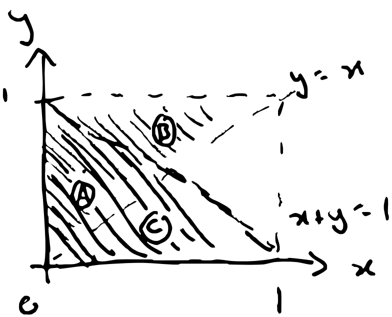
Problem Set Q1

i) $f(x,y) = \begin{cases} 2 & \text{for the region bounded by } x=0,\ y=1 \text{ and } y=x \\ 0 & \text{otw} \end{cases}$



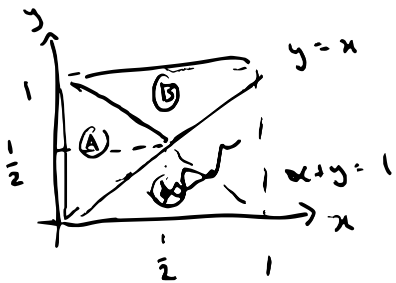
$P(X+Y<1)$

$\boxed{X+Y=1}$

$y = 1 - x$

- where $X+Y<1$: $x, y \in A \cup C$
- where $f(x,y) > 0$: $x, y \in A \cup B$

$$P(X+Y<1) = \int_{A} f(x,y)\,dx\,dy$$

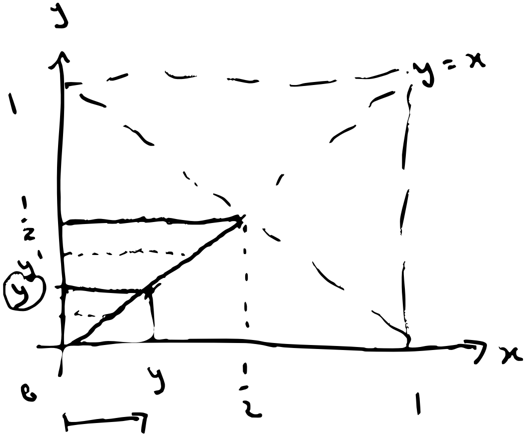


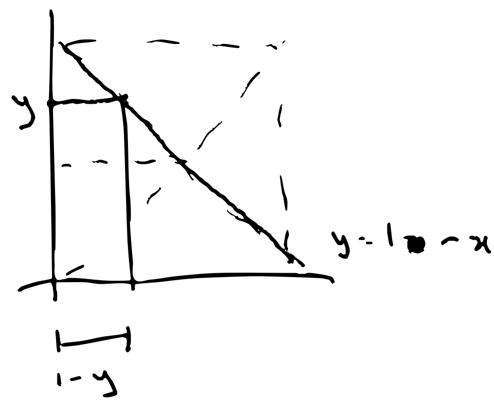
$$P(X+Y<1)$$

$$f(x,y) = \begin{cases} 2 & A \cup B \\ 0 & \text{otw} \end{cases}$$

$$= P(A)$$

$$= \int_0^{\frac{1}{2}} \int_0^{y} 2 \, dx \, dy + \int_{\frac{1}{2}}^{1} \int_0^{1-y} 2 \, dx \, dy$$





$$P(X+Y<1) = \int_0^{1/2}\int_0^{y} 2\,dx\,dy + \int_{1/2}^{1}\int_0^{1-y} 2\,dx\,dy$$

$$= \int_0^{1/2} \left[2x\right]_0^{y} dy + \int_{1/2}^{1} \left[2x\right]_0^{1-y} dy$$

$$= \int_0^{1/2} 2y\,dy + \int_{1/2}^{1} 2-2y\,dy$$

$$= \left[y^2\right]_0^{1/2} + \left[2y - y^2\right]_{1/2}^{1}$$

$$= \left(\frac{1}{4}\right) + \left(\cancel{2} - \cancel{1}\right) - \cancel{1} + \frac{1}{4}$$

$$= \frac{1}{2} \qquad \square$$

b) $\cancel{\text{f(z)}}$ $Z = X + Y$]

$$\underbrace{f(z) = \int f_{X,Y}(x, z-x)\, dx}$$ "CONVOLUTION FORMULA"

- WANT TO CHANGE

$$\underbrace{f(x) = g(u) \cdot |v^{-1}(u)|}$$

$$\Downarrow$$

- $f(\underline{x}) = g(\underline{u}) \cdot \det J \Rightarrow$

X, Y values in $\mathbb{N}$

$Z = X + Y$

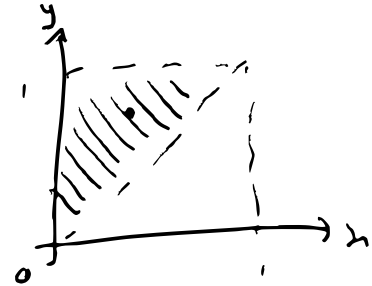
$$P(Z=5) = P(X=1, Y=4) + P(X=2, Y=3) + P(X=3, Y=2) + P(X=4, Y=1)$$

$$P(Z=z) = \sum_t P(X=t, Y=z-t)$$

$$= \int f_{X,Y}(t, z-t)\, dt$$

PTO $\to$

$$f(x,y) = \begin{cases} 2 & A \cup B \quad y \geq x \\ 0 & \text{otw.} \quad y < x \end{cases}$$



$Z = X + Y$ $\qquad x, y \in [0,1]$

$Z \in [0,2]$

$$f(z) = \int f_{x,y}(x, z-x)\, dx$$

$\boxed{Z \in [0,1]}$:

$z - x \geq x \Rightarrow x \leq \frac{1}{2} z$

$1 \geq x \geq 0$

$1 \geq z - x \geq 0$

$$\boxed{0 \leq x \leq \frac{1}{2} z}$$

$z \in [1,2]$:

$z - (z-1) = 1$

$$[z - 1 \leq x \leq \frac{1}{2} z]$$

If $z \in [0,1]$: $f(z) = $ "$\int f_{X,Y}(x, z-x)\,dx$"

$$= 0 + \int_0^{z/2} 2\,dx = \left[2x\right]_0^{z/2} = z$$

$z \in [1,2]$: $f(z) = \int_{z-1}^{z/2} 2\,dx + 0$

$$= \left[2x\right]_{z-1}^{z/2} = z - 2(z-1)$$

$\cancel{2\mathbb{I}(\cdot)}$

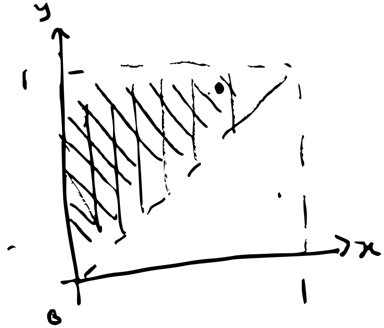
$$= z - 2z + 2$$

$$= 2 - z$$

$$\int_0^1 z\,dz + \int_1^2 2 - z\,dz = 1$$

$$\therefore\ f(z) = \begin{cases} z & z \in [0,1] \\ 2 - z & z \in [1,2] \end{cases}$$

$\square$



$\cancel{x=0}$

$\cancel{y=1}$

$\cancel{y=x}$

$$\boxed{\begin{array}{c} y \geq x \\ 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array}}$$

$\longrightarrow$

$z - x \geq x \Rightarrow x \leq \frac{1}{2} z$

$0 \leq x \leq 1$

$\left[\, 0 \leq z - x \leq 1 \,\right]$

1.5

$z \in [0,1]$:

$0 \leq x \leq \frac{1}{2} z$

$z \in [1,2]$:

$z - 1 \leq x \leq \frac{1}{2} z$

$$f(z) = \text{"} \int \underbrace{f_{x,y}(x, z-x)}\, dx \text{"}$$

2 if x, $\underset{y}{z - x}$

0 if not

PS 4

$X = Y^2$

$\cancel{f(X = x) = g(X^2 = y) \cdot J}$

$X_1, X_2 \sim N[0, \sigma^2)$

$$\boxed{\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_1 - X_2 \end{cases}}$$

pdf of (Y_1, Y_2)

$\Rightarrow$ prove Y_1, Y_2 are indep.

$$u\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$Y_1 + Y_2 = 2X_1$

$$\Rightarrow \begin{cases} X_1 = \frac{1}{2}(Y_1 + Y_2) = v_1(Y_1, Y_2) \\ X_2 = \frac{1}{2}(Y_1 - Y_2) = v_2(Y_1, Y_2) \end{cases}$$

$$g_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\tfrac{1}{2}(y_1+y_2),\ \tfrac{1}{2}(y_1-y_2)\right) \cdot J$$

$$\frac{1}{2}$$

$$J = \left| \det \begin{pmatrix} \partial v_1/\partial y_1 & \partial v_1/\partial y_2 \\ \partial v_2/\partial y_1 & \partial v_2/\partial y_2 \end{pmatrix} \right|$$

Jacobean Matrix

$$f: \mathbb{R}^n \to \mathbb{R} \qquad \nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$
$$\underline{x} \to f(\underline{x})$$

$$J: \underline{\mathbb{R}^n} \to \underline{\mathbb{R}^m}: \qquad \begin{bmatrix} [\nabla f_1]' \\ [\nabla f_2]' \\ \vdots \\ [\nabla f_m]' \end{bmatrix}$$

$$\underline{x} \to \begin{bmatrix} \boxed{f_1(\underline{x})} \\ \vdots \\ f_m(\underline{x}) \end{bmatrix}$$

$$J = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right|$$

$$= \left| -\frac{1}{4} - \frac{1}{4} \right|$$

$$= \frac{1}{2}$$

$$g_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\Big(\overbrace{\tfrac{1}{2}(y_1+y_2)}^{x_1}, \overbrace{\tfrac{1}{2}(y_1-y_2)}^{x_2}\Big)\cdot \frac{1}{2}$$

$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left[\left(\frac{y_1+y_2-2\mu_1}{2}\right)^2 + \left(\frac{y_1-y_2-2\mu_2}{2}\right)^2\right]\right)\cdot\frac{1}{2}$$

$$\frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left((x-\mu_1)^2+(x-\mu_2)^2\right)\right)$$

$$f_{X_1,X_2} \sim N\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\right]$$

$$=$$

$$g_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{4\sigma^2}\left[\left(y_1 - \frac{1}{2}(\mu_1+\mu_2)\right)^2 + \left(y_2 - \frac{1}{2}(\mu_1-\mu_2)\right)^2\right]\right)$$

$$= \frac{1}{4\pi\sigma^2} \underbrace{\exp\left(-\frac{1}{4\sigma^2}\left(y_1 - \frac{1}{2}(\mu_1+\mu_2)\right)^2\right)}_{} \cdot \underbrace{\exp\left(-\frac{1}{4\sigma^2}\left(y_2 - \frac{1}{2}(\mu_1-\mu_2)\right)^2\right)}_{}$$

$$\underbrace{y_1 \sim N\Big(}_{}\frac{1}{2}(\mu_1+\mu_2), (\sqrt{2}\sigma)^2\Big) \qquad \underbrace{y_2 \sim N}_{}\Big(\frac{1}{2}(\mu_1-\mu_2), \cdot\Big)$$

$$g(y_1, y_2) =$$

$$g(y) = f(v(y)) \cdot |v'(y)|$$

$$\boxed{x = v(y)}$$

$$y = v^{-1}(x)$$

$$G(y) = P(Y \leq y)$$

$$= P(v^{-1}(X) \leq y)$$

$$= P(X \leq v(y))$$

$$= F(v(y))$$

$$\boxed{g(y) = f(v(y)) \cdot v'(y)}$$

$\boxed{v \text{ inc.}}$

Inc. or Dec.

$\Downarrow$

Monotonic

$G(y) =$

$$P(v^{-1}(X) \leq y)$$

$$= P(X \geq v(y))$$

$$= 1 - F(v(y))$$

$$\boxed{g(y) = -f(v(y))v'(y)}$$

$$g(y) = f(v(y)) \cdot |v'(y)|$$

$$G(\underline{y}) = F(v(\underline{x}))$$

$$\int \cdots \int g(y)\, dy_1\, dy_2 \cdots = \int \cdots \int f(v(x))\, J\, dx_1\, dx_2 \cdots$$

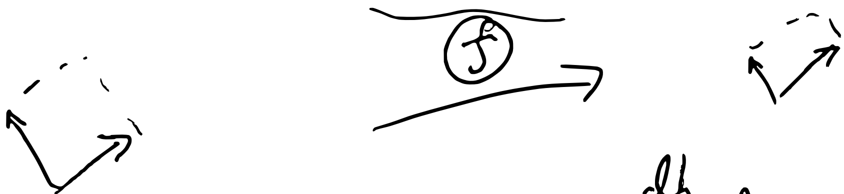
$$\int f(x)\, dx = \int f(h(u)) \cdot \boxed{\frac{dh}{du}}\, dx$$

$x = h(u)$

$$dy_1 = \frac{dy_1}{dx_1} dx_1 + \frac{dy_1}{dx_2} dx_2$$

$$dy_2 = \frac{dy_2}{dx_1} dx_1 + \frac{dy_2}{dx_2} dx_2$$

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} \\ \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$



$$\frac{dy}{dx_1} dx_1$$

$$\int_0^x f(y)\,dy = \int_0^x g(y)\,dy \quad \forall x \quad \Rightarrow \quad f(y) = g(y)$$

(a.s.)

QUIZ 4

$X \perp\!\!\!\perp Y \iff M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

a) Poisson

- "success or failure"
- constant P(success)
- count ~~#~~ successes

X, Y

$X+Y$ ✓

b) • counts how long until 1st success

X, Y

$X+Y$ X

d) $$\chi^2_r = \sum_{i=1}^{r} N(0,1)^2$$

$$\chi^2_p = \sum_{i=1}^{p} N(0,1)^2$$

$$\chi^2_r + \chi^2_p = \sum_{i=1}^{p+r} N(0,1)^2 \qquad \square$$

$$T = \frac{N(0,1)}{\chi^2_r}$$

$$F = \frac{\chi^2_r}{\chi^2_p}$$

1) $\boxed{f(x) = 2e^{-2x}}$ $x > 0$ $\boxed{\text{cdf } Y = 2X+4}$

V V^{-1}

$P(Y \le y) = P(2X+4 \le y)$

$= P\left(X \le \frac{y-4}{2}\right)$

$\boxed{= F\left(\frac{y-4}{2}\right)} = 1 - e^{-(y-4)}$

0

$\boxed{F(x) = \int_0^x 2e^{-2t}\,dt} = \left[-e^{-2t}\right]_0^x$

$= \left[-e^{-2x} + 1\right]$

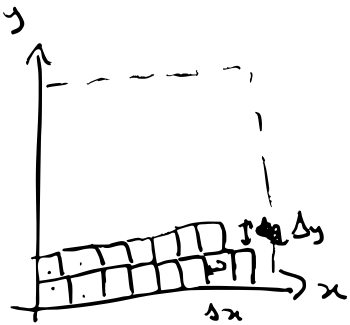
$= 1 - e^{-2x}$

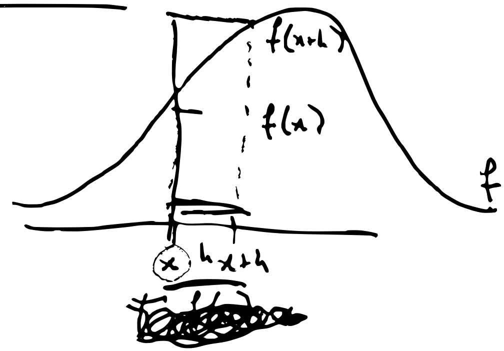
tonglinnan.github.io/EC400

$f(x) = 2e^{-2x}$ $x > 0$

$$F(x) = \int_0^x f(t)\,dt = 1 - e^{-2x}$$

$$\therefore\ F\left(\frac{y-4}{2}\right) = 1 - e^{-(y-4)}$$





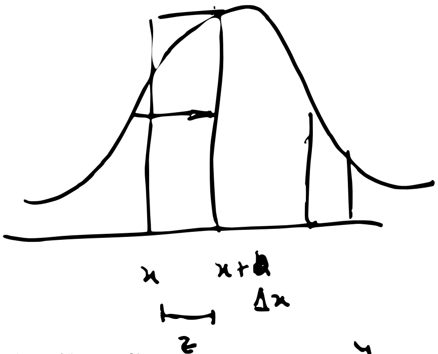
Riemann Integral

$x - f(x+h)$

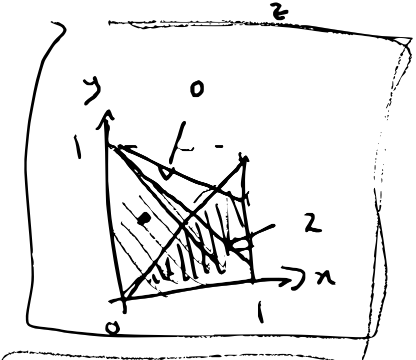
$\sum_{x} f(x+h)h$ $\sum_{x} f(x)h$

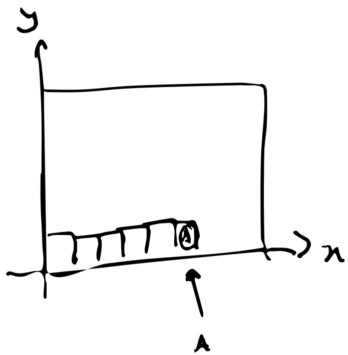
$$\sum_{x,y}\left[\sup_{x,y} f(x,y)\right]\Delta y\,\Delta x$$

$$\sum_{x}\left[\sup_{x} f(x)\right]\Delta x$$

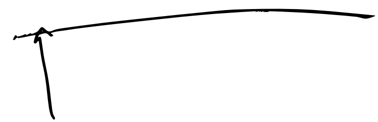


overestimate: $\displaystyle\sum_x \left[\sup_{z\in[x,\,x+\Delta x]} f(z)\right]\Delta x$





$\displaystyle\sum_{x,y}\left[\sup_{z\in A} f(z)\right]\Delta x\,\Delta y$



$\displaystyle\sum_y\sum_x = \sum_x\sum_y$

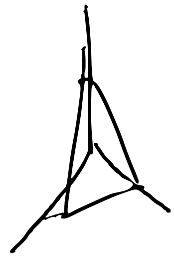
$\displaystyle P(X+Y<1) = \iint_{\boxed{x+y<1}} f(x,y)\,dx\,dy$

as $\Delta x,\ \Delta y \to 0$

$\displaystyle\int_X\int_Y f(x,y)\,dy\,dx$

$P(X+Y+Z<1)$

$X \leq Y$



fix z:

$f(x,y \mid z)$

$$\int_{A} f(x,y,z)\, dxyz = \int_{A} \boxed{f(x,y \mid z)}\, f(z)\, dxyz$$