

# The $z$ -Transform

Digital Signal Processing

October 2, 2025



# Motivation for the $z$ -Transform

- Until now, we've used the Fourier transform to analyze frequency content of **signals**.
- With the  $z$ -transform, we'll analyze **systems** (specifically, LTI systems).
- Helps determine properties of a system, such as stability, causality, frequency response, etc.
- Used to design LTI systems (filters).

# Review: Discrete Fourier Transform

## Definition

Let  $x[n]$  be a complex-valued, periodic signal with period  $L$ . The **discrete Fourier transform (DFT)** of  $x[n]$  is given by

**DFT analysis:**

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

**DFT synthesis:**

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

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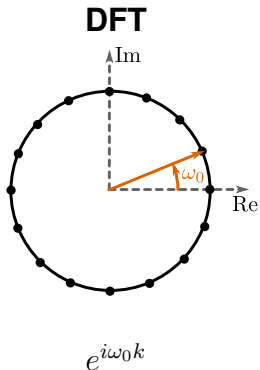
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# Transform Domains



# Discrete-Time Fourier Transform

## Definition (DTFT Analysis)

Let  $x[n]$  be a complex signal for  $-\infty < n < \infty$ . The **discrete-time Fourier transform (DTFT)** of  $x[n]$  is given by

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} e^{-i\omega n} x[n], \quad \text{for } \omega \in [-\pi, \pi).$$

- Note range of  $n$  is all of  $\mathbb{Z}$ .
- Note  $X(e^{i\omega})$  is defined on the **continuous** unit circle in  $\mathbb{C}$ .
- Can also think of  $X$  as a function of angular frequency,  $\omega$ .

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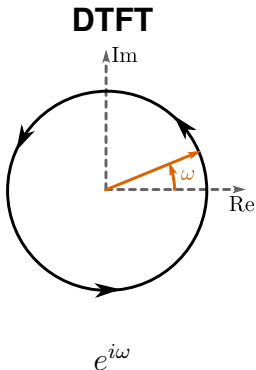
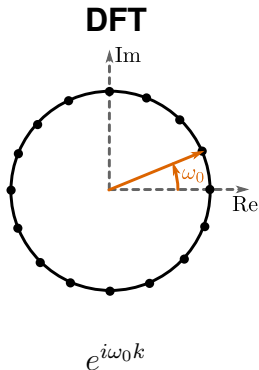
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# Transform Domains



## Definition (DTFT Synthesis)

Let  $X(e^{i\omega})$  be the DTFT of a signal  $x[n]$ . The **inverse discrete-time Fourier transform** is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega n} d\omega.$$

- Note this is an integral around the unit circle in  $\mathbb{C}$ .

# The $z$ -Transform

## Definition ( $z$ -Transform Analysis)

Given a complex discrete signal  $x[n]$ , its  **$z$ -transform** is given by

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n].$$

# The $z$ -Transform

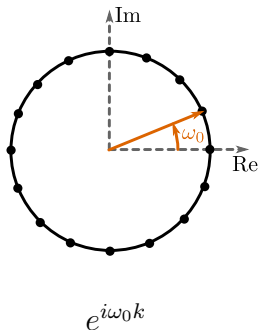
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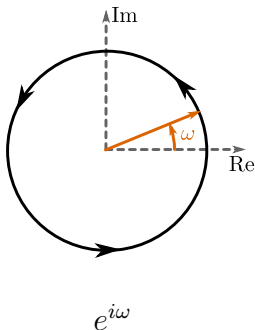
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# Transform Domains

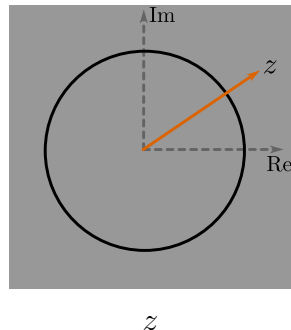
**DFT**



**DTFT**



**$z$ -Transform**



# Overloaded Notation!

Notice we reused the notation “ $X$ ” for the DFT, DTFT, and  $z$ -transform.

The context is clear from the input to  $X$ :

DFT	$X[k]$
DTFT	$X(e^{i\omega})$
$z$ -transform	$X(z)$



# Quick Side Note: Geometric Series

A geometric series looks like:

$$\begin{aligned}s &= 1 + r + r^2 + r^3 + \dots \\ &= \sum_{n=0}^{\infty} r^n\end{aligned}$$

The infinite sum evaluates to

$$s = \frac{1}{1-r}, \quad \text{for } |r| < 1.$$

This holds for  $r$  real or complex!

# DTFT Example: Right-Sided Exponential

Right-sided exponential signal, for some constant  $a \in \mathbb{C}$ , is:

$$\begin{aligned}x[n] &= \begin{cases} a^n & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases} \\ &= a^n u[n].\end{aligned}$$

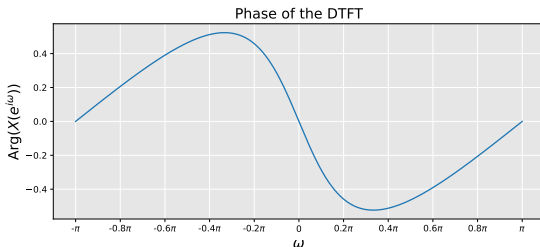
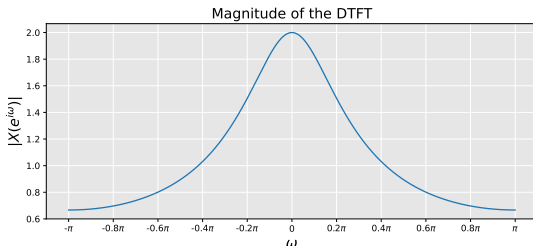
The DTFT is

$$\begin{aligned}X(e^{i\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-i\omega n} = \sum_{n=0}^{\infty} (ae^{-i\omega})^n \\ &= \frac{1}{1 - ae^{-i\omega}}, \quad \text{for } |ae^{-i\omega}| < 1, \text{ or } |a| < 1.\end{aligned}$$

Diverges for  $|a| \geq 1$ .

# Plotting the DTFT

$$x[n] = 0.5^n u[n] \quad \leftrightarrow \quad X(e^{i\omega}) = \frac{1}{1 - 0.5e^{-i\omega}}$$



# $z$ -Transform of Right-Sided Exponential

The  $z$ -transform of  $x[n] = a^n u[n]$  is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \\ &= \frac{1}{1 - az^{-1}}, \quad \text{for } |az^{-1}| < 1, \quad \text{or } |z| > |a| \\ &= \frac{z}{z - a}. \end{aligned}$$

# Region of Convergence (ROC)

## Definition (ROC)

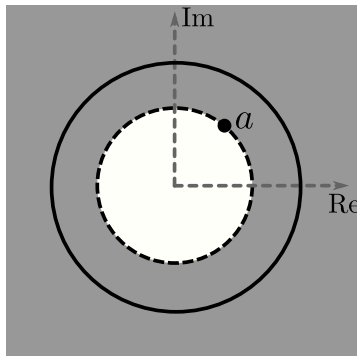
The **region of convergence** for the  $z$ -transform of a signal  $x[n]$  is defined as all points  $z \in \mathbb{C}$  where

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty.$$

# ROC for Right-Sided Exponential

For  $x[n] = u[n]a^n$ , we had

$$X(z) = \frac{z}{z - a}, \quad \text{for } |z| > |a|.$$



# Left-Sided Exponential

Left-sided exponential for a constant  $a \in \mathbb{C}$  is

$$\begin{aligned}x[n] &= \begin{cases} -a^n & \text{for } n < 0 \\ 0 & \text{for } n \geq 0 \end{cases} \\ &= -a^n u[-n - 1]\end{aligned}$$

The  $z$ -transform is

$$\begin{aligned}X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n - 1] z^{-n} = - \sum_{n=-\infty}^{-1} (a z^{-1})^n \\ &= - \sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m \\ &= 1 - \frac{1}{1 - a^{-1} z}, \quad \text{for } |a^{-1} z| < 1, \quad \text{or } |z| < |a|\end{aligned}$$

# Left-Sided Exponential

Further simplifying:

$$\begin{aligned}X(z) &= 1 - \frac{1}{1 - a^{-1}z} \\&= \frac{1 - a^{-1}z}{1 - a^{-1}z} - \frac{1}{1 - a^{-1}z} \\&= \frac{-a^{-1}z}{1 - a^{-1}z} \\&= \frac{z}{z - a}, \quad \text{for } |z| < |a|.\end{aligned}$$

Same as right-sided exponential, but different ROC!

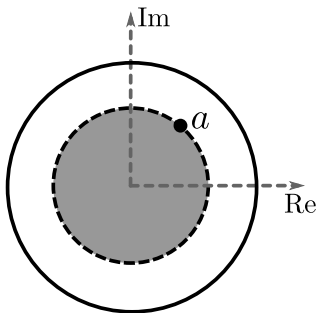
Right-sided ROC:  $|z| > |a|$ , Left-sided ROC:  $|z| < |a|$



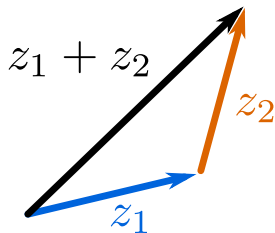
# ROC for Left-Sided Exponential

For  $x[n] = -a^n u[-n - 1]$ , we had

$$X(z) = \frac{z}{z - a}, \quad \text{for } |z| < |a|.$$



# Quick Side Note: Triangle Inequality



Complex Addition

Triangle inequality tells us:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Do this iteratively to get:

$$\left| \sum_{n=-\infty}^{\infty} z_n \right| \leq \sum_{n=-\infty}^{\infty} |z_n|.$$

# General ROC of $z$ -Transform

In general, we have:

$$\begin{aligned}|X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n}\end{aligned}$$

Note this only depends on the magnitude  $r = |z|$ , not the angle of  $z$ .

# General ROC of $z$ -Transform

ROC is an annulus:

$$0 \leq r_R < |z| < r_L \leq \infty$$

$r_R$ : right-side radius

$\sum_{n=0}^{\infty} |x[n]| r^{-n}$  diverges  
when  $r < r_R$ .

$r_L$ : left-side radius

$\sum_{n=-\infty}^{-1} |x[n]| r^{-n}$  diverges  
when  $r > r_L$ .

