EC2020 session 8

1) $f(x;\lambda) = \lambda e^{-\lambda x}$ for some $\lambda > 0$

Philosophy of MLE:

find $p(x;\theta)$ eg.

by independence

$$L = p(x_1, x_2 \dots x_n ; \lambda) = \prod_{i=1}^{n} p(x_i ; \lambda)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n \prod_{i=1}^{n} e^{-\lambda x_i}$$

$$\underline{e^A e^B = e^{A+B}}$$

$$= \lambda^n e^{-\lambda \sum x_i}$$

$$\log L = \log\left(\lambda^n e^{-\lambda \Sigma x_i}\right)$$

$$= \log(\lambda^n) + \log\left(e^{-\lambda \Sigma x_i}\right)$$

$$= n \cdot \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

$\log(AB) = \log A + \log B$

$\boxed{\begin{array}{l} L(x) \geq L(y) \text{ for all } y \\ \log L(x) \geq \log L(y) \\ \operatorname{argmax} L(\cdot) \\ = \\ \operatorname{argmax} \log L(\cdot) \end{array}}$

$\hat{\lambda}$

$$\frac{\partial}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Therefore: $\frac{n}{\hat{\lambda}} - \Sigma x_i = 0$

$$\hat{\lambda} = \frac{n}{\Sigma x_i} = \frac{1}{(\bar{x})}$$

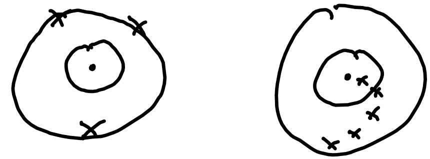
$\bar{x} = \frac{1}{n} \Sigma x_i$

$$\hat{\lambda} = \frac{1}{(\bar{x})} = \frac{n}{\sum_{i=1}^{n} x_i}$$ and $$\boxed{f(x;\lambda) = \lambda e^{-\lambda x} \quad \text{pdf.}}$$

$$E(x_i) = 1/\lambda$$

$$E(x) = \int x f(x)\, dx$$

is $\hat{\lambda}$ unbiased / consistent



unbiased consistent

U.B.: $E(\hat{\lambda}) = \lambda$ ▪

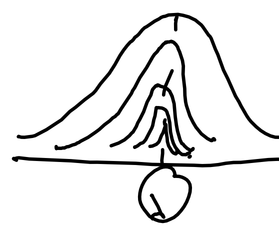
cons.: $\hat{\lambda} \xrightarrow{P} \lambda$ ▪

$$E g(x) \neq g E(x)$$

proper def: $\forall \varepsilon \quad P(|\hat{\lambda} - \lambda| > \varepsilon) \to 0$ as $n \to \infty$

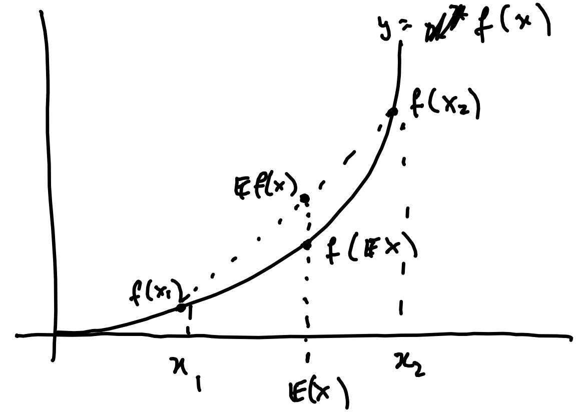
sufficient conditions: $E(\hat{\lambda}) \to \lambda$

and $Var(\hat{\lambda}) \to 0$



$$E(\hat{\lambda}) = \boxed{E\left(\frac{1}{\bar{x}}\right)} \not\Rightarrow \lambda = \boxed{\frac{1}{E(\bar{x})}}$$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum \frac{1}{\lambda} = \frac{1}{\lambda}$$



Jensen's inequality

$$\mathbb{E} f(x) > f\,\mathbb{E}(X)$$

$$\mathbb{E}\left(\frac{1}{X}\right) > \frac{1}{\mathbb{E}(X)}$$

$\cancel{\mathbb{E}x^2}$ $\mathbb{E} f(x)$

$\operatorname{plim} a_n = a$

$\Longleftrightarrow$

$a_n \xrightarrow{p} a$

$a_n \xrightarrow{p} a$

does that mean

$\frac{1}{a_n} \xrightarrow{p} \frac{1}{a}$

$$\frac{1}{n}\sum f(x_i) \xrightarrow{p} \mathbb{E} f(x_i) \qquad \underline{\text{LLN}}$$

by $\frac{1}{n}\sum x_i \xrightarrow{p} \mathbb{E} x_i$

$$\hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\left(\frac{1}{n}\sum x_i\right)} \xrightarrow{?} \frac{1}{\mathbb{E}(x_i)} = \frac{1}{(1/\lambda)} = \lambda$$

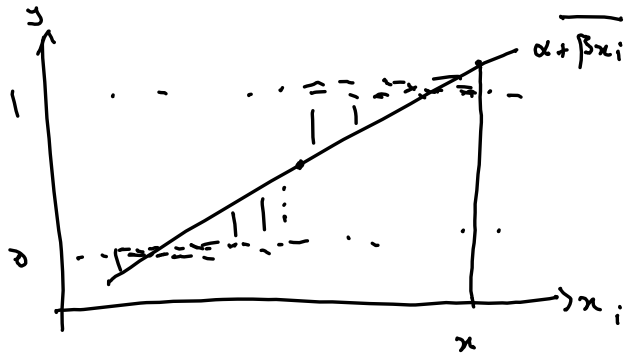
$y_i = \alpha + x_i' \beta + \varepsilon_i \qquad \varepsilon_i \sim N(0,1)$

$\Leftrightarrow$

$\cancel{P(y_i = y)}$ $\qquad y_i \sim N(\alpha + x_i \beta, 1)$

$\boxed{y_i \in \{0,1\}} \qquad y_i = \alpha + \beta x_i + \varepsilon_i$

LPM



Assuming that

$y_i = \alpha + \beta x_i + \varepsilon_i$ with $\mathbb{E}(\varepsilon_i | x_i) = 0$

$\Leftrightarrow \quad \mathbb{E}(y_i | x_i) = \alpha + \beta x_i$

$y_i \quad \cancel{\mathbb{E}(y_i | x_i) = \alpha + \beta x_i} + \mathbb{E}(\varepsilon_i | x_i)$

all GM ass.

$\mathbb{E}(\varepsilon_i | x_i) = 0$

$y_i \in \{0,1\}$

Then

$\mathbb{E}(y_i | x_i) = 1 \cdot P(y_i = 1 | x_i) + 0 \cdot \cancel{P(y_i = 0 | x_i)}$

$= P(y_i = 1 | x_i)$

$P(y_i = 1 | x_i) = \alpha + \beta x_i$

$Var(y_i | x_i) = p(x_i)(1 - p(x_i))$

motivation of logit / probit

step 1: assuming that

$$y_i = \alpha + \beta x_i + \varepsilon_i \quad \text{with} \quad \mathbb{E}(\varepsilon_i \mid x_i) = 0$$

$$\Leftrightarrow \quad \mathbb{E}(y_i \mid x_i) = \alpha + \beta x_i$$

step 2: if $y_i \in \{0,1\}$ then $\mathbb{E}(y_i \mid x_i) = P(y_i = 1 \mid x_i)$

$$\hookrightarrow \quad 1 \cdot P(y_i = 1 \mid x_i) + 0 \cdot P(y_i = 0 \mid x_i)$$

Therefore:

$$P(y_i = 1 \mid x_i) = \alpha + \beta x_i$$

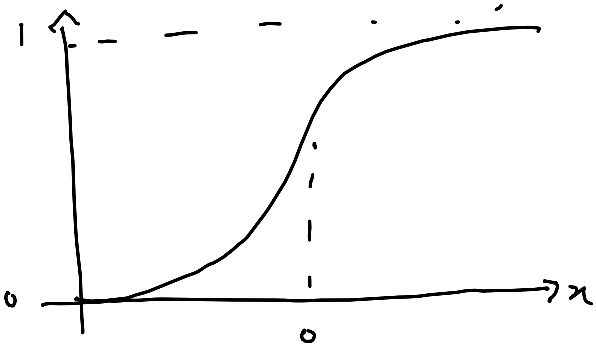
step 3:

imagine we have $F : \mathbb{R} \to [0,1]$

Then $P(y_i = 1 \mid x_i) = F(\alpha + \beta x_i)$

probit and logit

normal CDF



So

$$\boxed{P(y_i = 1 | x_i) = F(\alpha + \beta x_i)}$$

probit: F is CDF of a normal.

Logit: F is CDF of a logistic distribution

$$L = \prod_{i=1}^{n} p(y_i = 1 | x_i)$$

$$P(y_i = 0 | x_i) = 1 - F(\alpha + \beta x_i)$$

$$= p(y_i = 1 | x_i)^{m} \, p(y_i = 0 | x_i)^{n-m}$$

suppose I see $y_i = 1$ m times

$$= [F(\alpha + \beta x_i)]^{m} \cdot [1 - F(\alpha + \beta x_i)]^{n-m}$$

$$y_i = \alpha + \beta x_i + \varepsilon_i \qquad \beta = \frac{\partial \mathbb{E}(y_i | x_i)}{\partial x_i}$$

Partial effect at the average

$$\frac{\partial P(y_i = 1 | x_i)}{\partial x_i} = F'(\alpha + \beta \bar{x}_i) \cdot \beta$$

Average partial effect

$$\frac{1}{n} \sum F'(\alpha + \beta x_i) \cdot \beta$$