Maximum Likelihood Estimation

Week 2 - PSYC 5316

September 4, 2017

Recall

Last time, we gave a formal definition for a probability function. An example was the *binomial* distribution for N independent Bernoulli trials (e.g., coin flips):

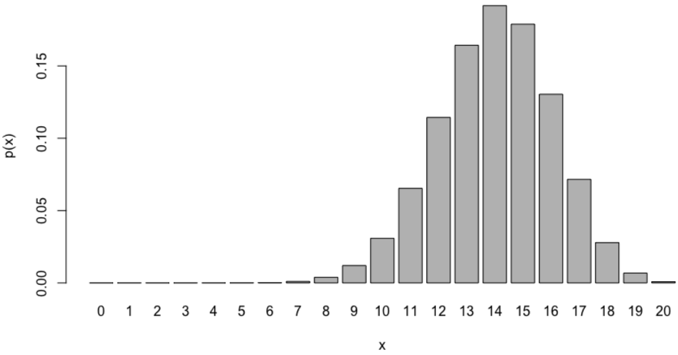
$$f(x \mid \theta) = \binom{N}{x}\theta^x(1-\theta)^{N-x}$$

where $x = \#$ of successes, and $\theta =$ probability of success.

Probability function

Suppose $N = 20$ and $\theta = 0.7$.

```
barplot(dbinom(0:20,size=20,prob=0.7),
        names.arg=0:20,
        ylab="p(x)",
        xlab="x")
```



Data and parameters

$$f(x \mid \theta) = \binom{N}{x} \theta^x (1 - \theta)^{N-x}$$

This function gives us the probability of data, *given* a specific parameter

Data and parameters

What if we switched these?

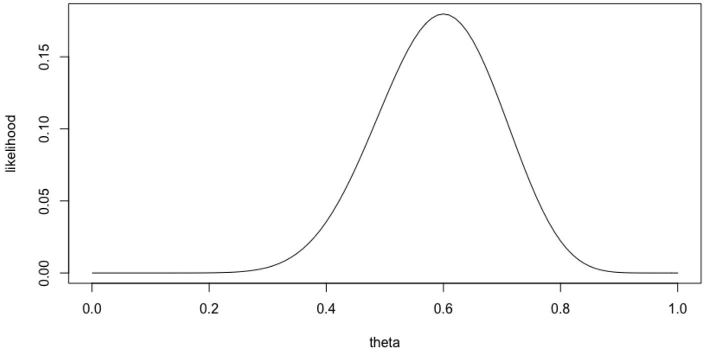
$$f(\theta \mid x) = \binom{N}{x} \theta^x (1-\theta)^{N-x}$$

This function then gives us the likelihood of a range of parameters, *given* a specific data point

Likelihood function

Suppose we observed 12 successes in 20 trials:

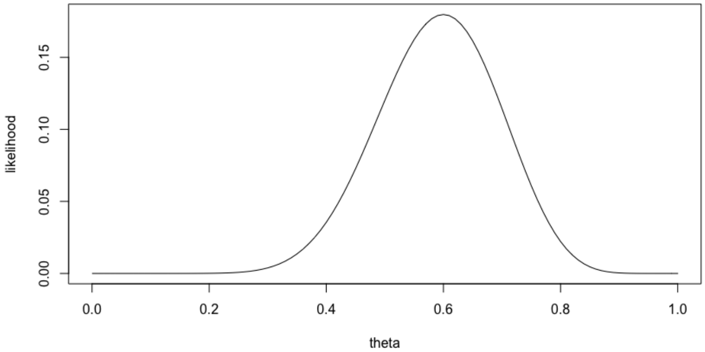
```
theta=seq(from=0, to=1, by=0.01)
plot(theta, dbinom(x=12, size=20, prob=theta),
     type="l",ylab="likelihood")
```



Likelihood function

Suppose we observed 12 successes in 20 trials:

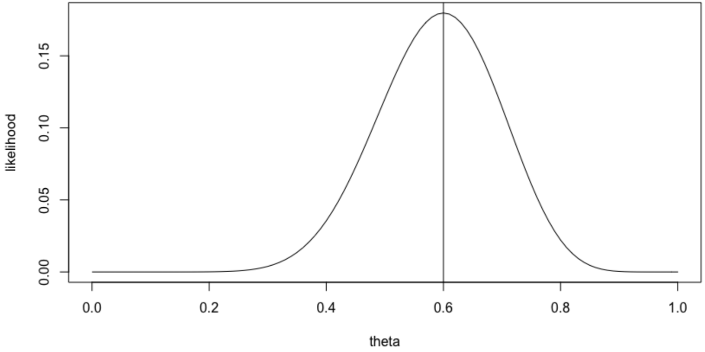
Natural question – what value of θ is most likely, given the data?



Likelihood function

Suppose we observed 12 successes in 20 trials:

Natural question – what value of θ is most likely, given the data?
Answer: $\theta = 0.6$



Maximum likelihood estimation

A key problem in statistical inference is how to infer from sample data to population parameters.

Maximum likelihood estimation is one solution to this problem

Maximum likelihood estimation

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Tutorial

Tutorial on maximum likelihood estimation

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Abstract

In this paper, I provide a tutorial exposition on maximum likelihood estimation (MLE). The intended audience of this tutorial are researchers who practice mathematical modeling of cognition but are unfamiliar with the estimation method. Unlike least-squares estimation which is primarily a descriptive tool, MLE is a preferred method of parameter estimation in statistics and is an indispensable tool for many statistical modeling techniques, in particular in non-linear modeling with non-normal data. The purpose of this paper is to provide a good conceptual explanation of the method with illustrative examples so the reader can have a grasp of some of the basic principles.

Maximum likelihood estimation

Basic workflow:

1. collect data
2. decide on a "model" for the data (e.g., binomial, normal, etc.)
3. define a likelihood function based on the underlying model
4. find the parameter value(s) that maximize the likelihood function

Three examples

We will do three examples of MLE:

1. binomial model
2. normal model
3. ex-Gaussian model

Example 1

Suppose that in a sequence of 20 coin flips, we observe 12 successes ("heads"). What is the maximum likelihood estimate for θ?

Step 1 – collect data

- done: we observed $x = 12$ successes

Example 1

Suppose that in a sequence of 20 coin flips, we observe 12 successes ("heads"). What is the maximum likelihood estimate for θ?

Step 2 – choose a model

- assume a binomial model

$$x \sim \text{Binomial}(\theta, n = 20)$$

Example 1

Suppose that in a sequence of 20 coin flips, we observe 12 successes ("heads"). What is the maximum likelihood estimate for θ?

Step 3 – define likelihood function

- Note: we will actually define the "negative log-likelihood" function

```
nll.binom <- function(data,par){
  return(-log(dbinom(data, size = 20, prob = par)))
}
```

Example 1

Suppose that in a sequence of 20 coin flips, we observe 12 successes ("heads"). What is the maximum likelihood estimate for θ?

Step 4 – find parameter that maximizes the likelihood function

- Note: we will actually minimize the negative log-likelihood

```
optim(par=0.5, fn=nll.binom, data=12)
```

Example 1

Suppose that in a sequence of 20 coin flips, we observe 12 successes ("heads"). What is the maximum likelihood estimate for θ?

```
> optim(par=0.5, fn=nll.binom, data=12) # look at "par" output..this is the MLE
$par
[1] 0.6

$value
[1] 1.716434

$counts
function gradient
      26       NA

$convergence
[1] 0

$message
NULL
```

MLE

Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 1: collect data

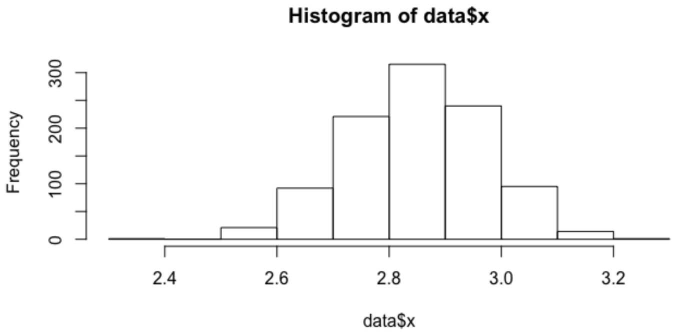
```
data <- read.csv("https://git.io/v58i8")
```

Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 1.5: before choosing a model, look at the distribution

```
hist(data$x)
```

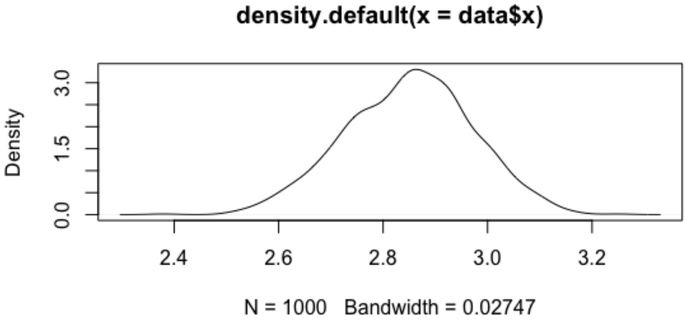


Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 1.5: before choosing a model, look at the distribution

```
plot(density(data$x))
```



Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 2: choose a model

- assume a normal model

$$x \sim \text{Normal}(\mu, \sigma)$$

Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 3: define likelihood function

```
nll.normal <- function(data,par){
  return(-sum(log(dnorm(data, mean=par[1], sd=par[2]))))
}
```

Example 2

The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

Step 4: find parameters that maximize the likelihood function

```
optim(par=c(1,0.1), fn=nll.normal, data=data$x)
```

Example 2

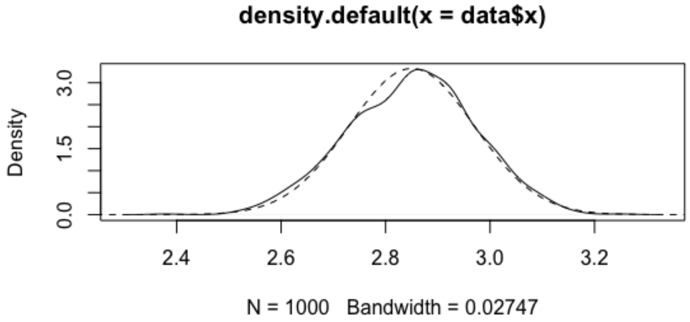
The following command will load a data set into R, consisting of 1000 observations. Fit the data with a normal model and find MLEs for mean and standard deviation.

```
> optim(par=c(1,0.1), fn=nll.normal, data=data$x) # look at par output; first number is mean, second
is sd
$par
[1] 2.8494538 0.1214792

$value
[1] -689.2001

$counts
function gradient
      85       NA

$convergence
[1] 0

$message
NULL
```

Example 2

As a check, we can see how well our model "fits" the data:

```
x=seq(-3,4,0.01)
plot(density(data$x))
lines(x,dnorm(x,mean=2.85, sd=0.12),lty=2)
```

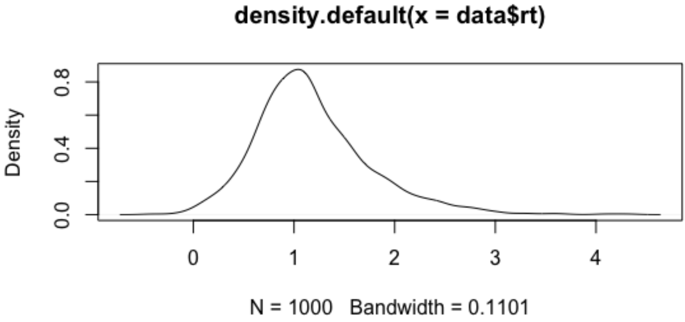


Example 3

Step 1: collect data

Here is a data set of 1000 response times (RTs) in a mental arithmetic task, followed by a plot of the data.

```
data <- read.csv("https://git.io/v58yI")
plot(density(data$rt))
```



Example 3

Step 2: choose a model

- distributions with long rightward tails can be modeled by an ex-Gaussian distribution, which is essentially a combination of the Gaussian (normal) distribution with an exponential distribution

Behavior Research Methods, Instruments, & Computers
1988, 20 (1), 54-57

Fitting the ex-Gaussian equation to reaction time distributions

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Two programs that can be used to determine the probability distributions of reaction times are detailed. The first program takes rank-ordered reaction times as input and outputs a file of quantized data. The second program uses a simplex procedure to estimate the parameters of the ex-Gaussian equation that provides the best description of the quantized data. The advantages of this type of data analysis are also discussed.

Example 3

Step 2: choose a model

- distributions with long rightward tails can be modeled by an ex-Gaussian distribution, which is essentially a combination of the Gaussian (normal) distribution with an exponential distribution

$$rt \sim \text{exGaussian}(\mu, \sigma, \tau)$$

- μ = mean of normal component
- σ = sd of normal component
- τ = rate of exponential component

Example 3

Step 3: define likelihood function

- ExGaussian is not already a function in R. So, we'll build it ourselves.
- note: just copy from the week2.R script!

```
dexg <- function(x, mu, sigma, tau){
  return((1/tau)*exp((sigma^2/(2*tau^2)) -
  (x-mu)/tau)*pnorm((x-mu)/sigma-(sigma/tau)))
}
```

Example 3

Step 3: define likelihood function

```
nll.exg <- function(data,par){
  return(-sum(log(dexg(data,
  mu=par[1],
  sigma=par[2],
  tau=par[3]))))
}
```

Example 3

Step 4: find parameters that maximize likelihood function

```
optim(par=c(0,0.1,0.1), fn=nll.exg, data = data$rt)
```

```
> optim(par=c(0,0.1,0.1), fn=nll.exg, data = data$rt)
$par
[1] 0.7151246 0.3363003 0.4646291

$value
[1] 794.2717

$counts
function gradient
     150       NA

$convergence
[1] 0

$message
NULL
```

Example 3

As before, let's check our model fit

```
x=seq(0,4,0.1)
plot(density(data$rt))
lines(x,dexg(x,mu=0.715, sigma=0.336, tau=0.465),lty=2)
```