Black-Litterman Portfolio Optimization with Linear Algebra in Python

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This paper presents a comprehensive implementation of the Black-Litterman portfolio optimization model using Python and linear algebra techniques. We combine traditional Markowitz mean-variance optimization with sentiment analysis of financial news to generate market views. The implementation leverages historical price data and news sentiment scores to construct optimal portfolio weights. Our results demonstrate how incorporating news sentiment as a source of market views can enhance the classical Black-Litterman framework. We provide a detailed mathematical foundation of the model and discuss practical considerations for implementation. The complete code implementation and analysis are made available through a public repository.

I. Introduction

When deciding on how to allocate a portfolio in the stock market, two of the most important factors are risk and return. Typically risk or volatility is measured by the variance of a stock. The problem becomes that many high-return stocks or assets also have high variance. In 1959, Harry Markowitz introduced a model that would allow for the optimization of a portfolio by balancing risk and return. This model is known as the Markowitz model [3] and it has become the basis for modern portfolio theory.

In our following paper, we will be implementing the Black-Litterman model [2] which is a modification of the Markowitz model using 3 risky assets.

II. Markowitz Model

We begin by optiaining the daily closing prices of GOOGL, AAPL, and AMZN from 2020-2024 and calculate the log returns. We use log returns because they are additive and easier to work with.

$$\text{Log Returns} = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

We define a portfolio as X such that:

$$x_A + x_B + x_C = 1 \tag{2}$$

Meaning that the sum of the percentage of the portfolio in each asset must equal 100%. The return of the portfolio is the following:

$$\mu_{p,x} = E[R_{p,x}] = x_A\mu_A + x_B\mu_B + x_C\mu_C \tag{3}$$

We find the expected return of each asset by taking the mean of the log returns. We find the covariance matrix by taking the covariance of the log returns.

$$E[\mathbf{R}] = E\left[\begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}\right] = \begin{pmatrix} E[R_A] \\ E[R_B] \\ E[R_C] \end{pmatrix} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} \tag{4}$$

$$\Sigma = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{BA} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{CA} & \sigma_{CB} & \sigma_C^2 \end{pmatrix} \tag{5}$$

To obtain the variance and standard deviation of the portfolio, we take the matrix multiplication of the portfolio weights and the covariance matrix and then the square root for the standard deviation.

$$\sigma_{p,x}^2 = \mathbf{w}^T \Sigma \mathbf{w} \tag{6}$$

$$\sigma_{p,x} = \sqrt{\sigma_{p,x}^2} \tag{7}$$

And we take the matrix multiplication of the weights and the expected returns to find the return of the portfolio.

$$\mu_{p,x} = \mathbf{w}^T \mu \tag{8}$$

To confirm our later results, we plot the standard deviation vs the expected return using 10,000 random points.

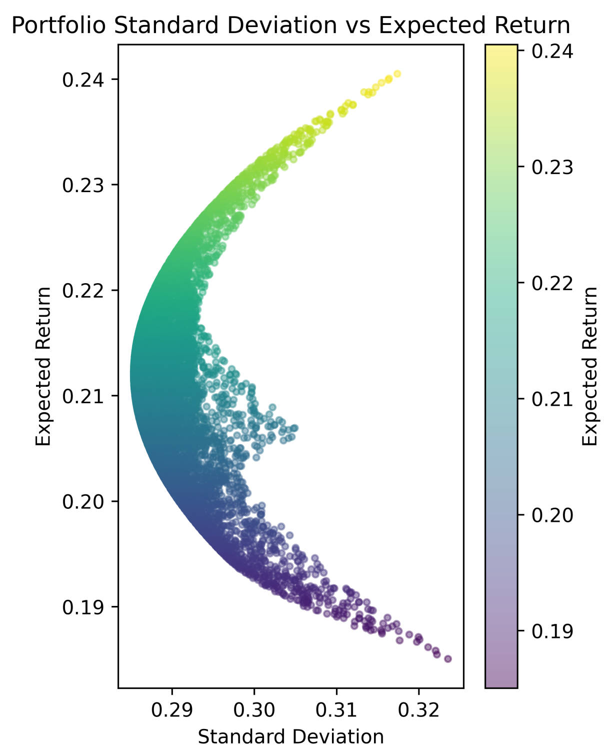

Fig. 1: Standard Deviation vs Expected Return

III. Global Minimum Variance Portfolio

We must find the minimum variance portfolio given the constraints:

$$\min \ \sigma_{p,x}^2 = \mathbf{w}^T \Sigma \mathbf{w} \tag{9}$$

Such that

$$m_A + m_B + m_C = 1 \tag{10}$$

If our goal is to minimize the variance, we can use the langrangian to to optimize the portfolio given a constraint. If we remember, the langrangian multiplier is defined as $\nabla f = \lambda \nabla g, \quad g = 0$ where f is the objective function and g is the constraint.

$$\mathcal{L} = \sigma_{p,x}^2 + \lambda(m_A + m_B + m_C - 1) \tag{11}$$

Taking the partial derivatives we get

$$\frac{\partial \mathcal{L}}{\partial w_A} = 2w_A\sigma_A^2 + 2w_B\sigma_{AB} + 2w_C\sigma_{AC} + \lambda = 0 \tag{12}$$
$$\frac{\partial \mathcal{L}}{\partial w_B} = 2w_B\sigma_B^2 + 2w_A\sigma_{AB} + 2w_C\sigma_{BC} + \lambda = 0 \tag{13}$$
$$\frac{\partial \mathcal{L}}{\partial w_C} = 2w_C\sigma_C^2 + 2w_A\sigma_{AC} + 2w_B\sigma_{BC} + \lambda = 0 \tag{14}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = w_A + w_B + w_C - 1 = 0 \tag{15}$$

This gives us a system of linear equations that can be written in matrix form:

$$\begin{bmatrix} 2\sigma_A^2 & 2\sigma_{AB} & 2\sigma_{AC} & 1 \\ 2\sigma_{AB} & 2\sigma_B^2 & 2\sigma_{BC} & 1 \\ 2\sigma_{AC} & 2\sigma_{BC} & 2\sigma_C^2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_A \\ w_B \\ w_C \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

Plotting this against the random points we get the following:

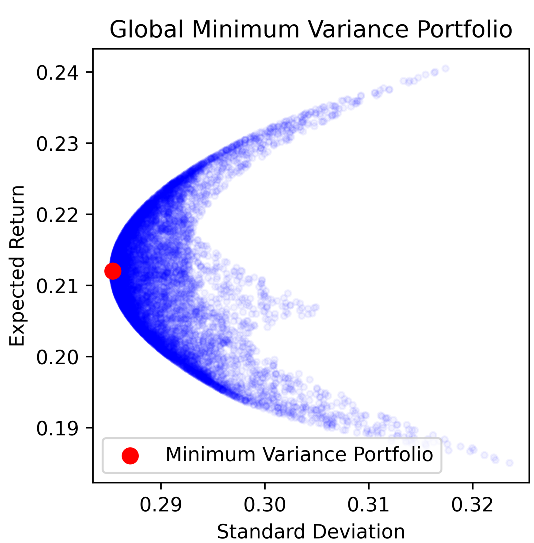

Fig. 2: Global Minimum Variance Portfolio

IV. Efficient Frontier

One consequence of the Markowitz model is the efficient frontier, the set of portfolios that offer the maximum return for a given level of risk, or minimum risk for a given level of return. In the next equations, we will be optimizing the efficient frontier of a function of the return.

$$\min \ \sigma_p^2 = \mathbf{x}'\Sigma\mathbf{x} \tag{17}$$

Subject to two constrainst:

$$\mathbf{x}'\mu = \mu_p \text{ (target return)} \tag{18}$$

$$x'1 = 1 \text{ (weights sum to 1)} \tag{19}$$

We can use the langrangian once again to optimize, but this time we have two constraints.

$$\mathcal{L}(x, \lambda_1, \lambda_2) = \mathbf{x}'\Sigma\mathbf{x} + \lambda_1(\mathbf{x}'\mu - \mu_p) + \lambda_2(1 - \mathbf{x}'1) \tag{20}$$

Taking the partial derivatives we get:

$$\frac{\partial \mathcal{L}(x, \lambda_1, \lambda_2)}{\partial \mathbf{x}} = 2\Sigma\mathbf{x} + \lambda_1\mu + \lambda_2\mathbf{1} = 0, \tag{21}$$
$$\frac{\partial \mathcal{L}(x, \lambda_1, \lambda_2)}{\partial \lambda_1} = \mathbf{x}'\mu - \mu_p = 0, \tag{22}$$
$$\frac{\partial \mathcal{L}(x, \lambda_1, \lambda_2)}{\partial \lambda_2} = \mathbf{x}'\mathbf{1} - 1 = 0. \tag{23}$$

We are then given a linear system, which we are able to solve:

$$\begin{bmatrix} 2\Sigma & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_p \\ 1 \end{bmatrix} \tag{24}$$

It can be shown that all portfolios on the efficient frontier are linear combinations of any two minimum variance portfolios [1].

Let α be any constant and define the portfolio $\mathbf{z}$ as a linear combination of portfolios $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y} \tag{25}$$

which results in:

$$\mathbf{z} = \begin{pmatrix} \alpha x_A + (1-\alpha)y_A \\ \alpha x_B + (1-\alpha)y_B \\ \alpha x_C + (1-\alpha)y_C \end{pmatrix} \tag{26}$$

By picking any two optimal portfolios and plotting the for different linear combinations $\alpha = (0, 1)$ we get this for the efficient frontier:

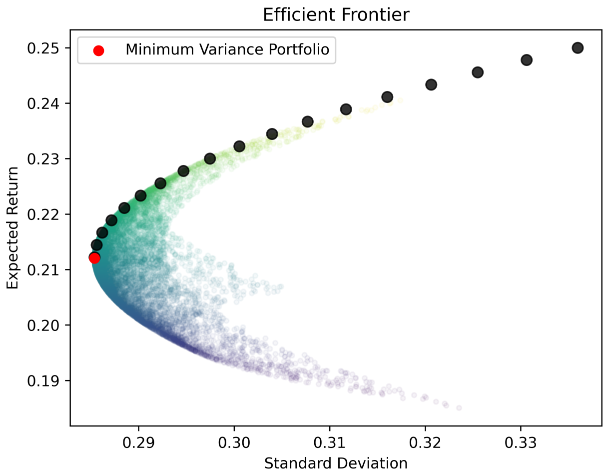

Fig. 3: Efficient Frontier

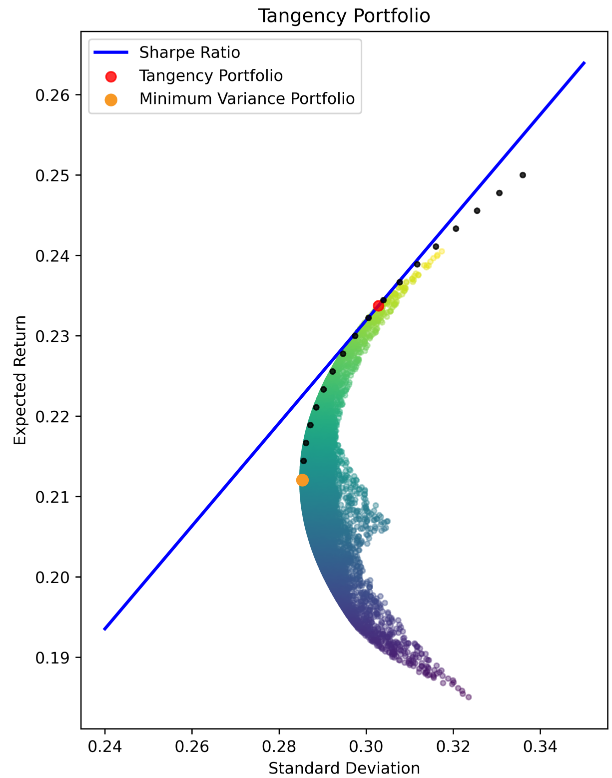

Fig. 4: Tangency Portfolio

V. Sharpe Ratio

The Sharpe Ratio is a measure used to evaluate the risk-adjusted return of an investment portfolio. It is calculated by subtracting the risk-free rate from the portfolio's return and then dividing the result by the portfolio's standard deviation. Mathematically, it is expressed as:

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p} \tag{27}$$

where R_p is the return of the portfolio, R_f is the risk-free rate, and σ_p is the standard deviation of the portfolio's returns. A higher Sharpe Ratio indicates a more attractive risk-adjusted return. To finish off the analysis, we can calculate the Sharpe Ratio for the optimal portfolios.

After lengthly calculations [1], optimizing said ratio, it can be shown that it can be computed as:

$$\mathbf{t} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - r_f \cdot \mathbf{1})} \tag{28}$$

where $\mathbf{t}$ is the tangency portfolio.

VI. Conclusion

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[2] F. Black and R. Litterman, "Global Portfolio Optimization," *Financial Analysts Journal*, vol. 48, no. 5, pp. 28–43, 1992.
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