Magneto-static Analysis of a Brushless DC Motor

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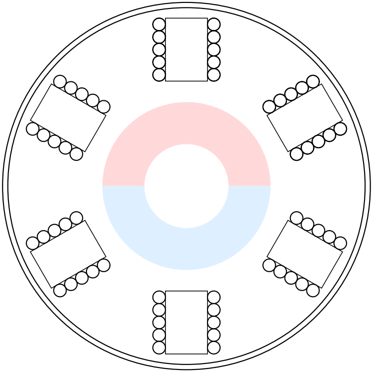
Overview

- Background
  Brushless DC Motors
- Theory
  Magneto-statics, Permanent Magnets, Non Linear Materials, Non Linear FEM
- Implementation
  Meshing, Matrix Assembly, Solvers, BLDC Problem
- Results
  Diagrams, Torques

Brushless DC Motors

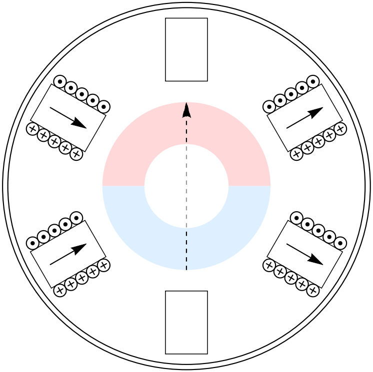
BLDC motors consist of coils surrounding a permanent magnet



8 coil, 6 pole BLDC motor

Brushless DC Motors

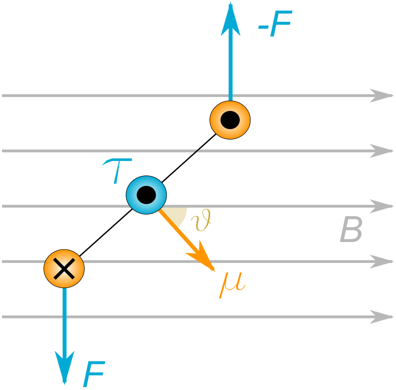
The inner disk (rotor) is driven by the magnetic field produced by the coils



Current and Magnetic Poles

Torque on a Magnetic Dipole

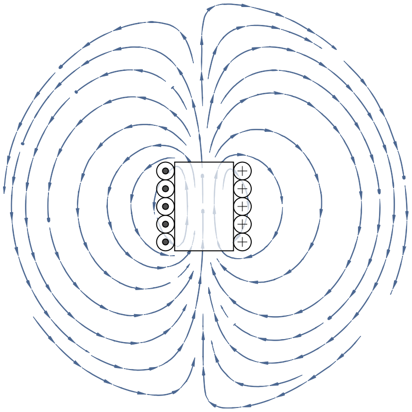
$$\tau \propto \vec{\mu} \times \vec{B}$$



Magnetic Dipole in a Uniform Field

Coils

Coils produce a field similar to that of a bar magnet



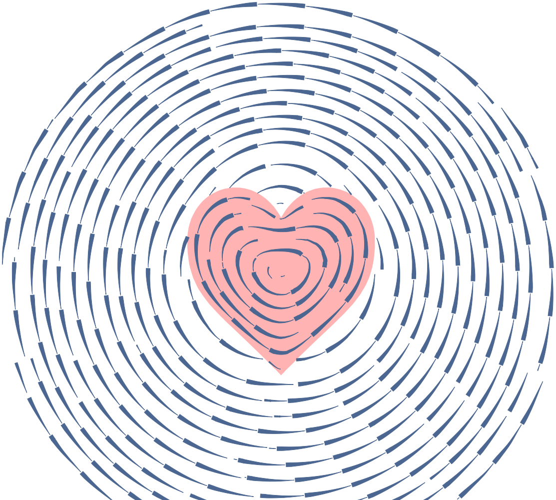
Field produced by a motor coil

Magnetostatics in a 2D Cross Section

If current is only perpendicular to the 2D plane, then

$$\nabla \times \nu B = J \text{ and } \nabla \times A = B$$

$$\nabla \times \nu \nabla \times A = J \iff \nabla \cdot \nu \nabla A = J$$



Permanent Magnets

Maxwell's Equation's for Permanent Magnets

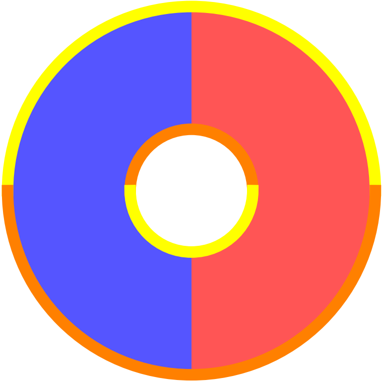
$$B = \mu H + \mu_0 M_r \text{ and } \nabla \times H = J \implies \nabla \times \nu B = J + \nabla \times \nu \mu_0 M_r$$

$$J_m \triangleq \nabla \times \nu \mu_0 M_r \implies \nabla \times \nu B = J + J_m$$

Permanent Magnets

We use a *2D-coil* model for permanent magnets.

- **Assumption**: A bar magnet has a similar field to a current carrying coil
- **Model**: Any magnet geometry can be approximated from gluing together curved bar magnets



Yellow lines are current sheet coming out of the page, Green are in

Torque

Force from Maxwell's Stress tensor

$$\vec{F} = \oint_s \sigma \cdot dS$$

$$\vec{F} = \oint_s \left[\frac{1}{\mu_0} \left(B_n B_t \right) \vec{t} + \frac{1}{2\mu_0} \left(B_n^2 - B_t^2 \right) \vec{n} \right] dS$$

Two dimensional simplification for the BLDC magnet

$$\tau = \frac{1}{\mu_0 \left(r_s - r_r \right)} \int_{r_r}^{r_s} \int_0^{2\pi} \left(r B_r B_\theta \right) dS$$

Since this is a two dimensional contour the output is torque per unit length

Non Linear FEM

How can we use FEM to solve problems in the form of

$$\nabla \cdot (\alpha(\|\nabla \phi\|) * \nabla \phi) = f(x, y)$$

First thing: Determine $\|\nabla \phi\|$ on a single element in terms of node values of ϕ: $\phi^e := (\phi_i, \phi_j, \phi_k)^\mathsf{T}$

$$\left\| \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) \right\| = \left\| \left(\frac{\partial \vec{N}}{\partial x}, \frac{\partial \vec{N}}{\partial y} \right)^\mathsf{T} \cdot \phi^e \right\|$$

$$\hat{\alpha}(\phi^e) := \alpha(\|\nabla \phi\|)$$

Solving the Non Linear System

Now our element matrix equation looks like:

$$F_e(\phi^e) = \mathbf{K}^e \cdot \phi^e * \hat{\alpha}(\phi^e) - \vec{b}^e = 0$$

No more linear solver :(we need to use Newton's Method

Consider the vector equation $f(\vec{x}) = 0$

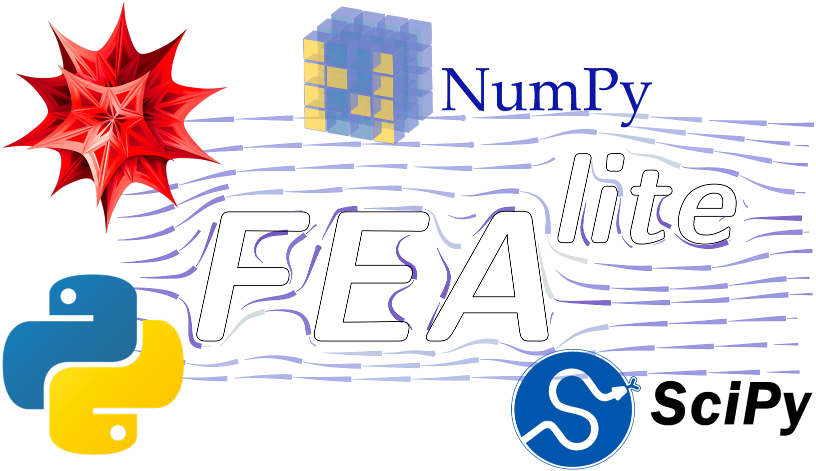
- Start with the guess $\vec{x} \approx \vec{x}_0$
- Expand in power series

$$f(\vec{x}) \approx f(\vec{x}_0) + \mathbf{J}_{f(\vec{x}_0)}(\vec{x} - \vec{x}_0) \approx 0$$

- Solve a linear system for $\vec{x}_0$ to find a better guess for $\vec{x}$. Iterate.

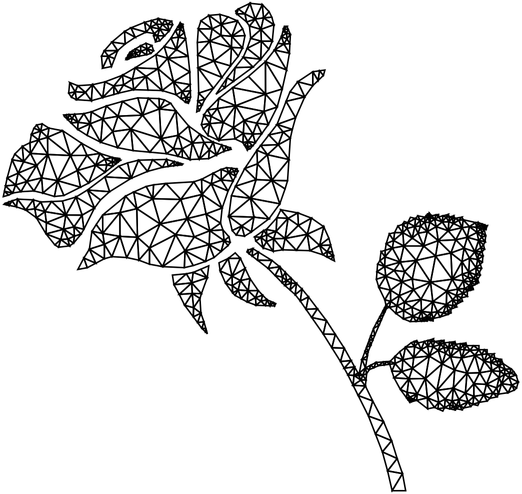
$$\mathbf{J}_{f(\vec{x}_n)}\left(\vec{x}_n - \vec{x}_{n+1}\right) = f(\vec{x}_n)$$
$$\mathbf{J}_{F_e(\phi^e_n)} = \mathbf{K}^e \left(\mathbf{I} * \hat{\alpha}(\phi^e_n) + (\phi^e_n) \cdot \nabla^{\mathsf{T}} \hat{\alpha}(\phi^e_n)\right)$$

Implementation: FEA*lite*



Mesh

- Generated with Mathematica
- Triangular Elements
- Linear Shape Functions $A = \sum_{i=1}^{3} N_i A_i$

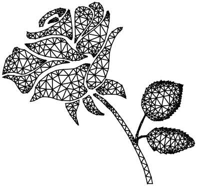


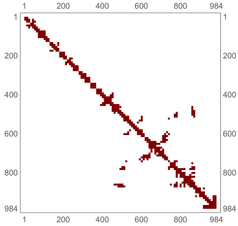
Matrix Assembly

Dense element matrices are assembled into a sparse global matrix by substituting equality constraints at shared nodes

$$\mathbf{K}_{i,j} = \sum_{e \in E(i,j)} \sum_{k=l=0}^{3} \mathbf{K}^e_{k,l} * \alpha(e)$$

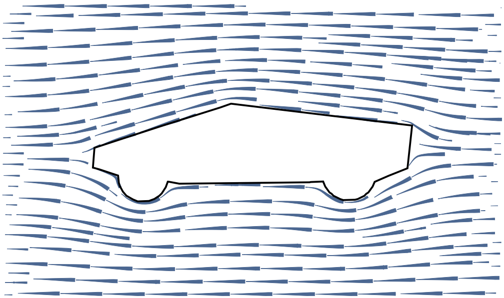
$e = \{v_a, v_b, v_c\}$ is an element, E is the element set, $E(i,j)$ is the set of elements $\{\, e \,|\, e \in E,\ (\{v_i\} \cup \{v_j\}) \subset e \}$, $\mathbf{K}^e$ is the local stiffness matrix of e, α is a function of the element properties.





Solvers

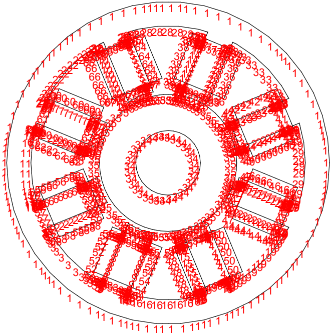
- Linear FEM:
  Sparse linear systems solved with
  `scipy.linalg.sparse.spsolve`[1]
- Non Linear FEM: Sparse non linear systems solved with `scipy.optimize.fsolve`[2]. Jacobian is analytically computed to speed up the solver.

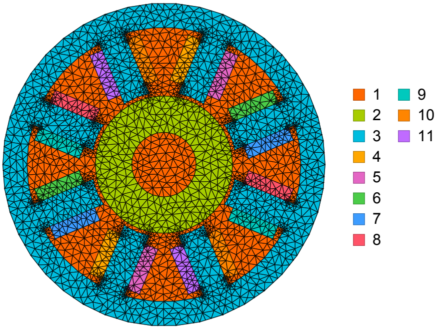


[1] Wrapper for SuperLU 4.0 (LU decomposition)
[2] Wrapper for MINIPACK hybrj (Newton's Method + Gradient Descent)

BLDC Simulation

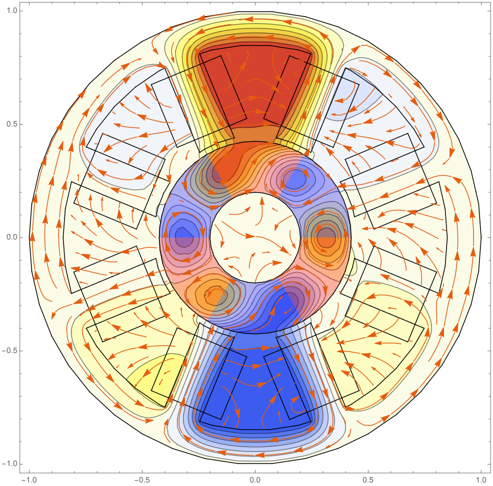
Mesh and Boundaries





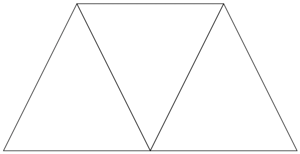
BLDC Simulation

Flux Plot



Appendix: Mesh Generation and Format

- Meshes are generated using the Mathematica 12 mesh generator
- Exported as:
  list of coordinates $[x, y]^v$
  list of mesh elements $[v_1, v_2, v_3, \text{marker}]^e$
  list of boundary elements $[v_1, v_2, \text{marker}]^{be}$



```
# Coordinates-5
0.0000  0.0000
1.0000  0.0000
0.5000  1.0000
2.0000  0.0000
1.5000  1.0000
# Triangle Elements-5
1       2       3       1
2       3       5       1
2       5       4       1
# Boundary Elements-5
1       2       1
2       4       1
4       5       1
5       3       1
3       1       1
```

Appendix: Matrix Assembly

Global stiffness matrix assembly code for reference

```
[(element[i], element[j], shp_fn.stiffness_matrix[i, j]
* alpha(marker)) for i in range(3) for j in range(3)
for element, shp_fn, marker in
zip(elements, shp_fns, markers)]
```

List of all (row, col, val) tuples, overlapping values are summed.

Appendix: Gradient of α

Given a function $\alpha(\|\nabla\phi\|)$ where

$$\|\nabla\phi\| = \left\| \left(\frac{\partial\vec{N}}{\partial x}, \frac{\partial\vec{N}}{\partial y} \right)^{\mathsf{T}} \cdot \begin{pmatrix} \phi_i \\ \phi_j \\ \phi_k \end{pmatrix} \right\|$$

$$\nabla\alpha = \frac{\alpha'(\|\nabla\phi\|)}{\|\nabla\phi\|} \left(\frac{\partial\vec{N}}{\partial x} \cdot \left(\frac{\partial\vec{N}}{\partial x} \right)^{\mathsf{T}} + \frac{\partial\vec{N}}{\partial y} \cdot \left(\frac{\partial\vec{N}}{\partial y} \right)^{\mathsf{T}} \right) \cdot \begin{pmatrix} \phi_i \\ \phi_j \\ \phi_k \end{pmatrix}$$

Appendix: Torque Integration

First make a transformation to polar coordinates

$$\begin{pmatrix} B_x(x,y) \\ B_y(x,y) \end{pmatrix} \rightarrow \begin{pmatrix} B_r(r,\theta) \\ B_\theta(r,\theta) \end{pmatrix}$$

The vector $(B_r,\ B_\theta)^\mathsf{T}$ is

$$\begin{pmatrix} \cos(\theta)B_x(r\cos(\theta), r\sin(\theta)) + \sin(\theta)B_y(r\cos(\theta), r\sin(\theta)) \\ \cos(\theta)B_y(r\cos(\theta), r\sin(\theta)) - \sin(\theta)B_x(r\cos(\theta), r\sin(\theta)) \end{pmatrix}$$

Torque per unit length is

$$\frac{1}{\mu_0(r_s - r_r)} \int_{r_r}^{r_s} \int_0^{2\pi} r^2 B_r B_\theta \mathrm{d}\theta \mathrm{d}r$$

This integral is easily approximated by sampling