Bayesian model comparison with informative hypotheses

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Consider the test scores from students in three different treatment conditions:

- Treatment 1 - read and reread
- Treatment 2 - read, then answer prepared questions
- Treatment 3 - read, then create and answer questions

Treatment 1	Treatment 2	Treatment 3
2	5	8
3	9	6
8	10	12
6	13	11
5	8	11
6	9	12
$M = 5$	$M = 9$	$M = 10$

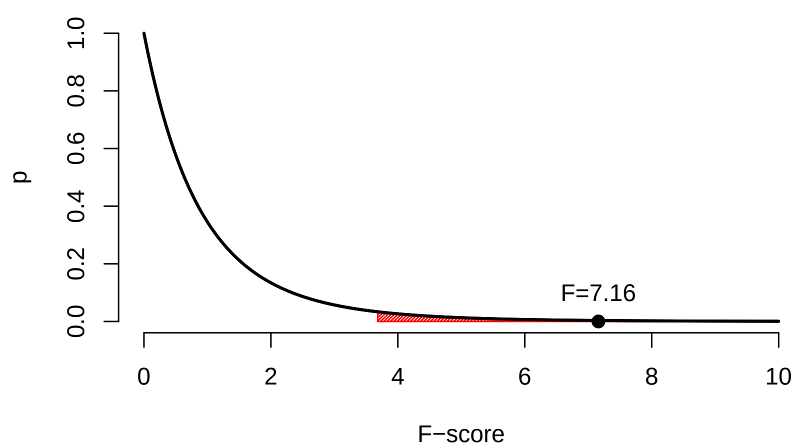
Typical question – are there differences among these condition means?

Standard approach - analysis of variance (ANOVA)

- model $Y_{ij} = \mu + \alpha_j + \varepsilon_{ij}$, where $\varepsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$
- assume "null hypothesis" $\mathcal{H}_0 : \alpha_j = 0$
- compute probability of observing data Y_{ij} under $\mathcal{H}_0$
- if data is $rare$ under $\mathcal{H}_0$, reject $\mathcal{H}_0$

ANOVA computations

source	SS	df	MS	F
between treatments	84	2	42	7.16
within treatments	88	15	5.87	
total	172	17		



Since our data Y_{ij} is rare under $\mathcal{H}_0$, we reject $\mathcal{H}_0$ as an implausible model restriction.

What does this tell us?

If we reject $\mathcal{H}_0 : \alpha_j = 0$, this tells us that $\alpha_j \neq 0$ for some j.

- which values of j?
- are they positive / negative?
- the alternative is rather **uninformative**

Informative hypotheses

Consider instead defining competing $informative$ models:

- $\mathcal{M}_1 : \mu_1 < \mu_2 < \mu_3$
- $\mathcal{M}_2 : \mu_2 < \mu_1 < \mu_3$
- $\mathcal{M}_3 : \mu_1 < \mu_3 < \mu_2$
- $\mathcal{M}_e : \mu_1, \mu_2, \mu_3$

Note:

1. each model tells a different story about effective study methods
2. typical ANOVA cannot differentiate between $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$

Goal - evaluate relative evidence for each model $\mathcal{M}_j$, in light of observed data $\boldsymbol{y}$

Bayes' Theorem:

$$\underbrace{\frac{p(\mathcal{M}_j \mid \boldsymbol{y})}{p(\mathcal{M}_k \mid \boldsymbol{y})}}_{\text{posterior odds}} = \underbrace{\frac{p(\mathcal{M}_j)}{p(\mathcal{M}_k)}}_{\text{prior odds}} \times \underbrace{\frac{p(\boldsymbol{y} \mid \mathcal{M}_j)}{p(\boldsymbol{y} \mid \mathcal{M}_k)}}_{\text{predictive updating factor}}$$

The predictive updating factor, or **Bayes factor**, tells us how much better $\mathcal{M}_j$ predicts our observed data compared to $\mathcal{M}_k$.

Computing Bayes factors for informative hypotheses

Theorem 1. *(Klugkist et al., 2005) Consider a model* $\mathcal{M}_1$ *nested within an encompassing model* $\mathcal{M}_e$ *via an inequality constraint. Then*

$$B_{1e} = \frac{F}{C}$$

where F *and* C *represent the proportions of the posterior and prior of the encompassing model, respectively, that are in agreement with the inequality constraint imposed by the nested model* $\mathcal{M}_1$.

Sample from prior:

Iteration	Prior					
	μ_1	μ_2	μ_3	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$
1	6.54	10.15	-1.78	0	0	0
2	22.60	-0.28	8.03	0	0	0
3	3.37	3.01	-0.63	0	0	0
4	-6.13	11.54	12.33	1	0	0
5	13.68	-0.61	1.50	0	0	0
6	27.83	7.43	6.79	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
5000	11.00	13.07	23.91	1	0	0
Sum				847	876	807
Proportion (C)				0.169	0.175	0.161

Sample from posterior:

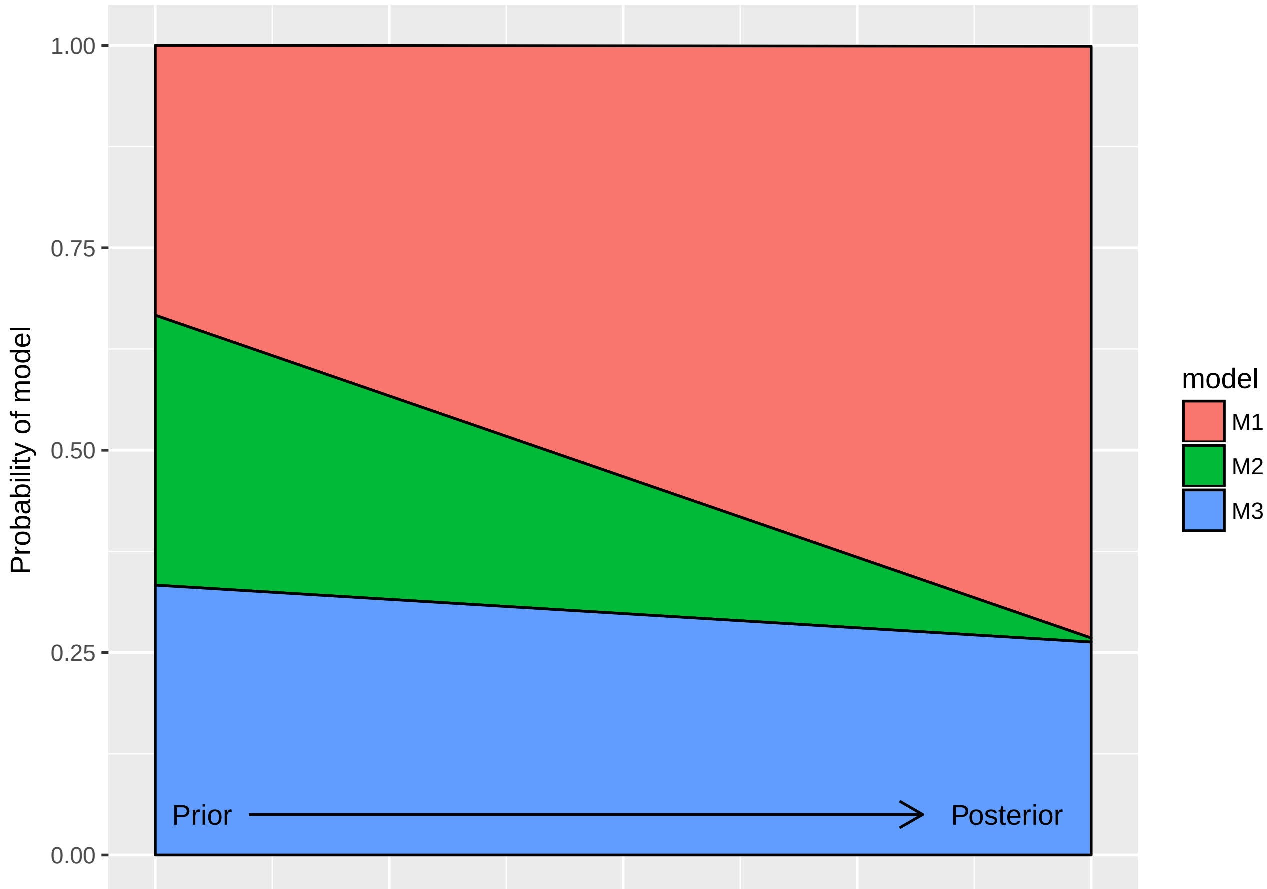
Iteration	Posterior					
	μ_1	μ_2	μ_3	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$
1	5.59	9.90	12.83	1	0	0
2	2.90	8.86	8.35	0	0	1
3	2.63	10.43	10.44	1	0	0
4	5.55	10.17	9.61	0	0	1
5	4.61	7.24	10.24	1	0	0
6	4.72	8.95	9.78	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5000	5.61	8.72	9.99	1	0	0
Sum				3674	29	1286
Proportion (F)				0.735	0.006	0.257
Proportion (C)				0.169	0.175	0.161
B_{je}				4.43	0.03	1.59

From these Bayes factors, we can compute *posterior model probabilities* (PMPs).

$$p(\mathcal{M}_j \mid \boldsymbol{y}) = \frac{B_{je}}{B_{1e} + B_{2e} + B_{3e}}$$

Model	F	C	B_{je}	PMP
$\mathcal{M}_1 : \mu_1 < \mu_2 < \mu_3$	0.735	0.169	4.43	0.731
$\mathcal{M}_2 : \mu_2 < \mu_1 < \mu_3$	0.006	0.175	0.03	0.005
$\mathcal{M}_3 : \mu_1 < \mu_3 < \mu_2$	0.257	0.161	1.59	0.263

“flow of model belief”



Thank you!

- Thanks to Tarleton Office of Research and Innovation for funding!
- slides available at `github.com/tomfaulkenberry/talks`
- more details in Faulkenberry, T. J. (2019). A tutorial on generalizing the default Bayesian t-test via posterior sampling and encompassing priors. *Communications for Statistical Applications and Methods, 26*(2), 1-22.
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