Lecture 5 - Bayesian t-tests

↳ recall that the t-test arises from situations where we want to compare a sample mean $\bar{x}$ to some hypothesized standard μ.

↳ one sample t-test

↳ paired samples t-test

↳ independent samples t-test

Let's consider an example:

Suppose we are interested in assessing effectiveness of some mathematics instruction program.

After implementing the program, we measure mathematical ability using the Scale for Advancing Mathematical Ability (SAMA) a national assessment with a known mean score of 50.

Our sample (who had the training) had a mean of 54.4 and a standard deviation of 10.

Did the training work?

Bayesian approach – build and compare two models on the population-level effect size δ

↳ where $\delta = \dfrac{\bar{x} - \mu}{\sigma}$ (population version of Cohen's d)

$\mathcal{H}_0 : \delta = 0$ (effect size is 0)

$\mathcal{H}_1 : \delta \neq 0$ (effect size is nonzero)

Goals:

(1) model comparison via Bayes factor $BF_{10} = \dfrac{p(\text{data} \mid \mathcal{H}_1)}{p(\text{data} \mid \mathcal{H}_0)}$

(2) if $\mathcal{H}_1$ is preferred, estimate effect size via the posterior distribution for δ under $\mathcal{H}_1$.

Remember – Bayesian inference requires us to specify a prior on the parameter of interest.

Recall: two types of "priors":

1) priors on <u>models</u>

2) priors on <u>parameters</u> within a given model

(1) <u>Priors on models</u> — before observing data, what is relative likelihood of competing models?

- common default: $p(\mathcal{H}_0) = p(\mathcal{H}_1) = 1/2$
  ↳ i.e., "1-1 prior odds"

- prior model probabilities are <u>updated</u> after observing data:

Posterior odds = prior odds × BF

$$\longrightarrow \boxed{p(\mathcal{H}_0 \mid \text{data}) = \frac{\text{posterior odds}}{1 + \text{posterior odds}}}$$

2. priors on parameters within a given model.

- model definitions: $\mathcal{H}_0: \delta = 0$

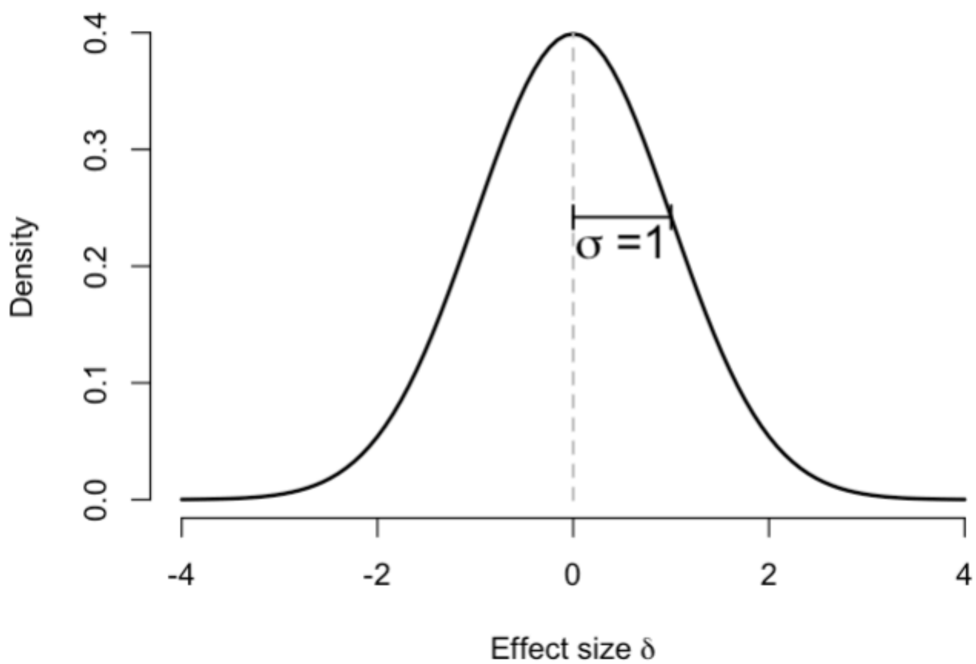
  $\mathcal{H}_1: \delta \neq 0$ ← what exactly do we mean here?

- we quantify our uncertainty about the effect size δ under $\mathcal{H}_1$ by placing a distribution on δ

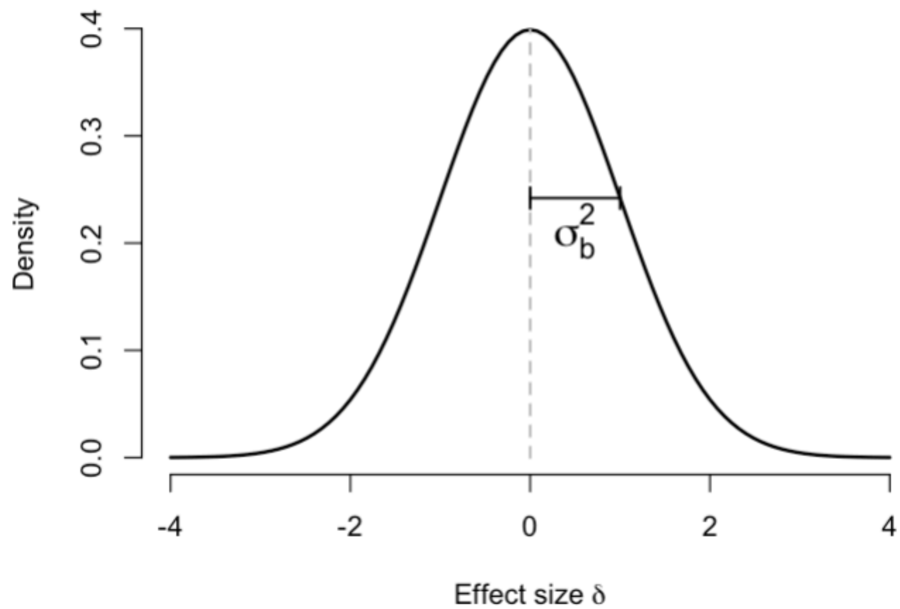
Note – unlike correlations, δ is unbounded. Thus, a uniform prior will not work.

Classic approach: $\delta \sim \text{Normal}(0,1)$

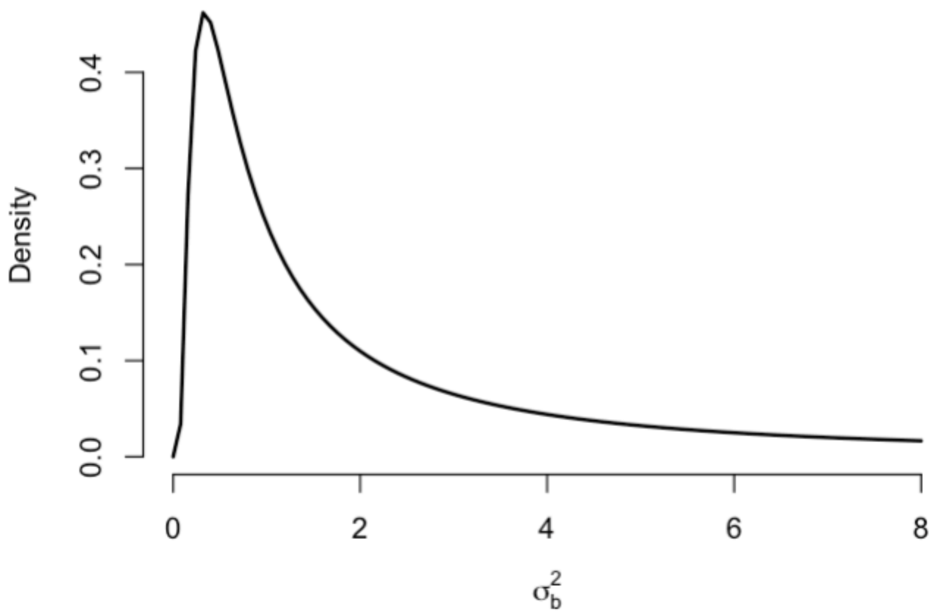
↳ "unit information prior"



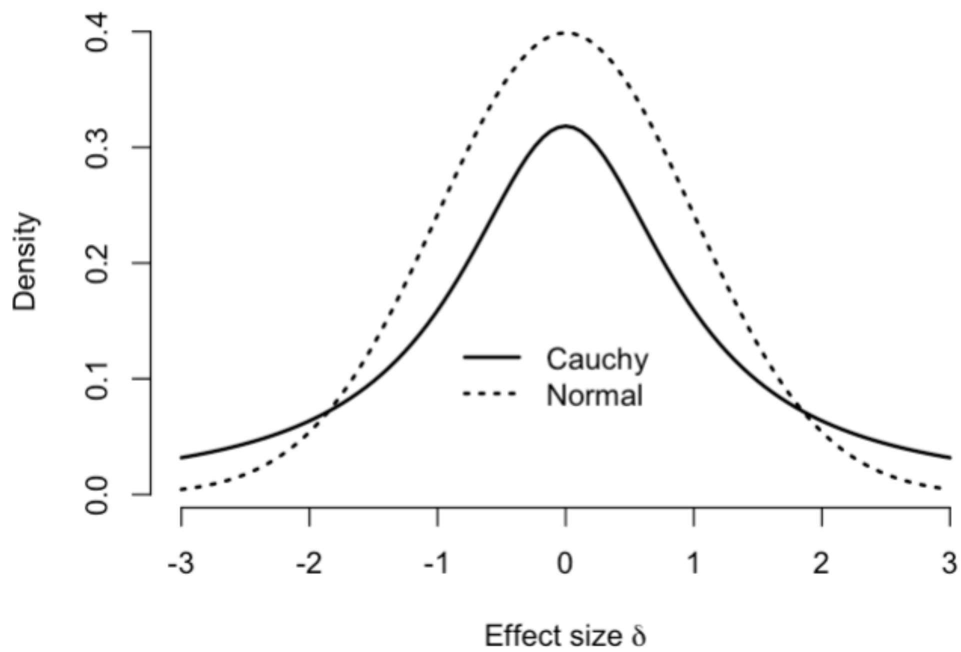
The prior can be made even more objective by letting the variance freely vary: $\delta \sim \text{Normal}(0, \sigma_\delta^2)$



Zellner & Siow (1980) further placed a prior on the variance component: $\sigma_\delta^2 \sim \text{Inverse } \chi^2(1)$



From this, Liang et al (2008) found that this hierarchical prior specification is equivalent to placing a Cauchy prior directly on δ:



Using this Cauchy prior specification, Rouder et al. (2009) showed that the Bayes factor can be computed directly from the t-score and degrees of freedom:

$$BF_{10} = C \sqrt{\left(1 + \frac{t^2}{\nu}\right)^{\nu+1}}$$

↳ where C must be computed by approximation

↳ this "JZS" Bayes factor is implemented in JASP.

Let's continue our working example. Recall that we tested $N = 65$ participants and observed a sample mean of 54.4 with SD = 10.

- compute $t = \dfrac{\bar{x} - \mu}{s/\sqrt{N}} = \dfrac{54.4 - 50}{10/\sqrt{65}} = 3.55$

- use JASP "Summary Statistics" module

Elements to report:

1. report results of hypothesis test

   - define H_0, H_1 and specify prior under H_1.

     "Under the null hypothesis we expect an effect size of 0. Thus, we define $H_0: \delta = 0$. The alternative hypothesis is two-sided, $H_1: \delta \neq 0$, and prior to observing data, we assumed that δ was distributed as a Cauchy distribution with scale $r = 0.707$."

   - report and interpret Bayes factor

     "We found a Bayes factor of $BF_{10} = 34.7$, which means that the observed data are approximately 35 times more likely under H_1 than H_0. This result indicates strong evidence in favor of H_1"

- calculate and report posterior model probability for preferred model.

  - from earlier,

$$p(\mathcal{H}_1 \mid \text{data}) = \frac{\text{posterior odds}}{1 + \text{posterior odds}}$$

$$= \frac{34.7}{35.7} = 0.972.$$

- "Assuming prior odds of 1-1 for $\mathcal{H}_1$ and $\mathcal{H}_0$, our observed data updated these odds to 34.7-to-1 in favor of $\mathcal{H}_1$. This is equivalent to a posterior model probability of $p(\mathcal{H}_1 \mid \text{data}) = 0.97$."

2. report results of parameter estimation
   - only if $\mathcal{H}_1$ is the preferred model!
   - specify parameter of interest and remind reader of prior under $\mathcal{H}_1$
     - "of interest is the posterior distribution for δ, the population-level effect size. Under $\mathcal{H}_1$, δ was assigned a Cauchy prior with scale $r = 0.707$."

- report the 95% credible interval.
  - "The posterior distribution for δ had a median of 0.422, with a central 95% credible interval that ranges from 0.171 to 0.675."