

# **Closed form Bayes factor techniques for measuring evidential value from analysis of variance models**

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The goal of this talk is to describe some methods for evaluating evidential value of data in analysis of variance models.

By *evidential value*, I mean the factor by which the prior odds is updated after observing data:

$$\underbrace{\frac{p(\mathcal{H}_1 | \mathcal{D})}{p(\mathcal{H}_0 | \mathcal{D})}}_{\text{posterior odds}} = \underbrace{\frac{p(\mathcal{H}_1)}{p(\mathcal{H}_0)}}_{\text{prior odds}} \times \underbrace{\frac{p(\mathcal{D} | \mathcal{H}_1)}{p(\mathcal{D} | \mathcal{H}_0)}}_{\text{predictive updating factor}}$$

Kass and Raftery (1995) called this predictive updating factor the **Bayes factor**

Motivating example: consider test scores from students in three instructional treatments:

Treatment 1	Treatment 2	Treatment 3
2	5	8
3	9	6
8	10	12
6	13	11
5	8	11
6	9	12
$M = 5$	$M = 9$	$M = 10$

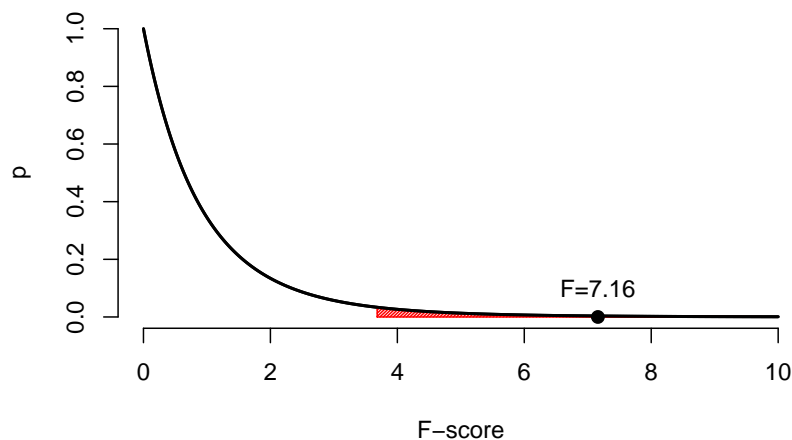
Typical question – are there differences among these condition means?

## Classical approach - analysis of variance (ANOVA)

- model  $Y_{ij} = \mu + \alpha_j + \varepsilon_{ij}$ , where  $\varepsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$
- assume “null hypothesis”  $\mathcal{H}_0 : \alpha_j = 0$
- compute probability of observing data  $Y_{ij}$  under  $\mathcal{H}_0$
- if data is *rare* under  $\mathcal{H}_0$ , reject  $\mathcal{H}_0$
- index “rareness” by computing  $F$  (a ratio of between-group and within-group variances)

## ANOVA computations

source	$SS$	$df$	$MS$	$F$
between treatments	84	2	42	7.16
within treatments	88	15	5.87	
total	172	17		



Since our data  $Y_{ij}$  is rare under  $\mathcal{H}_0$ , we reject  $\mathcal{H}_0$  as an implausible model restriction.

This ignores predictive adequacy of  $\mathcal{H}_1$ .

## The **Bayes factor**

$$B_{10} = \frac{p(\mathcal{D} \mid \mathcal{H}_1)}{p(\mathcal{D} \mid \mathcal{H}_0)}$$

tells us how much better  $\mathcal{H}_1$  predicts our observed data compared to  $\mathcal{H}_0$ .

Example: suppose  $B_{12} = 5$ . This means that the observed data are 5 times more likely under  $\mathcal{H}_1$  than  $\mathcal{H}_0$ .

Problem – computing Bayes factors is hard!

$$\begin{aligned} B_{10} &= \frac{p(\mathcal{D} \mid \mathcal{H}_1)}{p(\mathcal{D} \mid \mathcal{H}_0)} \\ &= \frac{\int f(\mathcal{D} \mid \theta_1, \mathcal{H}_1) \pi(\theta_1 \mid \mathcal{H}_1) d\theta_1}{\int f(\mathcal{D} \mid \theta_0, \mathcal{H}_0) \pi(\theta_0 \mid \mathcal{H}_0) d\theta_0} \end{aligned}$$

Today, I'll describe two approaches to making this computation easier

## 1. **BIC approximation** (Raftery, 1995; Wagenmakers, 2007; Masson, 2011)

Basic idea – if we construct 2nd order Taylor approximation of log-marginal likelihood of each  $\mathcal{H}_i$ , we get

$$BF_{01} \approx \exp\left(\frac{\text{BIC}(\mathcal{H}_1) - \text{BIC}(\mathcal{H}_0)}{2}\right),$$

where

$$\text{BIC}(\mathcal{H}_i) = n \ln\left(\frac{SSR}{SST}\right) + k \ln n.$$

In 2018<sup>1</sup>, I derived a simple formula that computes this BIC Bayes factor using only the ANOVA summary statistics:

$$BF_{01} \approx \sqrt{n^x \left(1 + \frac{Fx}{y}\right)^{-n}},$$

where

- $F$  is the observed  $F$ -ratio
- $x, y$  are the numerator/denominator df, respectively
- $n$  is the total number of observations.

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<sup>1</sup>Faulkenberry, T.J. (2018). Computing Bayes factors to measure evidence from experiments: An extension of the BIC approximation. *Biometrical Letters*, 55, 31-43



Using the summary statistics from the ANOVA ( $F = 7.16$ ,  $x = 2$ ,  $y = 15$ , and  $n = 18$ ), we get

$$\begin{aligned} BF_{01} &\approx \sqrt{n^x \left(1 + \frac{Fx}{y}\right)^{-n}} \\ &= \sqrt{18^2 \left(1 + \frac{7.16 \cdot 2}{15}\right)^{-18}} \\ &= 0.0432, \end{aligned}$$

Thus,  $BF_{10} = 1/BF_{01} \approx 1/0.0432 = 23.15$

## Bad approximation?

Sellke et al. (2001) showed that under a reasonable class of prior distributions for  $p$ -values, an upper bound for the Bayes factor can be computed directly from the  $p$ -value as

$$\begin{aligned} BF_{10} &\leq -\frac{1}{e \cdot p \ln(p)} \\ &= \leq -\frac{1}{e \cdot 0.0066 \cdot \ln(0.0066)} \\ &= 11.10. \end{aligned}$$

Thus, our BIC Bayes factor of 23.15 is quite an overestimate of the actual Bayes factor. **Can we compute an exact Bayes factor with only summary statistics?**

## 2. Pearson Type VI Bayes factor

In some new work<sup>2</sup>, I derived the following *exact* Bayes factor:

$$BF_{10} = \frac{\Gamma\left(\frac{x+1}{2}\right) \cdot \Gamma\left(\frac{y}{2}\right)}{\Gamma\left(\frac{x+y}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)} \left(\frac{y}{y + xF}\right)^{\frac{1-y}{2}}.$$

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<sup>2</sup>based on Wang & Sun (2014); preprint forthcoming on ArXiv

Using our example, we have

$$\begin{aligned} BF_{10} &= \frac{\Gamma\left(\frac{2+1}{2}\right) \cdot \Gamma\left(\frac{15}{2}\right)}{\Gamma\left(\frac{2+15}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)} \left( \frac{15}{15 + 2 \cdot 7.16} \right)^{\frac{1-15}{2}} \\ &= \frac{0.8662269 \cdot 1871.254}{14034.41 \cdot 1.772454} (0.5116)^{-7} \\ &= 7.268 \end{aligned}$$

Some observations:

- the resulting Bayes factor is reasonable w.r.t. the Sellke bound
- the expression is *analytic*, but not *closed form*

**Theorem 1.** *Given an ANOVA summary reported in standard form  $F(x, y)$  (i.e., where  $x$  equals the between-treatments degrees of freedom and  $y$  equals the residual degrees of freedom), the Bayes factor can be expressed in closed form as*

$$BF_{10} = C \sqrt{\left( \frac{y}{y + xF} \right)^{1-y}}$$

where  $C$  depends on the parity of  $x$  and  $y$ , as follows:

*Case 1: if  $x$  and  $y$  are even, then*

$$C = \frac{x! \left( \frac{y}{2} - 1 \right)!}{2^x \left( \frac{x}{2} \right)! \left( \frac{x+y}{2} - 1 \right)!}$$

*Case 2: if  $x$  is even and  $y$  is odd, then*

$$C = \frac{x!(y-1)!\left(\frac{x+y-1}{2}\right)!}{\left(\frac{x}{2}\right)!\left(\frac{y-1}{2}\right)!(x+y-1)!}$$

*Case 3: if  $x$  is odd and  $y$  is even, then*

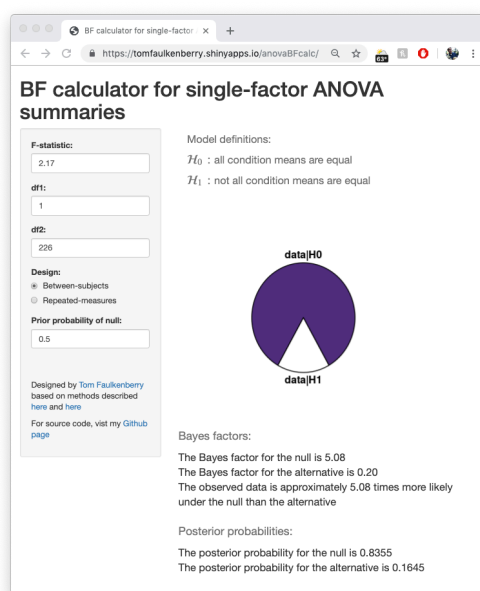
$$C = \frac{2^{x+y-1} \left(\frac{x+y-1}{2}\right)! \left(\frac{y}{2} - 1\right)! \left(\frac{x-1}{2}\right)!}{\pi(x+y-1)!}$$

*Case 4: if  $x$  and  $y$  are odd, then*

$$C = \frac{\left(\frac{x-1}{2}\right)!(y-1)!}{2^{y-1} \left(\frac{y-1}{2}\right)! \left(\frac{x+y}{2} - 1\right)!}$$

## Future goals

In a new paper<sup>3</sup>, I describe a web app that computes Bayes factors from ANOVA summaries – [https://tomfaulkenberry.shinyapps.io/anovaBFcalc](https://tomfaulkenberry.shinyapps.io/anovaBFcalc/)



- “Shiny app” (based on R)
- gives Bayes factors and posterior probabilities
- currently uses BIC approximation
- working on integrating exact Bayes factors

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<sup>3</sup>Faulkenberry, T. J. (2019). Estimating evidential value from analysis of variance summaries: A comment on Ly et al.(2018). *Advances in Methods and Practices in Psychological Science*, 2(4), 406-409

*Thank you!*

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