f|^2 \leq \delta^2 \tag{3.28}$$

for any possible $B_r(x_0) \subset \Omega$. In particular,

$$\fint_{B_4} |D^2u|^2 \leq \frac{1}{|B_4|} \int_{B_5(x_0)} |D^2u|^2 \leq \frac{|B_5|}{|B_4|} \leq 2^n, \tag{3.29}$$

and analogously,

$$\fint_{B_r} |f|^2 \leq 2^n \delta^2. \tag{3.30}$$

Let v be the solution to

$$\begin{cases} \Delta v = 0 & \text{in } B_4 \\ v = u - \overline{\nabla u}_{B_4} \cdot x - \overline{u}_{B_4} & \text{on } \partial B_4 \end{cases}, \tag{3.31}$$

where $\overline{u}_{B_4}$ denotes the mean of u in B_4 and $\overline{\nabla u}_{B_4}$, the mean of ∇u in B_4. Using the minimality of harmonic functions with respect to Dirichlet's energy, Poincaré's inequality and (3.29), we have that

$$\int_{B_4} |\nabla v|^2 \leq \int_{B_4} |\nabla u - \overline{\nabla u}_{B_4}|^2 \leq C \int_{B_4} |D^2u|^2 \leq C.$$

Since v is harmonic, we can use the estimates from Theorem 2.25 and Hölder's inequality to see that there is a constant N_0 so that

$$\|D^2 v\|_{L^\infty(B_3)} \le C\|\nabla v\|_{L^1(B_4)} \le C\|\nabla v\|_{L^2(B_4)} \le N_0. \tag{3.32}$$

Now we claim that $\{x \in B_1 \mid \mathcal{M}|D^2u|^2 > N_1^2\} \subset \{x \in B_1 \mid \mathcal{M}_{B_3}|D^2(u-v)|^2 > N_0^2\}$, where $N_1^2 = \max\{4N_0^2, 2^n\}$. First, note that if $y \in B_3$, then

$$\begin{aligned}|D^2u(y)|^2 &= |D^2u(y)|^2 - 2|D^2v(y)|^2 + 2|D^2v(y)|^2\\ &\le 2|D^2u(y) - D^2v(y)|^2 + 2|D^2v(y)|^2\\ &\le 2|D^2(u-v)(y)|^2 + 2N_0^2\end{aligned}$$

Now, let $x \in \{x \in B_1 \mid \mathcal{M}_{B_3}|D^2(u-v)|^2(x) \le N_0^2\}$. On one hand, if $r \le 2$, $B_r(x) \subset B_3$ and

$$\sup_{r\le 2} \fint_{B_r(x)} |D^2u|^2 \le 2\mathcal{M}_{B_3}|D^2(u-v)|^2(x) + 2N_0^2 \le 4N_0^2.$$

On the other hand, if $r > 2$, $B_r(x) \subset B_{2r}(x_0)$, and then we have that

$$\fint_{B_r(x)} |D^2u|^2 \le \frac{1}{|B_r|}\int_{B_{2r}(x_0)} |D^2u|^2 \le \frac{|B_{2r}|}{|B_r|} = 2^n.$$

Hence $\mathcal{M}|D^2u|^2(x) \le \max\{4N_0^2, 2^n\} = N_1^2$, which proves the claim. This way, we have that

$$\left|\{x \in B_1 \mid \mathcal{M}|D^2u|^2 > N_1^2\}\right| \le \left|\{x \in B_1 \mid \mathcal{M}_{B_3}|D^2(u-v)|^2 > N_0^2\}\right| \tag{3.33}$$

Then, using the weak 1-1 estimate for the Hardy-Littlewood maximal function (Theorem 2.13), we have that

$$\left|\{x \in B_3 \mid \mathcal{M}_{B_3}|D^2(u-v)|^2(x) > N_0^2\}\right| \le \frac{C}{N_0^2}\int_{B_3} |D^2(u-v)|^2 \tag{3.34}$$

Lastly, by the energy estimates (Theorem 3.1) on the function $u-v$ and (3.30), we have

$$\int_{B_3} |D^2(u-v)|^2 \le C\int_{B_3} f^2 \le C\int_{B_4} f^2 \le C\delta^2. \tag{3.35}$$

Therefore we combine (3.33), (3.34) and (3.35) so

$$\begin{aligned}\left|\{x \in B_1 \mid \mathcal{M}|D^2u|^2 > N_1^2\}\right| &\le \left|\{x \in B_1 \mid \mathcal{M}_{B_3}|D^2(u-v)|^2 > N_0^2\}\right|\\ &\le \frac{C}{N_0^2}\int_{B_4} f^2 \le \frac{C\delta^2}{N_0^2} = \epsilon|B_1|,\end{aligned}$$

for a δ that satisfies the last identity above. □

An immediate corollary of this lemma is the following.

Corollary 3.6.1. *There is a constant N_1 such that for all $\epsilon > 0$ there exists a $\delta > 0$ such that the following holds: Let $\Omega \subset \mathbb{R}^n$ and let B be any ball satisfying $6B \subset \Omega$. If u is a solution of $\Delta u = f$ in Ω with the condition $|\{\mathcal{M}|D^2u|^2 > N_1^2\} \cap B| \geq \epsilon|B|$, we have that*

$$B \subset \left\{\mathcal{M}|D^2u|^2(x) > 1\right\} \cup \left\{\mathcal{M}f^2 > \delta^2\right\}.$$

Another interesting corollary comes from using Lemma 3.6 on different sets or repeatedly.

Corollary 3.6.2. *There is a constant N_1 such that for all $\epsilon > 0$ there exists a $\delta > 0$ such that the following holds: Let $\Omega \subset \mathbb{R}^n$ and let B be any ball satisfying $6B \subset \Omega$. If u is a solution of $\Delta u = f$ in Ω with the condition $|\{\mathcal{M}|D^2u|^2 > N_1^2\} \cap B_1| \leq \epsilon|B_1|$, then for $\epsilon_1 = 10^n\epsilon$ we have:*

(a) $|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > N_1^2\right\}| \leq \epsilon_1 \left(|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > 1\right\}| + |B_1 \cap \left\{\mathcal{M}f^2 > \delta^2\right\}|\right)$

(b) $|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > N_1^2\lambda^2\right\}| \leq \epsilon_1 \left(|B_1 \left\{\mathcal{M}|D^2u|^2 > \lambda^2\right\}| + |B_1 \cap \left\{\mathcal{M}f^2 > \delta^2\lambda^2\right\}|\right)$

(c) $|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > (N_1^2)^k\right\}| \leq \sum_{j=1}^k \epsilon_1^j|B_1 \cap \left\{\mathcal{M}f^2 > \delta^2(N_1^2)^{k-j}\right\}|$
$+ \epsilon_1^k|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > 1\right\}|$

Proof. For the first item, set

$$C = B_1 \cap \left\{\mathcal{M}|D^2u|^2 > N_1^2\right\} \quad \text{and} \quad D = B_1 \cap \left(\left\{\mathcal{M}|D^2u|^2 > 1\right\} \cup \left\{\mathcal{M}f^2 > \delta^2\right\}\right)$$

By the Modified version of Vitali's lemma (Theorem 3.5) and Corollary 3.6.1, $|C| \leq 10^n\epsilon$. For the second item, it suffices to apply the first item to the equation $\Delta(\lambda^{-1}u) = \lambda^{-1}f$. For the last item, we apply the second one repeatedly. □

Lemma 3.6 and its Corollaries 3.6.1, 3.6.2 give us an understanding on how does the measure of the superlevel sets behave in B_1 and how can it be controlled. With that, we have (3.25), and thus, given any solution u to the Poisson equation in B_1, we can prove the Calderón-Zygmund inequality.

Theorem 3.7 (Calderón-Zygmund estimate, v2)**.** *Let $f \in L^p(B_1)$. Let u be a solution to $\Delta u = f$ in B_1. Then*

$$\|D^2u\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right) \tag{3.36}$$

Proof. Let $u \in C^\infty(B_1)$ be a function such that, if scaled down by a small constant, we can assume that

$$\left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 > N_1^2\right\}\right| \leq \epsilon|B_1|.$$

Without loss of generality, we may also assume that $\|f\|_p$ is small, for instance $\|f\|_{L^p(B_4)} = \delta$. We will see that $\mathcal{M}|D^2u|^2 \in L^{p/2}(B_1)$ and hence $D^2u \in L^p(B_1)$.

If we start with $f \in L^p$, then $\mathcal{M}f^2 \in L^{p/2}$. Note that by Lebesgue's differentiation theorem (Theorem 2.4):

$$\int_{B_1} \mathcal{M}|D^2u|^p = \int_{B_1} \sup_r \frac{1}{|B_r|} \int_{B_r} |D^2u|^p \geq \int_{B_1} |D^2u|^p.$$

Then:

$$\begin{aligned}
\int_{B_1} |D^2u|^p &\leq \int_{B_1} (\mathcal{M}|D^2u|^2)^{p/2} = p\int_0^\infty \lambda^{p-1} \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq \lambda^2\right\}\right| d\lambda \\
&= p\int_0^{N_1} \lambda^{p-1} \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq \lambda^2\right\}\right| d\lambda \\
&\qquad + p\sum_{k=1}^\infty \int_{N_1^k}^{N_1^{k+1}} \lambda^{p-1} \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq \lambda^2\right\}\right| d\lambda \\
&\leq N_1^p|B_1| + p\sum_{k=1}^\infty \int_{N_1^k}^{N_1^{k+1}} \lambda^{p-1} \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq \lambda^2\right\}\right| d\lambda \\
&\leq N_1^p|B_1| + \sum_{k=1}^\infty \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 > N_1^{2k}\right\}\right| \cdot N_1^{kp} \cdot N_1^p \\
&\leq N_1^p|B_1| + N_1^p \sum_{k=1}^\infty N_1^{kp} \sum_{j=1}^k \epsilon_1^j \left|\left\{x \in B_1 \mid \mathcal{M}f^2 \geq \delta^2 N_1^{2(k-j)}\right\}\right| \\
&\qquad + N_1^p \sum_{k=1}^\infty N_1^{kp}\epsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq 1\right\}\right| \\
&\leq N_1^p|B_1| + N_1^p \sum_{j=1}^\infty N_1^{jp}\epsilon_1^j \sum_{k\geq j} N_1^{(k-j)p} \left|\left\{x \in B_1 \mid \mathcal{M}f^2 \geq \delta^2 N_1^{2(k-j)}\right\}\right| \\
&\qquad + (N_1^p + 1) \sum_{k=1}^\infty N_1^{kp}\epsilon_1^k \left|\left\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq 1\right\}\right|
\end{aligned}$$

Now, we know that $|B_1| < +\infty$, so the first term is bounded. For the remaining terms, we choose ϵ_1 such that $N_1^p\epsilon_1 < 1$. This way, as $|\{x \in B_1 \mid \mathcal{M}|D^2u|^2 \geq 1\}| < \epsilon|B_1|$, the third term is also bounded. All it remains is to bound the second term. In order to do so, let us see that we can bound

$$\sum_{k\geq j} N_1^{(k-j)p}|\{x \in B_1 \mid \mathcal{M}f^2 \geq \delta^2 N_1^{2(k-j)}\}|.$$

We have that

$$\sum_{i=0}^\infty N_1^{ip} \left|\left\{x \in B_1 \mid \mathcal{M}f^2 \geq \delta^2 N_1^{2i}\right\}\right| = \sum_{i=0}^\infty N_1^{ip} \int_{B_1} \chi_{\{\mathcal{M}f^2 \geq \delta^2 N_1^{2i}\}}$$

$$
\begin{aligned}
&= \int_{B_1} \sum_{i=0}^{\infty} N_1^{ip} \chi_{\{\mathcal{M}f^2 \geq \delta^2 N_1^{2i}\}} \\
&= \int_{B_1} \sum_{i=0}^{N_1^i \leq \frac{(\mathcal{M}f^2)^{1/2}}{\delta}} N_1^{ip} \\
&\leq \int_{B_1} \left(\frac{(\mathcal{M}f^2)^{1/2}}{\delta^2} \right)^p \sum_{i=-\infty}^{0} N_1^{ip} \\
&\leq \int_{B_1} \frac{(\mathcal{M}f^2)^{p/2}}{\delta^p} \cdot \frac{N_1^p}{N_1^p - 1} \leq C\|f\|_{L^p}.
\end{aligned}
$$

In conclusion,

$$\int_{B_1} |D^2u|^p \leq C.$$

Then the case for $u \in W^{2,p}(B_1)$ follows from approximation. □

Remark 1. It is important to know that even though we state the estimates (3.36) in $B_{1/2}$, we can find an estimate for a bigger ball B_r, with $r \in (0,1)$. This can be done with a *covering argument*[1].

The covering argument goes as follows: we start in $B_{1/2}$, and we want to "scale" the argument to a bigger ball B_r. Then we cover B_r with smaller balls $B_\delta(x_i)$, for $x_i \in B_r$ and $\delta = \frac{1}{2}(1-r)$. This covering can be finite, so that $i \in \{1, ..., N\}$ for some $N := (r, n)$. Moreover, $B_{2\delta}(x_i) \subset B_1$.

By scaling and translation, we can apply the estimate (3.36) in each of the $B_{2\delta}(x_i)$, as $\Delta u = f$ in $B_{2\delta}(x_i) \subset B_1$. Therefore

$$
\begin{aligned}
\|D^2u\|_{L^p(B_\delta(x_i))} &\leq C(r) \left(\|f\|_{L^p(B_{2\delta}(x_i))} + \|u\|_{L^p(B_{2\delta}(x_i))} \right) \\
&\leq C(r) \left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)} \right).
\end{aligned}
$$

But as B_r can be covered by a finite number of these balls,

$$\|D^2u\|_{L^p(B_r)} \leq \sum_{i=1}^{n} \|D^2u\|_{L^p(B_\delta(x_i))} \leq NC(r) \left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)} \right). \tag{3.37}$$

So the estimate holds for any ball strictly contained in B_1.

[1] We state the covering argument for our case. In general the same reasoning can be done if the estimate is for a ball B_ρ with $\rho \in (0,1)$, as is done in [FR22] for the Schauder estimates.

3.2.4 Different statements, equivalent results

Throughout this section we have seen two different approaches to prove the Calderón-Zygmund inequality, but the reader may have realised that the statments of Theorems 3.2 and 3.7 are not identical. Recall the estimates given:

Theorem 3.2: $\|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$
Theorem 3.7: $\|D^2u\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right)$

The difference is obvious. It seems like $\|u\|_p$ is missing in the first estimate. Then one may ask themselves: how can they be equivalent? There is a whole term missing! Well, the main difference is that the first result, stated as in [GT01], is valid for u the Newtonian potential of the right-hand-side to the Poisson equation, while the second one, from [Wan03], is valid for any solution u to $\Delta u = f$ in B_1. This facts must be taken into account when proving the equivalence between those results, as not each solution to the Poisson equation is given by the Newtonian potential.

First let us see how Theorem 3.7 implies Theorem 3.2. Let u be the Newtonian potential of $f \in L^p(B_1)$. By Young's convolution theorem (Theorem 2.6) we already have that u is $L^p(B_1)$. Moreover, u solves $\Delta u = f$ in B_1. We define $u_r(x) := u(rx)$. Then

$$\Delta u_r = r^2 f_r, \tag{3.38}$$

where $f_r(x) = f(rx)$. By the second version of the estimate (Theorem 3.7), we have

$$\|D^2u_r\|_{L^p(B_{1/2})} \leq C\left(r^2\|f_r\|_{L^p(B_1)} + \|u_r\|_{L^p(B_1)}\right). \tag{3.39}$$

Note that

$$\|D^2u_r\|_{L^p(B_{1/2})} = r^2\|D^2u(rx)\|_{L^p(B_{1/2})} = r^2 \cdot r^{n/p}\|D^2u\|_{L^p(B_{r/2})}$$

and

$$\|f_r\|_{L^p(B_1)} = r^{n/p}\|f\|_{L^p(B_r)},$$

so the estimates (3.39) become

$$r^2 \cdot r^{n/p}\|D^2u\|_{L^p(B_{r/2})} \leq C\left(r^2 \cdot r^{n/p}\|f\|_{L^p(B_r)} + r^{n/p}\|u\|_{L^p(B_r)}\right). \tag{3.40}$$

Therefore

$$\|D^2u\|_{L^p(B_{r/2})} \leq C\left(\|f\|_{L^p(B_r)} + \frac{1}{r^2}\|u\|_{L^p(B_r)}\right), \tag{3.41}$$

and when we take $r \to \infty$ we have

$$\|D^2u\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.42}$$

which is the estimate given in Theorem 3.2.

Now let us see that starting from the first estimate we can obtain the second one. Let $u \in H^1(B_1)$ be a weak solution to $\Delta u = f$ in B_1 for some $f \in L^p(B_1)$. If we define $v = \Gamma * f$, we know that $v \in W^{2,p}(B_1)$ and that the difference $u - v$ satisfies

$$\Delta(u - v) = 0$$

in B_1. Therefore $u - v$ is harmonic, and thus, $C^\infty(B_1)$. Then, writing $u = (u - v) + v$ we see that u is also in $W^{2,p}(B_{1/2})$. Our next step is estimating its norm. We have that

$$\|u\|_{W^{2,p}(B_{1/2})} \leq \|u - v\|_{W^{2,p}(B_{1/2})} + \|v\|_{W^{2,p}(B_{1/2})} \tag{3.43}$$

For the second term on the right-hand-side, as v is the Newtonian potential of f, from the first estimates we have

$$\|v\|_{W^{2,p}(B_{1/2})} \leq C\|f\|_{L^p(B_1)}.$$

For the first term, as $u - v$ is harmonic, we can use Theorem 2.25 to see that

$$\begin{aligned}
\|u - v\|_{W^{2,p}(B_{1/2})} \leq \|u - v\|_{C^2(B_{1/2})} &\leq C\|u - v\|_{L^1(B_1)} \leq C\|u - v\|_{L^p(B_1)} \\
&\leq C\left(\|u\|_{L^p(B_1)} + \|v\|_{L^p(B_1)}\right) \\
&\leq C\left(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)}\right)
\end{aligned}$$

Plugging all this in (3.43), we end up with

$$\|D^2 u\|_{L^p(B_{1/2})} \leq C\left(\|f\|_{L^p(B_1)} + \|u\|_{L^p(B_1)}\right), \tag{3.44}$$

which is the estimate given in Theorem 3.7. In conclusion, the statements from Theorems 3.2 and 3.7 are equivalent.

CHAPTER 4

A nonlinear problem with linear tools

> *Understanding nonlinear partial differential equations is fundamental to predicting and controlling natural phenomena.*
>
> — **Claude-Louis Navier**
> *Mémoire sur les lois du mouvement des fluides*, 1822

In this chapter we aim to study the existence and regularity of nontrivial solutions to

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{4.1}$$

where u is a positive function and Ω a bounded, star-shaped domain. First we define what is a star-shaped domain.

Definition 4.1. *An open set Ω is called star-shaped with respect to 0 if for each $x \in \overline{\Omega}$, the line segment $\{\lambda x \,|\, 0 \leq \lambda < 1\}$ lies in Ω.*

Equation (4.1) arises as the Euler-Lagrange equation of the following constrained optimisation problem for the Dirichlet energy:

$$\min \left\{ \int_\Omega |\nabla u|^2 \;\middle|\; u \in H_0^1(\Omega) \text{ such that } \|u\|_{p+1} = 1 \right\} \tag{4.2}$$

We will denote the Dirichlet energy as

$$E(u) = \int_\Omega |\nabla u|^2 \tag{4.3}$$

and

$$\Lambda = \left\{ u \in H_0^1 \,|\, \|u\|_{p+1} = 1 \right\}. \tag{4.4}$$

Now let us derive the weak form of (4.2). For the equation, it is known that (4.3) yields the integral equation

$$\int_\Omega \nabla u \nabla \varphi = 0$$

for φ any test function. For the constrain, let $J(u) = \|u\|_{p+1}^{p+1} - 1$. Then for any test function $\varphi \in C_c^\infty(\Omega)$,

$$J(u+\epsilon\varphi) = \int_\Omega (u+\epsilon\varphi)^{p+1}\, dx - 1.$$

From the optimality condition, $\partial_\epsilon J(u+\epsilon\varphi)|_{\epsilon=0} = 0$, we get

$$\int_\Omega u^p \varphi = 0.$$

So the weak form associated to the minimisation problem (4.2) is

$$\int_\Omega \nabla u \nabla \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ such that } \int_\Omega u^p \varphi = 0.$$

Integrating by parts, we have

$$\int_\Omega \Delta u \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ such that } \int_\Omega u^p \varphi = 0,$$

that is, every φ orthogonal to u^p is also orthogonal to Δu, so we can write

$$\Delta u = \lambda u^p \tag{4.5}$$

Now we can consider $v = cu$ so that

$$\Delta v = c\Delta u = c\lambda u^p = c^{1-p}\lambda v^p, \tag{4.6}$$

and we can choose a c such that $|c^{1-p}\lambda| = 1$. Note that it must be $c^{1-p}\lambda = -1$. Indeed, if $c^{1-p}\lambda = 1$, we would have

$$\Delta v = v^p > 0, \tag{4.7}$$

as we are considering positive functions. Therefore v would be subharmonic with $v = 0$ on $\partial\Omega$, and by the weak maximum principle for subharmonic functions, we would have that $v = 0$ in Ω. Thus, $c^{1-p}\lambda = -1$, and the resulting equation is

$$-\Delta v = v^p. \tag{4.8}$$

Therefore, the minimisers of (4.2), in case they exist, will be solutions to (4.1) up to a multiplicative constant.

Let us see if the functional can be minimised under the desired integral constraint. That is, let us prove that that the minimisation problem

$$\min_{u\in\Lambda} E(u), \tag{4.9}$$

for E and Λ defined in (4.3) and (4.4) respectively, has a solution. It is known that E is weakly lower semi-continuous. We then take $\{u_j\} \subset \Lambda$ a minimising sequence such that

$$E(u_j) \to m = \inf\{E(u) \,|\, u \in \Lambda\}.$$

Then

$$\|u_j\|_{H_0^1} \leq C\|\nabla u_j\|_{L^2} = E(u_j) < +\infty,$$

since $u_j \in H_0^1(\Omega)$. Hence, there exists a subsequence $\{u_{j_k}\} \subset \{u_j\}$ such that $u_{j_k} \to \hat{u}$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, by reflexivity and Rellich-Kondrachov's embedding, respectively. Then

$$E(\hat{u}) \leq \liminf_{k\to\infty} E(u_{j_k}) = m. \tag{4.10}$$

To see that $\hat{u}$ is actually a minimiser, it remains to check that $\hat{u} \in \Lambda$. We do so checking that actually $u_{j_k} \to \hat{u}$ strongly in L^{p+1} via embeddings. It is straightforward to see that, by the Rellich-Kondrachov theorem (Theorem 2.22), the embedding

$$H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$$

is compact as long as

$$p+1 < \frac{2n}{n-2} \iff p < \frac{n+2}{n-2}. \tag{4.11}$$

Therefore, we have that $u_{j_k} \to \hat{u}$ strongly in $L^{p+1}(\Omega)$ when (4.11) holds. Thus

$$\|\hat{u}\|_{p+1} = \lim_{k\to\infty} \|u_{j_k}\|_{p+1} = 1,$$

as we wanted to see. So as $\hat{u} \in \Lambda$ and (4.10) holds, $\hat{u}$ is a minimiser to (4.9) and thus the claim holds for $p < \frac{n+2}{n-2}$.

As we see, we can guarantee the existence of a solution for (4.2), and hence for (4.1), as long as we have $1 < p < \frac{n+2}{n-2}$. Once we know that said solution exists, we can move on to study its regularity. Will u be just H_0^1? Will it be differentiable? And smooth? Moreover, another question remains. What happens when $p \geq \frac{n+2}{n-2}$? In this section we will focus on studying what happens when we are below and above $p = \frac{n+2}{n-2}$, known as the Sobolev critical exponent, and thus answer the stated questions.

Remark 2. It is clear that $1 < p < \frac{n+2}{n-2}$ only makes sense for $n > 2$. For the sake of simplicity, we will stick to this case. For the cases $n = 1$ and $n = 2$, the arguments used are exactly the same, but we obtain the desired solutions faster, as for $n = 1$ we must directly use Morrey's embedding; and for $n = 2$, we are at the critical case for the Sobolev embedding.

4.1 Case $1 < p < \frac{n+2}{n-2}$: regular solutions

Once we know that the problem (4.1) has a solution, we are concerned with its regularity. We have minimised the functional over Λ, so a priori we just know that $u \in H_0^1(\Omega)$. Now we ask ourselves whether u is $C^\infty(\Omega)$ or not. Before going on, we recall Sobolev and Morrey's embeddings:

$$\begin{array}{rlccc} \text{Sobolev:} & W^{l,p} \subset W^{k,p^*} & p^* = \frac{np}{n-(l-k)p} & l > k & p < n \\ \text{Morrey:} & W^{1,p} \subset C^{0,1-n/p} & & & p > n \end{array}$$

To prove that u is $C^\infty(\Omega)$, we will use repeatedly the Calderón-Zygmund inequality (Theorem 3.7). However, note that this result gives us interior estimates, as the domain is reduced. Therefore, we will consider the sequence of embedded domains

$$\{B^k\} = \left\{B_{\frac{1}{2}+\frac{1}{2^k}}\right\}$$

as well as a covering argument. This way, at the step $j+1$ the estimates will be given in the interior of B^j.

We start with $u \in H_0^1(B_1)$ (note that $B_1 = B^0$). By the Sobolev embedding, $u \in L^{q_0}(B_1)$, where

$$q_0 = \frac{2n}{n-2}.$$

Hence $u^p \in L^{q_0/p}(B_1) = L^{q_0'}(B_1)$, where $q_0' = q_0/p$. As $-\Delta u = u^p$, we may use the Calderón-Zygmund inequality (Theorem 3.7) to see that $u \in W^{2,q_0'}(B^1)$. Then, using the Sobolev embedding again we can say that $u \in L^{q_1}(B^1)$, for

$$q_1 = \frac{nq_0'}{n-2q_0'} = \frac{nq_0}{np-2q_0},$$

and by simple computation we see that $q_1 > q_0$ as long as $p < (n+2)/(n-2)$. Again, if $u \in L^{q_1}(B^1)$, then $u^p \in L^{q_1/p}(B^1) = L^{q_1'}(B^1)$ and by Calderón-Zygmund, $u \in W^{2,q_1'}(B^2)$. Repeating the same process, we see that $u \in L^{q_2}(B^2)$, where

$$q_2 = \frac{nq_1'}{n-2q_1'} = \frac{nq_1}{np-2q_1}.$$

We claim that $q_2 > q_1$. To see it, we consider the function

$$f(x) = \frac{nx}{np-2x}.$$

Let $q_{j+1} = f(q_j)$. It is clear that $u \in L^{q_j}(B^j)$. Moreover, f is increasing, and it is positive as long as $0 < x < np/2$. Then for such values of x, we will get a sequence of Sobolev order exponents by using recursively the Sobolev embedding. Having said so, in our case we have that

$$q_2 = f(q_1) > f(q_0) = q_0,$$

provided that q_0 and q_1 are in the desired range of values (we will later see what happens when we go above $np/2$). This way, if we keep repeating this reasoning, we end up with an increasing sequence

$$\{q_j\} = \left\{q_0, q_1 = f(q_0), q_2 = f^2(q_0), \ldots\right\}.$$

As the q_j are increasing, we can either have that $q_j \to \infty$ or that $q_{j+1} - q_j \to 0$. However, the second option would mean that the sequence converges to a fixed point, which is not possible. If it were, it we would have that

$$f(x) = x,$$

and this only holds for

$$x = 0 \quad \text{or} \quad x = \frac{n(p-1)}{2}.$$

Neither of them is possible, as both are smaller than q_0. Therefore, the sequence must be increasing.

Let q_N be the first q_j to satisfy $q_N \geq np/2$. Without loss of generality, we can assume $q_N > np/2$. If $q_N = np/2$, que could take $\tilde{q}_{N-1} = q_{N-1} - \epsilon$ to have $\tilde{q}_N = f(\tilde{q}_{N-1}) < np/2$ and apply the Sobolev embedding once more to get $\tilde{q}_{N+1} > np/2$. Therefore, if $q_N > np/2$, since $u \in W^{2,q_N}(B^{N+1})$, we can apply Morrey's embedding to have $u \in C^{0,\alpha}(B^{N+1})$ for $\alpha = 1 - q_N/n$.

Recall that we started with $u \in H_0^1(B_1)$. We have seen that u is actually Hölder continuous, and hence $u \in C^0(B^{N+1})$. But as u is continuous inside a bounded domain, $u \in L^\infty(B^{N+1})$, so $u \in W^{2,p}(B^{N+1})$ for all p.

Once we have controlled u, we move on to study ∇u. If we compute the gradient on both sides of (4.1), we end up with

$$-\Delta(\nabla u) = pu^{p-1}\nabla u. \tag{4.12}$$

Let us study the regularity of ∇u. Since $u \in W^{2,p}(B^{N+1})$ for all p, in particular we have that $\nabla u \in W^{1,2}(B^{N+1})$. We apply the same reasoning as before. By Sobolev's embedding, $\nabla u \in L^{q_0}(B^{N+1})$, with

$$q_0 = 2n/(n-2).$$

By Calderón-Zygmund's inequality, now we have

$$\|D^2\nabla u\|_{l_0} \leq C\|pu^{p-1}\nabla u\|_{q_0} \leq C\|\nabla u\|_{q_0},$$

as $u \in L^\infty(B^{N+1})$. Hence $\nabla u \in W^{2,q_0}(B^{N+2})$. We then apply again the Sobolev embedding to get $\nabla u \in L^{q_1}(B^{N+2})$, with

$$q_1 = \frac{nq_0}{n-2},$$

which by simple computation we see that it is always greater than q_0. Repeating the same arguments we used when studying u, we end up with a monotonously increasing sequence $\{q_j\}$. Then we take the first q_M such that $q_M > n/2$ and apply Morrey's embedding. This way, we will have that $\nabla u \in C^{0,\beta}(B^{M+1})$ for $\beta = 1 - q_M/n$, and in particular, $\nabla u \in C^0(B^{M+1})$.

To summarise, we have started with u a solution to (4.1), which we only knew to be in $H_0^1(B_1)$. Then, we have seen that $u \in C^0(B^{N+1})$ and moved on to see that $\nabla u \in C^0(B^{M+1})$. Therefore, u is actually $C^1(B^{M+1})$. The idea is to apply an analogous argument to $D^k u$ for all $k \geq 2$ to see that u is C^k in an appropriate domain. This way, we end up with $u \in C^\infty(B_{1/2})$.

4.2 Case $p > \frac{n+2}{n-2}$: non-existence of solutions

To prove that there are no solutions other than $u \equiv 0$ when we are above the critical exponent we rely on a *Pohozaev-type* identity. In general, the Pohozaev identity states that any bounded solution to the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.13}$$

must satisfy the following:

$$\int_\Omega 2nF(u) - (n-2)uf(u)\,dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)\,d\sigma(x), \tag{4.14}$$

where

$$F(u) = \int_0^u f(t)\,dt.$$

When we restrict to our case, that is, when $f(u) = u^p$, the identity (4.14) becomes

Proposition 4.2. *Let u be a bounded solution to (4.13) with $f(u) = u^p$. Then it must satisfy*

$$\left(\frac{n-2}{2} - \frac{n}{p+1}\right)\int_\Omega u^{p+1}\,dx = -\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 d\sigma(x). \tag{4.15}$$

For more information on the Pohozaev identity, we refer to [Ros17].

To prove Proposition 4.2 we need two results: first, a characterisation of star-shaped domains, which can be found for instance in the Chapter 9 of [Eva10]:

Lemma 4.3 (Normals to a star-shaped region)**.** *Assume $\partial\Omega$ is C^1 and Ω is star-shaped with respect to 0. Then*

$$x \cdot \nu(x) \geq 0$$

for all $x \in \partial\Omega$, where ν denotes the outward unit normal vector.

Then, we also need the following integration-by-parts type formula.

Lemma 4.4. *Let Ω be a star shaped domain where Ω is C^1. Let u be a $C^2(\Omega)$ function with $u = 0$ on $\partial\Omega$. Then*

$$2\int_\Omega (x \cdot \nabla u)\Delta u \, dx = (2-n)\int_\Omega u\Delta u \, dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (x \cdot \nu) d\sigma(x) \tag{4.16}$$

This lemma is proved using the following vector identity

$$\Delta(x \cdot \nabla u) = x \cdot \nabla \Delta u + 2\Delta u, \tag{4.17}$$

and the fact that for $x \in \partial\Omega$ it holds

$$x \cdot \nabla u = (x \cdot \nu)\frac{\partial u}{\partial \nu}. \tag{4.18}$$

With these two facts, we can now prove the formula (4.16).

Proof of Lemma 4.4. To prove it, we first integrate twice by parts and we use (4.17) and (4.18) when possible

$$\begin{aligned}
\int_\Omega (x \cdot \nabla u)\Delta u \, dx &= \int_{\partial\Omega} (x \cdot \nabla u)\frac{\partial u}{\partial \nu} \, d\sigma(x) - \int_\Omega \nabla(x \cdot \nabla u)\nabla u \, dx \\
&= \int_{\partial\Omega} (x \cdot \nabla u)\frac{\partial u}{\partial \nu} \, d\sigma(x) - \int_{\partial\Omega} u\frac{\partial(x \cdot \nabla u)}{\partial \nu} \, d\sigma(x) + \int_\Omega \Delta(x \cdot \nabla u)u \, dx \\
&= \int_{\partial\Omega} (x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma(x) + \int_\Omega (x \cdot \nabla\Delta u + 2\Delta u)u \, dx,
\end{aligned}$$

since $u = 0$ on $\partial\Omega$. Then, integrating by parts the second term of the right-hand-side, we have

$$\begin{aligned}
\int_\Omega (x \cdot \nabla\Delta u + 2\Delta u)u \, dx &= \int_\Omega (xu) \cdot \nabla\Delta u + 2u\Delta u \, dx \\
&= \int_{\partial\Omega} \cdot (x \cdot \nu)u\Delta u \, d\sigma(x) + \int_\Omega -\operatorname{div}(xu)\Delta u + 2u\Delta u \, dx \\
&= \int_\Omega -nu\Delta u - (x \cdot \nabla u)\Delta u + 2u\Delta u \, dx
\end{aligned}$$

Putting it all together, we end up with

$$\int_\Omega (x \cdot \nabla u)\Delta u \, dx = (2-n)\int_\Omega u\Delta u \, dx + \int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2 (x \cdot \nu)d\sigma(x) - \int_\Omega (x \cdot \nabla u)\Delta u \, dx,$$

which gives (4.16). □

Now we can prove Proposition 4.2.

Proof of Proposition 4.2. If we multiply both sides of $-\Delta u = u^p$ by u and integrate over Ω, we have

$$-\int_\Omega u\Delta u\, dx = \int_\Omega u^{p+1}\, dx. \tag{4.19}$$

If we multiply instead both sides by $(x \cdot \nabla u)$ and integrate over Ω, we have:

$$-\int_\Omega \Delta u(x \cdot \nabla u)\, dx = \int_\Omega u^p(x \cdot \nabla u)\, dx. \tag{4.20}$$

Integrating by parts, the right-hand-side can be rewritten as

$$\begin{aligned}\int_\Omega u^p(x \cdot \nabla u)\, dx &= \sum_i \int_\Omega u^p \cdot D_i u \cdot x_i\, dx = \sum_i \int_\Omega D_i \left(\frac{u^{p+1}}{p+1}\right) \cdot x_i\, dx \\ &= \sum_i \int_{\partial\Omega} x_i \left(\frac{u^{p+1}}{p+1}\right) \cdot e_i \cdot \nu\, d\sigma(x) - \int_\Omega \frac{u^{p+1}}{p+1}\, dx = -\frac{n}{p+1}\int_\Omega \frac{u^{p+1}}{p+1}\, dx\end{aligned}$$

Using this and the formula from Lemma 4.4 for the left-hand-side, the equation

$$-\int_\Omega \Delta u(x \cdot \nabla u)\, dx = \int_\Omega u^p(x \cdot \nabla u)\, dx$$

becomes

$$\frac{1}{2}\int_{\partial\Omega}(x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma(x) + \frac{(2-n)}{2}\int_\Omega u\Delta u\, dx = \frac{n}{p+1}\int_\Omega u^{p+1}\, dx.$$

Finally, using (4.19), we have

$$\frac{1}{2}\int_{\partial\Omega}(x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma(x) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_\Omega u^{p+1}\, dx,$$

or equivalently,

$$\left(\frac{n-2}{2} - \frac{n}{p+1}\right)\int_\Omega u^{p+1}\, dx = -\frac{1}{2}\int_{\partial\Omega}(x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma(x).$$

which is the Pohozaev-type identity (4.15). □

Now, with (4.15) we can prove the non-existence of nontrivial solutions to the problem (4.1). We know on one hand that $\|u\|_{p+1} = 1$ and $p > (n+2)/(n-2)$, so the left-hand-side is positive. On the other hand, because of Lemma 4.3, we have that the right-hand-side is non-positive. Therefore, the only possible solution to the equation in this case is $u \equiv 0$.

4.3 Some remarks on the case $p = \frac{n+2}{n-2}$

Now we briefly comment the case for the critical exponent $p = \frac{n+2}{n-2}$, which lies out of the scope of this project. In this case we have to be more careful. Using the same reasoning as for the case when we are above $p = \frac{n+2}{n-2}$ we have that

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

From here, one can deduce that $u \equiv 0$ using an analogue to Hopf's Lemma.

In this chapter we have restricted ourselves to star-shaped domains for simplicity's sake, but if we considered other types of domains, we could further study this case with other techniques. For instance, if we look for solutions in a ring, writing the equation in polar coordinates we have the following ODE

$$\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + u^p = 0, \tag{4.21}$$

which can be rewritten as a system of ODEs

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\dfrac{n-1}{r} v - u^p \end{cases}, \tag{4.22}$$

and then we can find nontrivial solutions by studying its qualitative behaviour.

CHAPTER 5

To nonlinearity and beyond

> *Keep in mind that there is in truth no central core theory of nonlinear PDE, nor can there be. The sources of PDE are so many - physical, probabilistic, geometric, etc. - that the subject is a confederation of diverse subareas, each studying different phenomena for different nonlinear PDE by utterly different methods.*
>
> — **Lawrence C. Evans**
> Lindenstrauss *et al.* (1994)

Throughout this project we have been concerned with the question of *regularity* of solutions to PDEs. This question was originated by Hilbert's nineteenth problem, which asked whether the solutions to the variational problem

$$J(u) = \int_\Omega L(\nabla u)\, dx \tag{5.1}$$

are real analytic. In our case, we have studied the L^p regularity theory for the Poisson equation, $\Delta u = f$, whose solutions are the minimisers of

$$I(u) = \int_\Omega |\nabla u|^2 - uf.$$

Mathematically speaking, we have studied just an $\epsilon > 0$ of the possible problems encompassed in (5.1). Nevertheless, having studied the Poisson equation in Chapters 2 and 3, and the example from Chapter 4, we have seen how the linear theory can be applied to a nonlinear equation. Unfortunately, this cannot be done for all nonlinear second order elliptic equations, thus we need to find other ways to study their regularity.

For example, can express *fully nonlinear elliptic equations* as

$$F(D^2u, \nabla u, u, x) = 0 \quad \text{in } \Omega,$$

or the simpler version

$$F(D^2u, x) = 0 \quad \text{in } \Omega.$$

In this setting, it is common to consider *viscosity solutions* (see Section 1 in [Caf89]).

As for linear elliptic equations, one can either look for $C^{1,\alpha}$ estimates or, as we have done in this project, look for L^p estimates. We have done so in Chapter 3 with the Calderón-Zygmund inequality, studied both in the case of the Newtonian potential (Theorem 3.2) and for a general solution to the Poisson equation (Theorem 3.7). We can further extend the Calderón-Zygmund theory to the nonlinear setting as done by Caffarelli in [Caf89].

Theorem 5.1 (Nonlinear $W^{2,p}$ estimates)**.** *Let u be a bounded viscosity solution of $F(D^2u, x) = f(x)$ in B_1. Assume further that the solutions w to the Dirichlet problem*

$$\begin{cases} F(D^2w, 0) = 0 & \text{in } B_r \\ \qquad\quad w = w_0 & \text{in } \partial B_r \end{cases}$$

satisfy the interior a priori estimate

$$\|w\|_{C^{1,1}(B_{r/2})} \leq Cr^{-2}\|w\|_{L^\infty(\partial B_r)}.$$

Let $n < p < \infty$ and assume that $f \in L^p$, and for some $\theta := \theta(p)$ sufficiently small

$$\sup_{B_1} \beta(x) \leq \theta(p),$$

where β is a function measuring the oscillations of F in the variable x. Then u is in $W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\sup_{\partial B_1} |u| + \|f\|_{L^p}\right).$$

There are many problems modelled by nonlinear PDEs: the motion of fluids, general relativity, the spread and development of cancerous tumours, the stock market... and these problems must be studied. Paraphrasing what Lawrence C. Evans said in the quote at the beginning of this chapter, there is not a solid theory of nonlinear PDEs, as they come from different backgrounds, represent different phenomena and have to be studied differently. Fortunately, we do have a solid background for linear equations, and we already see in Theorem 5.1 that the knowledge we have from Theorem 3.7 gives us a strong intuition of how things should look like in the nonlinear case. The same can be done for the $C^{1,\alpha}$ theory and many other results. Thus, having a solid background in linear equations is not only necessary, but key to fully understand nonlinear equations and, therefore, understand how do fluids move, how does the stock market change, how does cancer spread, and even to understand how does the universe behave.

There has been much work done since the second half of the twentieth century, but there is still much work to be done in the future, and *"the future has several names. For the weak, it is impossible; for the fainthearted, it is unknown; but for the valiant, it is ideal."*[1] We mathematicians must be valiant in order for the world to be understood.

[1]This quote is often attributed to Victor Hugo's *Les Misérables*. However, having read the book I do not recall the quote being there. There is a discussion on the internet where people say that it can either be from an older version or that it has been misattributed to him. There is even a person saying that it is from a speech Hugo gave in Jersey in 1854.

Bibliography

[GT01] Gilbarg, David, & Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. 2nd. Berlin, Springer, 2001.

[Wan03] Wang, Lihe, “A Geometric Approach to the Calderón-Zygmund Estimates”. *Acta Mathematica Sinica, English Series* 19 Jan. 2003, 381–396.

[HP70] Hawking, S. W., & R. Penrose, “The Singularities of Gravitational Collapse and Cosmology”. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 314.1519 1970, 529–548.

[Ber06] Serge Bernstein. “Sur la généralisation du problème de Dirichlet”. In: *Mathematische Annalen* 62 1906, pp. 253–271.

[Sch34] J. Schauder. “Über lineare elliptische Differentialgleichungen zweiter Ordnung.” In: *Mathematische Zeitschrift* 38 1934, pp. 257–282.

[CZ52] Calderon, A. P., & A. Zygmund, “On the existence of certain singular integrals”. *Acta Mathematica* 88 1952, 85–139.

[Ada75] Robert A. Adams. *Sobolev Spaces*. New York: Academic Press, 1975.

[Bre11] Brezis, Haim, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, Springer, 2011.

[Eva10] Lawrence C. Evans. *Partial Differential Equations*. 2nd. Vol. 19. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2010.

[FR22] Fernández-Real, Xavier, & Xavier Ros-Oton, *Regularity Theory for Elliptic PDE*. EMS Press, Dec. 2022.

[SS05] Elias M. Stein & Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton lectures in analysis. Princeton, NJ: Princeton University Press, 2005.

[HL30] Hardy, G. H., & J. E. Littlewood, “A maximal theorem with function-theoretic applications”. *Acta Mathematica* 54 1930, 81–116.

[Ros17] Ros-Oton, Xavier, *Boundary Regularity, Pohozaev Identities and Nonexistence Results*. Dec. 2017.

[Caf89] Caffarelli, Luis A., “Interior a priori estimates for solutions of fully nonlinear equations”. *Annals of Mathematics* 130.1 1989, 189–213.

[Gra08] Grafakos, Loukas, *Classical Fourier Analysis*. 2nd. 249. Graduate Texts in Mathematics. New York, Springer, 2008.

[Gra21] Gray, Jeremy, *Change and Variations: A History of Differential Equations to 1900*. Cham, Switzerland, Springer, 2021.

[BB02] Brezis, Haim, & Felix E. Browder, *Partial Differential Equations in the 20th Century*. New York, Springer, 2002.

[Rei70] Constance Reid. *Hilbert*. New York: Springer, 1970.