Magneto-static Analysis of a Brushless DC Motor

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Overview

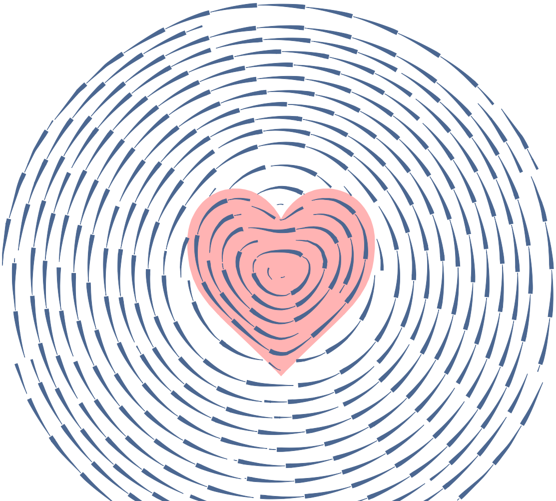
- Background
  Brushless DC Motors
- Theory
  Magneto-statics, Permanent Magnets, Non Linear Materials, Non Linear FEM
- Implementation
  Meshing, Matrix Assembly, Solvers, BLDC Problem
- Results
  Diagrams, Torques

Theory: Magnetostatics in a 2D Cross Section

If current is only perpendicular to the 2D plane, then

$$\nabla \times \nu B = J \text{ and } \nabla \times A = B$$

$$\nabla \times \nu \nabla \times A = J \iff \nabla \cdot \nu \nabla A = J$$



Theory: Permanent Magnets

Maxwell's Equation's for Permanent Magnets

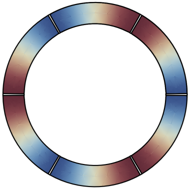
$$B = \mu H + \mu_0 M_r \text{ and } \nabla \times H = J \implies \nabla \times \nu B = J + \nabla \times \nu \mu_0 M_r$$

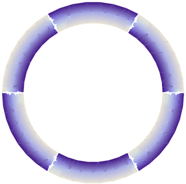
$$J_m \triangleq \nabla \times \nu \mu_0 M_r \implies \nabla \times \nu B = J + J_m$$

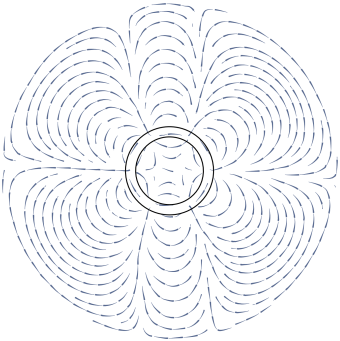
Permanent Magnets

We use a *2D-coil* model for permanent magnets.

- **Assumption**: A bar magnet has a similar field to a current carrying coil
- **Heuristic**: Any magnet geometry can be approximated from gluing together curved bar magnets







Theory: Non Linear FEM

How can we use FEM to solve problems in the form of

$$\nabla \cdot (\alpha(\|\nabla\phi\|) * \nabla\phi) = f(x, y)$$

First thing: Determine $\|\nabla\phi\|$ on a single element in terms of node values of ϕ: $\phi^e := (\phi_i, \phi_j, \phi_k)^\mathsf{T}$

$$\left\| \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right) \right\| = \left\| \left(\frac{\partial\vec{N}}{\partial x}, \frac{\partial\vec{N}}{\partial y} \right)^\mathsf{T} \cdot \phi^e \right\|$$

$$\hat{\alpha}(\phi^e) := \alpha(\|\nabla\phi\|)$$

Solving the Non Linear System

Now our element matrix equation looks like:

$$F_e(\phi^e) = \mathbf{K}^e \cdot \phi^e * \hat{\alpha}(\phi^e) - \vec{b}^e = 0$$

No more linear solver :(we need to use Newton's Method

Consider the vector equation $f(\vec{x}) = 0$

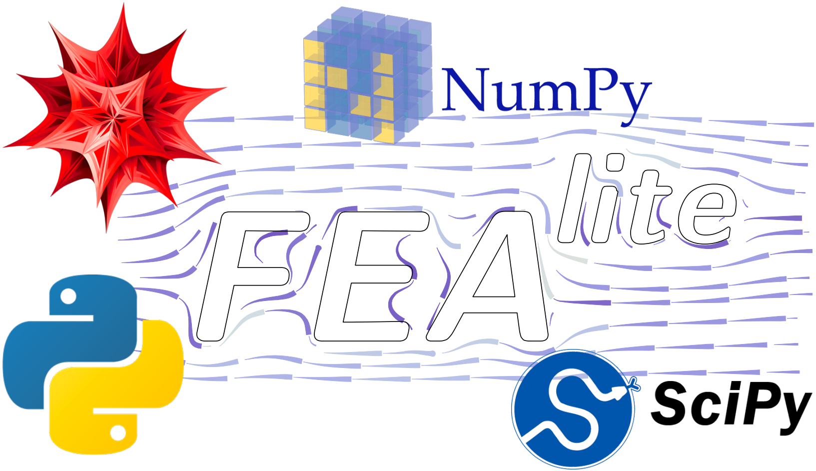
- Start with the guess $\vec{x} \approx \vec{x_0}$
- Expand in power series

$$f(\vec{x}) \approx f(\vec{x_0}) + \mathbf{J}_{f(\vec{x_0})}(\vec{x} - \vec{x_0}) \approx 0$$

- Solve a linear system for $\vec{x_0}$ to find a better guess for $\vec{x}$. Iterate.

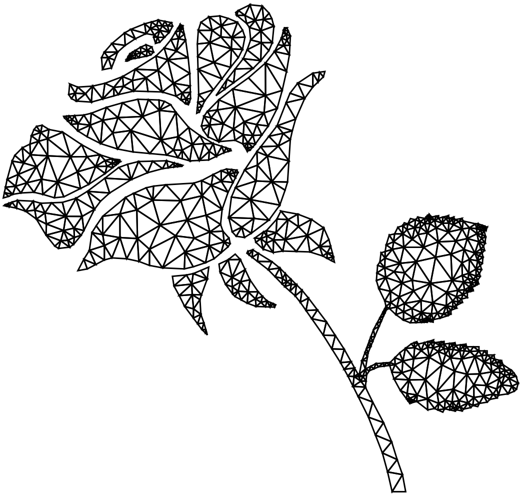
$$\mathbf{J}_{f(\vec{x}_n)}(\vec{x}_n - \vec{x}_{n+1}) = f(\vec{x}_n)$$
$$\mathbf{J}_{F_e(\phi_n^e)} = \mathbf{K}^e \left(\mathbf{I} * \hat{\alpha}(\phi_n^e) + (\phi_n^e) \cdot \nabla^{\mathsf{T}} \hat{\alpha}(\phi_n^e)\right)$$

Implementation: FEA*lite*



Mesh

- Generated with Mathematica
- Triangular Elements
- Linear Shape Functions $A = \sum_{i=1}^{3} N_i A_i$

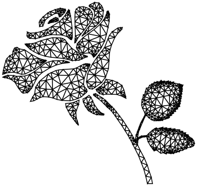


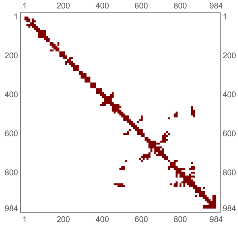
Matrix Assembly

Dense element matrices are assembled into a sparse global matrix using by substituting equality constraints at shared nodes

$$\mathbf{K}_{i,j} = \sum_{e \in E(i,j)} \sum_{k=l=0}^{3} \mathbf{K}^{e}_{k,l} * \alpha(e)$$

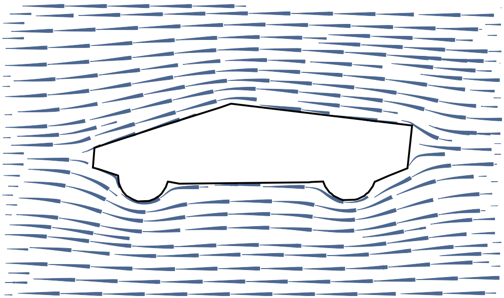
$e = \{v_a, v_b, v_c\}$ is an element, E is the element set, $E(i,j)$ is the set of elements $\{\, e \,|\, e \in E,\ (\{v_i\} \cup \{v_j\}) \subset e\}$, $\mathbf{K}^e$ is the local stiffness matrix of e, α is a function of the element properties.





Solvers

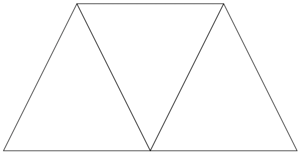
- Linear FEM:
  Sparse linear systems solved with
  `scipy.linalg.sparse.spsolve`[1]
- Non Linear FEM: Sparse non linear systems solved with `scipy.optimize.fsolve`[2]. Jacobian is analytically computed to speed up the solver.



[1] Wrapper for SuperLU 4.0 (LU decomposition)
[2] Wrapper for MINIPACK hybrj (Newton's Method + Gradient Descent)

Appendix: Mesh Generation and Format

- Meshes are generated using the Mathematica 12 mesh generator
- Exported as:
  list of coordinates $[x, y]^v$
  list of mesh elements $[v_1, v_2, v_3, \text{marker}]^e$
  list of boundary elements $[v_1, v_2, \text{marker}]^{be}$



```
# Coordinates-5
0.0000  0.0000
1.0000  0.0000
0.5000  1.0000
2.0000  0.0000
1.5000  1.0000
# Triangle Elements-5
1       2       3       1
2       3       5       1
2       5       4       1
# Boundary Elements-5
1       2       1
2       4       1
4       5       1
5       3       1
3       1       1
```

Appendix: Matrix Assembly

Global stiffness matrix assembly code for reference

```
[(element[i], element[j], shp_fn.stiffness_matrix[i, j]
    * alpha(marker)) for i in range(3) for j in range(3)
    for element, shp_fn, marker in
    zip(elements, shp_fns, markers)]
```

List of all (row, col, val) tuples, overlapping values are summed.

Appendix: Gradient of α

Given a function $\alpha(\|\nabla\phi\|)$ where

$$\|\nabla\phi\| = \left\| \left(\frac{\partial\vec{N}}{\partial x}, \frac{\partial\vec{N}}{\partial y}\right)^{\mathsf{T}} \cdot \begin{pmatrix} \phi_i \\ \phi_j \\ \phi_k \end{pmatrix} \right\|$$

$$\nabla\alpha = \frac{\alpha'(\|\nabla\phi\|)}{\|\nabla\phi\|} \left(\frac{\partial\vec{N}}{\partial x} \cdot \left(\frac{\partial\vec{N}}{\partial x}\right)^{\mathsf{T}} + \frac{\partial\vec{N}}{\partial y} \cdot \left(\frac{\partial\vec{N}}{\partial y}\right)^{\mathsf{T}} \right) \cdot \begin{pmatrix} \phi_i \\ \phi_j \\ \phi_k \end{pmatrix}$$