

More About Filter Design

Digital Signal Processing

November 11, 2025



Zero-Phase Filters

Say we want to design a filter with **zero phase delay**.

This is equivalent to the condition:

$$\text{Arg}(H(e^{i\omega})) = 0.$$

Or, in other words, or frequency response is a **real-valued function**:

$$H(e^{i\omega}) = A(\omega).$$

Inverse DTFT of Zero-Phase Condition

Because $h[n]$ is real, we have an even function: $A(\omega) = A(-\omega)$.

The DTFT is:

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) e^{i\omega n} d\omega && \text{definition of DTFT} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) (\cos(\omega n) + i \sin(\omega n)) d\omega && \text{expanding } e^{i\omega} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) \cos(\omega n) d\omega && A(\omega) \sin(\omega n) \text{ is odd} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) \cos(-\omega n) d\omega && \text{cosine is even} \\ &= h[-n] \end{aligned}$$

Impulse Response of Zero-Phase Filter

Fact

An LTI system has zero phase only if it has an even impulse response function, i.e.,

$$h[n] = h[-n]$$

Note: This means that the only interesting zero-phase filters are **not causal**.

Also, note: All zero-phase filters have an **odd** length (of non-zero coefficients).

Not All Even $h[n]$ Are Zero-Phase

If $h[n]$ is zero-phase, then it is even.

The impulse response $-h[n]$ is also even. It has frequency response:

$$-H(e^{-i\omega}) = -A(\omega) = A(\omega)e^{i\pi}.$$

So, it has phase $\text{Arg}(-H(e^{-i\omega})) = \pi$.

Linear-Phase Causal Filters

Let's say that we want a causal filter. Zero-phase is out of the question, so the next best thing is a linear phase response.

This implies:

$$\text{Arg}(H(e^{i\omega})) = -\omega\alpha,$$

for some positive constant α . Note, both the **phase delay** and **group delay** are equal to α .

In other words,

$$H(e^{i\omega}) = A(\omega)e^{-i\omega\alpha},$$

where $A(\omega) = |H(e^{i\omega})|$.

Linear-Phase as Shifted Zero-Phase

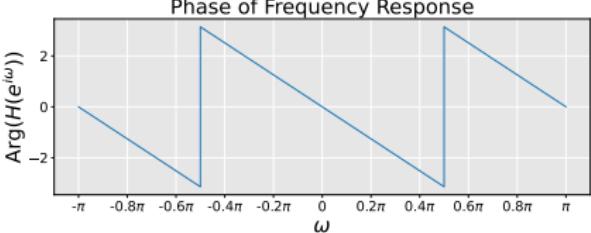
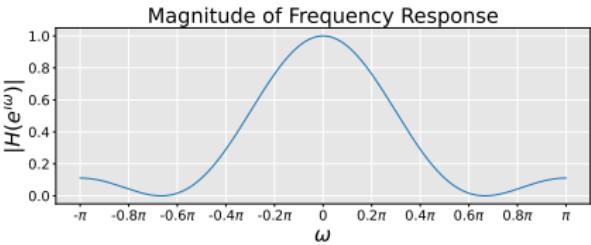
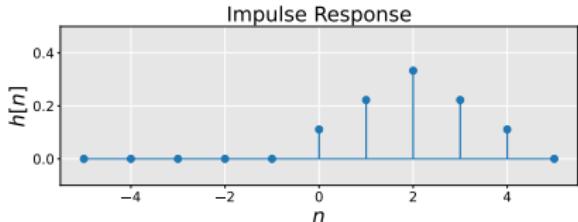
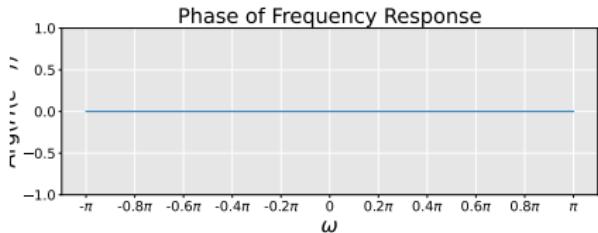
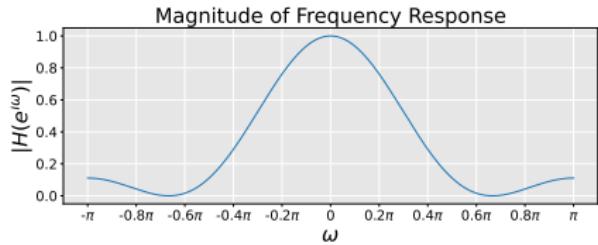
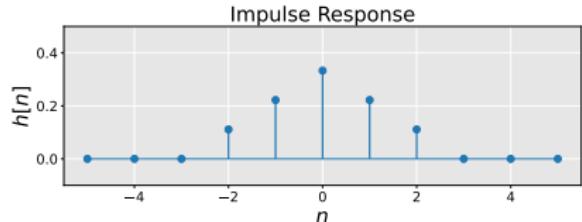
We can get a linear-phase causal filter by shifting the impulse response of a zero-phase filter.

Let $h_{\text{ZP}}[n]$ be a zero-phase filter with L non-zero coefficients. These will be at time points $-\frac{L-1}{2} \leq n \leq \frac{L-1}{2}$.

Then, $h[n] = h_{\text{ZP}}\left[n - \frac{L-1}{2}\right]$ is a causal filter with phase delay $\frac{L-1}{2}$. In other words, it has linear phase:

$$\text{Arg}(H(e^{i\omega})) = -\omega \frac{L-1}{2}.$$

Linear-Phase Causal Filter Example



Even-Length Linear-Phase Filters

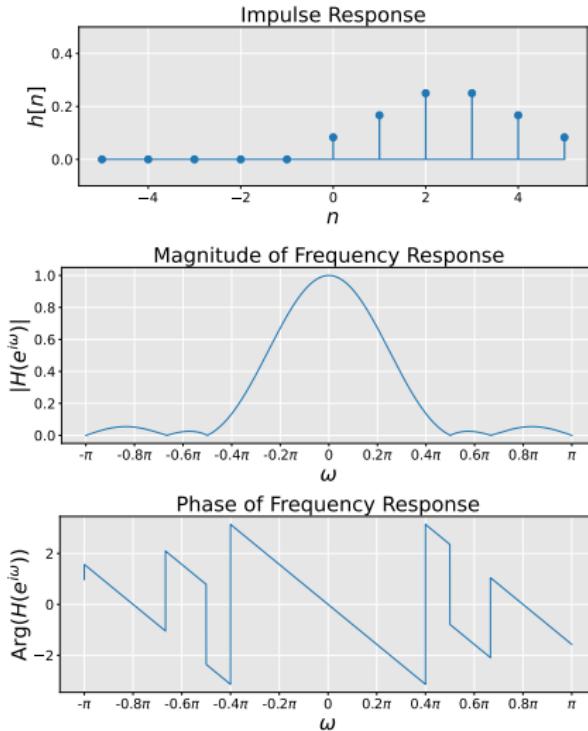
Shifting zero-phase filters can only give us odd-length filters.

These filters have symmetry:

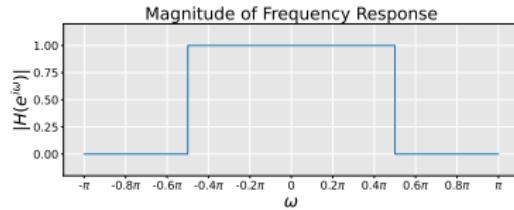
$$h[n] = h[L - n - 1].$$

We can also get an **even-length**, linear-phase, causal filter if we have a similar symmetry.

Even-Length Linear-Phase Example

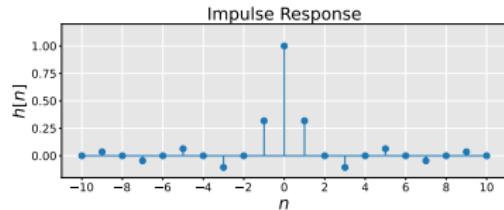


Review: Ideal Low-Pass Filter



$$\xrightarrow{\mathcal{F}^{-1}}$$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \omega_c, \\ 0 & \text{otherwise.} \end{cases}$$



$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}$$

- The ideal low-pass filter is a box in frequency domain.
- Inverse DTFT gives us a sinc impulse response.
- Non-causal and can't be implemented (infinite extent).

The Window Method

Construct a real-valued window function $w[n]$, such that:

$$w[n] = 0 \quad \text{for } n < -M \text{ and } n > M$$

$$w[n] = w[-n] \quad (\text{even function})$$

- ① Shift $h_{LP}[n]$ to $h_{LP}[n - M]$.
- ② Multiply by shifted window function $w[n - M]$ to get:

$$h[n] = w[n - M]h_{LP}[n - M]$$

The Window Method

Resulting frequency response of windowed sinc:

$$\begin{aligned} H(e^{i\omega}) &= \mathcal{F}\{w[n - M]h_{LP}[n - M]\} && \text{taking DTFT} \\ &= e^{-iM\omega} \mathcal{F}\{w[n]h_{LP}[n]\} && \text{shift property} \\ &= e^{-iM\omega} (W(e^{i\omega}) * H_{LP}(e^{i\omega})) && \text{convolution property} \end{aligned}$$

See Jupyter notebook: WindowFunctions.ipynb.

The Window Method

Note $W(e^{i\omega})$ and $H_{LP}(e^{i\omega})$ are both zero-phase.

Why? Because $w[n]$ and $h_{LP}[n]$ are even, real-valued functions.

So, $W(e^{i\omega}) * H_{LP}(e^{i\omega})$ has phase 0 or π everywhere.

Therefore $H(e^{i\omega}) = e^{-iM\omega} (W(e^{i\omega}) * H_{LP}(e^{i\omega}))$ is linear phase with group/phase delay of M .