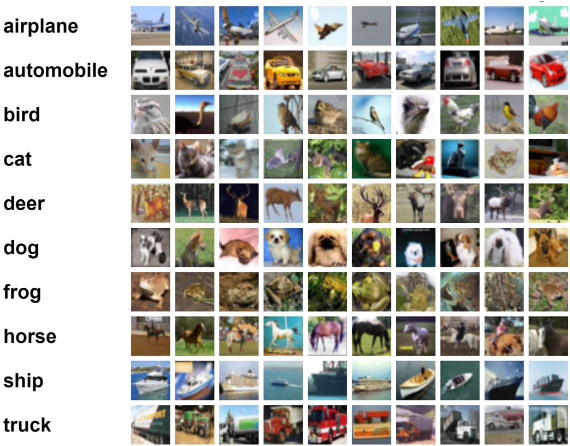
Introduction

Geometry of Data

August 22, 2023

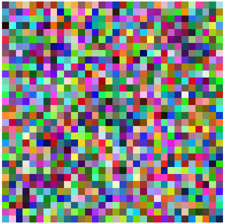
CIFAR-10



$32 \times 32 \times 3 = 3{,}072$ dimensions

10 classes

Uniform Random Images



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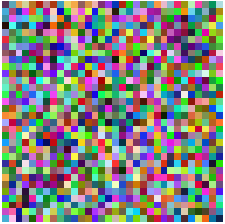

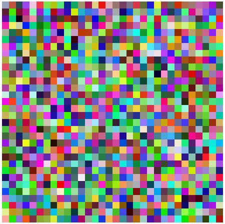

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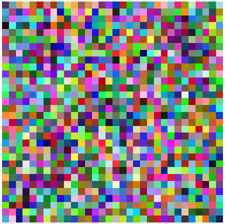

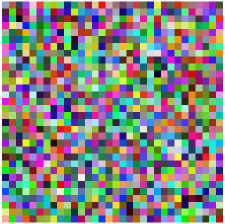

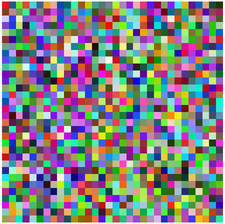

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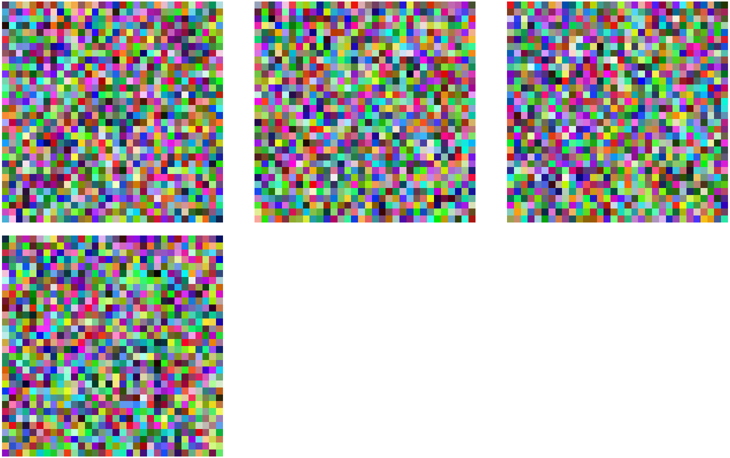

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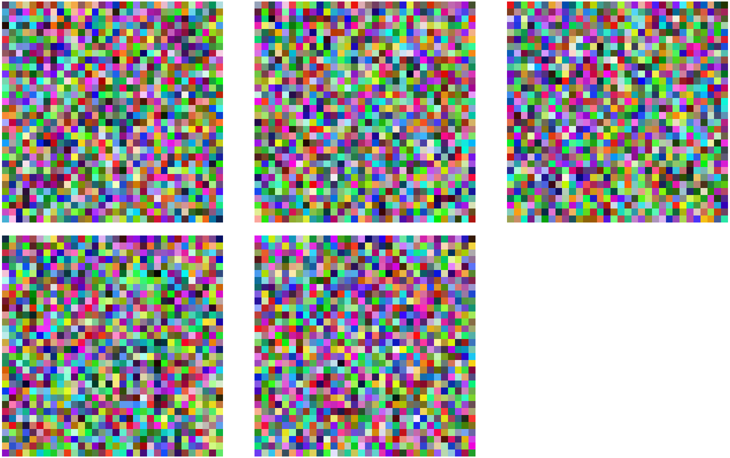

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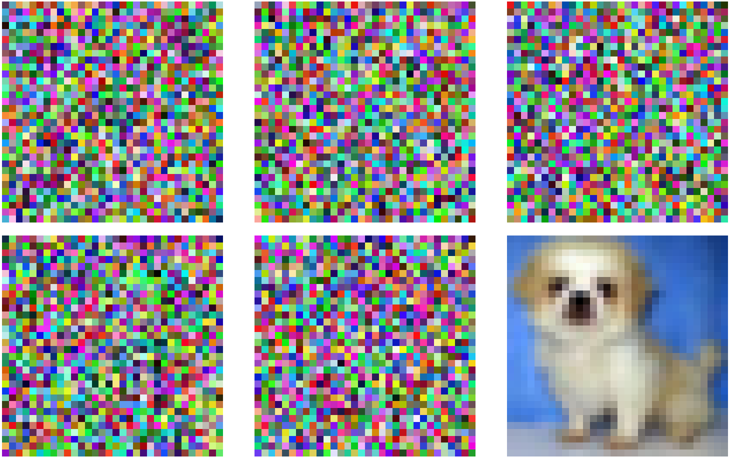

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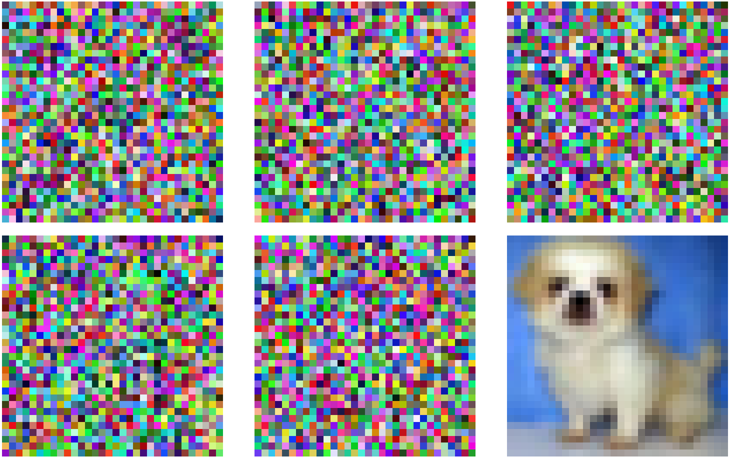
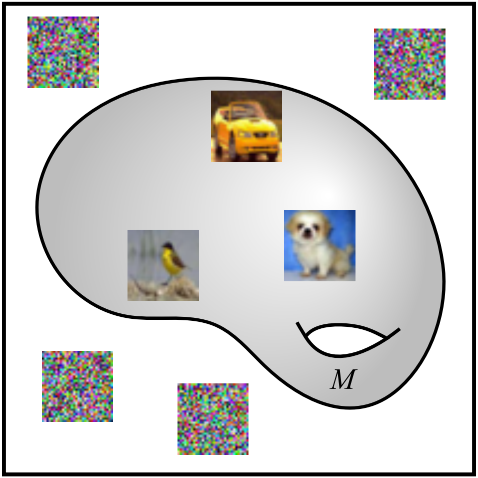


just kidding!

Manifold Hypothesis

Real data lie near lower-dimensional manifolds



Manifold Learning

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- Learn a model/representation for the data manifold

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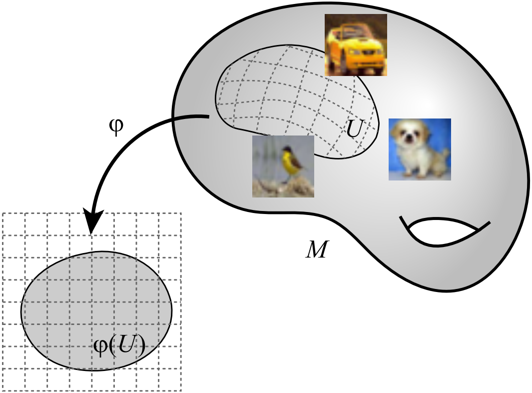

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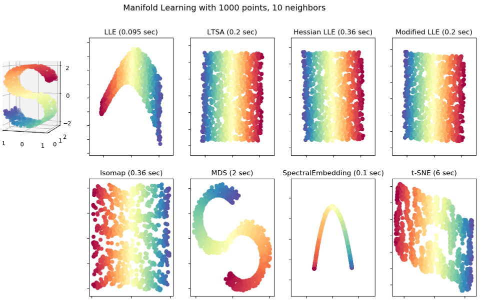
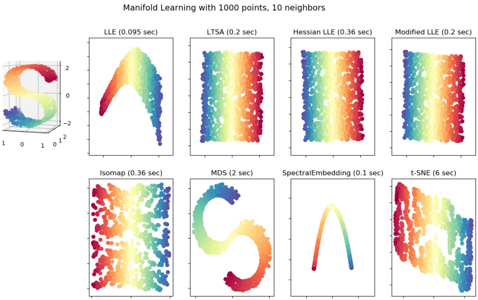
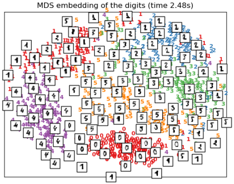


From scikit-learn.org

Manifold Learning





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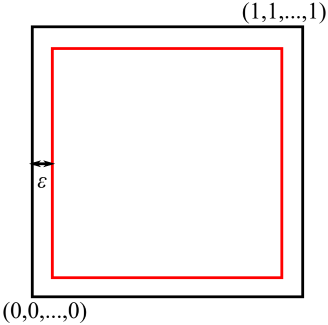
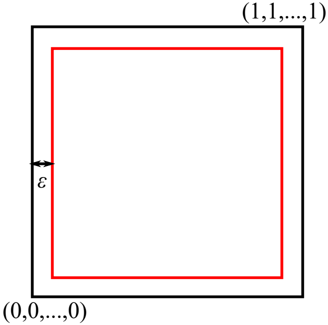
Volumes in High Dimensions



What is the volume of the unit d-cube shrunk by some small amount in each dimension?

Volumes in High Dimensions



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$$V = (1 - 2\epsilon)^d$$

Approaches 0 as $d \to \infty$

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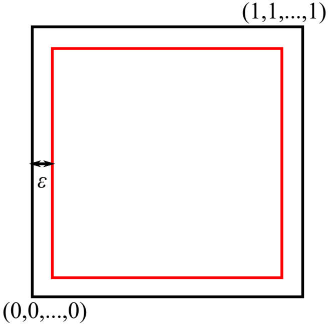

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Example: $256 \times 256 \times 3$ images, $\epsilon = \frac{1}{256}$

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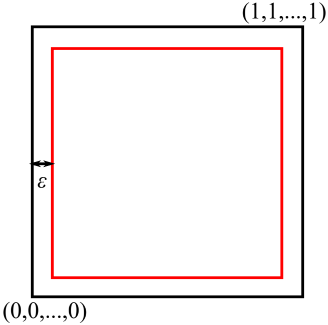

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Approaches 0 as $d \to \infty$

Example: $256 \times 256 \times 3$ images, $\epsilon = \frac{1}{256}$

$$V \approx 2.0 \times 10^{-670}$$

Distances in High Dimensions

Sample two points uniformly from the unit d-cube:
$X, Y \sim \mathrm{Unif}([0,1]^d)$

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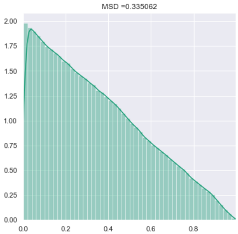
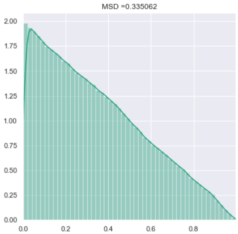
Sample two points uniformly from the unit d-cube:
$X, Y \sim \mathrm{Unif}([0, 1]^d)$

What is the distribution of the distance between them?
$D = \|X - Y\|$



$d = 1$



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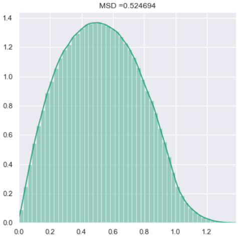
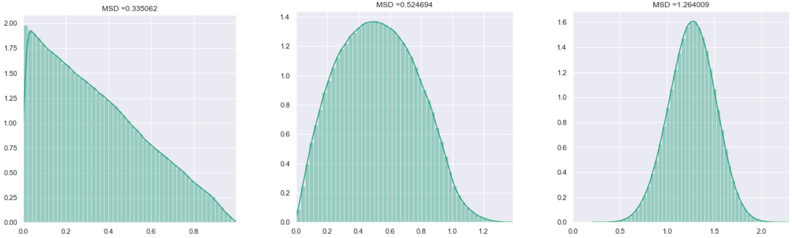
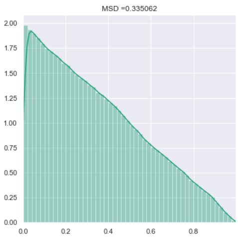


$d = 2$



$d = 1$ $\quad$ $d = 2$ $\quad$ $d = 10$



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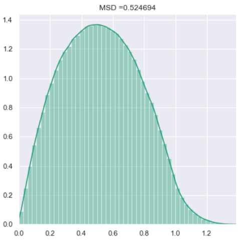
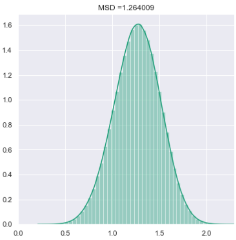
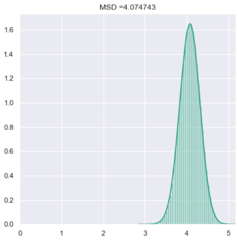


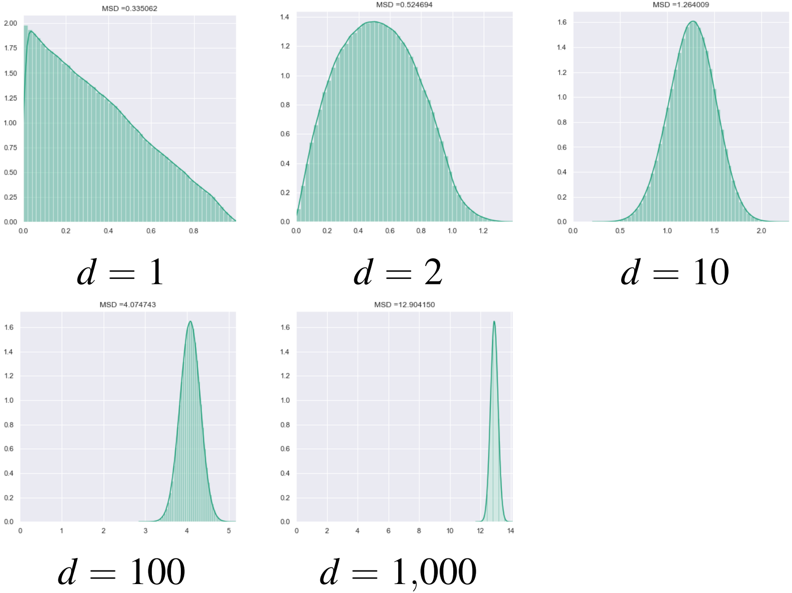
$d = 2$

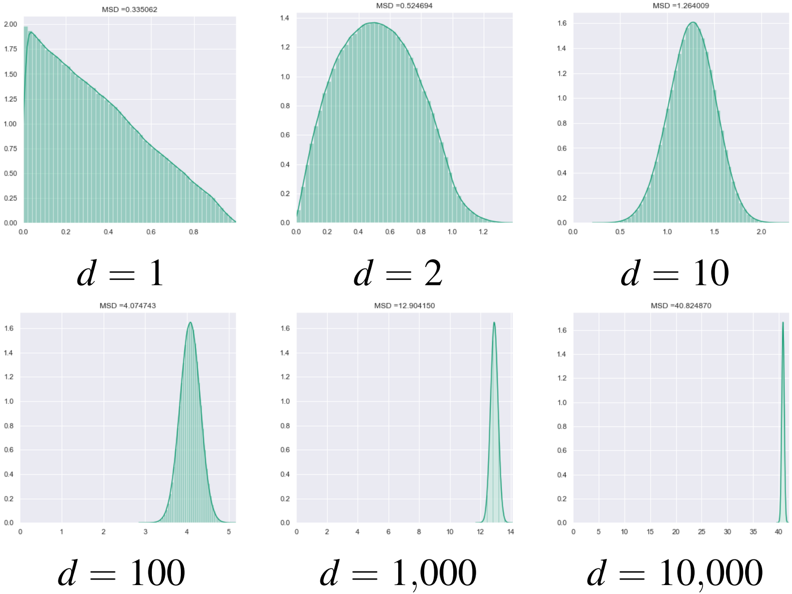


$d = 10$



$d = 100$


MSD =0.335062
$d = 1$
MSD =0.524694
$d = 2$
MSD =1.264009
$d = 10$
MSD =4.074743
$d = 100$
MSD =12.904150
$d = 1,000$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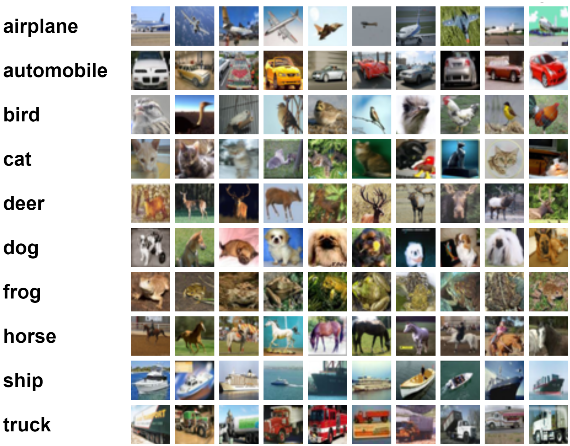
CIFAR-10



$32 \times 32 \times 3 = 3{,}072$ dimensions

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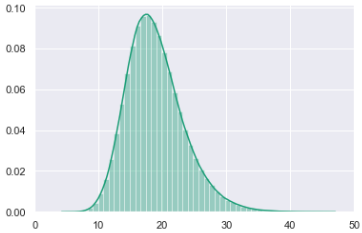
Distances in Real Data



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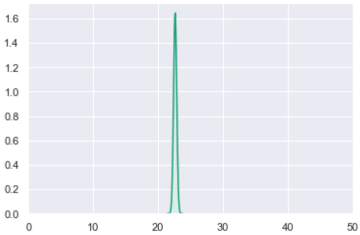

$\mathrm{Unif}([0,1]^{3072})$

Manifold-valued Data

- Manifold already known, not learned

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- Manifold arises from natural non-linear constraints on data

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- Manifold already known, not learned
- Manifold arises from natural non-linear constraints on data
- Linear data analyses (in fact, vector space operations) violate these constraints

Directional Data

Data living on a circle (S^1) or sphere (S^2), etc.

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- Orientation of molecules in protein structure

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- Direction of robot or autonomous vehicle

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- Orientation of molecules in protein structure
- Direction of robot or autonomous vehicle
- Position on the earth

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- Motion capture: orientation of joints

Directional Data

Data living on a circle (S^1) or sphere (S^2), etc.

- Orientation of molecules in protein structure
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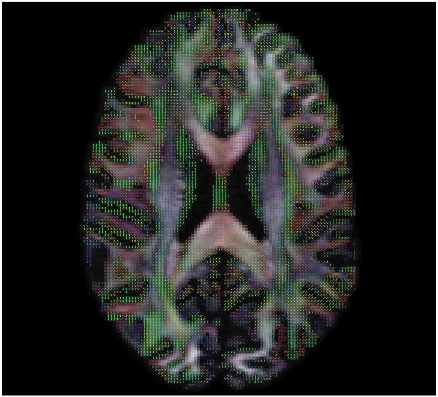
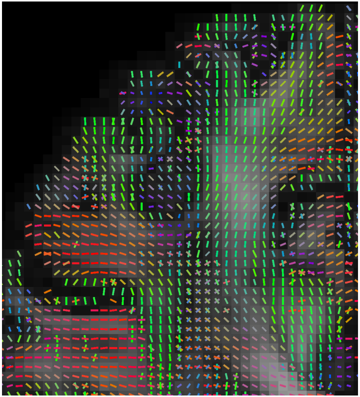
- Position on the earth
- Motion capture: orientation of joints
- Time (time of day, day of the year, etc.)

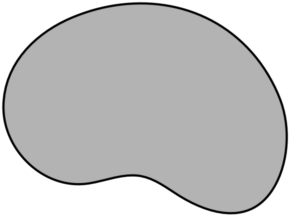
Directional Data: Diffusion MRI





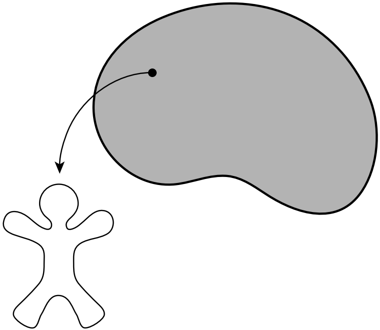
Voxel features are directions of axons in brain

Shape Manifolds



A shape is a point in a high-dimensional, nonlinear manifold, called a **shape space**.

Shape Manifolds



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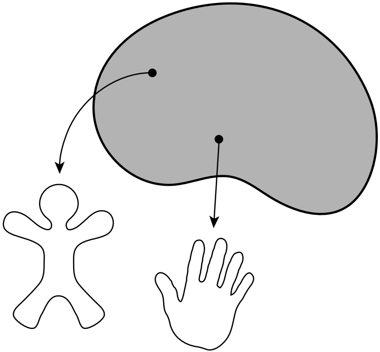

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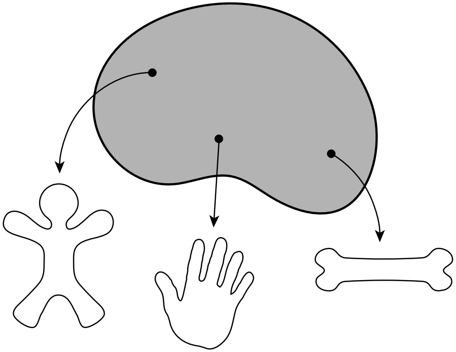

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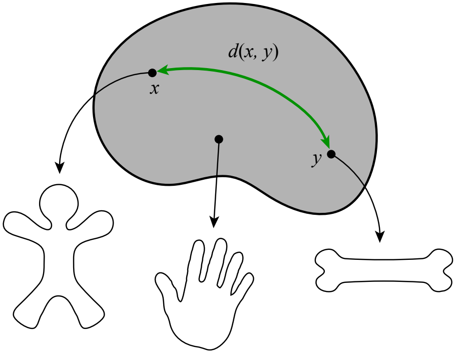
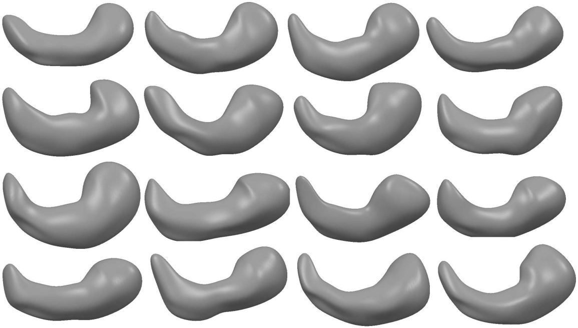


A metric space structure provides a comparison between two shapes.

Shape Statistics: Averages



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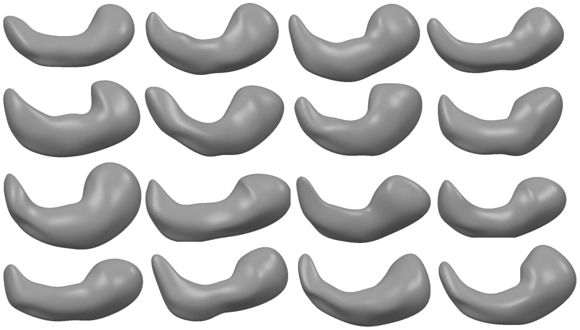
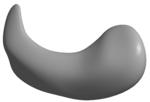
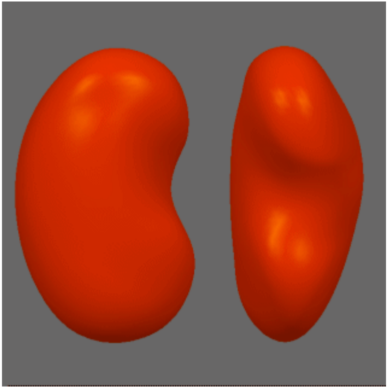
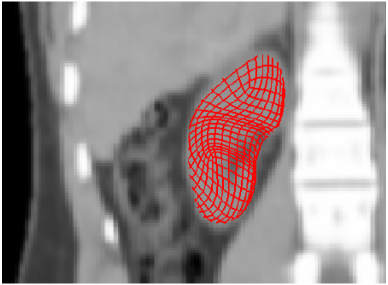


$\rightarrow$



Shape Statistics: Variability





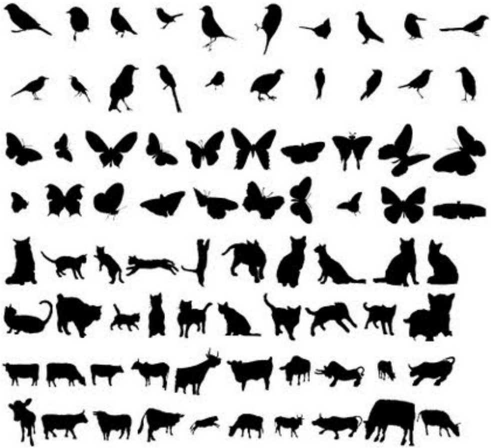
Shape priors in segmentation

Shape Application: Bird Identification

American Crow

Common Raven

Shape Statistics: Classification



http://sites.google.com/site/xiangbai/animaldataset

Information Geometry

Parameters of a probability model live on manifolds

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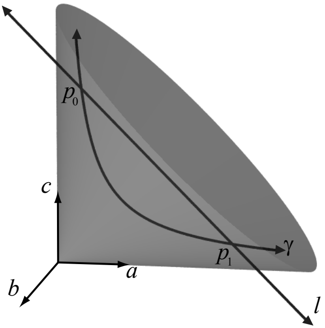
Parameters of a probability model live on manifolds

Example: covariance matrix of a 2D Gaussian distribution:



$\Sigma \in \mathrm{PD}(2)$ is of the form

$$\Sigma = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

$$ac - b^2 > 0, \quad a > 0.$$

(positive-definite constraint)