# Phase and Group Delay of Frequency Response

Digital Signal Processing

October 28, 2025

# Review: Transfer Function

Given an LTI system with impulse response $h[n]$:

$$y[n] = x[n] * h[n].$$

The **transfer function** is the $z$-transform of $h[n]$. It is given by the ratio:

$$H(z) = \frac{Y(z)}{X(z)},$$

where $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ and $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$.

# Review: Transfer Function of LCCDE

A linear constant-coefficient difference equation (LCCDE) is an LTI system of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

It's transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

# Review: Frequency Response

Looking at frequency response of an LTI:

$$Y(e^{i\omega}) = H(e^{i\omega})X(e^{i\omega})$$

Remember complex multiplication in Euler form:

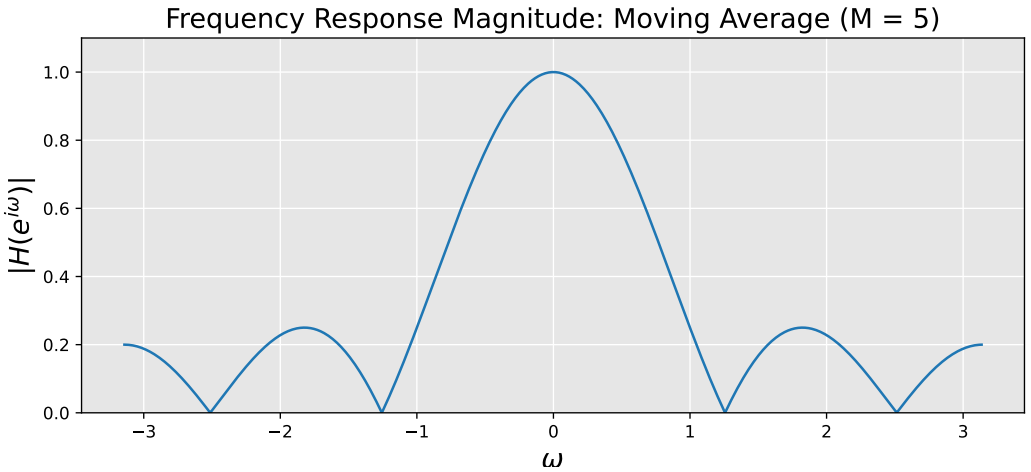
$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

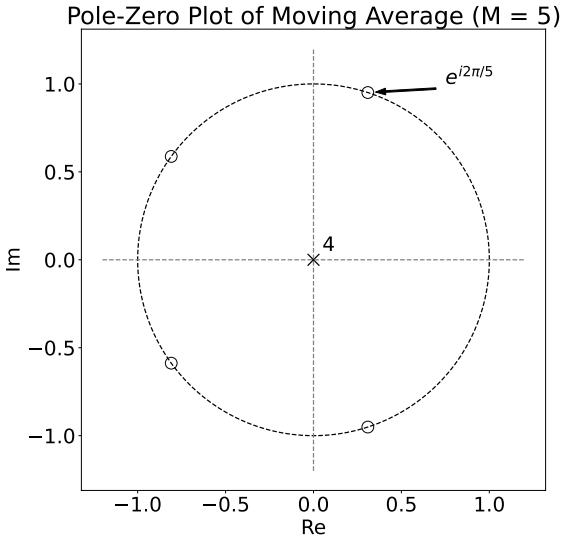
So, we have:

$$\text{Magnitude:} \quad |Y(e^{i\omega})| = |H(e^{i\omega})| \cdot |X(e^{i\omega})|$$

$$\text{Phase:} \quad \text{Arg}(Y(e^{i\omega})) = \text{Arg}(H(e^{i\omega})) + \text{Arg}(X(e^{i\omega}))$$

# Review: Moving Average (MA)

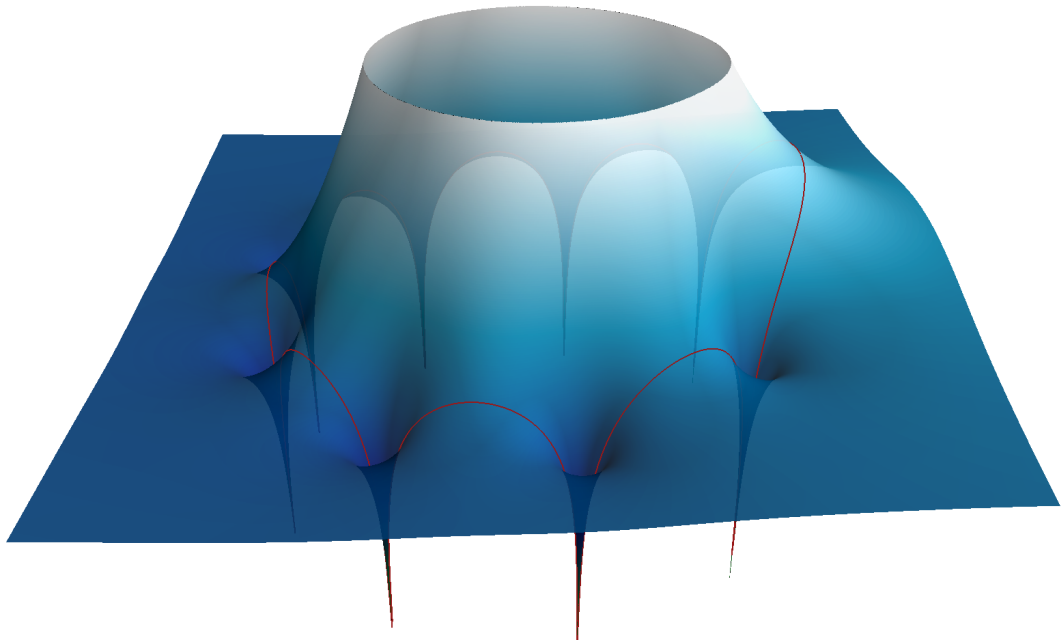



$$H(z) = \frac{1}{5}\sum_{k=0}^{4} z^{-k} = \frac{\prod_{k=1}^{4}(z - b_k)}{z^4},$$

where $b_k = e^{\frac{i2\pi k}{5}}$.

# Moving Average Transfer Function



Magnitude of full $z$-Transform, $|H(z)|$, as height function.

Frequency response is the red curve.

From https://tttapa.github.io/Pages/Mathematics/Systems-and-Control-Theory/Digital-filters/

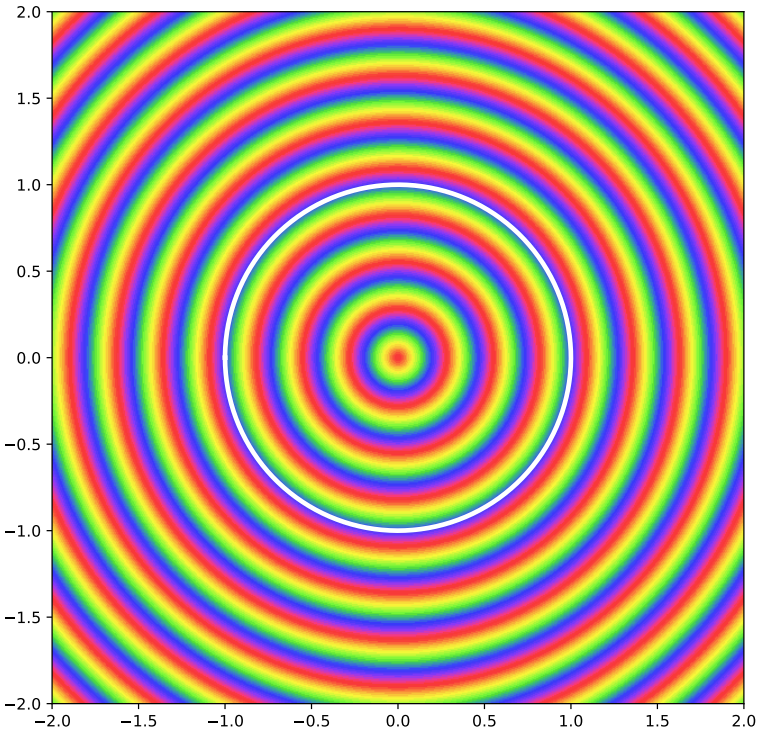
# Visualization of Complex Functions

Can't directly plot $H : \mathbb{C} \to \mathbb{C}$ (there are 4 dimensions!)

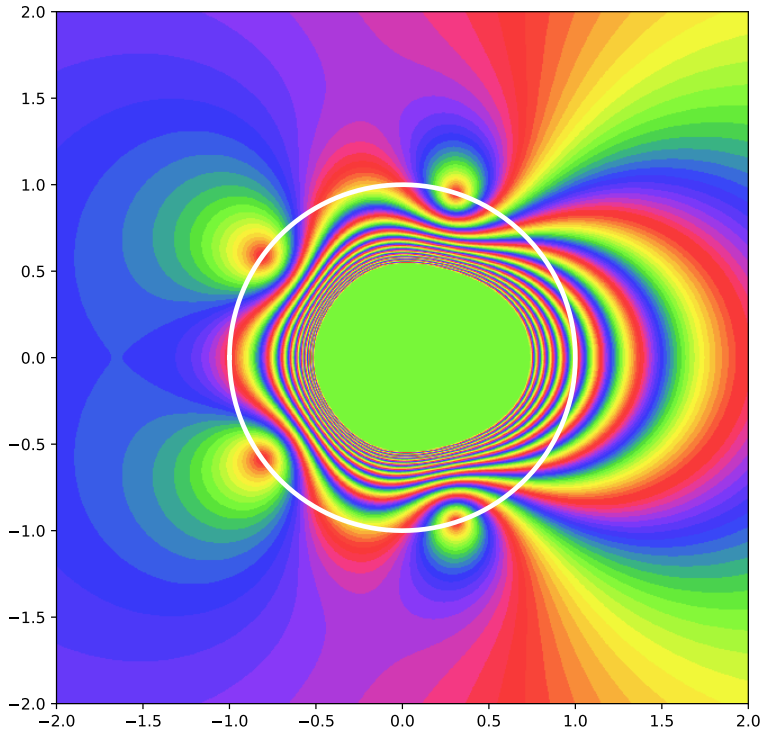
Let $f(z) = (R(z), G(z), B(z))$ be a map that assigns a color (red/green/blue) to every point $z \in \mathbb{C}$.

Then we can display $f(H(z))$ to visualize the complex function $H(z)$.

# Another Visualization of Magnitude

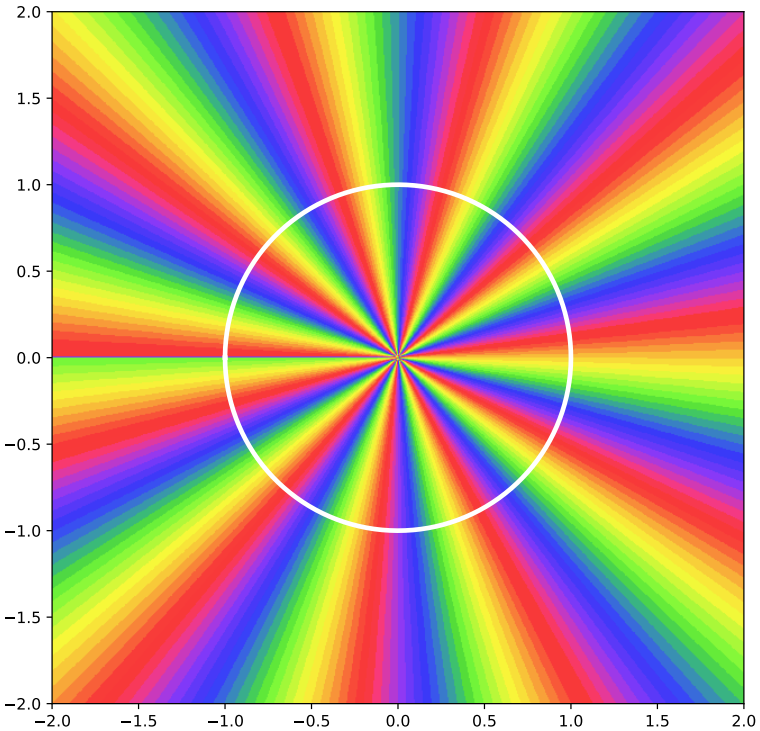



$|z|$

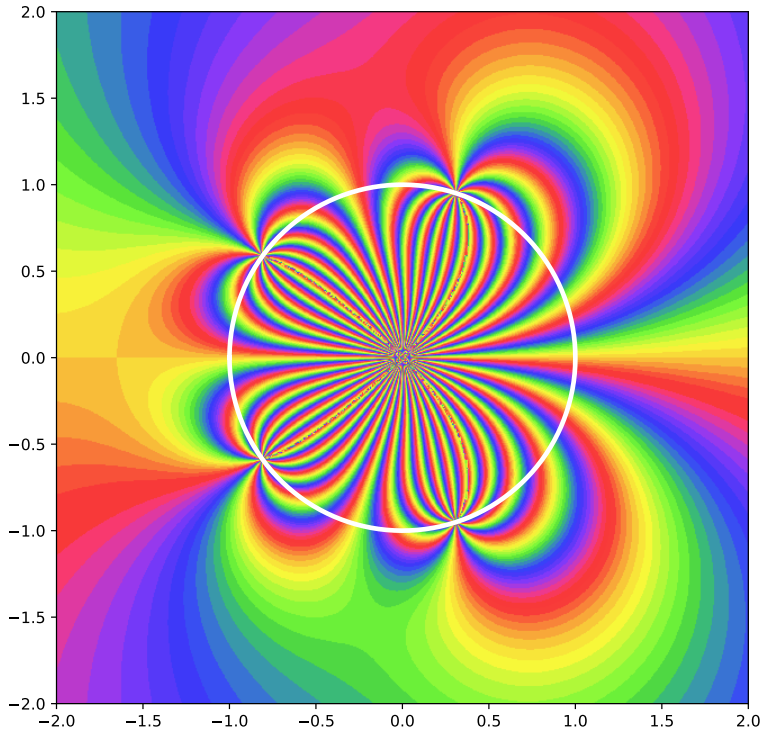



$|H(z)|$ for MA

# What About Phase?

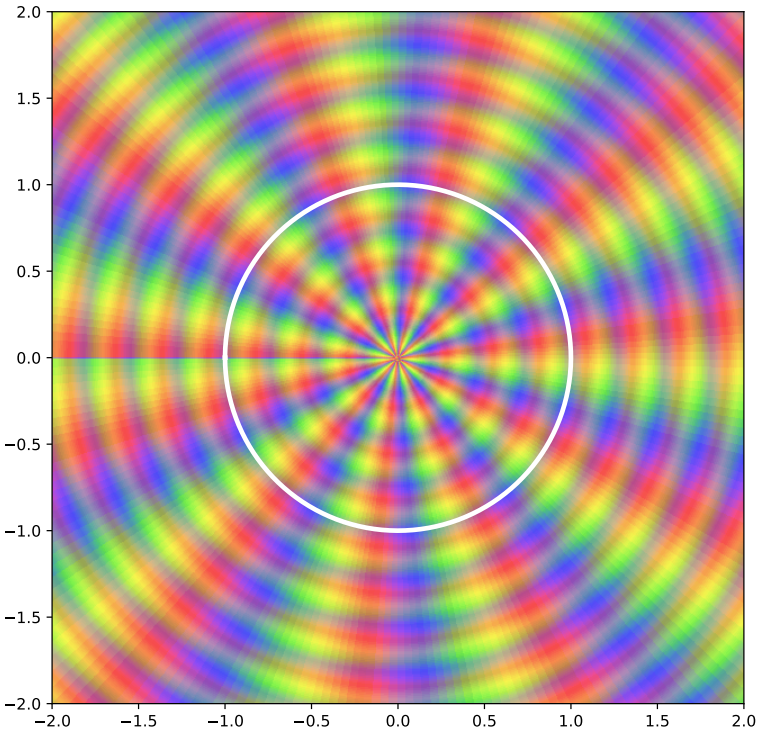



$\mathrm{Arg}(z)$

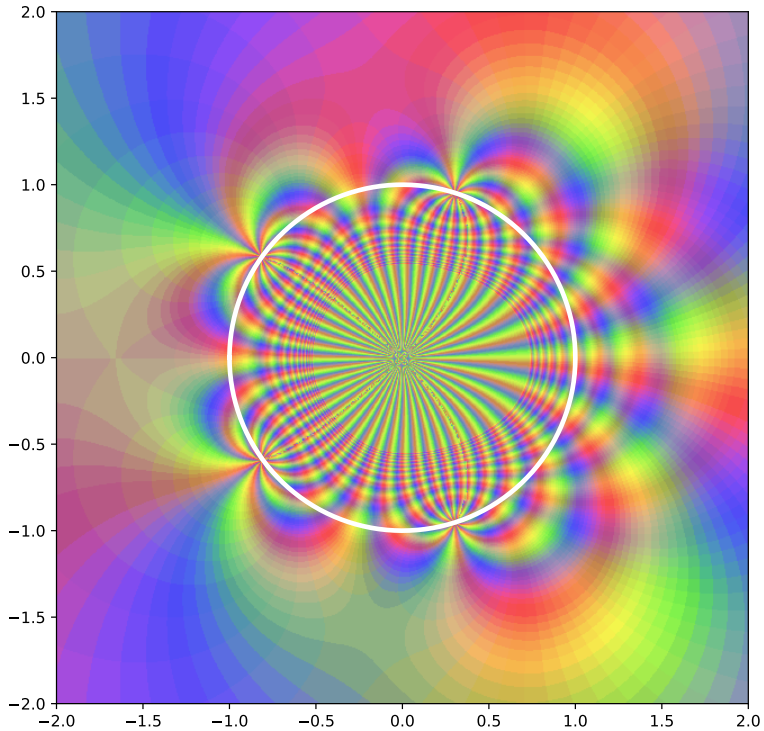



$\mathrm{Arg}(H(z))$ for MA

# Visualizing Magnitude and Phase



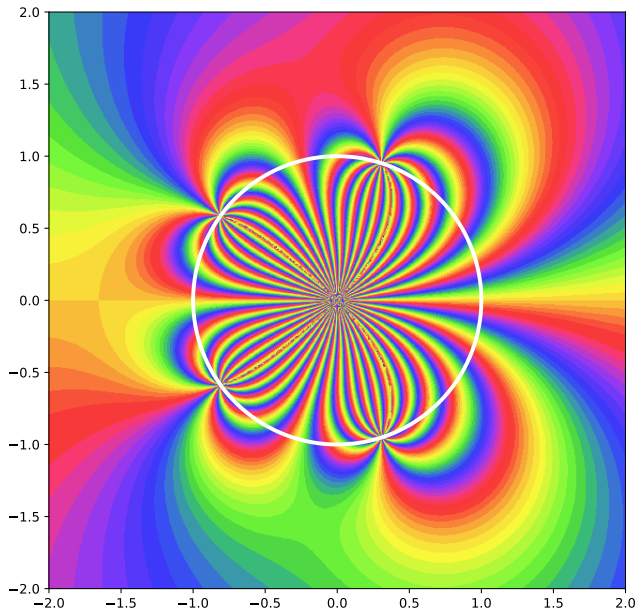

$z$

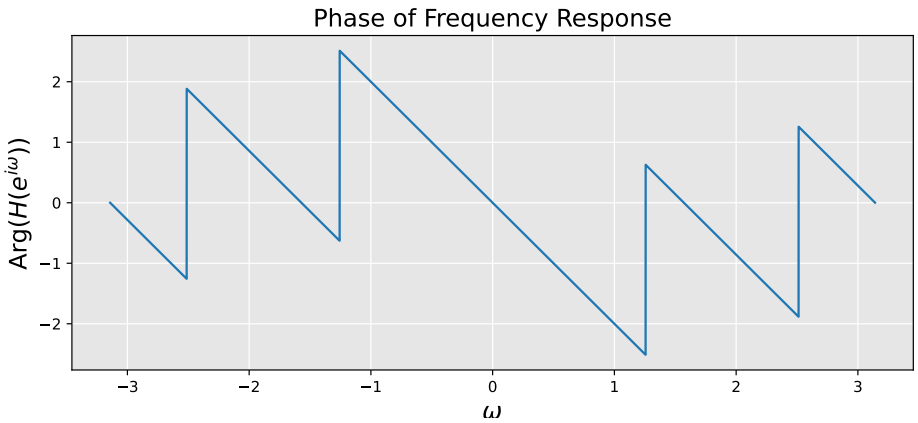



$H(z)$ for MA

# Phase Response of MA

Look at phase restricted to unit circle: $\mathrm{Arg}(H(e^{i\omega}))$

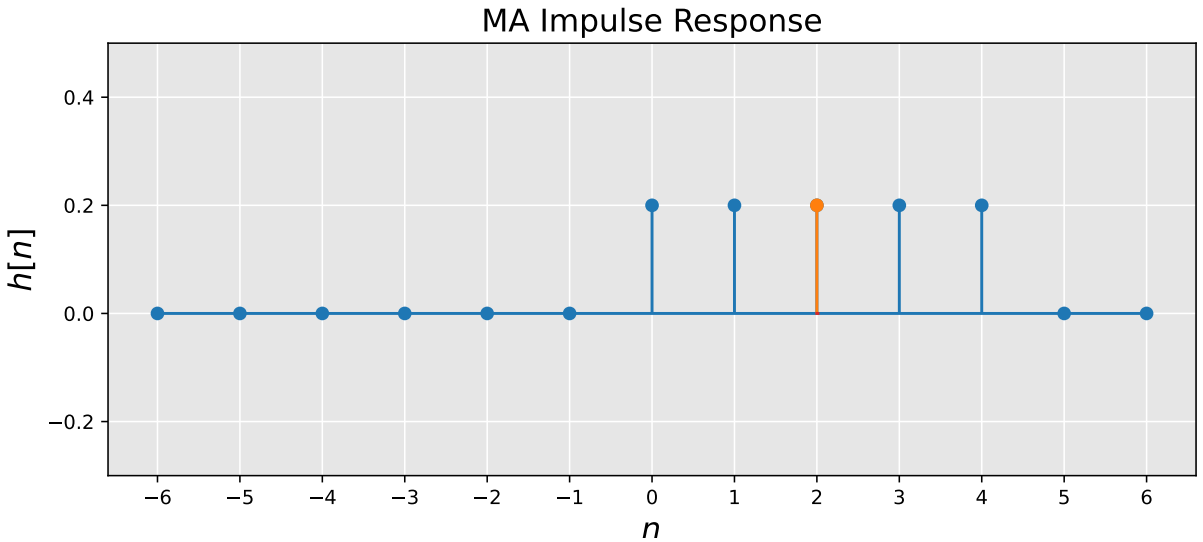



Note discontinuities (jumps by $\pi$) when we hit a zero.
Otherwise it is linear with slope = -2.

# Phase Response of MA

What is the significance of linear phase with slope = -2?




It tells us the average delay!

# Phase Response of Ideal Delay

Ideal Delay:

$$y[n] = x[n-k]$$

Its transfer function is:

$$H(z) = z^{-k}$$

Restricting to unit circle gives the frequency response:

$$H(e^{i\omega}) = e^{-i\omega k}$$

It has phase:

$$\mathrm{Arg}(H(e^{i\omega})) = -\omega k,$$

which is linear in $\omega$, with slope = $-k$.

# Phase Delay

This leads us to the following definition:

**Definition**

The **phase delay** of an LTI system at frequency $\omega$ is

$$\tau_{\text{ph}}(\omega) = -\frac{\text{Arg}(H(e^{i\omega}))}{\omega}.$$

# Another Way to Get Slope

If $\mathrm{Arg}(H(e^{i\omega})) = -\omega k$, I can also recover the slope, $-k$, as the **derivative** w.r.t. $\omega$. This leads to what is called the group delay.

## Definition

The **group delay** of an LTI system at frequency $\omega$ is

$$\tau_{\mathrm{gr}}(\omega) = -\frac{d}{d\omega}\mathrm{Arg}(H(e^{i\omega}))$$

# Why Two Types of Delays?

For an ideal delay, $y[n] = x[n-k]$, we have seen both phase and group delay are the same:

$$\tau_{\text{ph}}(\omega) = \tau_{\text{gr}}(\omega) = k.$$

More generally, these are the same whenever we have a **linear phase response** system:

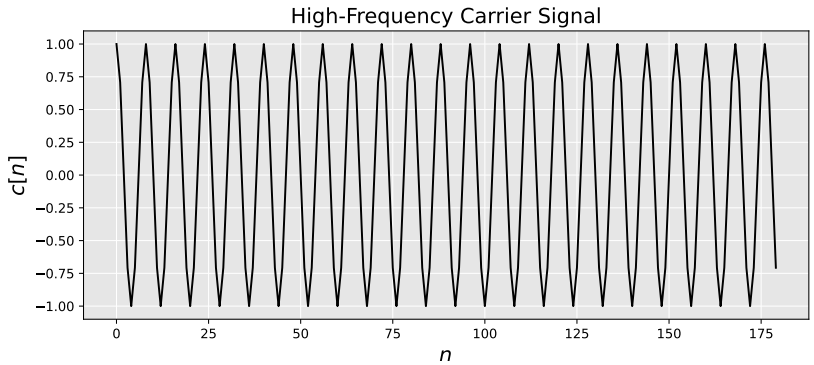
$$\text{Arg}(H(e^{i\omega})) = -\omega k.$$

But if the phase is not linear in $\omega$, these won't be the same:

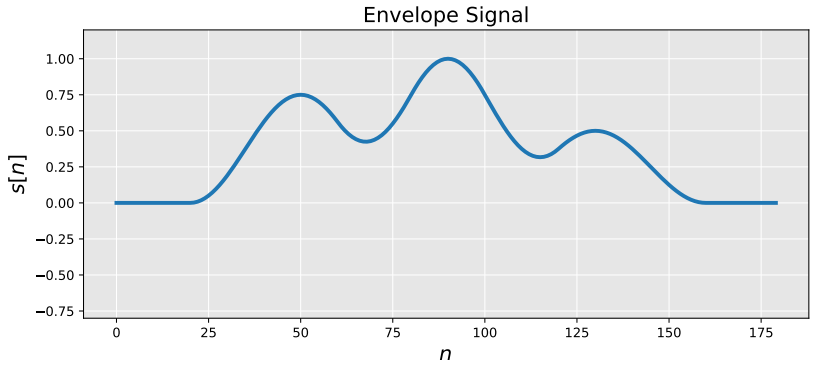
$$\tau_{\text{ph}}(\omega) = -\frac{\text{Arg}(H(e^{i\omega}))}{\omega} \neq -\frac{d}{d\omega}\text{Arg}(H(e^{i\omega})) = \tau_{\text{gr}}(\omega).$$

# Sinusoid Signal with Envelope

Consider a high-frequency sinusoid: $c[n] = \cos(\omega_0 n)$, which we'll call the **high-frequency carrier**:



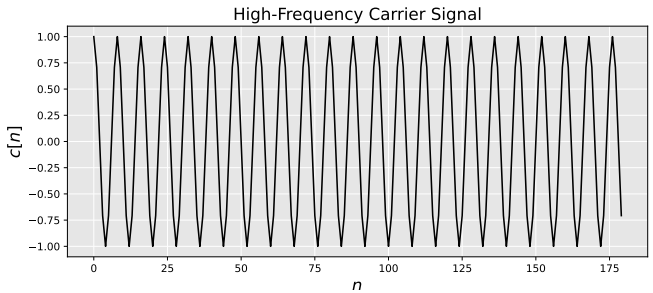

And some lower-frequency **envelope**, $s[n]$:

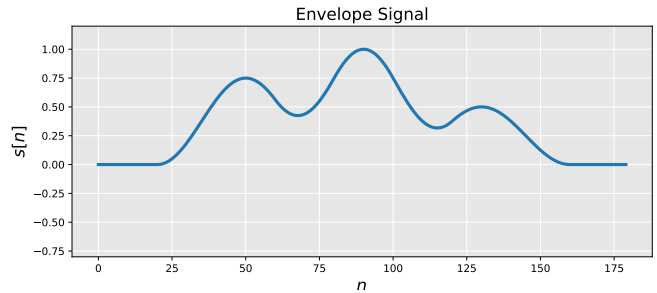


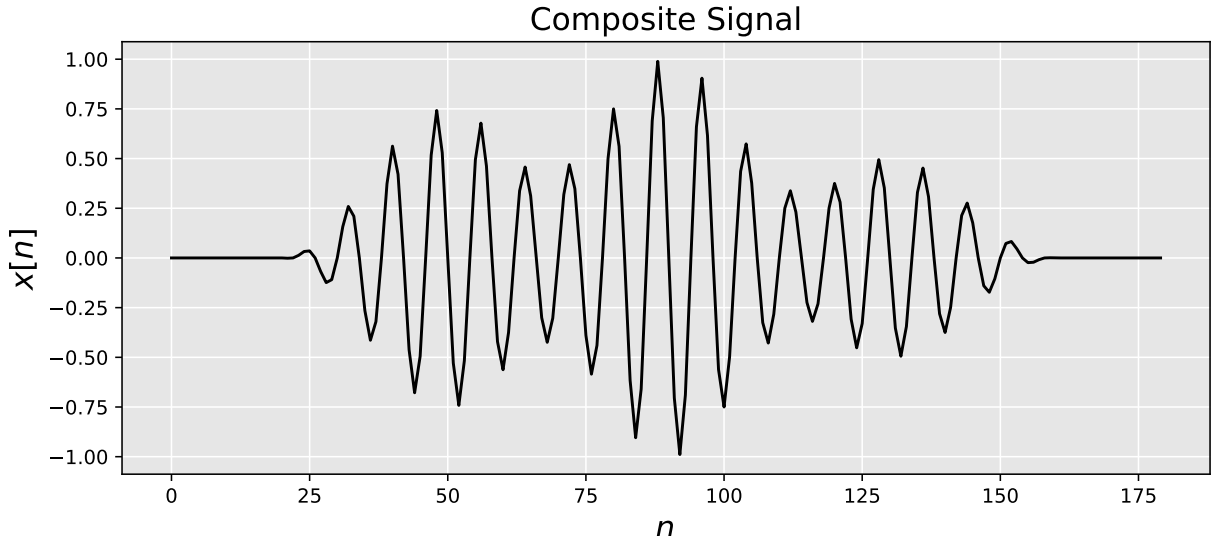
# Sinusoid Signal with Envelope

Now multiply them together to get the signal:

$$x[n] = s[n]\cos(\omega_0 n).$$



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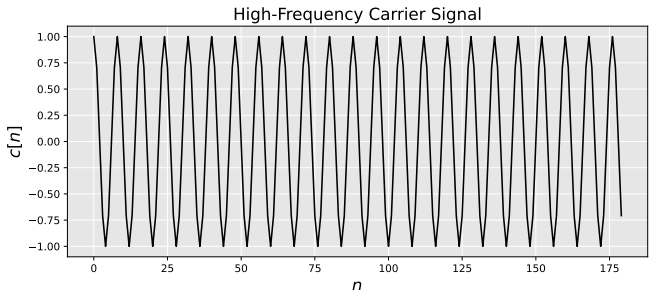
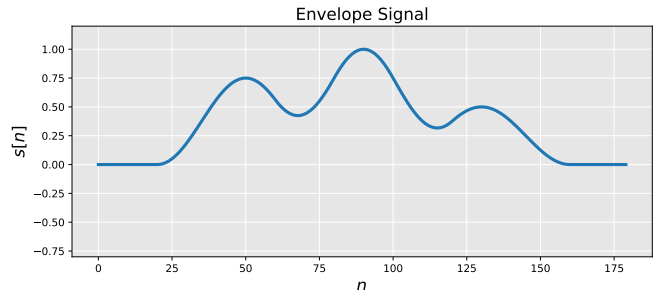
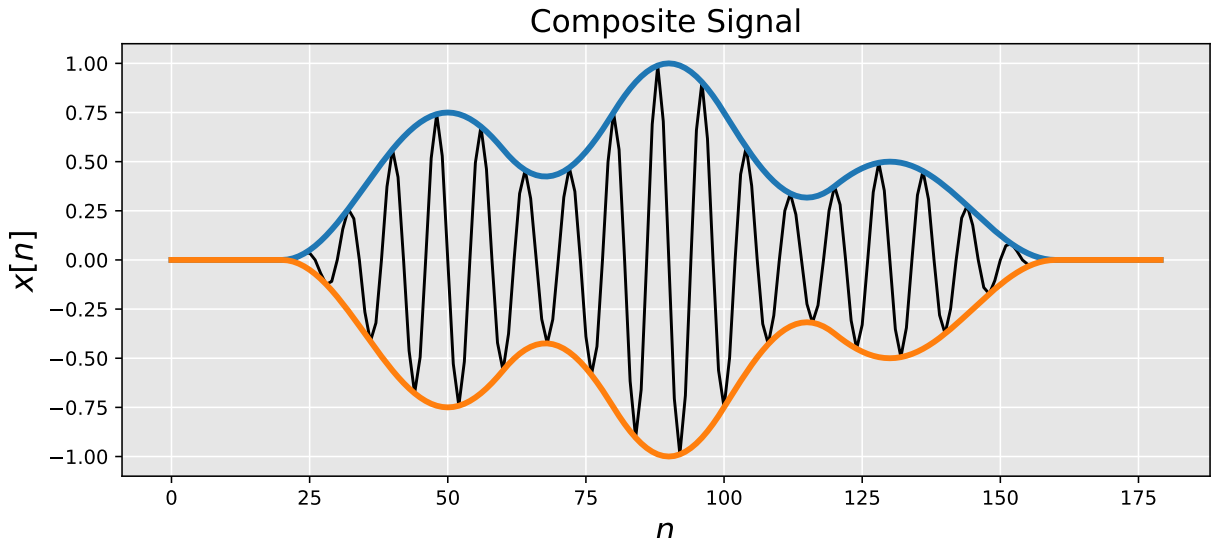


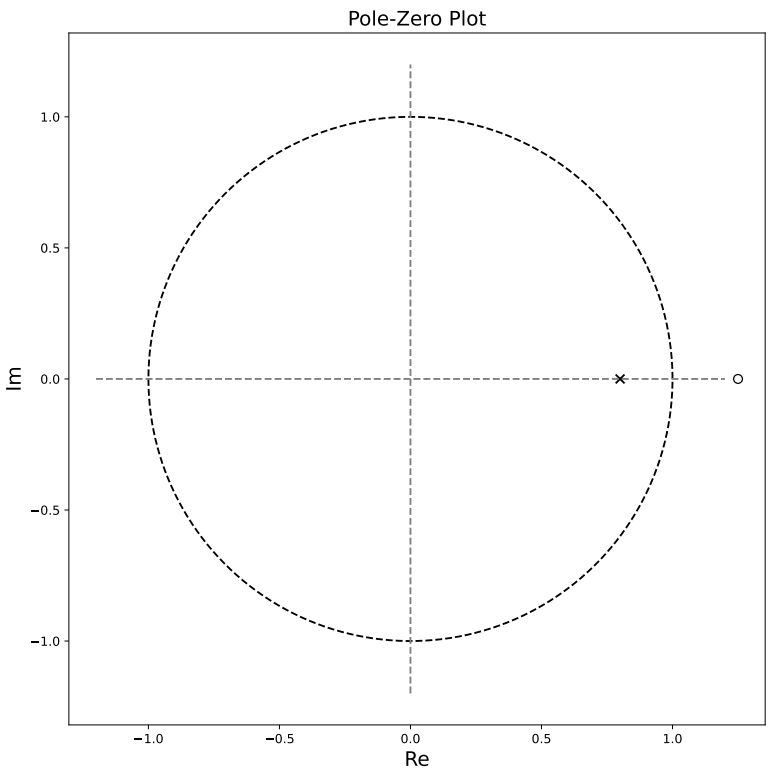



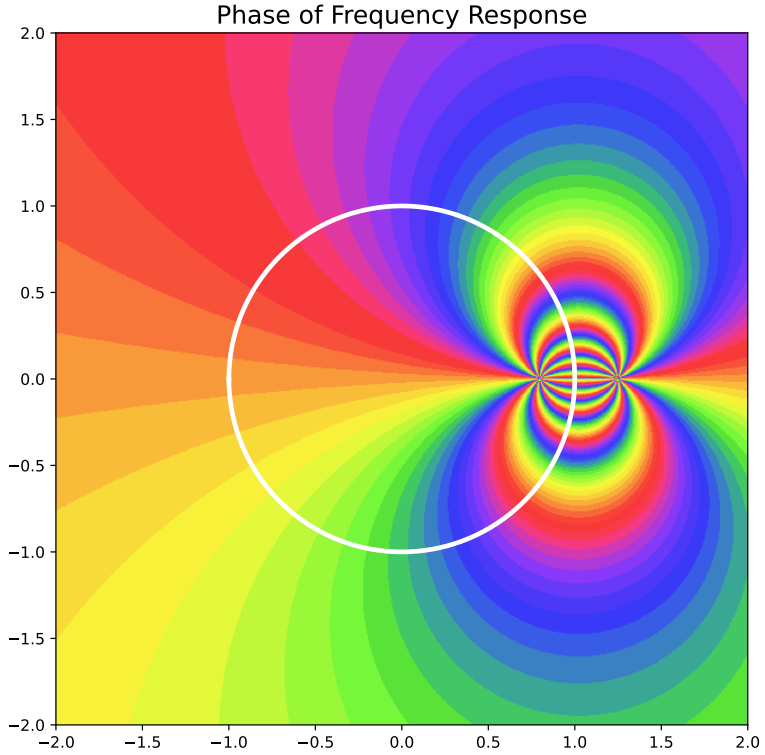
# Phase and Group Delay

Now when applying a general LTI system, $H(z)$, the phase delay will act on the high-frequency carrier, while the group delay will act on the envelope:

$$y[n] \approx |H(e^{i\omega_0}))|s[n - \tau_{\mathrm{gr}}(\omega_0)] \cos(\omega_0 n + \tau_{\mathrm{ph}}(\omega_0)).$$

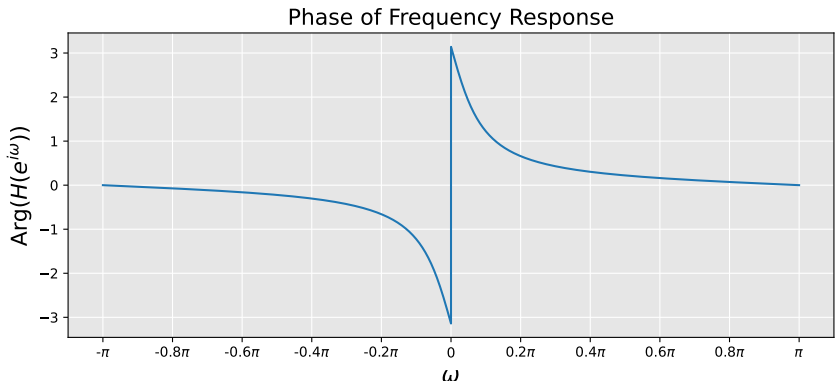
# Pole-Zero Inverse Pairs

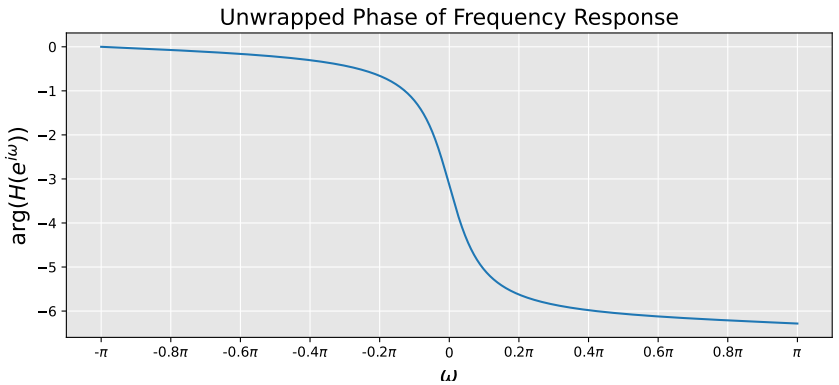


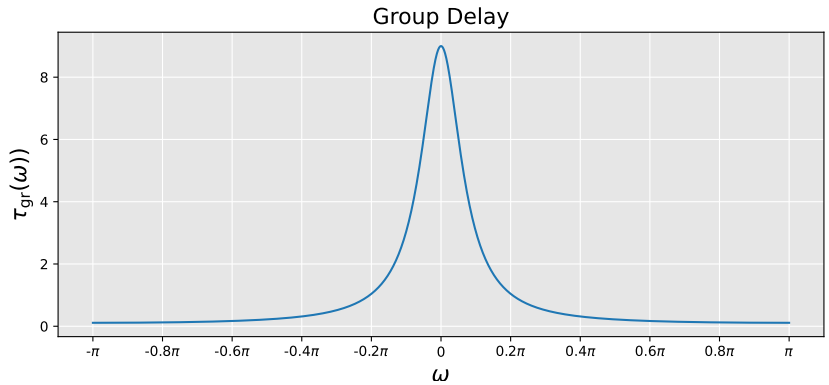



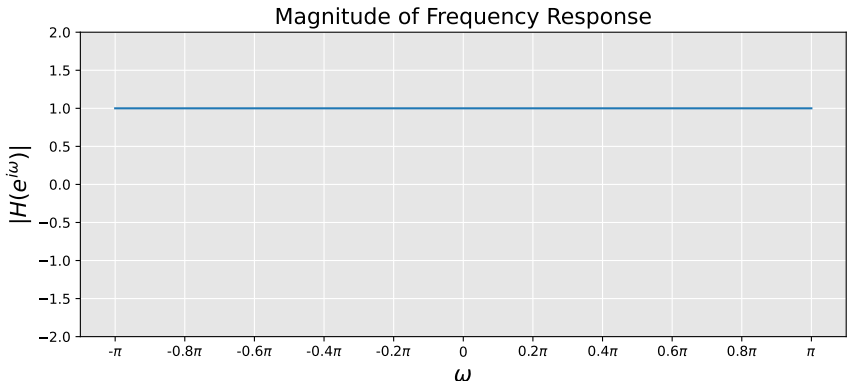

$$H(z) = \left(\frac{4}{5}\right) \frac{1 - \frac{5}{4}z^{-1}}{1 - \frac{4}{5}z^{-1}}$$

# Pole-Zero Inverse Pairs



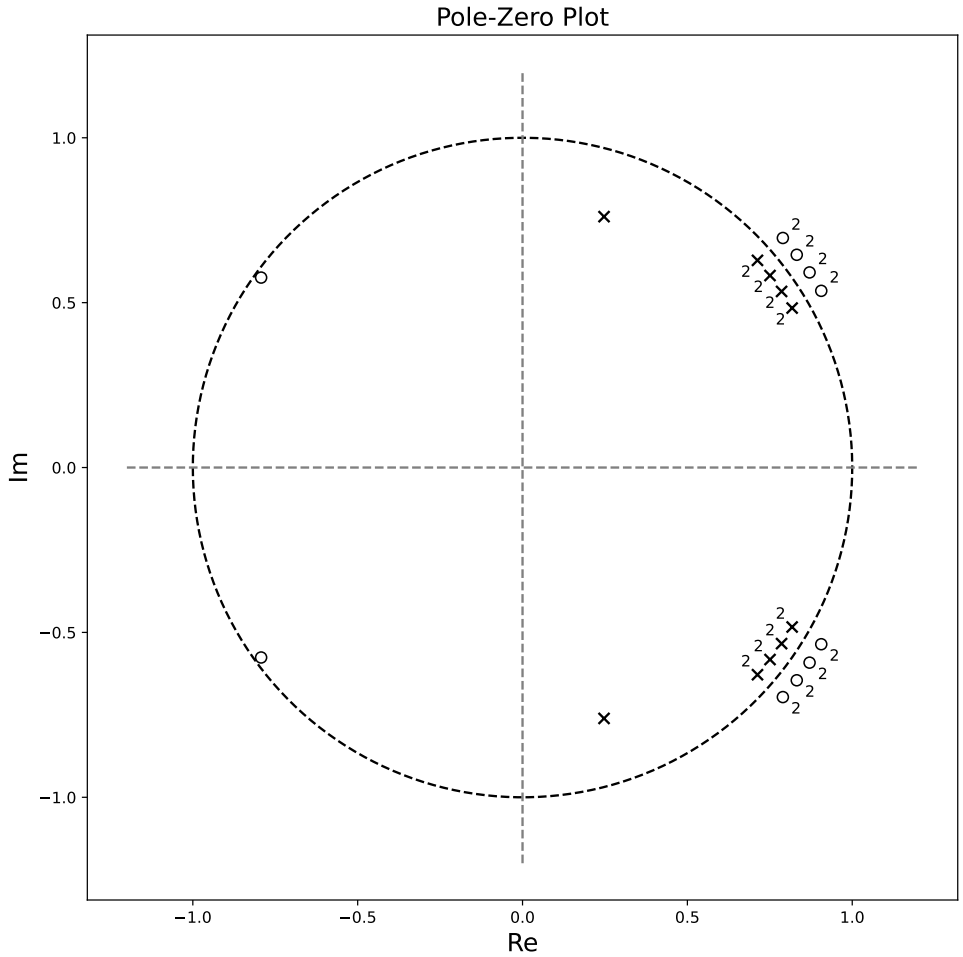




Pole-zero inverse pairs cause a narrowband group delay, while leaving the magnitude fixed.

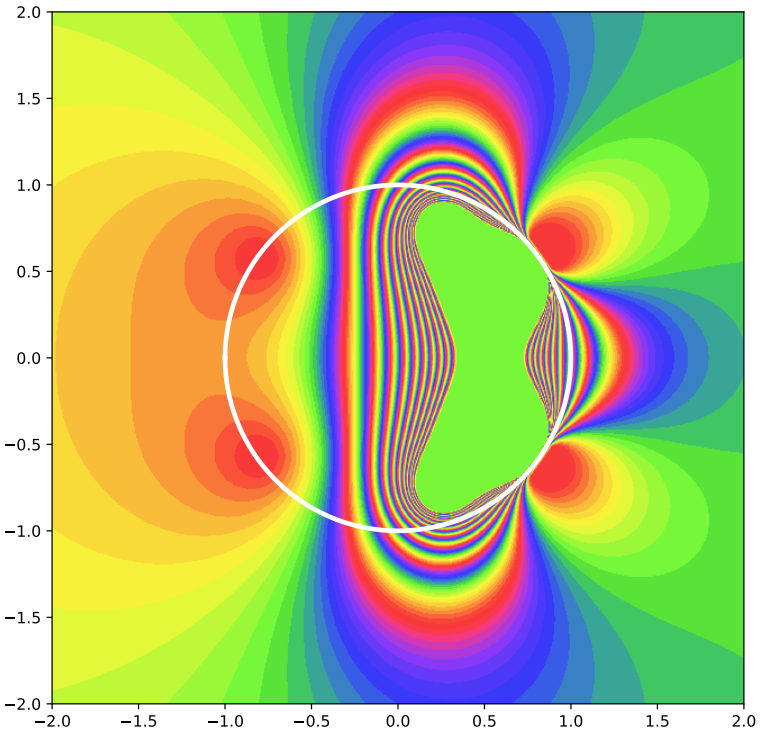
# Example[1]

Consider an LTI system with the following pole-zero plot:

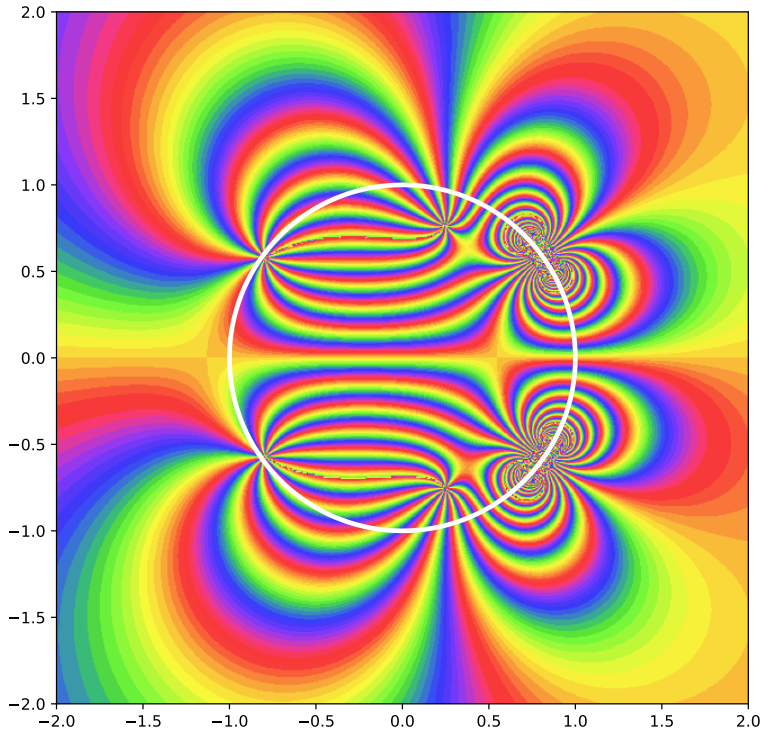



[1] From Oppenheim & Schafer, pg. 278

# Example: Transfer Function

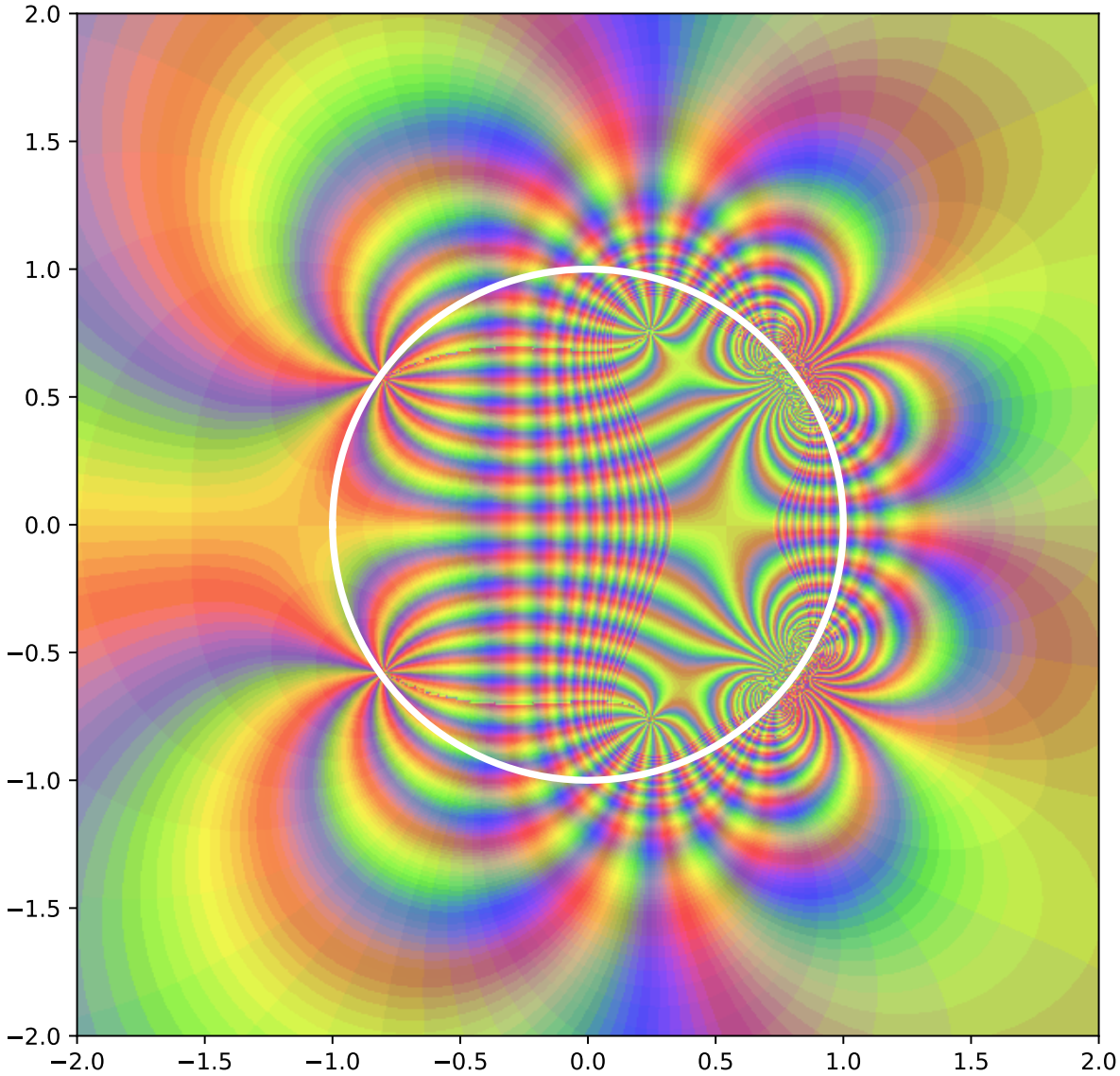



$|H(z)|$



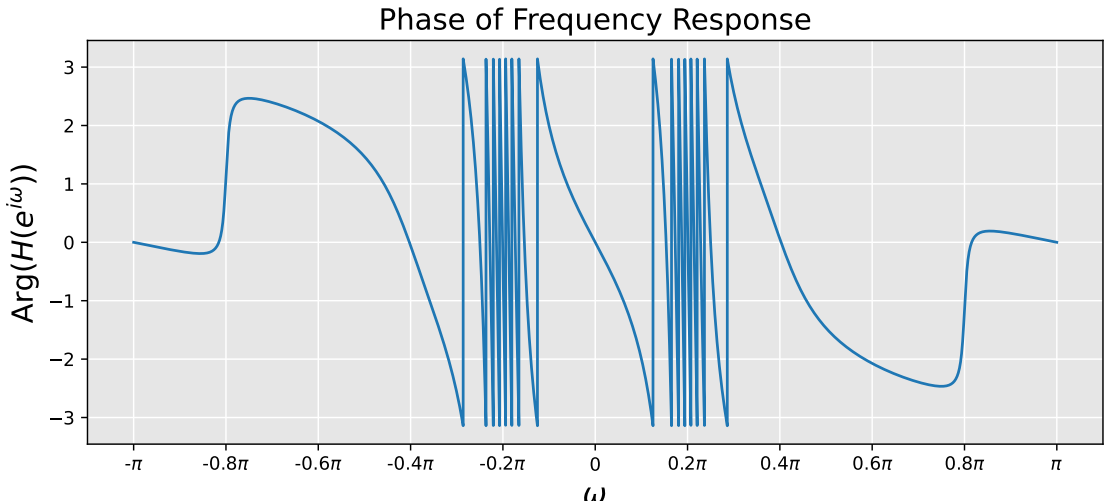

$\mathrm{Arg}(H(z))$

# Example: Transfer Function

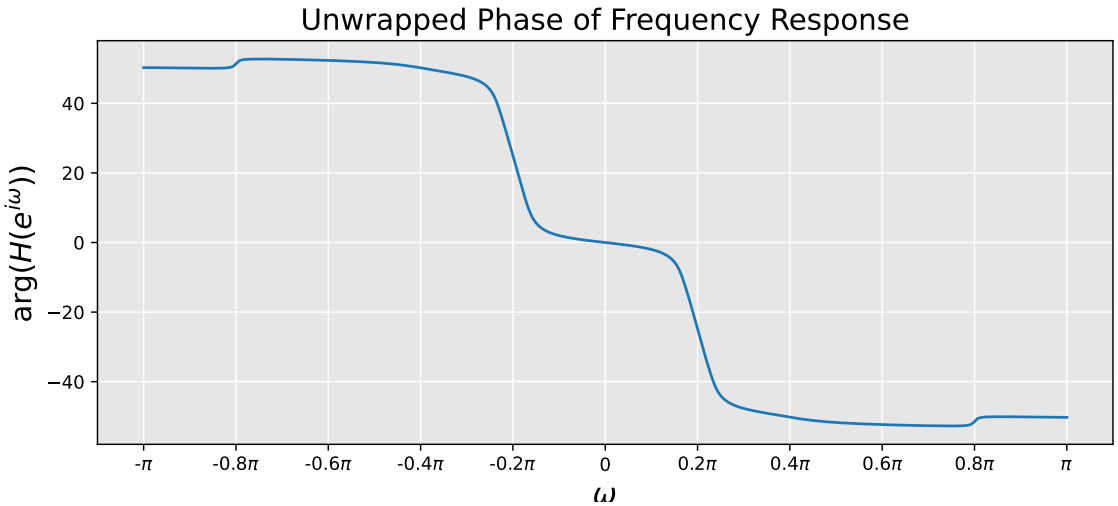


# Example: Phase of Frequency Response

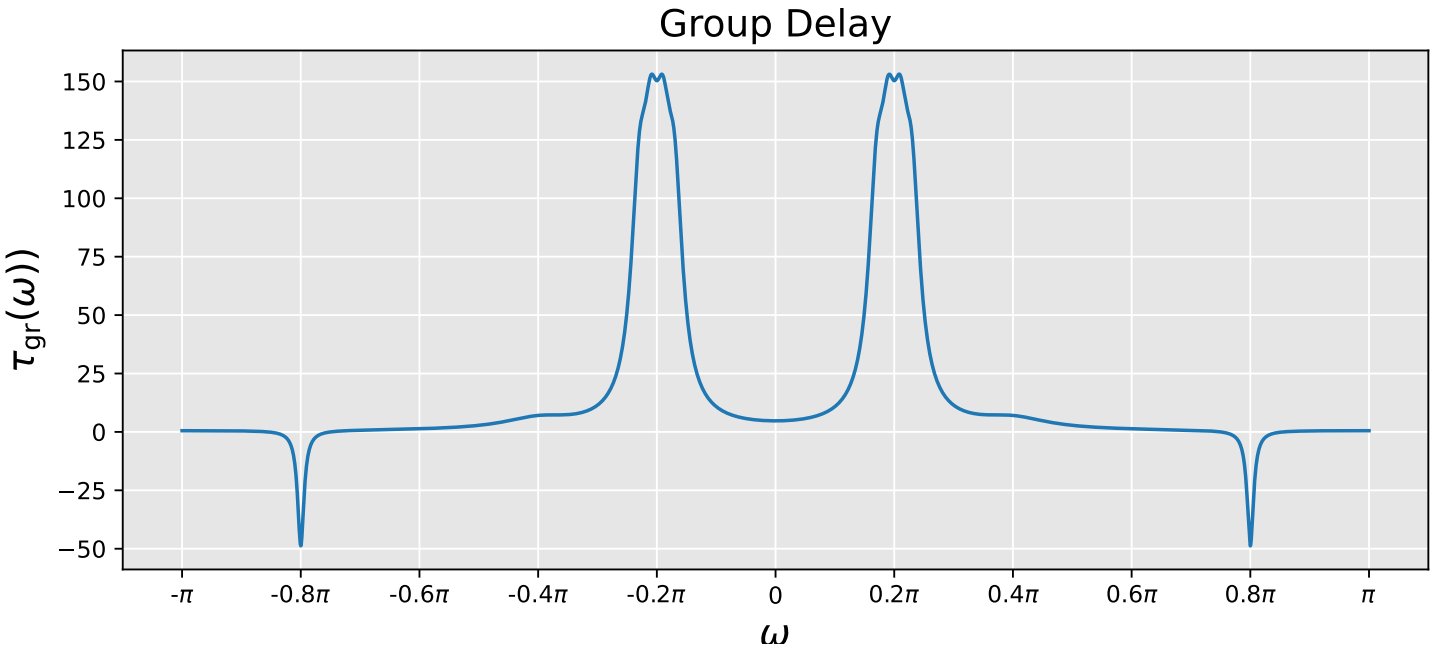
Wrapped:




Unwrapped:

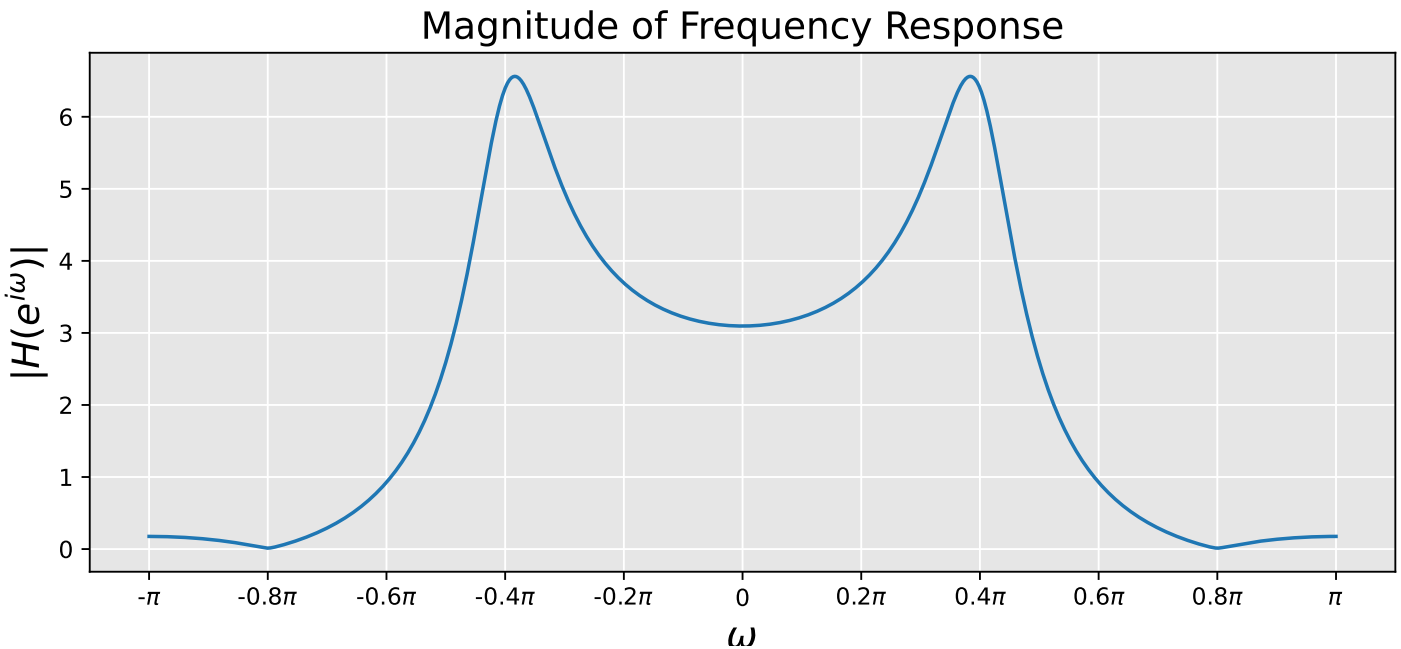


# Example: Group Delay




$$\tau_{\mathrm{gr}}(\omega) = -\frac{d}{d\omega}\mathrm{Arg}(H(e^{i\omega}))$$

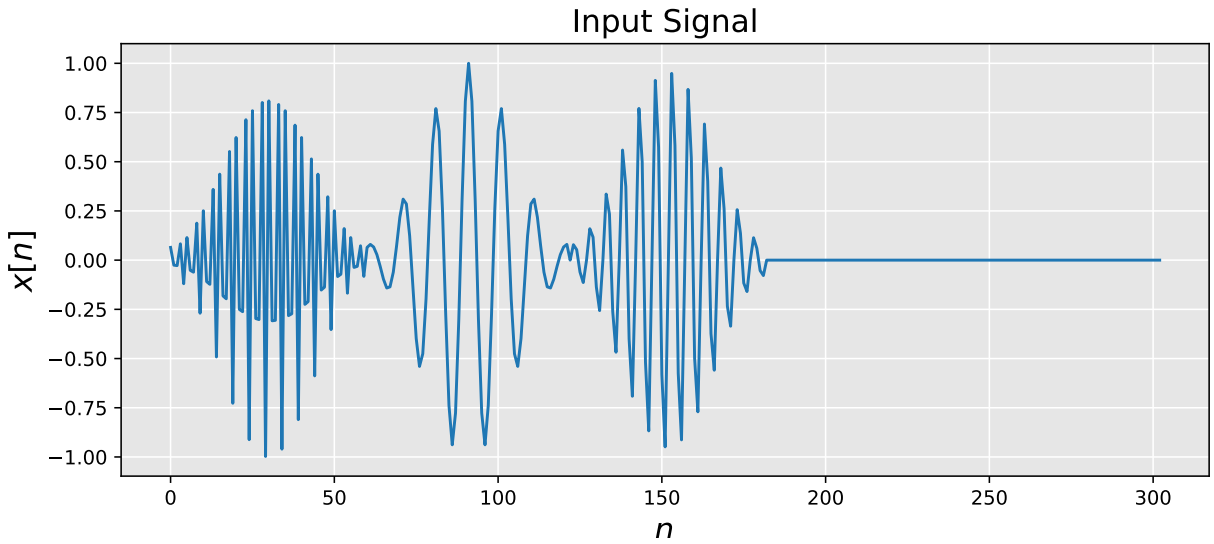
# Example: Magnitude of Frequency Response

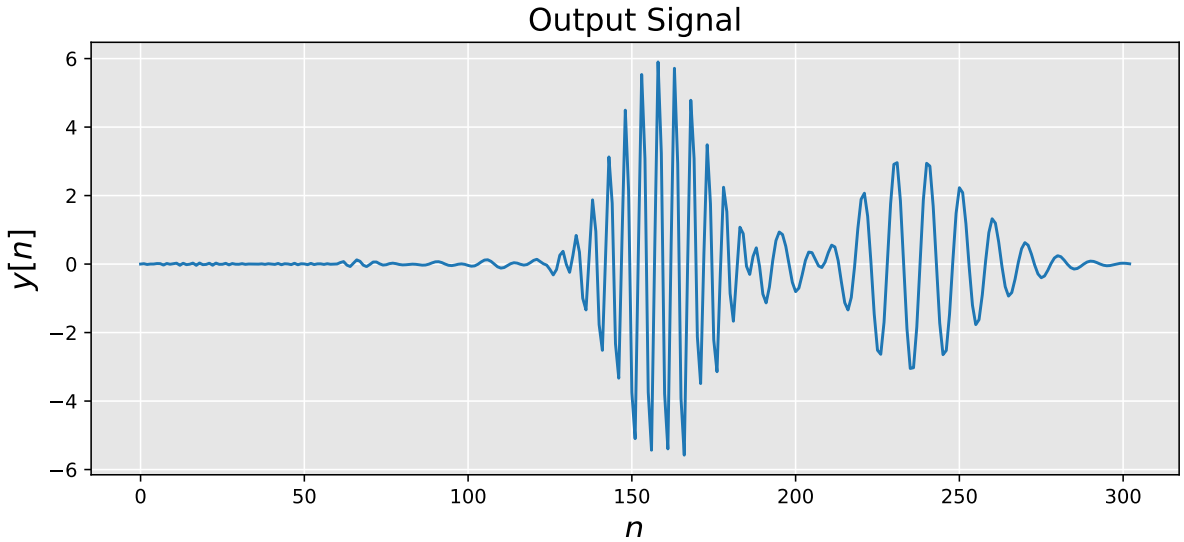


# Example: Input and Output

Input signal has three frequencies: $0.8\pi, 0.2\pi, 0.4\pi$

Input:




Output: