Searching for a Good Choice

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Abstract

How do agents make choices whenever they do not consider everything at once? This paper introduces the Blind Order-Conserving Search (BOCS) model of choice to help address this question. All of the model's parameters can be identified from choice data under an intuitive richness condition, and the classical utility model is nested as a special case. I prove necessary and sufficient conditions to falsify if a given family of consideration sets rationalize choice, and give a conjecture on necessary and sufficient conditions for any family of consideration sets to do so. I also show how the BOCS model provides novel explanations for empirical results such as the attraction and endowment effects, and explain how it can be used to test for a form of reference dependence.

Introduction

Ever since Samuelson introduced the theory of revealed preference to Economics (1938), numerous authors have challenged the notion of rationality. In particular, a major doubt is that agents cannot consider everything offered to them at once, with their attention organized instead into *consideration sets*. This paper presents a new way to model how an agent could form consideration sets and search through them to reach a choice.

There is strong evidence that consideration sets are relevant for consumer choice. Since their introduction by Wright and Barbour (1977), numerous studies (Wernerfelt and Hauser, 1990; Roberts and Lattin, 1991) have found that consumers do not consider every product available to them, especially when making repeat purchases (Hoyer, 1984). This is incompatible with standard theory, as an agent who does not consider every alternative at once might have incomplete preferences; a feature that cannot be captured by utility functions[1]. Moreover, certain methods of forming consideration sets, such as the model introduced in this paper, allow for intransitivities, which also cannot be captured. For this reason, the way in which consideration sets are formed and searched through has a significant theoretical importance.

Because of this, there have been many attempts to model consideration sets in the literature. Simon (1955) proposed a satisficing rule, where agents search until they reach a predetermined level

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[1]See a textbook such as Kreps (1988) for further detail

of utility. Others such as Masatlioglu and Nakajima (2013) and Caplin and Dean (2011) propose an iterating increase in consideration sets, and Masatlioglu et al. (2012) propose that consideration sets are formed by an attention filter instead of through search. A feature shared by all of these proposals is that choice is made with perfect recall; once a good is made a part of the consideration set, it will always be considered until the agent has made a choice.

The Blind Order-Conserving Search (BOCS) model that I present in this paper is instead a model of search without recall. Instead of a sequence of increasing consideration sets, the BOCS model supposes that information sets are disjoint[2]. In many choice problems, there are constraints that make this a better assumption than perfect recall. Perhaps most evidently, physical constraints can occur when the choice is being made over time and distance. A consumer choosing whether to buy items from shops on a different side of town is unlikely to visit previously-rejected items, as is a job-seeker in a loose labour market (whose opening might have been given to someone else), as is a singleton trying to find a spouse. Importantly, psychological constraints can also motivate why we might want to consider choice by search without recall. For example, agents with bounded memory such as those modelled by Mullainathan (2002) might not be able to remember what they have searched before, or might only search forward as a heuristic (see Tversky and Kahneman (1974)). While optimal search under imperfect recall has been studied (Janssen and Parakhonyak, 2008), the BOCS model is a descriptive theory of how search may work without recall.

As with all search models, the BOCS model assumes that agents have some beliefs about unseen alternatives, which determine whether or not the agent continues searching. The philosophy of the BOCS model is that removing an entire unseen consideration set from a menu is qualitatively different to removing one unseen item. In other words, the agent is tempted to continue searching by the consideration sets that they will be able to search in the future, rather than the items themselves. This is motivated by the idea that people have multiple stages of search - firstly over types of goods, which I refer to as *categories* in this paper, and then over goods themselves within a category. This is the property that makes the search 'blind' - it is *as if* the agents cannot see when a single item in a future consideration set has been removed from the menu, but they can see if the whole consideration set has been. In cases where the consideration sets are formed physically, this will likely make sense: a consumer who searches for a gift in different shops might not be able to see if a shop has stopped selling a given item until he looks inside it, but they likely would be able to tell *ex ante* if the entire shop is open or not. Furthermore, the model is 'order-conserving' so it has traction: the beliefs about how good unseen consideration sets will be does not evolve as the search process happens. If it did, then any behaviour could be rationalized by changing expectations arbitrarily.

Finally, the model also helps to rationalize intransitivities in choice sets; I show that the BOCS model allows for intransitive choices, but only if all violations of the Weak Axiom of Revealed Preference (WARP) take a certain form. The paper proceeds as follows: the next section gives the BOCS model formally, before exploring its properties and how its parameters can be revealed. I then present the model's representation theorem, assuming categories are known, before giving results on how it could be extended to cases when the categories are unknown. Finally, I explain various

[2] While ideally a model would like to be able to capture the whole spectrum of perfect to zero recall, such a model is difficult to gain testable implications from. As such this model focuses on the case of zero recall only.

applications of the model in rationalizing the attraction and the endowment effects. Proofs of major theorems are given in the text, and all lemmas are proven in the Appendix.

Model

We let X denote the finite set of all alternatives that the agent can choose from, and define a menu S to be a subset of X that they actually be choosing over. To capture the notion of consideration sets, X is partitioned into *categories* κ_i with K denoting the set of all categories. The projections of the categories onto the menu are known as *boxes* $\beta_i = S \cap \kappa_i$, with the set of all non-empty boxes denoted $B = \{\beta_1, ..., \beta_n\} \setminus \emptyset$. In this paper, lowercase Roman letters are goods (elements of X), lowercase Greek letters are usually boxes or categories, and uppercase Roman letters are sets of boxes or goods.

Agents have classical preferences over the goods in any given category and over the categories themselves - these are represented by the utility functions $u_i : \kappa_i \to \mathbb{R}$ for all $\kappa_i \in K$ and $\bar{v} : K \to \mathbb{R}$, respectively. Without loss of generality, all utilities are non-negative. In this model all preferences are strict, but this assumption could be weakened in future editions of this paper. We usually interpret $\bar{v}(\cdot)$ as the beliefs that the agent holds over how good an unseen category will be. However depending on the situation, $\bar{v}(\cdot)$ can also capture a search cost, discounting due to having to wait to make a choice, risk aversion, or many other factors that might influence the order of search, and whether or not the agent continues to search. While we do not need to do so explicitly, it often makes sense to think of $\bar{v}(\cdot)$ as the utility that the agent expects *ex ante* to get if they choose the best thing from the category, adjusted for the cost searching it. This would give $\bar{v}(\cdot)$ the form:

$$\bar{v}(\kappa_i) = g_i \left[\mathbb{E} \left(\max_{x \in \kappa_i \cap S} u_i(x) \right) \right] \qquad \forall \, \kappa_i \in K$$

Where $g_i(x) \leq x$ captures factors such as search costs. We can interpret the need for an expectation in a number of ways: the agent may have uncertainty over S (they do not know which goods will be available in a future category), or over $u_i(\cdot)$ (they do not know their preferences over goods in a future category until they look inside the category itself). Both features are captured in this model.

Importantly, agents search over the boxes actually available to them, and so will never consider searching an empty box. Agents therefore search using $v : B \to \mathbb{R}$, which is defined as:

$$v(\beta_i) := \begin{cases} \bar{v}(\kappa_i) & \text{if } \beta_i \neq \emptyset \\ 0 & \text{if } \beta_i = \emptyset \end{cases}$$

In this paper, we sometimes refer to $v(\beta_i)$ as the *ex-ante* utility of the box β_i, and $u(\beta_i) := \max_{x \in \beta_i} u_i(x)$ as the *ex-post* utility of β_i, as β_i is evaluated using $v(\cdot)$ before it is opened and using $u(\cdot)$ afterwards. As mentioned in the introduction, search will be blind, order-conserving, and without recall.

Given this, we can now state the model formally:

Definition 1. *A choice function $c(\cdot)$ has a Blind Order-Conserving Search (BOCS) representation if for some partition $K = \{\kappa_1, ..., \kappa_n\}$ of X there exist injective functions $u_i : \kappa_i \to \mathbb{R}$ and $\bar{v} : K \to \mathbb{R}$ such that for any menu $S \subseteq X$ the agent chooses according to the following algorithm:*

1. *Define β_i, B and $v : B \to \mathbb{R}$ as above*
2. *Let $\beta' = \arg\max_{\beta_i \in B} v(\beta)$ and $u(\beta') = \max_{x \in \beta'} u_{\beta'}(x)$*
3. *If $u(\beta') \geq v(\beta)$ for all $\beta \in B \setminus \beta'$, $c(S) = \arg\max_{x \in \beta'} u_{\beta'}(x)$. If not, remove β' from B and return to step 2*

We sometimes refer to this as the *BOCS algorithm*, and each cycle of step 2 - step 3 as an *iteration*. To illustrate how the model works, consider a simple example of a parent shopping for a birthday gift for their young child. Standing in front of a mall at the start of a busy sales day, they know that if they move on from one store, they will not be able to go and retrieve the item again, as someone else will have taken it. All of the stores sell different items, and they do not necessarily know which items each store will be selling in advance. In this case, K might be the set of stores in the mall and S is the set of goods which are actually available to buy that day. An illustration of this story is given in Figure 1. Three shops are open: β_1 is the one which they believe *ex ante* has the best gift inside of it, followed by β_2 and then β_3. In this example, the parent first searches β_1, causing them to now consider β_1 by the utility of the best good inside of it ($u(\beta_1)$) instead of their *ex ante* belief ($v(\beta_1)$). However, they believe that it is better to keep searching than stay with the best good in β_1 (as $v(\beta_2) > u(\beta_1)$), so they now consider β_2 by its u-utility instead of its v-utility. However this time, as $u(\beta_2) > v(\beta_3)$, $c(S) = \arg\max_{x \in \beta_2} u_2(x)$. The model does not

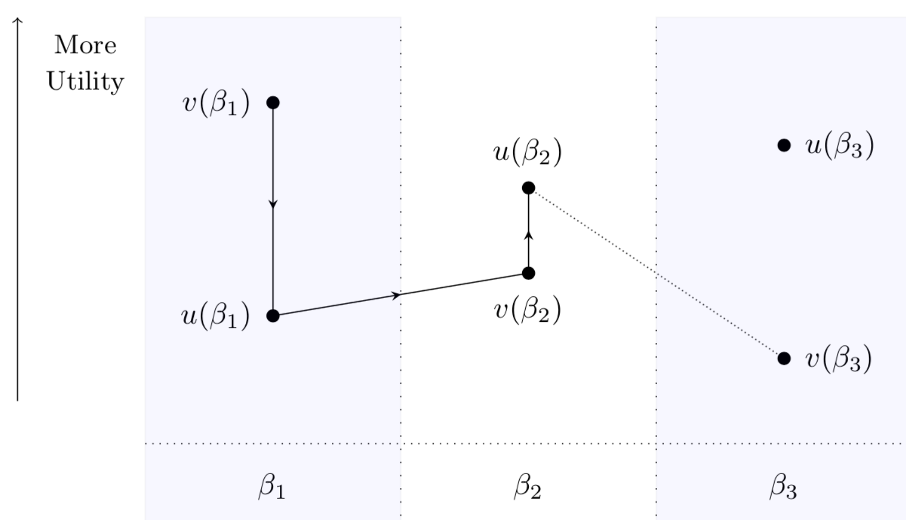

Figure 1: An example search pattern

make any comments about welfare in the general case as we do not assume that goods in different categories can be compared to one other. However when they can be, we can see that agents do not necessarily choose optimally: the best gift overall was in the store β_3, but the parent did not search it as they did not have high enough expectations about β_3. This reflects a situation seen during for example, online shopping, house-hunting, or job-searching, where people settle on items due to incorrect beliefs, risk aversion or search costs, and so do not choose optimally *ex post*.

The model is effectively one of two-stage choice with search. It could be easily extended to an n-stage model of search, where X is organized into categories of categories, etc. In our example, the parent might choose firstly which mall to shop at, and then the shops inside it, and then the aisles within the shops, etc. However for simplicity of analysis, I will develop the model with two stages in this paper.

Properties

WARP Violation

The BOCS model nests the classical model of choice as a degenerate case: when X is partitioned by only one category. However in general, the BOCS model violates the Weak Axiom of Revealed Preference (WARP), the cornerstone of classical choice. For choice functions, this is equivalent[3] to Sen's α:

$$\text{Sen's } \alpha: \qquad x \in T \subseteq S \text{ and } x = c(S) \implies x = c(T)$$

For some X, consider $S = \{x_1, x_2, x_3\}$ where $\kappa_i = \{x_i\}$ for $i = 1, 2, 3$, and let $T = \{x_1, x_3\}$. Suppose that:

$$\begin{array}{ccc} v(\beta_1) = 5 & v(\beta_2) = 4 & v(\beta_3) = 2 \\ u_1(x_1) = 3 & u_2(x_2) = 1 & u_3(x_3) = 3 \end{array}$$

Then it can easily be seen that $c(S) = x_3$, but $c(T) = x_1$, which is a violation of Sen's α and so WARP. More generally, search patterns of a 'diamond' shape like this cause WARP violations: since the agent has high beliefs about how good β_2 will be, they keep searching, but since β_2 is ultimately worse than they thought, they search again to β_3. However, when all of β_2 is removed from the menu, the agent never leaves β_1 in the first place, causing a WARP violation.

A further realization provides an intuition for when the agent keeps searching:

Remark 1. *Say for a menu S and a searched box β_i that $u(\beta_i) \geq v(\beta_i)$. Then $c(S) \in \beta_i$*

The proof is simple, as β_i is only searched if it has the highest $v(\cdot)$ utility out of the not-yet-rejected boxes. The agent does not continue its search if a box is better than they thought they would be *ex ante*. In other words, in order to continue searching, agents need to be disappointed in some sense by what they have seen so far. The BOCS model therefore has the most interesting implications when agents have overly-optimistic expectations, and when search costs are low.

[3] In general, WARP is equivalent to Sen's α and Sen's β. However, Sen's β is trivially satisfied for choice functions. See Sen (1971) for more details

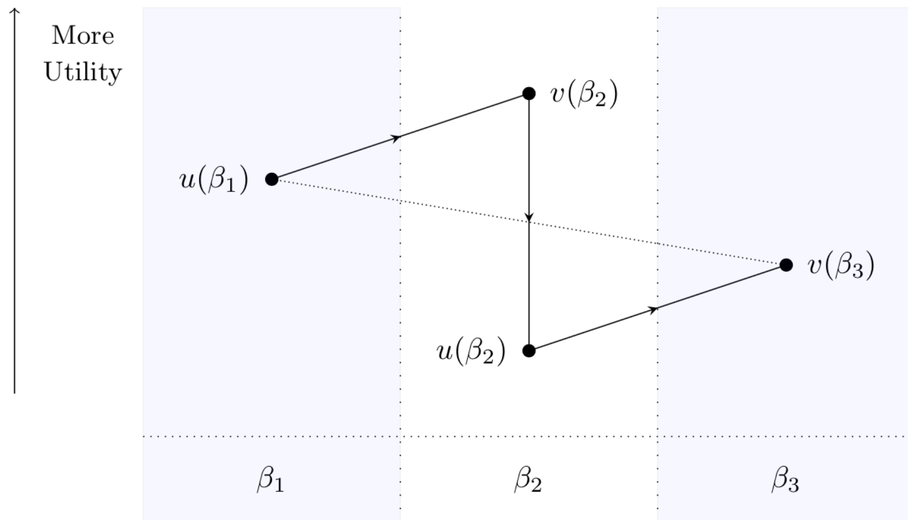

Figure 2: An example 'diamond' WARP violation

Revealing Parameters

All of the parameters of the BOCS model can be revealed from choice data alone under a simple assumption:

$$\text{Richness:} \qquad \forall \kappa_i \in K,\ \exists x \in \kappa_i \ \text{ s.t. } \ \bar{v}(\kappa_i) \le u_i(x)$$

In other words, the $\bar{v}$-utility of each category is no higher than the u-utility of the best good in that category. This arises naturally from the motivation given for $\bar{v}(\kappa_i)$ before: if inaccurate beliefs come from from uncertainty about S, we would expect that regardless of beliefs $\mathbb{E}(\max_{S\cap\kappa_i} u(x)) \le \max_{\kappa_i} u(x)$, and so $\bar{v}(\kappa_i) \le \max_{\kappa_i} u(x)$. Even for our second motivation, the richness condition would only not be satisfied if the agent believed that some category was better than anything that could possibly be in it, which would suggest a much greater kind of irrationality than we try to model in this paper.

In order to reveal parameters, we have to assume that the categories that the agent uses to search with are known. This is perhaps the most contentious assumption in this paper, though is reasonable in many situations: for example, where we might naturally think of different boxes as different shops in a mall or different pages on a website. Further work on a second representation theorem (presented later in this paper) would make it possible to mitigate this assumption.

As preferences within a category are classical, we can easily find the u-best goods in any two categories, which we denote $x_1 \in \kappa_1$ and $x_2 \in \kappa_2$. This also reveals $u_i(\cdot)$ by standard revealed preference tools. We now give the agent the menu $S = \{x_1, x_2\}$. If β_1 is searched first, then by richness we have that $u_1(x_1) \ge \bar{v}(\kappa_1) = v(\beta_1) > v(\beta_2)$, and so $c(S) = x_1$. Analogously, if β_2 is searched first then $c(S) = x_2$. Since exactly one of x_1 and x_2 is chosen, as all preferences are strict, the chosen good from S will be the good that was in the category which was searched first. Using this method, we can identify $\bar{v}(\cdot)$ entirely, which also gives the boxes' search order. Therefore we can identify all parameters in the BOCS model from choice data alone, assuming that K is known.

Representation

The BOCS model is only useful insofar as it helps to explain behavior that is observed in the real world, so it is important to know how it could be falsified. We present three axioms which put structure on how people search through their consideration sets to answer this question. We have a preliminary definition:

Definition 2. *The Search Order $\succ_k$ over K is defined as:*

$$\kappa_i \succ_k \kappa_j \iff c(\{c(\kappa_i), c(\kappa_j)\}) = c(\kappa_i)$$

By definition, $\succ_k$ is semi-connex and asymmetric as $c(\cdot)$ is always a singleton. This is a formal way to state how we might reveal the search order, as noted above. Our axioms are as follows:

A0[4] *Consistent Search Order:* $\succ_k$ is acyclic

This implies that $\succ_k$ is transitive, and so a strict total order over K[5].

A 1. *Costless Removal of Unseen Categories:*

$$c(S) \in \kappa_i \implies c(S) = c(S \setminus \kappa_j) \qquad \forall j > i$$

In other words, if the agent never reached a category, then they would choose the same thing as if it was never available in the first place. In the example of shopping for a gift: if the parent is happy with a gift from one store, they must prefer it to their expectations about other stores. Therefore, closing down an unexplored store would make it weakly worse for them to continue searching than it was before, so they would choose the same thing as they did before.

A 2. *Blind Search: Take any $x \neq c(S)$ with $x \in \kappa_j \cap S$. Then*

$$|\kappa_j \cap S| > 1 \implies c(S) = c(S \setminus x)$$

This axiom can be broken into two cases: if $c(S) \in \kappa_i$ then we separately consider the cases of $j > i$ and $j \leq i$. When $j > i$ this has a similar justification as A1: the agent does not expect to move on to κ_j, and so by removing something from κ_j there is no difference to choices. Another way to rationalize this is to say that the agents' preferences about which goods are in future boxes are unaffected by which goods are in already-seen boxes - they might not even realize if a good is removed from an unsearched category, and hence will make the same choice as before.

On the other hand, any unchosen good in κ_j with $j \leq i$ must have been something that the agent already rejected. Therefore, we can reasonably expect that the agent will continue to do the

[4] Make more proper to line up things later

[5] *Technical details:* suppose that $\succ_k$ was intransitive, such that for some distinct $a, b, c \in K$, $a \succ_k b$, $b \succ_k c$ and $a \not\succ_k c$. The last of which implies $c \succ_k a$ as $\succ_k$ is semi-connex, so a, b, c gives a cycle. Therefore $\succ_k$ acyclic implies $\succ_k$ transitive. $\succ_k$ is a strict total order as it is semi-connex, asymmetric and transitive

same as before if it is removed. In effect, A2 guarantees that WARP violations come from being tempted to move on to unseen boxes, rather than from any mechanism inside a box. This is the most contentious axiom of the model, but it is reasonable to suspect that agents have classical beliefs when consideration sets are defined to be small enough. A2 does not cover the case of when a whole consideration set is removed, as if removing x removes all of $\kappa_j \cap S$ then $\kappa_j \cap S = \{x\}$, so it would have cardinality 1.

A 3. *No Recall:*

$$\begin{cases} \kappa_j \cap S = \emptyset & \forall j < i \\ c(S) \notin \kappa_i \end{cases} \implies c(S) = c(S \setminus \kappa_i)$$

This axiom effectively says that if the agent does not choose from the first box they search, then they would make the same choice as if it was never available in the first place. This captures the idea of no recall.

A small remark is useful for proving results related to choices in this model:

Remark 2. *Specifying the choice at every iteration of the BOCS algorithm specifies exactly one overall choice from the menu.*

We now show the first main result of this paper: that A1-3 are necessary and sufficient to represent the BOCS model. Both sides of the proof are given separately.

Theorem 1. *If $c(\cdot)$ has a BOCS representation then it satisfies A1-3*

Proof. In all cases we follow on from the logic of remark 2.

A1: If $T = S \setminus \kappa_j$ for some $j > i$, then the first $i - 1$ steps are the same: the u-utilities of the all alternatives are the same, and the agent faces the same alternatives to move onto. Define $B_i = B \setminus \{\beta_1, ..., \beta_i\}$ In the i^{th} step, since $c(S) \in \kappa_i$, we have that $u_i(c(S)) \geq \max v(b)$ for all $b \in B_i$. As $B_i \setminus \beta_j \subseteq B_i$, the maximizer of $v(\cdot)$ over the B_i is greater than or equal to the maximizer over $B_i \setminus \beta_j$. Therefore $u_i(c(S)) \geq \max v(b) \geq \max v(b')$ for all $b \in B_i$ and $b' \in B_i \setminus \beta_j$, and hence $c(T) \in \kappa_i$. As all preferences are strict, $u_i(\cdot)$ has a unique maximizer, and so $c(S) = c(T) = c(S \setminus \kappa_j)$.

A2: Suppose $c(S) \in \kappa_i$ and $x \in \kappa_j \cap S$ with $j < i$. As $|\beta_j| = |\kappa_j \cap S| > 1$, removing x will not make β_j empty, and so the agent will still move from box $j - 1$ to j, as they face the same choice (between $\arg\max_{x \in S \cap \kappa_{j-1}} u_{j-1}(x)$ and $\bar{v}(\kappa_j)$). Moreover, the u_j-max good in β_j has a lower utility than when the menu contained x, and so the agent will still continue from box j to $j+1$. As no other part of the agent's preferences have changed, $c(S) = c(S \setminus x)$. Alternatively suppose that $i = j$. As $x \neq c(S)$, $u_i(c(S)) > u_i(x)$. Therefore the agent stops searching in the same box β_i, and chooses the same good from it as before $c(S)$, so $c(S) = c(S \setminus x)$. Finally suppose $j > i$. As removing x does not make β_j empty, $v(\beta_j)$ is the same as before. Therefore the agent still decides to stop in β_i and chooses the same good as before. In all cases, we have $c(S) = c(S \setminus x)$

A3: If $\beta_i = \emptyset$ then this is trivially true as $c(S) = c(S \setminus \emptyset)$ always. If $\beta_i \neq \emptyset$ and $\beta_j = \emptyset$ for all $j < i$ then β_i must be the first box searched in S. If $c(S) \notin \kappa_i$, or equivalently $c(S) \notin \beta_i$, then the BOCS procedure removes β_i from S and starts again. This is the same as saying that the search process effectively starts with $S \setminus \beta_i$, and therefore we have that $c(S) = c(S \setminus \beta_i) = c(S \setminus \kappa_i)$ □

Moreover, A1-3 are sufficient conditions to represent the BOCS model:

Theorem 2. *If $c(\cdot)$ satisfies A1-3 then it has a BOCS representation*

Proof. As K partitions X, each menu S generates a unique family of submenus $\{S_1, ..., S_m\}$ defined by:

$$S_i = (\kappa_i \cap S) \ \cup \ \bigcup_{j>i} \{\kappa_j \ : \ \kappa_j \cap S \neq \emptyset\} \qquad \forall i = 1, ..., m$$

where m is the index of the category containing $c(S)$. Intuitively, each S_i represents what the agent will choose between at iteration i of the algorithm: they can either choose a good $x \in \beta_i$ or they can move on to a non-empty box β_j with $j > i$. Similarly, we define $\{X_1, ..., X_m\}$ as the family of submenus generated by X. We now define a function:

$$c^i(S_i) = \begin{cases} c(S) & c(S) \in \kappa_i \\ \underset{k_j \in S_i}{\arg\max} \succ_k & \text{otherwise} \end{cases}$$

Intuitively, $c^i(S_i)$ represents what the agent chooses at iteration i. It is defined so that it satisfies the important property:

$$c(S) = x \in \kappa_i \iff c^i(S_i) = x \in \kappa_i \text{ and } c^j(S_j) \notin \kappa_j \quad \forall j < i \tag{*}$$

We can see that $c^i(S_i)$ is always non-empty: if $c(S) \notin \kappa_i$, then as $c(S) \in \kappa_m \cap S \neq \emptyset$, we have that $m > i$ and so $\kappa_m \in S_i$. As $\succ_k$ is a strict total order by [A0] and K is finite, $\arg\max_K \succ_k$ is always non-empty. Therefore $c^i(\cdot)$ is a well-defined choice function over X_i.

We now show the main part of the proof: that under A0-3, $c^i(\cdot)$ satisfies WARP. To link $c_i(\cdot)$ to our assumptions (which are about $c(\cdot)$) we need to any given submenu at iteration i $T_i \subseteq S_i$ to an overall submenu $T \subseteq S$. Due to some algebra we find that for any $T_i \subseteq S_i$:

$$T = \Big[T_i \cap \kappa_i\Big] \ \cup \ \bigcup_{j>i} \{\kappa_j \cap S \ : \ \kappa_j \in T_i\}$$

generates T_i at iteration i. We now consider three cases:

Case 1. If T_i removes a set of categories, then as all categories κ_j in S_i have $j > i$, $c(S) = c(T)$ by [A1], so if $c^i(S_i) \notin K$, then $c^i(S_i) = c^i(T_i)$. Alternatively if $c^i(S_i) \in K$ then $c^i(S_i)$ would not be removed by T_i, and so $c^i(S_i) = \arg\max \succ_k = c^i(T_i)$

Case 2. If T_i removes some, but not all, goods in S_i, then $c(S) = c(T)$ by [A2], and so $c^i(S_i) = c^i(T_i)$

Case 3. If T_i removes all goods in S_i, then we must have had that $c^i(S_i) \in K$. Therefore $c^i(S_i) = \arg\max_{S_i} \succ_k = \arg\max_{T_i} \succ_k = c^i(T_i)$

As any subset of S_i is made up of applying a mix of cases 1, 2 and 3, we see that $c^i(S_i) = c^i(T_i)$ for any T_i containing $c^i(T_i)$. In other words, $c^i(\cdot)$ satisfies Sen's α and so WARP. Hence, there exist functions $\bar{u}_i : X_i \to \mathbb{R}$ for which $c^i(S_i) = \underset{x \in S_i}{\arg\max}\, \bar{u}_i(x)$ for every i. Since preferences according to $c^i(\cdot)$ over categories $\kappa \in K$ are ordered the same way for every i (because they are ordered according to $\succ_k$), we can specify without loss of generality that $\bar{u}_i(\kappa)$ is the same across all i for any given $\kappa \in K$, and relabel $\bar{v}(\kappa) := \bar{u}_i(\kappa)$. As $\succ_k$ is a strict total order, $\bar{v}(\cdot)$ will be injective. Finally, define u_i to be the restriction of $\bar{u}_i$ to goods.

[A3] allows the agent to solve the RHS of (*) by an iterative process. An equivalent way to finding $c(S)$ in one step is to search κ_1, and then remove κ_1 from the menu if $c^1(S_1) \notin \kappa_1$, and iterate this process until there is a solution, because removing κ_1 from S will not affect the preferences over S_i for $i > 1$. Since $c^i(S_i) = \arg\max_{x \in \beta_i} u_i(x)$ if and only if:

$$\max_{x \in \beta_i} u_i(x) \geq \max_{\kappa_j}\{\bar{v}(\kappa_j) : j > i \text{ and } \kappa_j \cap S \neq \emptyset\} = \max_{\beta_j : j > i} v(\beta_j)$$

the agent is choosing equivalently to a BOCS model with parameters u_i, v and K. $\square$

Therefore, the BOCS model can be falsified if and only if one of A1-3 does not hold. The three axioms are quite strong when taken together, as they rule out all WARP violations not in the shape of Figure 2. However, for suitably-defined consideration sets the axioms seem reasonable.

Note for now that we can re-cast A1, A2 in a way that is useful in later proofs.

Lemma 1. *$A1, A2 \iff A1', A2'$ where:*

A1': $c(S) \in \kappa_i \implies c(S) = c(S \setminus x) \quad \forall x \in \kappa_j$ for some $j \geq i$ with $x \neq c(S)$

A2': $c(S) \in \kappa_i \implies c(S) = c(S \setminus T)$ where $T \subset \kappa_j \cap S$ for some $j < i$

Towards a Second Representation Theorem

The theorem above is useful when K and $\succ_k$ are known up to a small number of possibilities, but we can not expect to know much about categories in the vast majority of cases. Theoretically, the BOCS model could be falsified by checking whether every possible partition of X can be used as a family of categories to rationalize $c(\cdot)$, but this is so computationally intensive that it could rarely be used in practice[6]. We therefore want a second representation theorem which gives conditions under which *any* set of categories K and ordering on them $\succ_k$ can rationalize choice.

To build towards the result, we firstly restrict attention to the case where $|K| = 3$. From the proof of theorem 2, there exists a relation $\succ_i$ for every category κ_i which satisfies WARP and represents the choice of the agent over all of the goods in the current category and moving onto another category. Intuitively, we can see that when $|K| = 3$, we all of the goods are a member of exactly one of $\alpha_1, ..., \alpha_6$, where:

[6]Extra Details: If we let $|X| = \xi$, then the number of ways to partition X is the ξ^{th} Bell number B_ξ, which can be calculated as the ξ^{th} derivative of $\exp(\exp(t) - 1)$ evaluated at $t = 0$. The Bell numbers grow very quickly: $B_{15} > 1.9 \times 10^8$ and $B_{25} > 4.4 \times 10^{17}$ (see the Online Encyclopedia of Integer Sequences (2019)). A rough calculation shows that the world's fastest supercomputer running continuously with perfect choice data would not finish checking all categories of a set $|X| \approx 30$ before the expected end of the universe. This is especially damaging as we would expect the BOCS model to be more useful when there are multiple consideration sets, which may require X to be large.

κ_1	κ_2	κ_3
α_1	α_4	α_6
$\bar{v}(\kappa_2)$	$\bar{v}(\kappa_3)$	
α_2	α_5	
$\bar{v}(\kappa_3)$		
α_3		

The columns of the table represent categories $\kappa_1, \kappa_2, \kappa_3$, and within each category κ_i the higher goods have a higher $\succ_i$ ranking. $\alpha_1, ..., \alpha_6$ are sets representing positions that each good in X could be in.

I now present a series of lemmas to give some intuition about the significance of each group, and of the way that categories are formed.

Lemma 2. *Suppose that $c(\cdot)$ has a BOCS representation and $\exists S, x$ s.t. $c(S \setminus x) \neq c(S) \neq x$. Then $[c(S \setminus x)] \succ_k [x] \succ_k [c(S)]$*

In other words, any WARP violations have implications for how the original choice, the good that was removed, and the new choice must be ordered. Intuitively, this captures the idea that WARP violations are 'diamond'-shaped in this model. Recall that $[y]$ is the category containing y for all $y \in X$ and that $\succ_k$ represents the search order over K.

Lemma 3. *$c(x, y, z) = z$ and $c(x, z) = x$ if and only if $x \in \alpha_2$, $y \in \alpha_5$ and $z \in \alpha_6$*

In other words, all WARP violations take place using goods that are in α_2, α_5 and α_6, and if a menu contains at least one good in each set then there exists a submenu of it where we can generate a WARP violation.

We also want to have some intuition about α_1 and α_4. To do so, we define the relation $\succ_{12}$ over $\alpha_1 \cup \alpha_4$ as:

$$x \succ_{12} y \iff x \succ_1 y \text{ or } x \succ_2 y \text{ or } [x \in \alpha_1 \text{ and } y \in \alpha_2] \qquad \forall x, y \in \alpha_1 \cup \alpha_4$$

Intuitively, $\succ_{12}$ is the ranking of elements of $\alpha_1 \cup \alpha_4$ such that the α_1 elements come first, in $\succ_1$ order, and then the α_4 elements in $\succ_2$ order. $\succ_{12}$ is complete and transitive over $\alpha_1 \cup \alpha_4$. We can see that if there is an element of $\alpha_1 \cup \alpha_4$ in a menu S, then $c(S) \in \alpha_1 \cup \alpha_4$. More formally, we can see that:

Lemma 4. *For any menu S, if $S \cap (\alpha_1 \cup \alpha_4) \neq \emptyset$ then $c(S) = \underset{S \cap (\alpha_1 \cup \alpha_4)}{\arg\max} \succ_{12}$*

From lemma 4 it follows that there can be no WARP violations if there is an element of α_1 or α_4 in a menu. This is because if $S \cap (\alpha_1 \cup \alpha_4) \neq \emptyset$ then $c(S) = \arg\max S \cap (\alpha_1 \cup \alpha_4) \succ_{12}$. Hence if z is an unchosen good, $c(S) \in S \setminus z$, and so $c(S) = c(S \setminus z)$ since $\succ_{12}$ satisfies WARP.

Through lemmas 3 and 4, we can think of the sets $\alpha_1, ..., \alpha_6$ in terms of the ways in which they contribute towards forcing, or preventing, a WARP violation. While I have not analyzed α_3, it is clear from the table that an element of α_3 is only chosen if none of $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ or α_6 have an element that is in the menu, which means that there is little of it to be analyzed.

Thinking of each of the sets in terms of how they allow or prevent a WARP violation motivates a theorem on the uniqueness of categories, which serves as a basis for a representation theorem. Note that a set of categories K over X is uniquely determined by the set $\alpha_1, ..., \alpha_6$ that each good is in. I refer to these sets in a set of categories K using the normal labels and their equivalents in a set of categories $\hat{K}$ as $\hat{\alpha}_1, ..., \hat{\alpha}_6$. The theorem effectively says that elements of α_1 and α_4 can be arbitrarily moved between κ_1 and κ_2 in a way that maintains the $\succ_{12}$ order. It is stated formally as follows:

Theorem 3. *Suppose that $c(\cdot)$ has a BOCS representation which has three categories only. Then $(u_i, \bar{v}, K)$ represents $c(\cdot)$ if and only if $(\hat{u}_i, \hat{\bar{v}}, \hat{K})$ represents $c(\cdot)$, where:*

- *$\alpha_1 \cup \alpha_4 = \hat{\alpha_1} \cup \hat{\alpha_4}$ and $\succ_{12}$ is the same over both sets*
- *α_1 and α_4 contain an element such that they satisfy the richness condition.*
- *$\hat{\alpha_2} = \alpha_2$, $\hat{\alpha_5} = \alpha_5$, $\hat{\alpha_6} = \alpha_6$*
- *$u_i = \hat{u}_i$ and $\bar{v} = \hat{\bar{v}}$ in all other cases*

Proof. We can see that the conditions imply that $\alpha_3 = \hat{\alpha_3}$ since all other elements of K and $\hat{K}$ are specified, and so the only potential difference between K and $\hat{K}$ is the ordering of elements into α_1 and α_4. In order to show equivalence, we must show that $(u_i, \bar{v}, K)$ and $(\hat{u}_i, \hat{\bar{v}}, \hat{K})$ give the same choice for any menu S

Firstly suppose there exists an $x \in S$ such that $x \in \alpha_1 \cup \alpha_4$. Then $c(S) = \arg\max_{S \cap (\alpha_1 \cup \alpha_4)} \succ_{12} = \arg\max_{S \cap (\hat{\alpha_1} \cup \hat{\alpha_4})} \succ_{12} = \hat{c}(S)$. If there does not exist such an x, then the two representations of S are identical as $\alpha_i = \hat{\alpha_i}$ for all $i \in \{2, 3, 5, 6\}$, and the utilities are the same. Therefore in all cases, $c(S) = \hat{c}(S)$ □

A much less important lemma also gives some intuition on the uniqueness of u_i and $\bar{v}$ themselves:

Lemma 5. *$(u_i, \bar{v}, K)$ represents $c(\cdot)$ if and only if $(\hat{u}_i, \hat{\bar{v}}, K)$ represents $c(\cdot)$, where for all $i = 1, ..., n$, $x \in X$ and $\kappa_i \in K$:*

$$\hat{u}_i(x) = f_i \circ u_i(x)$$
$$\hat{v}(\kappa_i) = f_{i-1} \circ \bar{v}(\kappa_i)$$

and $\{f_0, f_1, ..., f_n\}$ are strictly increasing functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f_0 \circ \bar{v}(\kappa_1) > f_1 \circ \bar{v}(\kappa_2) > ... > f_{n-1} \circ \bar{v}(\kappa_n) \quad (**)$$

Given we have only considered representations of $c(\cdot)$ with three categories, we need to see how $c(\cdot)$ is represented when there are more than three categories. Due to A1-3, it turns out that it extends very naturally:

Lemma 6. *(u, v, K) represents $c(\cdot)$ for all menus S if and only if $(u, v, \{\kappa_1, \kappa_2, \kappa_3\})$ represents $c(\cdot)$ over all menus $S' \subseteq \kappa_1 \cup \kappa_2 \cup \kappa_3$ for all $\kappa_1, \kappa_2, \kappa_3 \in K$*

Theorem 3 tells us that we can arbitrarily change around elements of α_1 and α_4 in any way so long as they keep the same order, and the model still satisfies the richness condition needed to identify parameters. This is quite a strong condition: it requires that the elements which lead to WARP violations ($\alpha_2, \alpha_5, \alpha_6$ by lemma 4) are kept in the same order, and that elements of α_1, α_4 cannot move around too much. While I cannot yet present a full representation theorem, I conjecture two constrictions to capture this intuition.

Firstly, I define a relation which captures the ordering of the boxes. $\succ$ is defined as the smallest relation on X such that:

$$\exists S, x \text{ s.t. } c(S) \neq c(S \setminus x) \neq x \implies c(S \setminus x) \succ x \succ c(S)$$

B 1. *Uni-Directional WARP violations: $\succ$ is acyclic*

Intuitively, the BOCS model captures WARP violations that occur because removing a good x can mean that agents do not consider as much of the menu as they did before. More specifically, removing x effectively lowers the expected benefits of searching, as they had expected that the best thing they would get if they kept searching was x. This places a natural order on how the agent searches their menu, which $\succ$ captures. The assumption that $\succ$ is acyclic forces a kind of consistency in the way that agents search; it prevents them from sometimes searching κ_i before κ_j and other times searching κ_j before κ_i. This is quite an intuitive criterion: it seems very unlikely that the mother in our example would be so influenced by what she saw in shop β_1 that she would now first look in β_2 instead of β_3 when she previously thought that β_3 would be better *ex ante* than β_2. Similarly, it adds a consistency to how psychological constraints are constructed and searched.

Secondly, in order to capture the idea that the $\succ_{12}$ has to be maintained, another axiom would be along the lines of:

B 2. *Consistent 'best' goods: $\bigcup_{j>i} \succ_{i,j}$ is acyclic*

From this I present a conjecture of the second representation theorem:

Conjecture. *There exists a representation (u_i, v, K) of $c(\cdot)$ if and only if $c(\cdot)$ satisfies B1 and B2*

The justification for this conjecture is as follows. By having an acyclic $\succ$ relation, we would be able to consistently order all objects of α_2, α_5 and α_6 type without cycles. Moreover, an axiom such as B2 would allow us to consistently order all of the sets of the α_1 or α_4 form. This effectively makes B1 a condition on the parts of categories which cause WARP violations, and B2 is a condition on the parts where there will never be WARP violations if they exist. With both orders being acyclic, both of their transitive closures are antisymmetric. I conjecture that we could group together objects of $\bigcup \succ_{ij}$ in a way such that if $(x, y) \in \bigcup \succ_{ij}$ then we do not have that y is in an earlier group than x, with the group order being acyclic. Then, I conjecture that since the group order and $\succ$ would be acyclic, we could connect up the two relations, and extend them to a total order $\succ_k$ over X, where the indifference curves of $\succ_k$ are the categories and $\succ_k$ itself represents the search order. I have two remarks about B1-B2 that make me relatively confident that the conditions for the second representation theorem are similar to B1 and B2:

Remark 3. *B1 is necessary in a BOCS model*

This can be quite easily seen from lemma 2. Since $\succ$ is the smallest relation such that if x causes a WARP violation over the menu S then $c(S \setminus x) \succ x \succ c(S)$, this lemma tells us that if $x \succ y$ then $[x] \succ_k [y]$, so a cycle over objects in $\succ$ would also be a cycle over their categories in $\succ_k$. However, since $\succ_k$ is a strict total order over K, we know that $\succ_k$ is acyclic, and so $\succ$ is acyclic in a BOCS model.

Remark 4. *B2, or a variant of it, might be satisfied for all choice functions*

We can see if there are only three categories then B2 is satisfied: by its definition, $\succ_{12}$ is irreflexive, complete and transitive, and so it is acyclic by a standard proof. As this is the only relation of this type when there are three categories, this is B2. Drawing a table for four categories manually shows that the four relations generated by each group of three categories can be placed together acyclically. A proof by induction on the number of categories in K might be able to show that B2 is always satisfied.

However, it is likely to be a variant of B2 rather than B2 itself which is needed in a representation theorem. The proof mentioned above would be tautological: it would work using induction on the number of categories K, but generate categories by using an extension of the group order over $\alpha_1 \cup \alpha_4$, which only exists if B2 is satisfied. More likely, a related axiom to B2 that captures the conditions in theorem 3 would be needed to find a second representation theorem. Such work will be undertaken in future editions of this paper.

Applications

After establishing the properties of the BOCS model, it is important to show where it is useful: it is a more general version of the classical utility model that can rationalize several empirical phenomenon that the classical model cannot. I present results on the attraction effect, the endowment effect, how the BOCS model compares to models of reference dependence, and how a form of reference dependence can be tested for.

The Attraction Effect

Suppose that a consumer is looking to buy clothes from a second-hand store, and faces the following menu T. The table below shows (price, condition) pairs of the u_i-bets goods in each β_i:

	[Competitor] β_1	[Decoy] β_2	[Target] β_3
Price (p):	\$3.00	\$2.50	\$1.50
Condition (c):	2 units	1 unit	1.5 units

Let $S = T \backslash \beta_2$. Suppose further that she prefers goods which are in better condition and are cheaper, *ceteris paribus*. Moreover in this example, she searches from highest to lowest price, possibly as a heuristic for the goods which are in the best condition. If only S is available, then after searching β_1 she would have to weigh up the choice between a high-priced good which is in good condition (the good in β_1) or a low-priced good in bad condition (β_3). However, if the menu faced is instead T - with the decoy goods β_2 added into the consideration set - then she might find that the she continues searching β_2 where she did not before, and ends up choosing from β_3.[7]. This is known as the attraction effect, which was first explored by Huber et al. (1982), and has been demonstrated empirically many times since then[8]. In essence, it creates a WARP violation by adjusting the consumer's reference: as the decoys are better than the competitors in one dimension (here: price), but worse than the target in both, the target is seen as more attractive, and so is more likely to be purchased.

The explanation that the BOCS model gives for the attraction effect is slightly unorthodox; it is generated by a mismatch between *ex ante* and *ex post* valuations for a good, here represented by u_i and v. By remark 1, I interpreted search in the BOCS model as coming from a form of disappointment. Hence, the attraction effect occurs because the addition of the decoy goods makes the agent more optimistic about goods which have not been searched yet.

The model also provides some hypotheses about when the attraction effect should be more pronounced. Firstly since it is caused by a 'diamond' WARP violation, we would expect the attraction effect to be more pronounced when there are large differences between u_i and v. This is exactly in agreement with Huber et al. (2014), whose commentary notes from decades of studies that the attraction effect is strongest when consumers are unfamiliar with the goods being offered. Furthermore, it is clear from the model that the way in which the consumer categorizes the menu and searches the categories should be important in determining the size of the effect. In particular:

Remark 5. *There are no attraction effects if* $|K| \leq 2$

In other words, at least three boxes are needed to create an attraction effect. If interpret boxes as consideration sets, then this suggests that the attraction effect should be larger in markets with highly differentiated products, as then we might reasonably expect that the consumer would partition X more finely. For the same reason, we would expect the attraction effect to be stronger when consumers are searching with more general heuristics (such as wanting to buy 'some kind of food' rather and a specific type). From lemma 2, we can more accurately say that the targets, the decoys and the competitors need to be considered in different categories. As such, we would only expect the attraction effect to occur when the decoys are sufficiently differentiated from the competitors and the targets.

Remark 6. *The decoy effect is the same for any number of decoys, provided there is at least one*

In other words, one decoy is enough to tempt the consumer into continuing to search. This means that we would not expect to see a stronger attraction effect when several decoys are added to the market. If we did, this would be evidence that individuals do not search 'blind' - in other words, in a way that respects A1'.

[7]This can be done here for example by supposing that for $i = 1, 2, 3$, $v(\beta_i) = p$ and $u(\beta_i) = c - \frac{1}{10}p$

[8]See Ratneshwar et al. (1987) for a summary of the empirical literature.

Remark 7. *A welfare-reducing attraction effect is only supported if it is sufficiently costly to revisit boxes*

While the BOCS model considers search without recall, a simple extension to partially costly search shows that if the agent can pay a fee δ to revisit a box they have already seen, then there exists a $\hat{\delta} > 0$ such that for all $\delta < \hat{\delta}$ there is no attraction effect. An implication of this is that the attraction effect cannot exist in models with sequentially increasing consideration sets, such as those studied by Masatlioglu and Nakajima (2013), as they are models of costless recall and so effectively have $\delta = 0$. The BOCS model therefore predicts that the attraction effect is most prominent when decisions are difficult or impossible to reverse, such as buying an item with limited supply, or when there are physically constraints like those mentioned in the introduction.

The Endowment Effect

Suppose that a family is deciding whether or not to move house. The houses $h \in H$ which they are considering are represented as (quality, price) pairs, and classical preferences over them are represented by a utility function $u : H \times \mathbb{R} \to \mathbb{R}$, where $u(h, p)$ is decreasing in p. Importantly, their current house h_0 is an element of H, as they have the choice of deciding not to move at all.

In order to calculate the willingness to pay (WTP) and willingness to accept (WTA) for each house, we consider different models for K. If the family did not own h_0 and was looking to move into the neighbourhood, then their WTP for h_0 would be the price that makes them indifferent between h_0 and $\bar{h}$, the u-best good in $H \setminus h_0$. More formally:

$$p_{wtp} : \qquad u(h_0, p_{wtp}) = u(\bar{h}, \bar{p})$$

where $\bar{p}$ is the price of $\bar{h}$. If instead the house were owned by the family, we might suspect that they first decide whether or not to stay with their current house, and then if they decide to sell their house they then decide which other house to buy. This can be modelled as the family considering the set of houses using the categories $K = \{\kappa_1, \kappa_2\}$, where $\kappa_1 = \{h_0\}$ and $\kappa_2 = H \setminus h_0$. In this case, WTA would be the price of h_0 that would make the family indifferent between keeping their house and moving, and so:

$$p_{wta} : \qquad u(h_0, p_{wta}) = \bar{v}(\kappa_2)$$

From this, we find that iff $u(\bar{h}, \bar{p}) < \bar{v}(\kappa_2)$, then $u(h_0, p_{wtp}) < u(h_0, p_{wta})$ and so:

$$p_{wta} > p_{wtp}$$

which is the endowment effect, notably explored by Kahneman et al. (1991). As with the attraction effect, the BOCS model provides an explanation for the endowment effect due to consideration sets and a mismatch between *ex ante* and *ex post* beliefs about a category. There could be many explanations for why $u(\bar{h}, \bar{p}) < \bar{v}(\kappa_2)$ in situations such as these. In this example, risk aversion about $\bar{h}$ being taken off of the market after selling h_0 might be one factor. Less classically, we might suspect that the agent has fundamentally different preferences over $H \setminus h_0$ while they own a house compared to when they do not; for example, they might undervalue the fact that a house $h \in H \setminus h_0$

is in the catchment area of a good school while they are living in h_0 - they might only take this factor fully into account when their children have been taken out of the school next to h_0.

This and other potential features of the BOCS model give it a symmetric interpretation to its justification using reference points. In Kahneman, Knetsch and Thaler's classic paper (1990), the authors offer students the chance to trade pens for mugs and vice versa to elicit the endowment effect. They conclude that the endowment effect occurs because students systematically add value to their endowments; they begin to value a mug more than they did before simply because they have been assigned it. The interpretation of the BOCS model in this context is that mug-endowed agents are not fully considering what it would be like to have a pen while they still have a mug. In other words, they are not systematically *over*-valuing mugs because they become attached to them, but are systematically *under*-valuing pens because they are not fully considering them while they still have mugs.

While these two concepts sound like essentially the same thing (up to a scaling of the utility function), an experimental intervention could distinguish between the two. An example might be to first ask subjects to write down their WTP for mugs and pens by asking them hypothetically whether or not they would prefer the object or a certain amount of money, with the knowledge that one of the decisions would be randomly implemented (to make this mechanism incentive compatible). This experiment should then give all subjects endowed with mugs or pens the opportunity to trade with one another. Finally, for all those who did not trade, the experiment could them the same hypothetical exercise as earlier, where the subjects are also aware that one of the decisions will be implemented randomly. The experiment would distinguish between players who did not trade because they became attached to their object (reference-dependent), and those who did not trade because they did not fully consider the other objects (search-biased). Reference-dependent agents endowed with mugs would have the same WTP for pens in both rounds of questions, as only their valuation of mugs changes. On the other hand, search-biased agents endowed with mugs would have a lower WTP for pens in the second round, as the fact that they are endowed with a mug makes them value the pens less. Both types could be distinguished from classical agents by the proportion of trades carried out in the intermediate stage. An experiment such as this could serve as a further contribution to our understanding of the endowment effect, if data did not strongly reject the BOCS model as a way to understand how subjects make choices.

Testing

The set of categories that rationalize choices is not unique in general: if $c(\cdot)$ satisfies WARP then having all of the goods in X in one category, or having them all in separate categories with $u_i(x) = \bar{v}(x)$ both rationalize $c(\cdot)$. However, we can use some of the results before in the section on the second representation theorem to falsify a form of reference dependence: whether a reference point is in the first category considered. In order to do this, we need the reference point to be both observable (as it is not determined endogenously) and for it to be actually a member of X. For this reason, we focus on testing status quo bias: the tendency for what an agent currently has to be considered first. We can also reasonably assume that the status quo q is observable in many cases. We are therefore interested in testing:

$$H_0 : q \in \kappa_1 \quad \text{vs.} \quad H_1 : q \notin \kappa_1$$

The following lemma provides a way to falsify H_0:

Lemma 7. *Suppose a choice function $c(\cdot)$ has a BOCS representation. If:*

$$\exists S \text{ s.t. } c(S \setminus q) \neq c(S) \neq q \tag{C1}$$

or:

$$\exists T, x \in T \text{ s.t. } x \neq q = c(T) \text{ and } c(T \setminus x) \neq q \tag{C2}$$

then H_0 is falsified.

The test agrees with some of the intuition of the properties a good should have if it is considered first. C1 says that removing it should not cause preferences over other goods to systematically change, which makes sense in the same way that A3 does; if an agent does not like a good q then they often would effectively start again as if q were not there. Moreover, C2 says that if the agent would choose the status quo q, then their choice cannot be changed by removing another item. This is also intuitive: if q is the first option that the agent searches, removing another item would likely mean they still do not want to keep searching. Lemma 7 does not test that $\kappa_1 = \{q\}$, but such a test could easily be constructed by testing the joint hypotheses that $q \in \kappa_1$ and that no other goods are in the same category as q.

Conclusion

The BOCS model presented here is clearly not applicable to all situations where agents have limited consideration, as sometimes the assumption of search without recall is unreasonable. However, it is highly applicable to cases where search costs are high or when agents do not have fully complete or transitive preferences. The model can be interpreted as a dual to that of reference dependence, generating much of the same effects and nesting classical choice as a special case. Further research can be done to relax the model's assumptions of strict preference, to prove a second representation theorem, and to test its different interpretation to classic experiments such as that by Kahneman, Knetsch and Thaler. Ultimately, this paper contributes to the literature by adding another perspective on bounded rationality, which can help us get closer to understanding choice, even in very complicated situations.

Appendix (Extra Proofs)

Remark 1: Say for a menu S and a searched box β_i that $u(\beta_i) \geq v(\beta_i)$. Then $c(S) \in \beta_i$

Proof. As β_i is searched, $v(\beta_i) \geq v(\beta_j)$ for all $j \neq i \in B \setminus \bigcup_{k \leq i} \beta_k$. Therefore, $u(\beta_i) \geq v(\beta_i) \geq v(\beta_j)$, and so by definition, $c(S) = \arg\max_{x \in \beta_i} u_i(x) \in \beta_i$ □

Lemma 1: $A1, A2 \iff A1', A2'$

Proof. (If:) Presume that A1', A2' are satisfied. Then:

A1: As $j > i$, $c(S) \notin \kappa_j$. Therefore by repeated application of A1', $c(S) = c(S \setminus \bigcup_{x \in \kappa_j} x) = c(S \setminus \kappa_j)$, which is A1. **A2:** For $j \geq i$ and $x \neq c(S)$, $c(S) = c(S \setminus x)$ always, by A1'. Moreover for $j < i$, we always have that $x \neq c(S)$ as $c(S) \in \kappa_i$. If $|\kappa_j \cap S| > 1$, then $x \subset \kappa_j \cap S$. Therefore by A2', $c(S) = c(S \setminus x)$, and so A2 is satisfied

(Only if:) Presume A1, A2 are satisfied. Then:

A1': For $i = j$, $x \neq c(S)$ and $x \in \kappa_i \cap S$ imply that $|\kappa_i \cap S| > 1$. Therefore by A2, $c(S) = c(S \setminus x)$. Moreover for $j > i$, if $|\kappa_j \cap S| > 1$ then by the same argument $c(S) = c(S \setminus x)$. If instead $|\kappa_j \cap S| = 1$ then $\kappa_j \cap S = \{x\}$. Therefore by A1, $c(S) = c(S \setminus \kappa_j) = c(S \setminus x)$. Together, these results give us A1'. **A2':** If $T \subset \kappa_j \cap S$ then we must have that $|\kappa_j \cap S| > 1$. Therefore by repeated application of A2, $c(S) = c(S \setminus_{x \in T} x) = c(S \setminus T)$, which is A2'

And hence A1 and A2 are satisfied if and only if A1' and A2' are satisfied. □

Lemma 2: Suppose that $c(\cdot)$ has a BOCS representation and $\exists S, x$ s.t. $c(S \setminus x) \neq c(S) \neq x$. Then $[c(S \setminus x)] \succ_k [x] \succ_k [c(S)]$

Proof. Define $[y]$ as the category containing y for any $y \in X$. Take any S, x such that $c(S \setminus x) \neq c(S) \neq x$. If $[c(S)] \succ_k [x]$ or $[x] = [c(S)]$, then from A1' we have that $c(S) = c(S \setminus x)$ which is a contradiction, and so $[x] \succ_k [c(S)]$. We now also show that $[c(S \setminus x)] \succ_k [x]$. Suppose that $c(S \setminus x) \in [x]$, and therefore $|[x]| > 1$, making $\{x\} \subset [x]$. As $[x] \succ_k [c(S)]$, we have that $c(S) = c(S \setminus x)$ from A2. This is a contradiction and so $c(S \setminus x) \notin [x]$. Hence either $[c(S \setminus x)] \succ_k [x]$ or $[x] \succ_k [c(S \setminus x)]$ as $\succ_k$ is a strict total order over K. Suppose that $[x] \succ_k [c(S \setminus x)]$, and we define $S' := S \setminus \bigcup_{[y] \succ_k [x]} [y]$ - in other words, removing all boxes before $[x] \cap S$ so that $[x] \cap S$ is the $\succ_k$-first box in S'. Then by a repeated application of A3, $c(S) = c(S')$. Moreover, since we assume that $[x] \succ_k [c(S \setminus x)]$, we also have that $c(S \setminus x) \notin \bigcup_{[y] \succ_k [x]} [y]$, and therefore by A3 we also have that $c(S \setminus x) = c(S' \setminus x)$. In case $[x] \cap S$ is not a singleton, we define $S'' = S' \setminus ([x] \cap (S \setminus x))$ - in other words, with all elements of $[x] \cap S$ removed apart from x itself. As $[x] \cap S''$ is the $\succ_k$-first box in S'', and we already know that $[x] \succ_k [c(S)]$, we know from A2' that $c(S'') = c(S')$ and $c(S'' \setminus x) = c(S' \setminus x)$. As $[x] \cap S'' = \{x\}$, we therefore have from another application of A3 that $c(S'') = c(S'' \setminus x)$. Therefore, $c(S) = c(S') = c(S'') = c(S'' \setminus x) = c(S' \setminus x) = c(S \setminus x)$, which is a contradiction. Hence we do not have $[x] \succ_k [c(S \setminus x)]$ and so $[c(S \setminus x)] \succ_k [x]$. In other words:

$$c(S \setminus x) \neq c(S) \neq x \implies [c(S \setminus x)] \succ_k [x] \succ_k [c(S)]$$

□

Lemma 3: $c(x,y,z) = z$ and $c(x,z) = x$ if and only if $x \in \alpha_2$, $y \in \alpha_5$ and $z \in \alpha_6$

Proof. **If:** Suppose $x \in \alpha_2$, $y \in \alpha_5$ and $z \in \alpha_6$. Then we know that $x \in \kappa_1$, $y \in \kappa_2$, $z \in \kappa_3$, $\bar{v}(\kappa_3) < u_1(x) < \bar{v}(\kappa_2)$ and $u_2(y) < \bar{v}(\kappa_3)$. When faced with the menu $\{x,y,z\}$, all boxes are non-empty. As $u_1(x) < \bar{v}(\kappa_2)$, the agent continues to β_2. Moreover, $u_2(y) < \bar{v}(\kappa_3)$, so the agent continues to β_3, and hence $c(x,y,z) = z$. However when faces with the menu $\{x,z\}$, β_2 is empty, and so since $\bar{v}(\kappa_3) < u_1(x)$ we know that $c(x,z) = x$.

Only if: Suppose $\exists x,y,z$ such that $c(x,y,z) = z$ but $c(x,z) = x$. Since this is a WARP violation, we know from lemma 2 that $[x] \succ_k [y] \succ_k [z]$. Since $|K| = 3$ and $\succ_k$ is a strict total order over categories, we must have that $x \in \kappa_1$, $y \in \kappa_2$ and $z \in \kappa_3$. If $u_1(x) > \bar{v}(\kappa_2)$ then $c(x,y,z) = x$, which is a contradiction and so $x \notin \alpha_1$. Similarly, if $u_1(x) < \bar{v}(\kappa_3)$ then $c(x,z) = z$, which is a contradiction and so $x \notin \alpha_3$. Moreover, $u_2(y) \geq \bar{v}(\kappa_3)$ then $c(x,y,z) = y$, which is a contradiction, so we know that $y \notin \alpha_4$. The only possibilities that survive this logic are that $x \in \alpha_2$, $y \in \alpha_5$ and $z \in \alpha_6$, which was what we wanted to show. $\square$

Lemma 4: For any menu S, if $S \cap (\alpha_1 \cup \alpha_4) \neq \emptyset$ then $c(S) = \underset{S\cap(\alpha_1\cup\alpha_4)}{\arg\max} \succ_{12}$

Proof. We note firstly that such a unique $\arg\max \succ_{12}$ exists because $\succ_{12}$ is complete, irreflexive and transitive. If $x \in \alpha_1$ then $u_1(x) > \bar{v}(\kappa_2)$ and so $\max_{y \in \kappa_1 \cap S} u_1(y) > \bar{v}(\kappa_2)$, meaning that $c(S) = \arg\max_{S\cap\alpha_1} \succ_1$. If instead $S \cap \alpha_1 = \emptyset$ but $x \in S$ then we find that $\nexists y \in \kappa_1 \cap S$ such that $u_1 > \bar{v}(\kappa_2)$ and so the agent searches κ_2. By the same argument as above, we can then see that $c(S) = \arg\max_{S\cap\alpha_2} \succ_2$. Hence in summary, if $S \cap (\alpha_1 \cup \alpha_4) \neq \emptyset$ then:

$$\begin{aligned} c(S) &= \begin{cases} \arg\max_{S\cap\alpha_1} \succ_1 & \text{if } \exists x \in \alpha_1 \cap S \\ \arg\max_{S\cap\alpha_4} \succ_2 & \text{otherwise} \end{cases} \\ &= \underset{S\cap(\alpha_1\cup\alpha_4)}{\arg\max} \succ_{12} \end{aligned}$$

Which was what we wanted to prove. $\square$

Lemma 5: Uniqueness of u_i, v

Proof. Using the same idea as in the proof of Theorem 2, we think of the BOCS model's choice $c(S)$ as a sequential choice over sets $c'(S_i)$, where S_i consists of $\kappa_i \cap S$ and all non-empty categories after κ_i. By (**), $\bar{v}$ and $\hat{v}$ order the categories $\kappa_i \in K$ in the same way, and so the agent will search them in the same order. Moreover, at each iteration i of the algorithm and for all $x \in \kappa_i \cap S$, the agent chooses

$$\begin{aligned} c'(S_i) &= \arg\max\{\ \bar{v}(\kappa_{i+1})\ ,\ u_i(x)\ \} \\ &= \arg\max\{\ f_i \circ \bar{v}(\kappa_{i+1})\ ,\ f_i \circ u_i(x)\ \} \\ &=: \arg\max\{\ \hat{v}(\kappa_{i+1})\ ,\ \hat{u}_i(x)\ \} \end{aligned}$$

As f_i is strictly increasing. This means that the agent searches the boxes in the same order and makes the same choice at each iteration, which is sufficient for $\hat{u}_i$ and $\hat{v}$ to represent $c(S)$ by the proof of Theorem 2. $\square$

Lemma 6: (u, v, K) represents $c(\cdot)$ for all menus S if and only if $(u, v, \{\kappa_1, \kappa_2, \kappa_3\})$ represents $c(\cdot)$ over all menus $S' \subseteq \kappa_1 \cup \kappa_2 \cup \kappa_3$ for all $\kappa_1, \kappa_2, \kappa_3 \in K$

Proof. Let $\bigcup_1^3 \kappa_i := \kappa_1 \cup \kappa_2 \cup \kappa_3$.

Only If: For any $\kappa_1, \kappa_2, \kappa_3$, choose any S'. Since (u, v, K) represent $c(\cdot)$ for all menus S, which includes S', (u, v, K) represent $c(\cdot)$ for S'. As $S' \subseteq \bigcup_1^3 \kappa_i$, we know that $K \setminus \bigcup_1^3 \kappa_i$ is irrelevant to the choice. Therefore as S' was arbitrary, $(u, v, \{\kappa_1, \kappa_2, \kappa_3\})$ represents $c(\cdot)$ over any S'.

If: Take any S and choose three categories $\kappa_1, \kappa_2, \kappa_3$ such that $\kappa_1, \kappa_2, \kappa_3$ have no non-empty boxes between them in the $\succ_k$ ordering and $c(S) \in \bigcup_1^3 \kappa_i$. Let S' be the restriction of S to $\bigcup_1^3 \kappa_i$. Consider all categories $\kappa_j \in K \setminus \bigcup_1^3 \kappa_i$ such that $\kappa_j \succ_k \kappa_1$. As $c(S) \in \bigcup_1^3 \kappa_i$, $c(S) \notin \kappa_j$, and so by repeated application of A3 we find that $c(S) = c(S \setminus \{\kappa_j \,|\, \kappa_j \succ_k \kappa_1\}) = c(S \setminus \{\kappa_j \cap S \,|\, \kappa_j \succ_k \kappa_1\})$. Moreover, by a similar application of A1, we find that $c(S) = c(S \setminus \{\kappa_j \,|\, \kappa_3 \succ_k \kappa_j\}) = c(S \setminus \{\kappa_j \cap S \,|\, \kappa_3 \succ_k \kappa_j\})$. As $\kappa_1, \kappa_2, \kappa_3$ have no non-empty boxes between them, we know that

$$\{\kappa_j \cap S \,|\, \kappa_j \succ_k \kappa_1\} \;\cup\; \{\kappa_j \cap S \,|\, \kappa_3 \succ_k \kappa_j\} \;=\; \{\kappa_j \cap S \,|\, j \neq 1, 2, 3\}$$

and hence by applying both of the previous results,

$$c(S) = c(S \setminus \{\kappa_j \cap S \,|\, j \neq 1, 2, 3\}) = c(S')$$

Therefore for any S there exist a triple of categories $\kappa_1, \kappa_2, \kappa_3$ such that $c(S)$ is represented by $(u, v, \{\kappa_1, \kappa_2, \kappa_3\})$. Therefore if $(u, v, \{\kappa_1, \kappa_2, \kappa_3\})$ represents $c(\cdot)$ for all $\kappa_1, \kappa_2, \kappa_3 \in K$ then (u, v, K) represents $c(\cdot)$. □

Remark 5: There are no attraction effects if $|K| \leq 2$

Proof. Suppose there is an attraction effect (and so a WARP violation) in the BOCS model: in other words, $\exists S, x$ such that $c(S \setminus x) \neq c(S) \neq x$. By the proof of Theorem 3, we know that $[c(S \setminus x)] \succ_k [x] \succ_k [c(S)]$. In particular, no two of $c(S)$, x and $c(S \setminus x)$ are in the same category as each other since $\succ_k$ is a strict total order over K, so $|K| \geq 3$. □

Remark 6: One decoy good has the same effect as many

Proof. Suppose we define a decoy good as an element $d \in \beta_2$, where if $S \neq S \cap \kappa_2$ then $c(S) \notin \beta_2$ and $c(S) \neq c(S \setminus \beta_2)$. In particular, for any S, $c(S) \notin \beta_2$, which implies that $c(S) \notin \beta_2 \setminus d$. Therefore by repeated application of A2, $c(S) = c(S \setminus (\beta_2 \setminus d))$, and hence any $\beta_2 \ni d$ will have the same attraction effect as it would do if the menu had $\beta_2 = \{d\}$ □

Remark 7: A welfare-reducing attraction effect is only supported if it is sufficiently costly to revisit boxes

Proof. Suppose the agent can pay $\delta > 0$ to revisit a box which they have already opened, and that in this case the v-utility of an already-opened box β updates to $u(\beta)$. As the addition of β_2 to S is welfare-reducing, $u(\beta_1) < u(\beta_3)$. Therefore, $c(S) \in \beta_1$ for any $\delta < \hat{\delta} := u(\beta_1) - u(\beta_3)$, since $u(\beta_1) - \delta > u(\beta_3)$. Moreover for $\delta \geq \hat{\delta}$, $c(S) \in \beta_3$. Therefore the attraction effect holds if and only if search costs are high enough. □

Lemma 7: Testing for reference dependence

Proof. Suppose that C1 is true. By definition, $c(S \setminus q) \succ q \succ c(S)$ and so $[c(S \setminus q)] \succ_k [q] \succ_k [c(S)]$ by lemma 2. This means that q cannot be in κ_1 since there is a category before $[q]$ in the $\succ_k$ ordering. Moreover suppose that C2 is satisfied. By a similar argument, $c(T \setminus x) \succ x \succ c(T) = q$, and so $[c(T \setminus x)] \succ_k [x] \succ_k [c(T)] = [q]$. Therefore q cannot be in κ_1, and so H_0 is falsified. $\square$

References

Caplin, A. and Dean, M. (2011), 'Search, choice, and revealed preference', *Theoretical Economics* .

Hoyer, W. D. (1984), 'An Examination of Consumer Decision Making for a Common Repeat Purchase Product', *Journal of Consumer Research* **11**(3), 822–829.

Huber, J., Payne, J. W. and Puto, C. (1982), 'Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis', *Journal of Consumer Research* **9**(1), 90–98.

Huber, J., Payne, J. W. and Puto, C. (2014), 'Let's be honest about the attraction effect', *Journal of Marketing Research* **51**, 520–525.

Janssen, M. and Parakhonyak, A. (2008), 'Consumer search with costly recall', *Tinbergen Institute Discussion Paper No. 08-002/1* .

Kahneman, D., Knetsch, J. L. and Thaler, R. H. (1990), 'Experimental Tests of the Endowment Effect and the Coase Theorem', *Journal of Political Economy* **98**(6), 1325–1348.

Kahneman, D., Knetsch, J. L. and Thaler, R. H. (1991), 'Anomalies: The endowment effect, loss aversion, and status quo bias', *The Journal of Economic Perspectives* **5**(1), 193–206.

Kreps, D. M. (1988), *Notes on the Theory of Choice*, Westview Press, Inc., Boulder, Colorado.

Masatlioglu, Y. and Nakajima, D. (2013), 'Choice by Iterative Search', *Theoretical Economics* **8**, 701–728.

Masatlioglu, Y., Nakajima, D. and Ozbay, E. Y. (2012), 'Revealed attention', *American Economic Review* **102**(5), 2183–2205.

Mullainathan, S. (2002), 'A memory-based model of bounded rationality', *The Quarterly Journal of Economics* **117**(3), 735–774.

Online Encyclopedia of Integer Sequences (2019), 'The bell numbers: A000110'.
URL: *https://oeis.org/A000110*

Ratneshwar, S., Shocker, A. D. and Stewart, D. W. (1987), 'Toward Understanding the Attraction Effect: The Implications of Product Stimulus Meaningfulness and Familiarity', *Journal of Consumer Research* **13**(4), 520–533.

Roberts, J. H. and Lattin, J. M. (1991), 'Development and testing of a model of consideration set composition', *Journal of Marketing Research* **28**(4), 429–440.

Samuelson, P. A. (1938), 'A note on the pure theory of consumers' behaviour', *Economica* **17**(5), 61–71.

Sen, A. K. (1971), 'Choice Functions and Revealed Preference', *The Review of Economic Studies* **38**(3), 307–317.

Simon, H. A. (1955), 'A Behavioral Model of Rational Choice', *The Quarterly Journal of Economics* **69**(1), 99–118.

Tversky, A. and Kahneman, D. (1974), 'Judgment under uncertainty: Heuristics and biases', *Science* **185**(4157), 1124–1131.

Wernerfelt, B. and Hauser, J. R. (1990), 'An evaluation cost model of consideration sets', *Journal of Consumer Research* **16**(4), 393–408.

Wright, P. and Barbour, F. (1977), 'Phased decision strategies: Sequels to an initial screening', *Stanford Graduate School of Business Working Paper 353* .