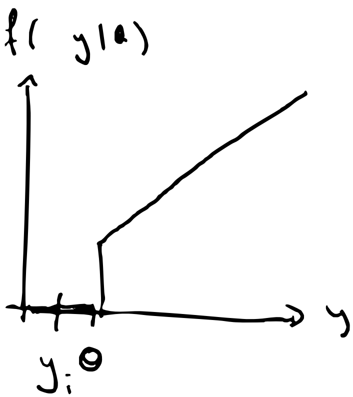
QUIZ 1

MLE $\quad f(y_i; \theta) = \dfrac{2y}{1-\theta^2} \qquad 0 \leq \theta \leq y \leq 1$

$y_i \in \{0.7,\ 0.83,\ 0.92,\ 0.86,\ 0.43,\ 0.21\}$



- $\boxed{\theta \leq y_i = 0.21}$
- $\ln L$

$= \sum_{i=1}^{n} \ln\left(\dfrac{2y_i}{1-\theta^2}\right)$

$= \sum_{i=1}^{n} \ln(2y_i) - \ln(1-\theta^2)$

$= n \cdot \ln 2 - n \ln(1-\theta^2) + \sum_{i=1}^{n} \ln y_i$

$\xrightarrow{\hat{\theta}_{MLE} = 0.21}$ $\boxed{\dfrac{\partial \ln L}{\partial \theta} > 0}$

$\underset{\theta}{\operatorname{argmax}}\ f(y|\theta)$

QUIZ 2

R , W

$$\boxed{p \in \{1/5,\ 1/3,\ 1/2,\ 3/4\}}$$

4: R W W W $\quad$ 1R 3W

MLE of p.

~~$E\, x_i \varepsilon_i = 0$~~

X = # Reds

$\Rightarrow X \sim B(4, p)$

~~Unconstrained MLE $\hat{p} = \frac{1}{4}$~~

$P(X = 1 \mid p)$ for $p \in \{1/5, 1/3, 1/2, 3/4\}$

$$P(X=1 \mid p = \tfrac{1}{5}) = \underbrace{\binom{4}{1}}_{4} \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^3 = 4 \cdot \frac{1^1 \times 4^3}{5^4}$$

$$P(X=1 \mid p=\tfrac{1}{3}) = \binom{4}{1}\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^3.$$

$$P(X=1 \mid p=\tfrac{1}{2}) = \binom{4}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^3.$$

$$P(X=1 \mid p=\tfrac{3}{4}) = \binom{4}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^3.$$

$$P(X=1 \mid p=\tfrac{1}{5})$$

~~Q~~

$$\hat{p}_{MLE} = \tfrac{1}{5}. \quad \square$$

QUIZ 3

$n = 6$

$\downarrow$

$\{Y_1 \cdots Y_6\}$

$f(y \mid \theta) = \frac{1}{3} \qquad 0 \le y \le 3$

$$\boxed{\hat{\theta} = Y_{max}}$$

$\cancel{\theta = 3.}$

$$P(\hat{\theta} \in [\theta - 0.3, \theta + 0.3])$$

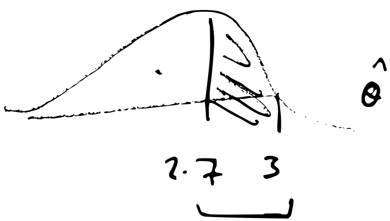
$$P(2.7 \le \hat{\theta} \le 3.3)$$

$$=$$

$$P(2.7 \le \hat{\theta} \le 3)$$

$$=$$

$$P(\hat{\theta} \le 3) - P(\hat{\theta} \le 2.7)$$



$$\boxed{1 - P(\hat{\theta} \le 2.7)}$$

$$\left(P(Y_{max} \le 2.7) = P(Y_1 < 2.7 \wedge Y_2 < 2.7 \ldots\ldots \right)$$

$\downarrow$ Independence

Random sample

$\iff$ iid

$$= \prod_{i=1}^{6} P(Y_i < 2.7)$$

$P(A \cap B)$
$= P(A)P(B)$

$\downarrow$ Identically distributed

$$= P(Y_1 < 2.7)^6$$

$$Y_1 \sim U[0,3]$$

$$P(Y_1 < 2.7) = \int_0^{2.7} \frac{1}{3}\, dy$$

$$= \left[\frac{1}{3} y \right]_0^{2.7}$$

$$= \frac{2.7}{3} = \frac{9}{10}$$

$$P(\hat{\theta} \in [2.7, 3.3])$$

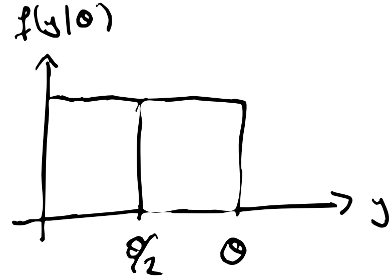
$$= 1 - \left(\frac{9}{10}\right)^6 \approx 0.47. \quad \square$$

QUIZ 4

$y: \{17, 92, 46, 39, 56\}$

$$f(y|\theta) = \frac{1}{\theta} \qquad 0 \le y \le \theta$$

$$\boxed{\underbrace{\mathbb{E}(y|\hat{\theta}_{MM})}_{\frac{\theta}{2}} = \underbrace{\frac{1}{n}\sum y_i}_{50}}$$



$$\mathbb{E}(y) = \int_0^\theta \frac{y}{\theta}\,dy$$

$$= \frac{1}{\theta}\left[\frac{1}{2}y^2\right]_0^\theta = \frac{1}{2}\frac{\theta^2}{\theta} - 0 = \frac{\theta}{2}$$

$$\frac{\hat{\theta}}{2} = 50 \qquad \hat{\theta} = 100$$

$\square$

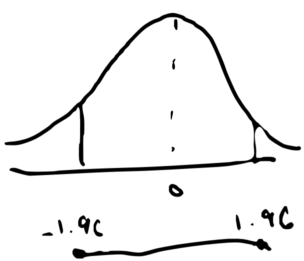
QUIZ 5

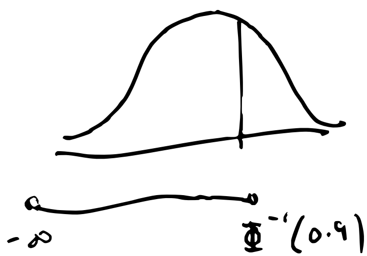
$X \sim N[\theta, \sigma^2]$ $\qquad \theta, \sigma^2 \qquad n = 9 \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 1$

$$S = \left[\frac{1}{n}\sum x_i^2 - \bar{x}^2\right]^{1/2} = 3$$

$$P(X \in C) = 90\%$$

'A CI' $\qquad$ 'The CI'





'The CI'

CI $\Leftrightarrow$ Test

"The CI" $\Leftrightarrow$ "Most powerful" test.

$P(X \in C) = 95\%$

$\bar{x} = 1 \qquad s = 3 \qquad n = 9$

$T = \frac{1}{3/\sqrt{9}} = 1$

$H_0: \theta = 0$

$H_1: \theta \neq 0$

$\frac{\bar{x}}{s/\sqrt{n}}$

$[\bar{x} - sz, \bar{x} + sz]$

$P(\bar{X} \in (-\infty, 2.86]) = 0.95$

$\Leftrightarrow$

$P(\bar{X} > 2.86) = 0.05$

" $P(\bar{x} > 2.86) = 0.05$ "

$$P\left(\underbrace{\frac{\bar{x} - \bar{\bar{x}}}{s/\sqrt{n}}}_{T} > \underbrace{\frac{2.86 - \bar{\bar{x}}}{s/\sqrt{n}}}_{} \right)$$

1

1

$$X \sim N(5, 7)$$

$$\downarrow$$

$$\frac{X - 5}{\sqrt{7}} \sim N(0,1)$$

$$P(\bar{x} > 2.86) = P\left(T > \frac{2.86 - 1}{1}\right)$$

$$= P(T > 1.86)$$

$$\doteq 5\%$$

$\square$

$$P(-5.91 \le \bar{X} \le 7.91)$$

$$= P\left(\frac{-5.91-1}{1} \le T \le \frac{7.91-1}{1}\right)$$

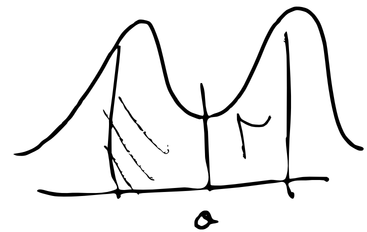
$$= P(-6.91 \le T \le 6.91)$$

$$= P(T \le 6.91) - P(T \le -6.91)$$

$$= P(T \le 6.91) - [1 - P(T \le 6.91)]$$

$$= 2 \cdot \underline{P(T \le 6.91)} - 1 \ne \underline{5\%}$$

$$\theta : [\hat{\theta} - \sim , \hat{\theta} + \sim]$$



QUIZ 6

$Y_1 \dots Y_n$

$$f(y|\theta) = \frac{1}{\theta} e^{-y/\theta} \quad \theta > 0 \quad y \geq 0$$

exp$\left(\frac{1}{\theta}\right)$

$\hat{\theta}_1 = Y_1$

$\hat{\theta}_2 = n \cdot \min Y_i$

UnBiased:

$$\mathbb{E}\,\hat{\theta} = \theta$$

THINK OF AN ARCHERY TARGET

UNBIASED:

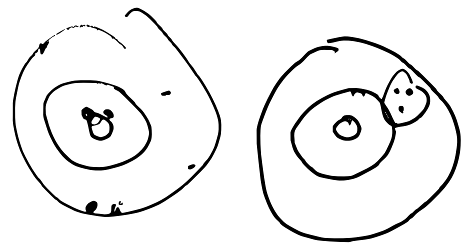
ARROW HIT MIDDLE ON AVERAGE

Efficient

EFFICIENT:

ARROWS CLOSELY PACKED TOGETHER

$$MSE = Bias^2 + Var$$



U　　E

$$f(y|\theta) = \frac{1}{\theta} e^{-y/\theta}$$

$$F(y|\theta) = 1 - e^{-y/\theta}$$

$$f_{Y_{min}}(y) = n \cdot f(y) \left[1 - F(y)\right]^{n-1}$$

$$f_{Y_{min}}(y) = n \cdot \left[\frac{1}{\theta} e^{-y/\theta}\right] \cdot \left[e^{-y/\theta}\right]^{n-1}$$

$$= \frac{n}{\theta} e^{-y/\theta} \cdot e^{(-y/\theta)(n-1)}$$

$$= \frac{n}{\theta} e^{-y/\theta \cdot n}$$

$$= \left(\frac{n}{\theta}\right) e^{-\left(\frac{n}{\theta}\right) y}$$

$$e^{A+B} = e^{A} e^{B}$$

$$\boxed{\begin{aligned} \hat{\theta}_1 = Y_{(1)} &\sim \exp\left(\frac{1}{\theta}\right) \\ \hat{\theta}_2 = n \cdot Y_{min} &\sim \exp\left(\frac{n}{\theta}\right) \cdot n \end{aligned}}$$

$$\boxed{\exp(\lambda) = \lambda e^{-\lambda y}}$$

$\hat{\theta}_1 \sim \exp\left(\frac{1}{\theta}\right)$ $\qquad$ $\hat{\theta}_2 \sim n \cdot \exp\left(\frac{n}{\theta}\right)$

$\mathbb{E}(\exp(\lambda)) = \frac{1}{\lambda}$ $\qquad$ $f(y|\lambda) = \lambda e^{-\lambda y}$

$y > 0$

$$\left.\begin{aligned} \mathbb{E}\,\hat{\theta}_1 &= \theta \\ \mathbb{E}\,\hat{\theta}_2 &= \frac{\theta}{n} \cdot n = \theta \end{aligned}\right\} \text{Both unbiased}$$

$\mathrm{Var}(\exp(\lambda)) = \frac{1}{\lambda}$ $\qquad$ $\mathrm{Var}(\hat{\theta}_1) = \theta = \mathrm{Var}(\hat{\theta}_2)$

equally efficient.

1.

QUIZ 7

- $\boxed{X_4 \sim N(\mu, \sigma^2)}$

$n = 3$ $\quad \bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$ $\quad X_4$

- $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{3}\right)$

$\frac{1}{3}(\underbrace{X_1 + X_2 + X_3}_{N(3\mu, 3\sigma^2)})$

$P(\bar{X} - 2\sigma \le X_4 \le \bar{X} + 2\sigma)$

$\boxed{E(X_4 - \bar{X}) = 0}$

$X_4 \perp \bar{X} \Rightarrow \text{cov}(\cdot,\cdot) = 0$

$$\text{Var}(X_4 - \bar{X}) = \text{Var}(X_4) + \text{Var}(\bar{X}) + \cancel{\text{cov}(X_4, \bar{X})}$$

$$= \sigma^2 + \frac{\sigma^2}{3} = \boxed{\frac{4\sigma^2}{3}}$$

$$P(\bar{x} - 2\sigma \le X_4 \le \bar{x} + 2\sigma)$$

$$= P(-2\sigma \le X_4 - \bar{x} \le 2\sigma)$$

$$= P\left(-\frac{2\sigma}{(2\sigma/\sqrt{3})} \le \frac{X_4 - \bar{x}}{2\sigma/\sqrt{3}} \le \frac{2\sigma}{(2\sigma/\sqrt{3})} \right)$$

$$Var(X_4 - \bar{x}) = \frac{4\sigma^2}{3}$$

$$\sqrt{Var(\ldots)} = \frac{2\sigma}{\sqrt{3}}$$

$$= P\left(-\sqrt{3} \le \frac{X_4 - \bar{x}}{2\sigma/\sqrt{3}} \le \sqrt{3} \right)$$

$$X_4 - \bar{x} \sim N\left(0, \frac{4\sigma^2}{3}\right)$$

$$\frac{X_4 - \bar{x}}{2\sigma/\sqrt{3}} \sim N(0,1)$$

$$= P(-\sqrt{3} \le Z \le \sqrt{3})$$

$$= P(Z \le \sqrt{3}) - P(Z \le -\sqrt{3})$$

$$= 2P(Z \le \sqrt{3}) - 1 \quad \square$$

$$f(x \mid \theta) = \frac{x^{k-1} e^{-x/\theta}}{\Gamma(k)\,\theta^{k}}$$

(k, θ)

$k > 0$, $\theta > 0$

$$\Gamma(k) = \underbrace{\int_0^\infty t^{k-1} e^{-t}\, dt}_{\mathbb{E} X^{k-1} \quad X \sim \exp(1)}$$

$$\hat{\theta}, \hat{k} = \underset{\theta, k}{\operatorname{argmax}}\ L(x \mid \theta, k)$$

$$L = \prod_{i=1}^{n} \frac{x_i^{k-1} e^{-x_i/\theta}}{\Gamma(k)\,\theta^{k}}$$

Independence

$P(A \cap B) = P(A)P(B)$

$\{X_i\}$ iid

Optimization

$$x_0 = \underset{x}{\operatorname{argmax}}\ f(x)$$

$\iff$

$f(x_0) \geq f(x)$ $\forall x$

$\forall x$

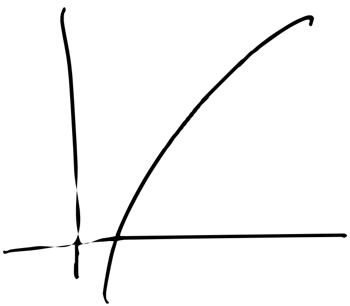
$g(f(x_0)) \geq g(f(x)) \iff$

$\iff$

$$x_0 = \underset{x}{\operatorname{argmax}}\ g(f(x))$$

$\boxed{g \text{ increasing}}$

$$\boxed{y = \ln x}$$



$$\frac{\partial}{\partial x} \ln x = \frac{1}{x} > 0$$

$$e^{A+B} = e^{A} \cdot e^{B}$$

$$L = \prod_{i=1}^{n} \frac{x_i^{k-1} \; e^{-x_i/\theta}}{(\Gamma(k)\,\theta^{k})}$$

$$\ln\left(\frac{1}{(\Gamma(k)\theta^{k})^{n}}\right) = -n\left[\ln(\Gamma(k)\theta^{k})\right] = -n\ln[\Gamma(k)] - nk\ln(\theta)$$

$$= \frac{1}{(\Gamma(k)\theta^{k})^{n}}\left[x_1^{k-1} e^{-x_1/\theta} \cdot x_2^{k-1} e^{-x_2/\theta} \cdots\right]$$

$$= \frac{1}{(\Gamma(k)\theta^{k})^{n}} \boxed{\prod_{i=1}^{n} x_i^{k-1}} \cdot \boxed{e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}}$$

$$\rightarrow \boxed{\log L(k,\theta) = -nk\ln(\theta) - n\cdot\ln(\Gamma(k)) + (k-1)\sum_{i}\ln(x_i) - \frac{1}{\theta}\sum x_i}$$

$\bar{X}$

- $\boxed{\frac{1}{n}\sum x_i = \mathbb{E}(X \mid \hat{\theta}, \hat{k})}$

$X \sim \text{Gamma}(k, \theta)$

$\mathbb{E}X = \theta k$

$\text{Var}(X) = \theta^2 k$

$\cancel{\frac{1}{n}\sum x_i^2 = \mathbb{E}(X^2 \mid \hat{\theta}, \hat{k})}$

- $s^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 = \text{Var}(X \mid \hat{\theta}, \hat{k})$

$$\boxed{\begin{cases} s^2 = \hat{\theta}^2 \hat{k} \\ \bar{X} = \hat{\theta}\hat{k} \end{cases}}$$

$$\bar{X} = \hat{\theta}\hat{k}$$

$$S^2 = \hat{\theta}^2 \hat{k}$$

$$\frac{S^2}{\bar{X}} = \frac{\hat{\theta}^2 \hat{k}}{\hat{\theta}\hat{k}} = \hat{\theta} \quad \square$$

$$\bar{X} = \frac{S^2}{\bar{X}} \hat{k}$$

$$\therefore \quad \frac{\bar{X}^2}{S^2} = \hat{k} \quad \square$$

Better est.

Most assumptions

Least

MLE

↓

OLS

↓

kernel reg.

$$\mathbb{E} X = \theta k$$

$$Var(X) = \theta^2 k$$

$$y_i = x_i' \beta + \varepsilon_i$$

$$\mathbb{E}\, x_i \varepsilon_i = 0$$

$$y_i = f(x_i) + \varepsilon_i$$

f continuous.