A hierarchical Bayesian model of individual differences in the size-congruity effect

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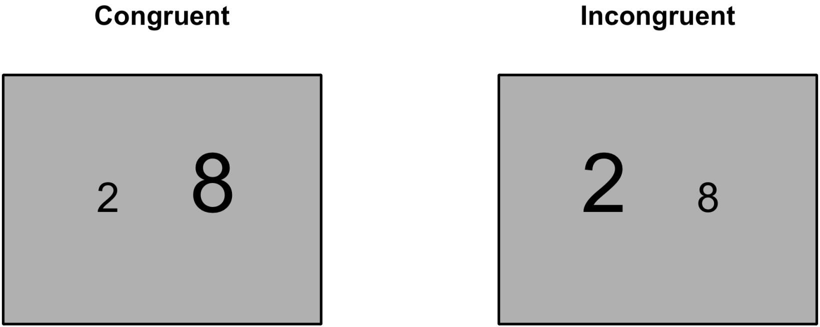
MCLS Symposium: Current directions in symbolic number processing

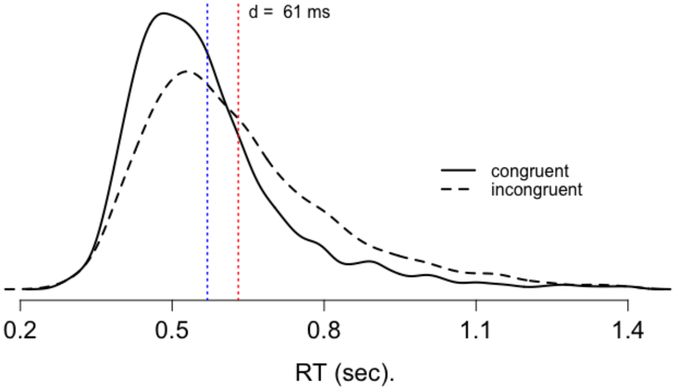
Size congruity effect

Typical laboratory task: choose the **physically larger** digit



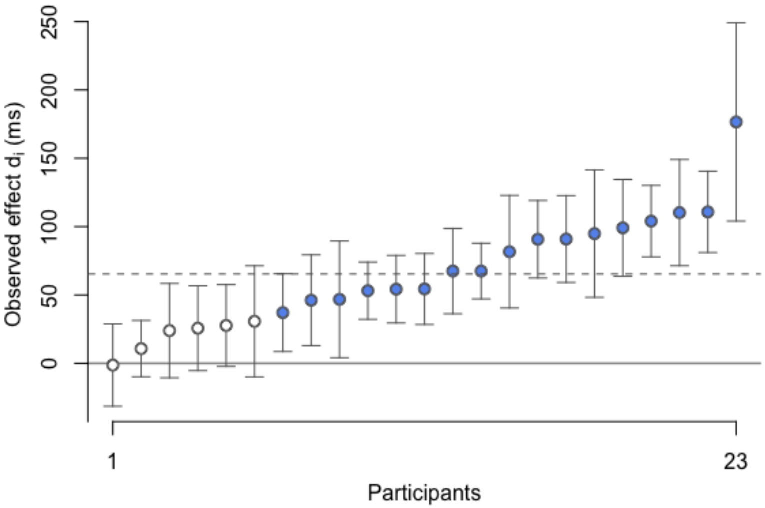
Size congruity effect

Typical result – mean RT larger for *incongruent trials*



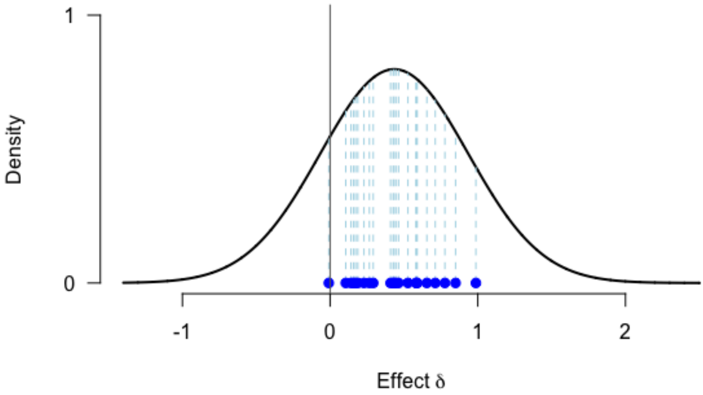
Size congruity effect

What about individual effects?



Individual differences

Suppose these observed effects d_i are drawn from population of *true* effects δ. What is the structure of this population?



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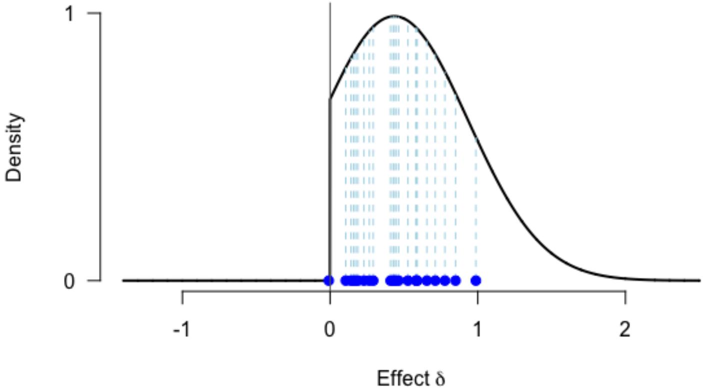


A new question

Does everybody *exhibit the size congruity effect?*

- if *yes*, then SCE is obligatory, resistant to strategic control, ...
- if *no*, then SCE is complex, malleable, ...

Importantly, both answers have downstream consequences for processing architecture of numerical cognition

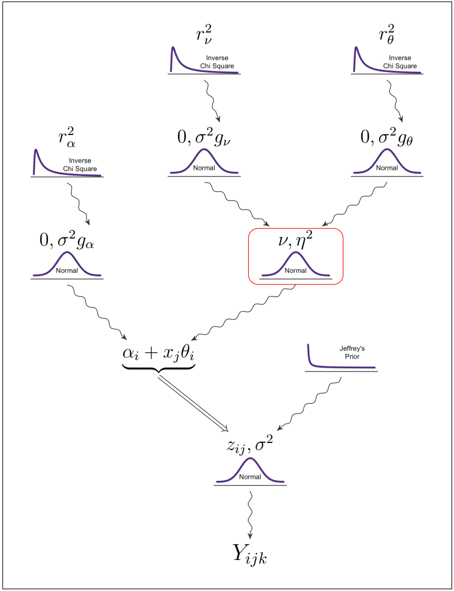
A new question

Does everybody *exhibit size congruity effect?*

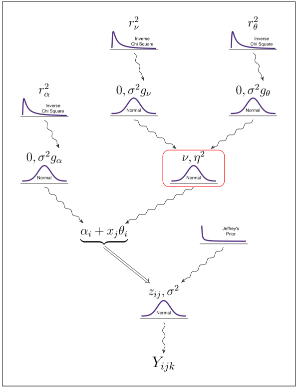
How to answer:

- build models of individual difference structures in SCE (e.g., Haaf & Rouder, 2017)
- adjudicate the models via Bayesian model comparison

Hierarchical structure



Hierarchical structure



Basic idea (Haaf & Rouder, 2017):

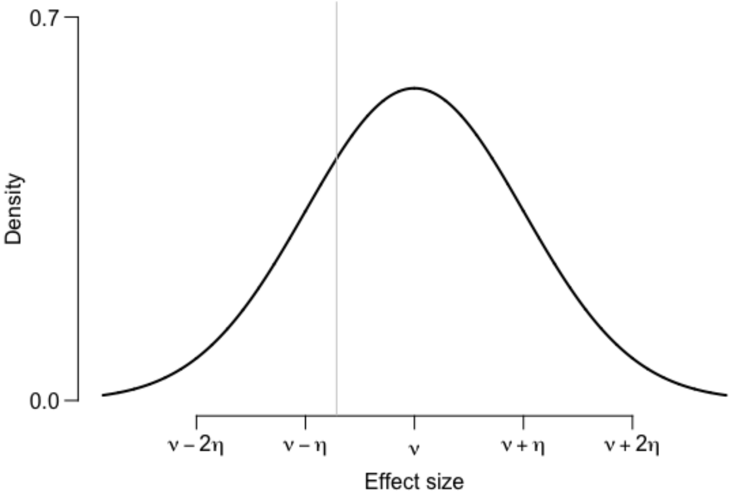
- model RTs as a random-effects linear model with effect parameter θ_i ($i = 1 \ldots, N$)
- assume (as baseline) that θ_i is drawn from a normal distribution with mean ν and variance η^2
- use g-priors (Zellner & Siow, 1980), specifying *a priori* scale on variance of overall effect and individual variability around effect
- define competing models by **constraining** effect parameter θ_i

Four competing models

1. Unrestricted model, $\mathcal{M}_u$
2. Positive-effects model, $\mathcal{M}_+$
3. Common-effect model, $\mathcal{M}_1$
4. Null-effect model, $\mathcal{M}_0$

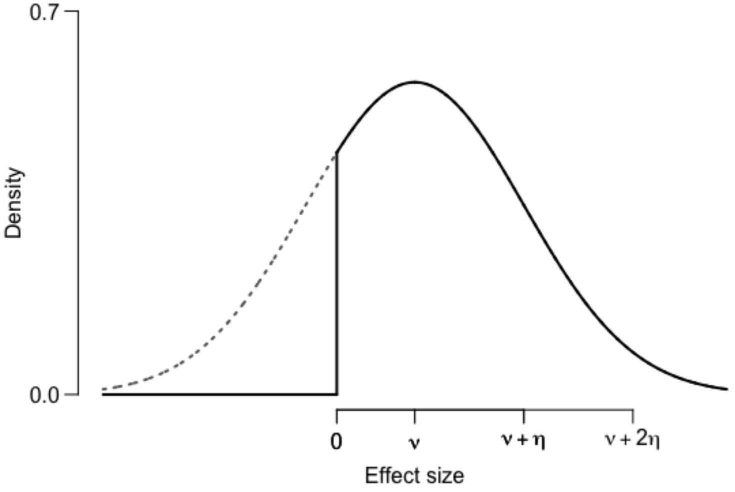
Unrestricted model

$\mathcal{M}_u : \theta_i \sim \text{Normal}(\nu, \eta^2)$



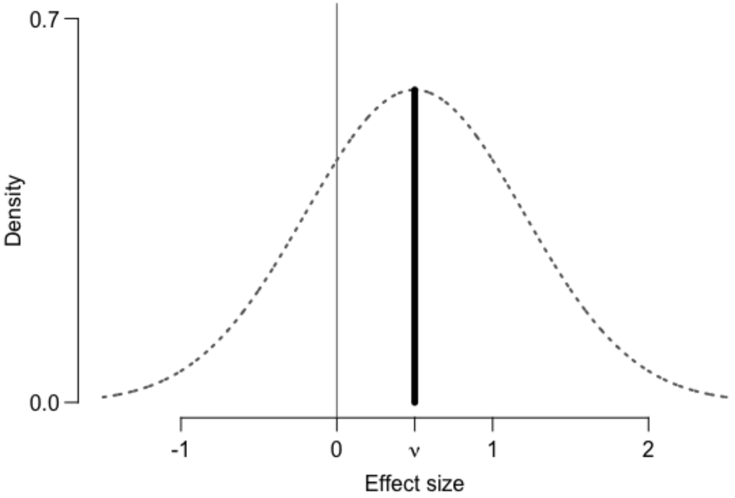
Positive-effects model

- $\mathcal{M}_+ : \theta_i \sim \text{Normal}_+(\nu, \eta^2)$



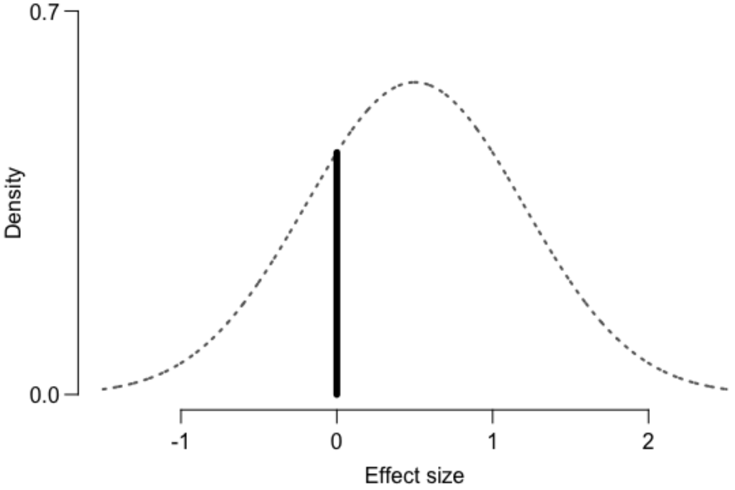
Common-effect model

- $\mathcal{M}_1 : \theta_i = \nu$



Null-effect model

- $\mathcal{M}_0 : \theta_i = 0$



Model comparison

We use the **Bayes factor**, which indexes how well the observed data $\boldsymbol{Y}$ are predicted under one model relative to another:

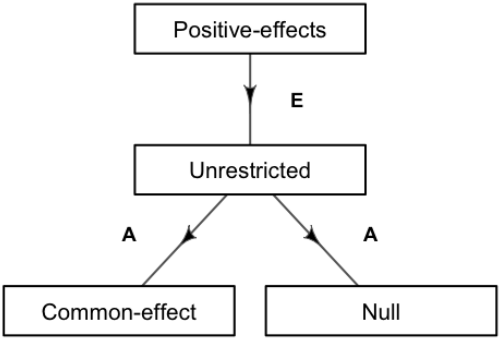
$$B_{ab} = \frac{p(\boldsymbol{Y} \mid \mathcal{M}_a)}{p(\boldsymbol{Y} \mid \mathcal{M}_b)}$$

Ex: suppose $B_{ab} = 10$. This means:

- the observed data are 10 times more likely under $\mathcal{M}_a$ compared to $\mathcal{M}_b$
- "10-to-1 evidence for $\mathcal{M}_a$ over $\mathcal{M}_b$"

Bayes factor computations

So $B_{ab} = \dfrac{p(\boldsymbol{Y} \mid \mathcal{M}_a)}{p(\boldsymbol{Y} \mid \mathcal{M}_b)}$. How do we compute this?



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$$p(\boldsymbol{Y} \mid \mathcal{M}) = \int_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} p(\boldsymbol{Y} \mid \boldsymbol{\xi}, \mathcal{M}) p(\boldsymbol{\xi} \mid \mathcal{M}) d\boldsymbol{\xi}$$

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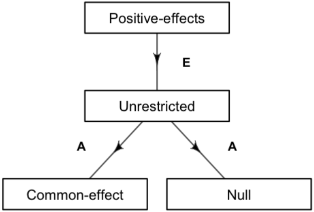
Problem: for our models $\mathcal{M}$, the parameter vectors $\boldsymbol{\xi}$ look like

$$\boldsymbol{\xi} = (\mu, \sigma^2, \nu, \alpha_1, \ldots, \alpha_N, \theta_1, \ldots, \theta_N, g_\alpha, g_\nu, g_\theta)$$

so the integral is carried out in $\mathbb{R}^{2N+6}$.

For $N = 35$, this would be a 76-dimensional integral!

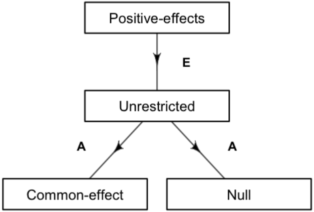
Bayes factor computations



A = *analytic approach*

- Zellner & Siow (1980); Rouder et al. (2012)
- place g-priors on individual intercepts and effect parameters
- everything *except* the g-parameters integrates symbolically
- g-parameters can be well approximated with MCMC sampling
- techniques coded into BayesFactor package in R

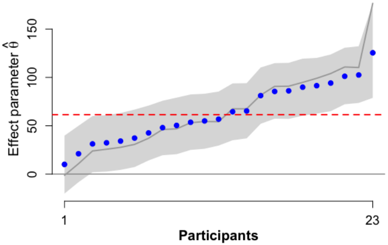
Bayes factor computations



E = *encompassing approach*

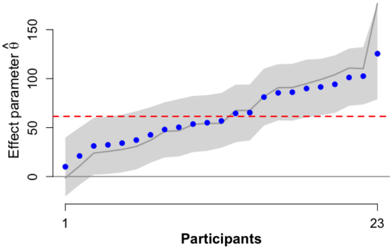
- Klugkist et al. (2005)
- generalization of Savage-Dickey density ratio
- $B_{+u} = \dfrac{P(\boldsymbol{\theta} > 0 \mid \boldsymbol{Y}, \mathcal{M}_u)}{P(\boldsymbol{\theta} > 0 \mid \mathcal{M}_u)}$
- probabilities computed as fraction of MCMC samples from unrestricted model that are **positive** for all individuals (both in the prior and posteriori)

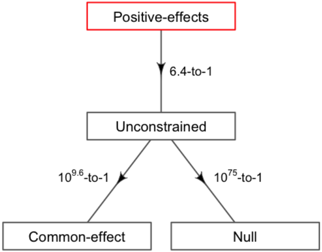
Results - Exp 1



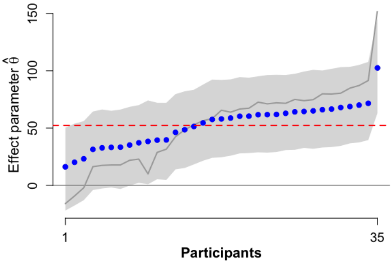
- Red line = estimated effect θ from $\mathcal{M}_1$
- Blue dots = individual effect estimates θ_i
- Gray line = estimates from mean differences d_i
- Gray area = 95% credible intervals

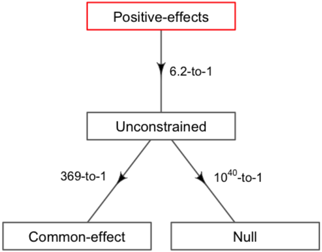
Results - Exp 1





Results - Exp 2





Sensitivity to prior specifications

Experiment 1:

r_ν	r_θ	$\mathcal{M}_0$	$\mathcal{M}_1$	$\mathcal{M}_+$	$\mathcal{M}_u$
$\frac{1}{6}$ (50 ms)	$\frac{1}{10}$ (30 ms)	5.6e-77	4.2e-11	*	0.16
$\frac{1}{12}$ (25 ms)	$\frac{1}{20}$ (15 ms)	1.8e-76	7.7e-11	*	0.16
$\frac{1}{12}$ (25 ms)	$\frac{1}{5}$ (60 ms)	1.9e-77	8.1e-12	*	0.05
$\frac{1}{3}$ (100 ms)	$\frac{1}{20}$ (15 ms)	1.9e-76	1.9e-10	*	0.39
$\frac{1}{3}$ (100 ms)	$\frac{1}{5}$ (60 ms)	3.0e-77	3.1e-11	*	0.16

Note: Bayes factors computed against the "winning" model, denoted by *

Sensitivity to prior specifications

Experiment 2:

r_ν	r_θ	$\mathcal{M}_0$	$\mathcal{M}_1$	$\mathcal{M}_+$	$\mathcal{M}_u$
$\frac{1}{6}$ (50 ms)	$\frac{1}{10}$ (30 ms)	4.3e-41	0.0004	*	0.17
$\frac{1}{12}$ (25 ms)	$\frac{1}{20}$ (15 ms)	1.2e-40	0.0007	*	0.16
$\frac{1}{12}$ (25 ms)	$\frac{1}{5}$ (60 ms)	2.6e-41	0.0002	*	0.06
$\frac{1}{3}$ (100 ms)	$\frac{1}{20}$ (15 ms)	1.4e-40	0.0017	*	0.42
$\frac{1}{3}$ (100 ms)	$\frac{1}{5}$ (60 ms)	4.1e-41	0.0005	*	0.19

Note: Bayes factors computed against the "winning" model, denoted by *

Summary points

Does everybody *exhibit the size congruity effect?*

- if *yes*, then SCE is obligatory, resistant to strategic control, . . .
- if *no*, then SCE is complex, malleable, . . .

Importantly, both answers have downstream consequences for processing architecture of numerical cognition

- what does this say about *early vs. late* interaction debate (e.g., Faulkenberry et al., 2016; Sobel et al., 2016; 2017; Faulkenberry, Vick, & Bowman, in press)

Summary points

Some other benefits:

- Bayes factors easy to interpret
- hierarchical structure removes trial noise from individual estimates
- common effect (CE) model provides important self-check:
    - if CE model is best, is our design adequate to capture individual differences
- Might be good approach to disentangle competing theories of mental arithmetic
    - Does everyone exhibit size-by-format interaction?
    - Does everyone reflect fast counting in small addition problems?

Thank you!

- Thanks to Tarleton Office of Research and Innovation for funding!
- slides available at `github.com/tomfaulkenberry/talks`
- Twitter: `@tomfaulkenberry`
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